Chapter 5

Dynamics and Bifurcations of Single Year Class Maps.

5.1 Introduction.

In this chapter we will concentrate on the single year class dynamics, i.e., at a given year we have individuals of the same age. We write a model for a semelparous population consisting of one year class in the following form:

\[
\begin{align*}
N_0(t+k) &= N_{k-1}(t+k-1) \cdot h_{k-1}(N_{k-1}(t+k-1)) \\
N_{k-1}(t+k-1) &= N_{k-2}(t+k-2) \cdot h_{k-2}(N_{k-2}(t+k-2)) \\
& \vdots \\
N_i(t+1) &= N_{i-1}(t) \cdot h_i(N_{i-1}(t)).
\end{align*}
\]

The number of newborns \( N_0(t+k) \) in a year \( t+k \) is proportional to the number of \((k-1)\)-years old individuals in the previous year and the proportionality factor \( h_{k-1} \) is the expected number of offspring of a \((k-1)\)-years old individuals after one year. Similarly, \( N_i(t+i), i = 1, \ldots, k-1 \) is the number of individuals of age \( i \) at a year \( t+i \) and it is proportional to the number \( N_{i-1} \) of the previous age class in the previous year with the factor \( h_{i-1}(N_{i-1}) \) which is a survival probability. It is a decreasing function of the age class number \( N_{i-1} \) that reflects the fact that the larger the population is, the stronger is the competition. We make the same assumption about the function \( h_{k-1}(N_{k-1}) \) arguing that the expected number of offspring is proportional to the survival probability in the last year of life. (See Chapter 1 for a more detailed description of the modelling procedure).

This model can be rewritten in the form of a one-dimensional composite map

\[
N_0(t+k) = f_{k-1} \circ \cdots \circ f_0(N_0(t)),
\]

where \( f_i(x) = xh_i(x) \), or

\[
x \mapsto f_{k-1} \circ \cdots \circ f_0(x).
\]

This chapter is a version of [Davydova]
This composition-map or, a SYC- (for Single Year Class), map is the object of our study. A formal definition is given in Section 5.2 as is a (partial) classification of SYC-maps.

In Section 5.2 we introduce a parametrization of SYC-maps. Let a SYC-map be a composition of unimodal functions of the same form. Each of these functions is characterized by one parameter, the relative (w.r.t. the $x$-scale) height of its peak. Then the resulting composite map has $k$ parameters. In terms of these parameters we can discuss bifurcations. If $k = 1$, we observe well-known cascades of period-doublings. We do not consider this case and refer to the excellent review of [Thunberg]. In Section 5.3 we recover the cyclic symmetry of Lemma 2.3 applied to SYC-maps.

In Section 5.4 we look for fixed points of SYC-maps. We find that for small values of the parameters, a SYC-map has a unique fixed point, while the maximal number of nontrivial fixed points depends on the order of the map and equals $2^k - 1$. Also in this section we consider a SYC-map which is composed of increasing functions, and show that it has a unique globally stable nontrivial fixed point. This simple result leads however to an important conclusion in modelling context. In order to avoid complex behaviour in the model, one should choose monotone nonlinearities whenever possible (e.g. Beverton-Holt like, see (1.6)).

Of our particular interest is the case $k = 2$. Analysis of local and global bifurcations of 2-SYC-maps constitutes the body of this chapter (Sections 5.5-5.8). In Section 5.5 we show that a large class of the 2-SYC-maps possesses a cusp bifurcation of spring type. We believe that this bifurcation is an organizing center of the bifurcation diagram of a 2-SYC-map.

Section 5.6 is devoted to the detailed analysis of local bifurcations of fixed points in 2-Ricker-maps. The results are summarized in the bifurcation diagram of Figure 5.2.

In Section 5.7 we consider global aspects of the dynamics of the 2-SYC-maps such as the maximal number of attractors, global stability of a unique nontrivial fixed point and homoclinic bifurcations. Application of these results to the 2-Ricker-maps yields a much better understanding of the dynamics and helps to produce a more complete bifurcation diagram (Figure 5.7). A detailed description of this diagram, combining analytical results and numerical insights, is presented in Section 5.8. Moreover we discuss four numerical bifurcation diagrams of a 2-Ricker-map in this section: Figure 5.8 and the three figures on the back cover of this book produced with the aid of the packages CONTENT [Kuznetsov & Levitin] and DYNAMICS [Nusse & Yorke: Dynamics], on which one observes such interesting phenomena as windows of successive periods and cascades of cusp bifurcations. The challenge is to describe the sequence of these cusps (see also [Branner]).
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The bifurcation diagram has an intrinsic symmetry which, perhaps, gives a clue to the solution of this problem.

We conclude with a couple of remarks concerning general \( k \)-SYC-maps. The symmetry gives some intuition concerning the dynamics for higher values of \( k \) (Section 5.9). But a detailed description of bifurcations is quite problematic just because the dimension of a parameter space is more than 2.

Most sections begin with a short description of the results followed by subsections containing precise statements.

5.2 What is a single year class map?

The aim of this section is to introduce some notation and to give a formal definition of a single year class map. We also give alternative formulations and parametrizations of the map.

**Definition 5.1.** A single year class map of order \( k \), \( k \geq 2 \), also called a \( k \)-SYC-map, is a one-dimensional map:

\[
x \mapsto f_{k-1} \circ \ldots \circ f_0(x),
\]

\( (5.1) \)

defined on \([0, \infty)\), where for \( i = 0, 1, \ldots, k - 1 \)

\[
f_i(x) = x h_i(x),
\]

\( (5.2) \)

with \( h_i \) defined on \([0, \infty)\), positive and bounded. We call \( x \mapsto f_i(x) \) a building block of \((5.1)\).

**Remark.** As both the functions \( h \) and \( f \), with \( f(x) = x h(x) \), are useful in various formulations below, we shall switch freely between them. So, whenever we use the symbols \( h \) and \( f \) (with or without a specific index), they are related by \((5.2)\).

Note that from the interpretation in the introduction it follows that the functions \( h_i \) for \( i = 0, \ldots, k - 2 \) are bounded by 1 from above because they are survival probabilities. We do not put this restriction into the Definition 5.1, to keep it more general and, in particular, to allow for various forms of rescaling.

Sometimes it is convenient to write the map \((5.1)\) as

\[
x \mapsto C(f)(x),
\]

\( (5.3) \)

using a composition operator

\[
C(f) = f_{k-1} \circ \ldots \circ f_0,
\]

with

\[
f = (f_0, \ldots, f_{k-1}).
\]
5.2.1 Classes of SYC-maps

In this subsection we introduce several classes of building block functions $f$ defining a SYC-map. The way we do it is somewhat formal and if the reader is not very interested in the technical details, (s)he can skip this. All the notation from this subsection which we use later on, is summarized in the Definitions 5.2 and 5.3.

First of all, we always require continuity for the functions $f$. The class $H_1$ consists of functions $f$ such that the corresponding $h$ is a function decreasing to zero. The interpretation of it is that the survival probability decreases due to competition and there is no Allee effect (roughly speaking, this effect is that at very low densities survival and reproduction increase with population density, for instance because it becomes easier to find mates).

The class $H_2$ is a subset of $H_1$ and consists of functions $f$, which first increase to a maximum and then decrease to zero. So, these functions are unimodal but the domain is unbounded. The interpretation of the class $H_2$ is that there is a population number for which the number of individuals in the next age-class is maximal. Clearly, if $f$ belongs to $H_1$ it need not belong to $H_2$. In principle, it is possible to construct a function $h$ such that $x \mapsto f(x)$ has several or no extrema. For example, if a function $h$ is of the form $\frac{1}{1+x^2}$ (Beverton-Holt nonlinearity), it is decreasing, but the corresponding $f$ is strictly increasing. We define the class $H_3$ so that $f$ belongs to $H_3$ if and only if $f$ is increasing but $h$ is decreasing; we deal with such maps in Section 5.4.2.

Note that the functions $f$ of class $H_2$, as well as their compositions, are bounded. Hence, despite the fact that the domain of $f$ is $[0, \infty)$, after one iteration step we are confined to a bounded interval.

Next we introduce an important class $H_S$, characterized by smoothness and negative Schwarzian derivative (see Appendix A). We define

$$H_S = \{f : f \in H_2 \cap C^3, Sf < 0 \text{ for all } x; \text{such that } f'(x) \neq 0\}.$$  

(The symbol $S$ denotes the Schwarzian derivative in this chapter unlike the other chapters where it denotes the cyclic shift, see also a remark in Section 5.3.)

Note that the Ricker map

$$x \mapsto \lambda xe^{-x}$$

as well as the Hassell map

$$x \mapsto \frac{\lambda x}{(1 + x)^\beta},$$
(which are both widely used in modelling), have negative Schwarzian [Thunberg].

To conclude the description of the various classes we give formal definitions.

**Definition 5.2.** A continuous function $f$ defined on $[0, \infty)$ such that $f(0) = 0$ and $f(x) > 0$ for $x > 0$, is of class

- $H_1$ if the function $h$, such that $f(x) = xh(x)$, is strictly decreasing and $\lim_{x \to \infty} h(x) = 0$;
- $H_2$ if it is $H_1$ and there exists a point $c \in (0, \infty)$ such that it is strictly increasing on $(0, c)$ and strictly decreasing on $(c, \infty)$;
- $H_3$ if it is $H_1$ and strictly increasing on $[0, \infty)$;
- $H_S$ if it is $H_2$, $C^3$ and $Sf(x) < 0$ for all $x \neq c$.

Frequently one considers a SYC-map with all the functions $f_i$ from the same class. So the following definition is reasonable.

**Definition 5.3.** We say that a SYC-map (5.1) is $H_j$, $j \in \{1, 2, 3, S\}$, if all the functions $f_i$, $i = 0, \ldots, k - 1$, defining the map are of class $H_j$.

**Remark.** A result that we will often use is that a function which is a composition of functions with negative Schwarzian has itself negative Schwarzian. However a composition of $H_j$ functions with $j \in \{1, 2, 3, S\}$, is not necessarily an $H_j$ function!

### 5.2.2 Parametrization of a SYC-map.

We shall parametrize a SYC-map in order to describe bifurcations. The idea is very simple. In examples like the well-known quadratic family $x \mapsto \lambda x(1 - x)$ and the Ricker family $x \mapsto \lambda x e^{-x}$, the bifurcation parameter $\lambda$ is a multiplicative factor. A SYC-map is a composition of such functions. For each building block we choose a multiplicative factor as a parameter (if the functions are unimodal, this factor gives the relative height of their peaks with respect to the $x$-scale). Thus a SYC-map of order $k$ will have $k$ parameters.

Let us write a function $h_i$ in the following form

$$h_i(x) = s_i \hat{h}_i(x)$$

$$s_i = h_i(0).$$
Note that since $h_i$ is bounded (see Definition 5.1), $h_i(0) < \infty$. If $h_i$ is also decreasing, then the function $\hat{h}_i$ reaches its maximum 1 at zero.

The biological meaning of $s_i$ for $i = 0, \ldots, k - 2$ is a survival probability from age $i$ to age $i + 1$ if there is no density dependence, for example, there is no competition in the population. The value $s_{k-1}$ is the expected number of offspring of a $(k - 1)$-year old individual in the next year, again in the case with no density dependence.

This form of the functions $h_i$ suggests the following definition. We call a positive function $h$ normalized if $h(0) = 1$.

Using the normalized functions $\hat{h}_i(x)$ and omitting hats we can rewrite the SYC-map (5.1) as follows

$$x \mapsto R_0 x \Pi(x), \quad (5.4)$$

with

$$R_0 = s_0 \ldots s_{k-1}$$

$$\Pi(x) = h_0(x) h_1 \left( f_0(x) \right) h_2 \left( f_1 \circ f_0(x) \right) \ldots h_{k-1} \left( f_{k-2} \circ \ldots \circ f_0(x) \right),$$

where $f_i(x) = s_i x h_i(x), i = 0, \ldots, k - 1$.

Note that the function $\Pi$ is a product of normalized functions $h_i$. If all these functions are decreasing, then the product $\Pi$ (as well as all $h_i$), has maximum 1 at zero.

The parameter

$$R_0 = h_0(0) \ldots h_{k-1}(0),$$

which is known in population dynamics as the basic reproduction ratio, is the expected number of offspring per newborn individual if this individual experiences no competition.

Of special importance is the case when the functions $h_i$ are of the following two-parameter class

$$h_i(x) = s_i h(g_i x) \quad (5.5)$$

where $h$ is a normalized function which has the additional property that $|h'(0)| = 1$ (note that this second normalization amounts to a proper choice of the parameters $g_i$, see also (2.9)). We call a positive function $h$ defined on $[0, \infty)$ doubly normalized if it is normalized and $|h'(0)| = 1$. For example, the functions $e^{-x}$ and $\frac{1}{1+e^{x}}$ are doubly normalized.

The interpretation of the parameters $g_i$ is sensitivity of individuals of age $i$ to competition (or, in general, to density dependence). The larger the value of $g_i$ is, the less individuals survive the competition.
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We make the following rescaling and reparametrization

\[ g_0 x \mapsto x \]

\[ p_i = s_i \frac{s_{i+1}}{s_i}, \quad i = 0, \ldots, k - 1, \]

where the indices are taken modulo \( k \). Then we can rewrite the map (5.1) as

\[ x \mapsto (p_{k-1} f) \circ (p_{k-2} f) \circ \ldots \circ (p_0 f)(x), \]

i.e. the functions \( f_i \) are given by

\[ f_i(x) = p_i f(x). \]

We call (5.7) the parametric SYC-map. It is \( H_j \) if the function \( f \) is of class \( H_j \). The factor \( R_0 \) is in this case the product of the \( p \)'s:

\[ R_0 = p_0 \ldots p_{k-1}. \]

In principle, one can choose either \( \{p_0, \ldots, p_{k-1}\} \) or \( \{R_0, p_0, \ldots, p_{k-2}\} \) as parameters of a SYC-map. There is a slight difference. If you choose the first set, then a symmetry property of the map is easy to formulate (see the next section). The alternative set helps to find explicit expressions for bifurcation curves (Section 5.6). Moreover it allows to decrease the order of a SYC-map by a restriction to one of the coordinate hyper-planes \( p_i = 0 \) of the parameter space. This is impossible in the first case because the map reduces to \( x \mapsto 0 \) on the coordinate planes. So, throughout the chapter, we assume

\[ (p_0, \ldots, p_{k-1}) \in \{(p_0, \ldots, p_{k-1}) : p_i > 0, i = 0, \ldots, k - 1\} \]

\[ (R_0, p_0, \ldots, p_{k-2}) \in \{(R_0, p_0, \ldots, p_{k-2}) : R_0 > 0, p_i \geq 0, i = 0, \ldots, k - 2\} \]

In the sequel we consider often the case \( k = 2 \), i.e., we deal with a family of parametric 2-SYC-maps which we write here as

\[ x \mapsto p_1 f \circ p_0 f. \]

We choose often the function \( h \) in the form \( h(x) = e^{-x} \). The building blocks are then the well-known Ricker maps:

\[ x \mapsto p_i xe^{-x} \]

and the 2-SYC-map (which we call 2-Ricker-map) is

\[ x \mapsto R_0 xe^{-\nu(x,p_0)}, \]

\[ \nu(x, p_0) = x(1 + p_0 e^{-x}). \]
5.3 Symmetry

An ingredient we need to formulate the symmetry property is a cyclic shift of indices. We define it as follows

\[
S \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{k-1} \end{pmatrix} = \begin{pmatrix} f_{k-1} \\ f_0 \\ \vdots \\ f_{k-2} \end{pmatrix}
\]

(In other chapter of this thesis we denote the cyclic shift by the regular symbol \( S \), but in this section we use \( S \) in order to distinguish from the Schwarzian derivative, see p. 122.)

**Proposition 5.4.** The dynamics generated by a SYC-map (5.1) does not change under the cyclic shift of indices.

**Proof.** We will use the form (5.3) of the map.

Write

\[
\begin{align*}
\psi_1 &= f_0 \\
\psi_2 &= f_{k-2} \circ \cdots \circ f_0,
\end{align*}
\]

then

\[
\begin{align*}
\psi_1 \circ C(f) &= f_0 \circ f_{k-1} \circ \cdots \circ f_0 = C(S^{-1}f) \circ \psi_1 \\
\psi_2 \circ C(f) &= f_{k-2} \circ \cdots \circ f_0 \circ f_{k-1} \circ f_{k-2} \cdots \circ f_0 = C(Sf) \circ \psi_2
\end{align*}
\]

Let, for given \( x_0 \), the sequence \( \{ x_n \} \) be defined recursively by

\[
x_{n+1} = C(f)(x_n).
\]

Similarly, for given \( y_0 \), let \( \{ y_n \} \) be defined by

\[
y_{n+1} = C(Sf)(y_n).
\]

If \( y_0 = \psi_2(x_0) \) then, by induction, \( y_n = \psi_2(x_n) \). And, similarly, if \( x_0 = \psi_1(y_0) \) then \( x_n = \psi_1(y_n) \). So \( \psi_1 \) and \( \psi_2 \) map orbits to orbits. \( \square \)

Note that neither \( \psi_1 \) nor \( \psi_2 \) is necessarily an homeomorphism (the functions need not be monotone) so the equivalence of the map (5.1) and its shifted versions is not the standard one from the theory of dynamical systems.

**Corollary 5.5.** The dynamics generated by a parametric SYC-map (5.7) does not change under the cyclic shift of indices

\[
(p_0, p_1, \ldots, p_{k-1}) \mapsto (p_{k-1}, p_0, \ldots, p_{k-2}) \quad (5.10)
\]
Corollary 5.6. A bifurcation diagram of a parametric SYC-map (5.7) in the parameter space \((p_0, p_1, \ldots, p_{k-1})\) is symmetric, i.e., invariant under the cyclic shift (5.10). In particular, a bifurcation diagram of a parametric 2-SYC-map is invariant under the reflection \((p_0, p_1) \mapsto (p_1, p_0)\).

If we choose \(\{p_0, \ldots, p_{k-2}, R_0\}\) as a parameter set, we can formulate the symmetry property as well. We will do it for the case \(k = 2\), but it is easy to generalize.

Corollary 5.7. A bifurcation diagram of a parametric 2-SYC-map in the parameter space \((R_0, p_0)\) is invariant under the reflection \((R_0, p_0) \mapsto (R_0, \frac{R_0}{p_0})\).

5.4 Fixed points.

A SYC-map has always a trivial fixed point. In this section we are looking for nontrivial fixed points. The question of stability we leave for later sections.

Using the alternative form (5.4) of a SYC-map, we can say that all nontrivial fixed points have to satisfy the equation

\[
\Pi(x) = \frac{1}{R_0},
\]

(5.11)

In various propositions below we estimate maximal and minimal number of nontrivial fixed points. In particular, we show that if the factor \(R_0\) is less than one, an \(H_1\) SYC-map has no nontrivial fixed points. The biological interpretation of this fact is very simple: if the expected number of offspring per individual is less than one even under ideal conditions, a population goes extinct.

For \(R_0 = 1\) we have a transcritical bifurcation. For \(R_0\) slightly bigger than one, a SYC-map has one nontrivial fixed point.

\(H_3\) SYC-maps have a unique nontrivial fixed point for all \(R_0 > 1\) while for other SYC-maps the number of fixed points can vary. They appear and disappear in fold bifurcations. The maximal number of fixed points depends on the order of a SYC-map and for \(H_S \ k\)-SYC-maps it is \(2^k - 1\).

However in the following subsection we will show that, firstly, for \(R_0 - 1 > 0\) small enough and, secondly, for all but one \(p\)’s small enough, any \(H_S \ k\)-SYC-map has a unique nontrivial fixed point. Therefore in the neighbourhood of the axes of the parameter space \((R_0 - 1, p_0, \ldots, p_{k-2})\) or, alternatively, \((p_0, \ldots, p_{k-1})\) the SYC-map has a unique fixed point. In some cases this point is also globally stable (we show it for \(k = 2\), Section 5.7).

The biological interpretation of this fact is rather important: for \(R_0\) slightly bigger than 1 the population is stable. It is also stable (even for \(R_0\) large) if all but one parameters \(p_i\) are small, for example, if the survival
probabilities are very low, but the number of offspring per reproducing individual is large. Or, if sensitivity to competition $g_i$ in one of the age classes is much higher than in all the others.

We use the following terminology: if $F$ is a one-dimensional map with fixed point $\bar{x}$, we call the derivative $F'(\bar{x})$ the multiplier of $\bar{x}$.

5.4.1 Small $R_0$ and small $p$’s.

Proposition 5.8. If $R_0 \leq 1$ an $H_1$ SYC-map $x \mapsto F(x)$ has no nontrivial fixed points and the trivial fixed point is a global attractor.

Proof. For all $x > 0$ $|F(x)| < |x|$. Hence $x = 0$ is the unique global attractor. □

Proposition 5.9. Given an $H_1$ SYC-map, there exists $R_0$ such that for any $R_0 \in (1,\bar{R}_0)$ this map has a unique nontrivial fixed point.

Proof. The function $\Pi(x)$ decreases from $\Pi(0) = 1$ to a local minimum $m$. Hence for $1 < R_0 < \frac{1}{m}$ the map has a fixed point. In the rest of its domain the function $\Pi(x)$ can have several local maxima. Notice that all these maxima are strictly less than 1. Denote by $M$ the largest of these maxima. Take $\bar{R}_0 = \frac{1}{M}$.

□

Theorem 5.10. Let $R_0 > 1$. A parametric $H_S$ SYC-map given by (5.7) has at most one nontrivial fixed point if for $i = 0, \ldots, k - 1$ $p_i \leq \frac{c}{f(c)}$ where $c$ is a critical point of $f$. And the multiplier of this fixed point is less than 1.

In order to prove this theorem we first formulate a lemma and a corollary of it, which will also be useful later on.

Lemma 5.11. Let a $C^3$ function $F$ be increasing and assume that $SF < 0$ for all $x$ such that $F'(x) \neq 0$. Then the map $x \mapsto F(x)$ has at most one fixed point with multiplier larger or equal to 1.

Proof. In this and some other proofs we exploit the following simple fact. If a fixed point of the map, defined by a $C^1$ function, has a multiplier larger (less) than 1, then its neighbouring fixed points (if they exist, of course) have multipliers less (larger) or equal to 1.

Let there exist two fixed points $x_1 < x_2$ with multipliers strictly larger than 1. Then there should be a fixed point $x_3$ between them with $F'(x_3) \leq 1$. Let $G(x) = F(x) - x$, then $G(x_1) = G(x_3) = G(x_2) = 0$ and $G(x) > 0$ on $(x_1, x_3)$ and $G(x) < 0$ on $(x_3, x_2)$. Hence there exist $y_1 \in (x_1, x_3)$ and $y_2 \in (x_3, x_1)$ such that $G'(y_1) = G'(y_2) = 0$ and $G'(x) < 0$ on $(y_1, y_2)$. Hence there exists $z$ such that $G''(z) = F''(z) = 0$.
and, in addition, \( G'''(z) = F'''(z) \geq 0 \). Since \( SF < 0 \), \( \frac{F'''(z)}{F'(z)} < 0 \). Hence \( F'''(z) \neq 0 \) and \( F'(z) < 0 \); but \( F \) is, by the condition of the lemma, an increasing function. So, we get a contradiction.

Let now \( x_1 \) have multiplier 1. Then there are two possible cases: there is still a point \( x_3 \) between \( x_1 \) and \( x_2 \) with multiplier \( F'(x_3) \leq 1 \) and we are back to the previous proof, or there is no point between \( x_1 \) and \( x_2 \). In this last case \( G(x) < 0 \) on \( (x_1, x_2) \) and there is a \( y_2 \) with \( G(y_2) = 0 \). Noticing that \( G'(x_1) = 0 \), we are back to the previous proof again.

**Corollary 5.12 (of Lemma 5.11).** Let a \( C^3 \) function \( F \) with \( SF < 0 \) (for all \( x \) such that \( F'(x) \neq 0 \)) be increasing on intervals \( I_0, I_2, I_4, \ldots \) and decreasing on \( I_1, I_3, I_5, \ldots \). Then it has at most one fixed point with multiplier larger or equal to 1 on each interval \( I_{2m}, m = 0, 1, \ldots \). And between any two successive of these points there is a fixed point with multiplier strictly less than 1.

Now let us prove Proposition 5.10.

**Proof of Proposition 5.10.** Recall that \( c \) is a unique critical point of maximum because \( f \) belongs to \( H_2 \).

The range of the function \( f_0 = p_0 f \) is \([0, p_0 f(c)]\). This interval is at the same time the domain of the function \( f_1 \) if we consider the composition \( f_1 \circ f_0 \). By the condition of the proposition \( p_0 f(c) \leq c \), hence \( f_1 \) is increasing on its domain.

Reasoning in exactly the same way we can say that the composition \( f_{k-1} \circ \ldots \circ f_1 \) is an increasing function. Hence the composition \( f_{k-1} \circ \ldots \circ f_0 \) is increasing on \([0, c] \) and decreasing on \((c, \infty)\).

Since \( R_0 > 1 \), 0 is a fixed point with multiplier larger than 1 and by Corollary 5.12 the proposition is proved.

**Corollary 5.13 (of Theorem 5.10).** A parametric \( H_S \) 2-SYC-map (5.8) has at most one nontrivial fixed point if \( 1 < R_0 \leq \left( \frac{c}{f(c)} \right)^2 \). And the multiplier of this point is less than 1.

**Proof.** If \( R_0 \) is in the given interval, then one of the parameters \( p_0 \) or \( p_1 \) or both are less or equal to \( \frac{c}{f(c)} \).

### 5.4.2 Monotone maps.

The dynamics of an \( H_3 \) map are very simple. We have the following theorem.

**Theorem 5.14.** Let \( R_0 > 1 \). The \( H_3 \) SYC-map has a unique nontrivial fixed point which is globally stable.
Proof. Any composition \( f_j \circ \ldots \circ f_0 \) of strictly increasing functions \( f_i \) is again a strictly increasing function. Hence for any \( j \), \( h_{j+1}(f_j \circ \ldots \circ f_0) \) is a strictly decreasing function with \( \lim_{x \to \infty} h_{j+1}(x) = 0 \) and therefore their product \( \Pi \) is also a decreasing function and \( \lim_{x \to \infty} \Pi(x) = 0 \). Since \( \Pi(0) = 1 \) and \( \frac{1}{\Pi(0)} < 1 \) the equation (5.11) has a unique positive solution \( x_0 \).

Since \( x < x\Pi(x) < x_0 \) for \( x \in (0, x_0) \) and the opposite holds for \( x \in (x_0, \infty) \) the fixed point \( x_0 \) is globally stable.

\[ \square \]

Remark. What we needed for this theorem is only strictly decreasing (to zero) and strictly increasing \( x \mapsto x\Pi(x) \).

5.4.3 The maximal number of fixed points.

Theorem 5.15. Let \( R_0 > 1 \). An \( H_S \) SYC-map of order \( k \) has at most \( 2^k - 1 \) nontrivial fixed points. In particular, for \( k = 2 \) the map has at most three nontrivial fixed points.

In order to prove this theorem we first formulate the following lemma.

Lemma 5.16. The function \( f_{k-1} \circ \ldots \circ f_0 \), where all \( f_i \) are of class \( H_2 \), has at most \( 2^k - 1 \) critical points: \( 2^{k-1} - 1 \) points of minimum and \( 2^{k-1} \) points of maximum. Moreover, if the function has the maximal number of critical points, all the maxima are equal to the maximum of \( f_{k-1} \) and it has a minimum in the critical point \( c_0 \) of \( f_0 \).

Proof. Let \( c_i \) be the critical points of the functions \( f_i \).

Let \( k = 2 \). The function \( f_0 \) maps both \( (0, c_0) \) and \( (c_0, \infty) \) to \( (0, f_0(c_0)) \) and it is one-to-one on each of the intervals. If \( f_1 \) has its maximum on \( (0, f_0(c_0)) \), the composition \( f_1 \circ f_0 \) has two points of maximum on \( (0, \infty) \), moreover, both maxima are equal to the maximum of \( f_1 \), and as a consequence there is a minimum in between (in the point \( c_0 \)).

Let \( k = 3 \). Clearly, \( f_2 \circ f_1 \) has the same property, i.e., at most two maxima and a minimum in between. If all the critical points lie in the interval \( (0, f_0(c_0)) \), then the composition of the three functions has \( 2 + 2 \) maxima (two on the interval \( [0, c_0) \) and two on \( (c_0, \infty) \)) and \( 1 + 1 + 1 \) minima (the first minimum is on \( (0, c_0) \), the second on \( (c_0, \infty) \) and the third is in the point \( c_0 \)).

By induction, if \( f_j \circ \ldots \circ f_1 \) has \( 2^{(j-1)} \) points of maxima and \( 2^{(j-1)} - 1 \) points of minima and all of them lie in \( (0, f_0(c_0)) \), then \( f_j \circ \ldots \circ f_0 \) has \( 2 \cdot 2^{j-1} \) points of maxima and \( 2 \cdot (2^{j-1} - 1) + 1 \) points of minima.

\[ \square \]
Now the proof of the Theorem 5.15 is very simple.

**Proof of Theorem 5.15.** From Corollary 5.12 we see that the larger the number of critical points is, the larger the number of fixed points of a map can be. Let $c_1, ..., c_{2^{k-1}-1}$ be points of minimum of $f_{k-1} \circ ... \circ f_0(x)$ and $C_1, ..., C_{2^{k-1}}$ be points of maximum.

On $[0, C_1]$ 0 is the fixed point with multiplier larger than one, on all other intervals $[c_m, C_{m+1}]$ there can be at most one such a point. In addition, there are points with multipliers less than one between them and to the right of the largest of them. So, in total there are $2(2^{k-1} - 1) + 1 = 2^k - 1$ nontrivial fixed points.

5.5 Cusp bifurcation and fold curves. $k = 2$.

We have already said that the fixed points appear and disappear in pairs in a fold bifurcation. Namely two neighbouring fixed points (one with a multiplier $> 1$ and another with multiplier $< 1$) move to each other as a parameter varies. At the moment of the fold bifurcation they collide creating a fixed point with multiplier equal to one. And after that they disappear.

A fixed point can collide with its right or its left neighbour. If these two bifurcations happen simultaneously (i.e., for some fixed parameter values) we have a cusp bifurcation: three fixed points collide. This bifurcation is of codimension 2 while the fold bifurcation is of codimension 1. This means that two parameters should be tuned to find the cusp bifurcation. (For more details see, e.g., [Kuznetsov].)

In this section we consider parametric 2-SYC-maps. They have a lot in common with maps with two extrema and, in particular, with cubic maps [Branner, Skjolding et al.]. In [Branner] the author considers odd functions with two extrema. He describes various bifurcations of codimension 1 and 2 of $n$-periodic points and gives ordering of periods.

We find in this section that a family of parametric 2-SYC-maps has a cusp point on its symmetry axis $p_0 = p_1$ (see Section 5.3) and two symmetric fold curves originate from this point. This is a local result in a neighbourhood of the cusp point. But for an arbitrary fold curve in the parameter plane $(p_0, p_1)$ we can say that it is either invariant under the parameter reflection $(p_0, p_1) \rightarrow (p_1, p_0)$, or there should exist another symmetric fold curve which is obtained by this reflection.

For an $H_S$ family of 2-SYC-maps the cusp is of spring type [Broer et al., Section 4.1.2]. The normal form for this bifurcation is

$$x \mapsto b + (1 + a)x - x^3.$$
And a bifurcation diagram of such a family in the neighbourhood of the cusp point can be found in [Kuznetsov, Section 9.2]. In our case of 2-SYC-maps this structure is presented in the Figure 5.1. Namely, we have a wedge in a parameter plane consisting of two fold curves originating from the cusp point. Inside the wedge the map has three nontrivial fixed points, outside it has only one. (A detailed description of Figure 5.1 is given in Theorem 5.25.)

The cusp point \((x, p, \bar{p})\) lies on the symmetry axis \(p_0 = p_1\). It "coincides" with a flip point \((x, \bar{p})\) of a building-block map \(x \mapsto p f(x)\). (In the flip point the multiplier of a fixed point is \(-1\). As a result of this bifurcation the fixed point loses its stability and, generically, a two-cycle appears around it.)

5.5.1 Fold curves. General case.

First we give a definition of a fold curve which is convenient in our case.

**Definition 5.17.** A two-parameter family of \(C^2\) maps \(x \mapsto F(x, p_0, p_1)\) (or, alternatively, \(x \mapsto F(x, R_0, p_0)\)) has a fold point at \((x, p_0, p_1)\) if the conditions

\[
\begin{align*}
F(x, p_0, p_1) &= x \\
F_x(x, p_0, p_1) &= 1
\end{align*}
\] (5.12)

are satisfied. We call a continuous branch of solutions to (5.12) a fold curve in the space \((x, p_0, p_1)\) if

\[
F_{xx}(x, p_0, p_1) > 0 \text{ or } F_{xx}(x, p_0, p_1) < 0
\]

on this branch. Also we call the corresponding curve in the parameter space \((p_0, p_1)\) parametrized by \(x\) a fold curve in \((p_0, p_1)\).

We adopt the convention that if the second derivative changes the sign, we move to another fold curve.

In a smooth enough family of 2-SYC-maps (5.8) a fold point should satisfy

\[
\begin{align*}
p_0 p_1 h(x) h(p_0 f(x)) &= 1 \\
p_0 p_1 f'(x) f'(p_0 f(x)) &= 1.
\end{align*}
\] (5.13)

We can rewrite (5.13) as follows

\[
\begin{align*}
y &= p_0 f(x) \\
x &= p_1 f(y) \\
p_0 f'(x) p_1 f'(y) &= 1.
\end{align*}
\] (5.14)

The second derivative at a fold point is

\[
F_{xx}(x, p_0, p_1) = p_0 p_1 G(x, p_0 f(x), p_0)
\]
with
\[ G : (x, y, p) \mapsto f''(x) f'(y) + p(f'(x))^2 f''(y) \]

We observe immediately the symmetry of Section 5.3.

**Proposition 5.18.** If there exists a fold curve of a parametric 2-SYC-map (5.8) in \((p_0, p_1)\), there exists also a symmetric fold curve given by the transformation \((p_0, p_1) \mapsto (p_1, p_0)\). Moreover, if \((\bar{x}, \bar{p}_0, \bar{p}_1)\) is a fold point, \((\bar{p}_0 f(\bar{x}), \bar{p}_1, \bar{p}_0)\) also is a fold point.

**Proof.** The system (5.14) is invariant under \((x, y, p_0, p_1) \mapsto (y, x, p_0, p_1)\). Therefore if \((x, p_0, p_1) = (\bar{x}, \bar{p}_0, \bar{p}_1)\) is a fold point, \((y, p_1, p_0) = (\bar{p}_0 f(\bar{x}), \bar{p}_1, \bar{p}_0)\) also is a fold point. Hence a fold curve (if it exists) must have a symmetric one with respect to \((p_0, p_1) \mapsto (p_1, p_0)\).

**Proposition 5.19.** Consider two symmetric fold curves in \((x, p_0, p_1)\). Let \((x, p_0, p_1)\) and \((y, p_1, p_0)\) be symmetric fold points on these curves, i.e., \(y = p_0 f(x)\). Then the derivatives \(f'(x)\) and \(f'(y)\) both are non zero and have the same sign along the curves. Moreover, if the sign is plus, the second derivative \(F_{xx}\) has the same sign on both the fold curves; otherwise, \(F_{xx}\) has a different sign on each curve.

**Proof.** The first property follows from the last equation of (5.14) and from continuity of fold curves.

We notice that for the symmetric fold points
\[ G(y, x, p_1) = p_1 f'(y) G(x, y, p_0). \]

Hence
\[ F_{xx}(y, p_1, p_0) = p_1 f'(y) F_{xx}(x, p_0, p_1) \]

and the second assertion is proved.

A trivial case of a fold curve is when \(x = y = 0\).

**Proposition 5.20.** A curve \(p_0 p_1 = 1\) (or \(R_0 = 1\)) is a fold curve for an \(H_1\) parametric 2-SYC-map (5.8). It corresponds to the fold points \((0, p_0, \frac{1}{p_0})\).

**Proof.** We notice that \(f'(x) = h(x) + x h'(x)\) and \(f''(x) = 2h'(x) + x h''(x)\). The function \(h(x)\) is doubly normalized, i.e., \(h(0) = 1\) and \(h'(0) = -1\) (\(h(x)\) is decreasing). Having this we get that \((x, p_0, p_1) = (0, p_0, \frac{1}{p_0})\) satisfies (5.14) for all \(p_0 > 0\), and \(F_{xx}(0, p_0, \frac{1}{p_0}) = -2(p_0 + 1) < 0\).
Clearly, the curve $p_0 p_1 = 1$ is symmetric to itself. The derivative $f'(0) = 1$ is positive. And, indeed, the second derivative in a fold point does not change the sign along this curve. This curve corresponds to the situation when a nontrivial fixed point "emerges" from a trivial one. This is not a very interesting case, we have already considered the behaviour of an (even general $k$-) SYC-map in the neighbourhood of this bifurcation (Section 5.4).

Now we turn our attention to the case of nontrivial fold points. Let a fold curve in $(x, p_0, p_1)$ intersect the symmetry plane $p_0 = p_1$ at a point $(\bar{x}, \bar{p}, \bar{p})$. Then a symmetric fold curve should intersect it at a point $(\bar{p} f(\bar{x}), \bar{p}, \bar{p})$. If the derivative $f'(x)$ is negative at some fold point, it should be negative along both the symmetric fold curves and the second derivative $F_{xx}$ has a different sign on each curve (Proposition 5.19). If, in addition, $\bar{x} = \bar{p} f(\bar{x})$, i.e. the fold curves intersect $p_0 = p_1$ at the same point, the second derivative $F_{xx}(\bar{x}, \bar{p}, \bar{p})$ should be zero by continuity. This is exactly the case of a cusp point. Motivated by this, we will prove that there exists a cusp point on the symmetry axis.

5.5.2 Cusp point.

Definition 5.21. A smooth enough two–parameter family $x \mapsto F(x, p_0, p_1)$ (or, alternatively, $x \mapsto F(x, R_0, p_0)$) has a cusp point at $(x, p_0, p_1)$ if (with all the derivatives evaluated at the point $(x, p_0, p_1)$)

$$
F(x, p_0, p_1) = x \\
F_x = 1 \\
F_{xx} = 0
$$

and the bifurcation is generic if

$$
F_{xxx} \neq 0 \quad (a) \\
F_{p_0} F_{x p_1} - F_{p_1} F_{x p_0} \neq 0 \quad (b)
$$

(see [Kuznetsov, p. 398]). The cusp is of spring type [Broer et al., Section 4.1.2] if

$$F_{xxx} < 0.$$ 

Theorem 5.22. A smooth enough family of parametric 2-SYC-maps (5.8) possesses a cusp bifurcation at a point $(\bar{x}, \bar{p}, \bar{p})$ with $\bar{x} \neq 0$ and

$$
\bar{p} = \frac{1}{h(\bar{x})}
$$

if and only if there exists $\bar{x}$ such that

$$
\frac{\bar{x} h'(\bar{x})}{h(\bar{x})} = -2.
$$
5.5. CUSP BIFURCATION AND FOLD CURVES. $K = 2$.

The bifurcation is generic and of spring type if the map is $H_S$ and

$$\frac{d}{dx} \left( \frac{xh'(x)}{h(x)} \right) \bigg|_{x=\tilde{x}} \neq 0$$

(5.20)

Proof. A cusp point is a fold point with the additional property of zero second derivative. Using $f(x) = xh(x)$, we rewrite the equations (5.14) for a fold point $(x_0, p_0, p_1)$ in the following form

$$y = p_0 f(x)$$

$$x = p_1 f(y)$$

$$\phi(x) + \phi(y) + \phi(x)\phi(y) = 0$$

(5.21)

with $\phi(x) = \frac{xh'(x)}{h(x)}$ (recall that $h(x)$ is positive for all $x$). If $x = y$, then $p_0 = p_1 = \tilde{p}$. This is the case of intersection of fold curves with the symmetry plane $p_0 = p_1$ at the same point $(\tilde{x}, \tilde{p}, \tilde{p})$. (We have discussed this situation in the end of the previous subsection.) In this case either $\phi(\tilde{x}) = 0$ or $\phi(\tilde{x}) = -2$. The first case corresponds to the fold curve $p_0p_1 = 1$ which we have already considered in Proposition 5.20. So, we concentrate on the case $\phi(\tilde{x}) = -2$.

Since $\phi(x) + 1 = \frac{f'(x)}{f(x)}$, the derivative $f'(\tilde{x})$ is negative. Using (5.15), we have

$$F_{xx}(\tilde{x}, \tilde{p}, \tilde{p})(1 - \tilde{p}f'(\tilde{x})) = 0$$

and thus the second derivative is zero.

Finally, if $x = y$, $\tilde{x} = \tilde{p}f(\tilde{x})$, hence $\tilde{p} = \frac{1}{h(\tilde{x})}$. So, under the conditions (5.18) and (5.19) we have a cusp point at $(\tilde{x}, \tilde{p}, \tilde{p})$.

Let now the map be $H_S$. The Schwarzian of $F'$ in the cusp point is just $F_{xxx}$ because $F_{xx} = 0$ and $F_x = 1$. It is negative, hence the first genericity condition (5.17a) is satisfied and the cusp bifurcation is of spring type.

Now we should check the second genericity condition (5.17b). It demands some calculations. We omit the argument $\tilde{x}$ of the functions below.

$$F_{p_0} = p_1 f'(p_0 f) f$$

$$F_{p_1} = f(p_0 f)$$

$$F_{xp_0} = p_1 p_0 f''(p_0 f) f f' + p_1 f'(p_0 f) f''$$

$$F_{xp_1} = p_0 f'(p_0 f) f'.$$

In the cusp point we have $\tilde{p}f(\tilde{x}) = \tilde{x}$. The condition (5.17b) is satisfied if

$$\tilde{p}^2 \left( (f')^3 f - f(f''f' + h(f'))^2 \right) \neq 0.$$
Noticing that $p^2 \neq 0$, $f'(\bar{x}) \neq 0$ and $f(\bar{x}) \neq 0$, we divide by all these expressions to get

$$(f')^2 - f''f - h f' \neq 0$$

Finally, substituting $(f'(\bar{x}))^2 = f'(\bar{x})(h(\bar{x}) + \bar{x}h'(\bar{x}))$ and dividing by $h^2(\bar{x}) \neq 0$, we have

$$\frac{d}{dx} \left( \frac{f'(x)}{h(x)} \right) \bigg|_{x=\bar{x}} \neq 0,$$

which is indeed equivalent to (5.20).

**Remark.** Note that the conditions (5.19)–(5.20) are satisfied for the 2-Ricker family $h(x) = e^{-x}$ and the 2-Hassell family $h(x) = \frac{1}{(1+x)^\beta}$ with $\beta > 2$. It is interesting that the function $\frac{xh'(x)}{h(x)} = \frac{d\ln h(x)}{d\ln x}$ has a biological meaning, it is the so-called elasticity of survival [Caswell] (see also p. 40).

The condition (5.20) can be viewed as $\phi'(\bar{x}) \neq 0$. From the last equation of (5.21) we see that we can use the Implicit Function Theorem to find a branch of solutions $y = g(x)$ passing through $(x, y) = (\bar{x}, \bar{x})$. Moreover, from the symmetry with respect to $(x, y) \rightarrow (y, x)$ the function $g$ is equal to its inverse, i.e. $g = g^{-1}$ in a neighbourhood of $\bar{x}$. This solution corresponds to symmetric fold curves originating from the cusp point.

The same result we have from the normal form theorem [Kuznetsov, Theorem 9.1]. Moreover, by proving Theorem 5.22 we have made a step towards discussion of stability properties. Namely, from the normal form it follows that in a neighbourhood of a generic cusp point, a map has either one stable fixed point (outside the wedge in Figure 5.1) or two stable fixed points with an unstable one in between (inside the wedge) or two points: one is stable and the other with multiplier 1 (on the fold curves).

### 5.5.3 Cusp point is a flip point in the building block map.

**Definition 5.23.** A $C^1$ map $x \mapsto F(x, p)$ has a flip point at $(x, p)$ if

\[
\begin{align*}
F(x, p) &= x \\
F_x(x, p) &= -1.
\end{align*}
\]

**Proposition 5.24.** If a building block map $x \mapsto p f(x)$ has a flip point at $(\bar{x}, \bar{p})$, the corresponding 2-SYC-map (5.8) has a cusp point at $(\bar{x}, \bar{p}, \bar{p})$.

**Proof.** The conditions for a building block map to have a flip point are

\[
\begin{align*}
\bar{p} f(\bar{x}) &= \bar{x} \\
\bar{p} f'(\bar{x}) &= -1.
\end{align*}
\]
They can be rewritten as

\[
\bar{p} = \frac{1}{h(\bar{x})} \quad f'(\bar{x}) + h(\bar{x}) = 0.
\]

And the last equation is equivalent to \( \phi(\bar{x}) = -2 \). From the proof of Theorem 5.22 these expressions are precisely the conditions (5.18) and (5.19) for the cusp point.

5.6 Local dynamics of the 2-Ricker map.

We now consider the 2-Ricker-map (5.9) which we rewrite here, omitting the index 0 of \( p_0 \) (so \( p = p_0 \), as

\[
x \mapsto R_0 xe^{-\nu(x,p)},
\]

\[
\nu(x,p) = x(1 + pe^{-x}).
\]

First we find explicit expressions for the fold curves and the cusp point in the parameter space \((R_0, p)\). Then we consider local stability of the fixed points.

The linear stability is determined by the multiplier. If the absolute value of the multiplier is strictly less than 1, the fixed point is locally asymptotically stable. A fold bifurcation takes place when the multiplier equals +1. If the multiplier passes the value \(-1\), we have a flip (or period-doubling) bifurcation: the (dis)appearance of a two-cycle around the fixed point.

We conjecture that the bifurcation diagram we obtained (Figure 5.2) for the 2-Ricker-map (5.22) is representative (in the qualitative sense) for general \( H_2 \) SYC-maps of order 2. In particular, in [Broer et al, Section 4.1.2] there is a whole collection and even classification of areas around a cusp point and in most of the cases the organization of fold and flip curves around a cusp of spring type is like we find in the special case of the Ricker family.

5.6.1 Fold curves. Ricker case.

Theorem 5.25. Let \( R_0 > 1 \).

i) For \( R_0 > e^4 \) and \( p \in (p_-, p_+) \), where

\[
p_{\pm} = e^{x_{\pm}} \left( \frac{\ln R_0}{x_{\pm}} - 1 \right),
\]

(5.23)
Figure 5.1: Fold curves and a cusp point for the 2-Ricker-map (5.22). We have a wedge consisting of the two fold curves originating from the cusp point. Inside the wedge the map has three nontrivial fixed points, outside it has only one. On the fold curves two of the three fixed points collide. In the cusp point the all three points collide.

\[ x_{\pm} = \frac{\ln R_0 \pm \sqrt{(\ln R_0)^2 - 4 \ln R_0}}{2}, \]  

the map (5.22) has three nontrivial fixed points, the middle of which is always unstable.

ii) For \( R_0 > e^4 \) and \( p = p_{\pm} \) there are two nontrivial fixed points. Moreover, if \( p = p_+ \) the upper fixed point is \( x_+ \) and its multiplier equals 1. If \( p = p_- \) the lower fixed point is \( x_- \) and its multiplier is 1.

iii) In all other cases there is a unique nontrivial fixed point. Moreover, if \( R_0 = e^4 \) and \( p = e^2 \) this fixed point is \( x = 2 \). It is a generic cusp point of spring type (Definition 5.21).

The proof of the theorem is given in Appendix B. The fold–bifurcation curves \( p_{\pm}(R_0) \), given by (5.23)–(5.24) are strictly monotone increasing for \( R_0 \geq e^4 \).

**Remark.** For a fixed value of \( R_0 \), the transformation \( p_+ f(x_+) \) sends a fixed point \( x_+ \) corresponding to a fold point \( (R_0, p_+) \) with \( F''(x_+) < 0 \), to the other fixed point \( x_- \) corresponding to another fold point \( (R_0, p_-) \) with, correspondingly, \( F''(x_-) > 0 \) and vice versa. From the symmetry of Corollary 5.7 it follows that \( p_+ = \frac{R_0}{p_-} \) and that the cusp point lies on the symmetry axis \( p = \sqrt{R_0} \).

### 5.6.2 Local stability of the fixed points.

Let us now consider the stability properties of the fixed points. Similarly to fold curves (Definition 5.17), we give a definition for flip curves on which the
multiplier of a fixed point equals \(-1\).

**Definition 5.26.** A two-parameter family of $C^1$ maps $x \mapsto F(x, R_0, p)$ has a flip point at $(x, R_0, p)$ if the conditions

$$
F(x, R_0, p) = x \quad \text{(a)} \\
F_x(x, R_0, p) = -1 \quad \text{(b)}
$$

are satisfied. We call a continuous branch of solutions to (5.25) parametrized by $x$ a flip curve in the space $(R_0, p)$.

We do not add extra conditions on the second derivative of $F$ as in the fold case. However, we formulate the following lemma.

**Lemma 5.27.** Let $(\bar{x}, \bar{p})$ be a flip point of a smooth enough one-parameter family of maps $x \mapsto F(x, p)$. Let, in addition,

$$
SF(\bar{x}, \bar{p}) < 0 \quad \text{(a)} \\
F_{xp}(\bar{x}, \bar{p}) < 0. \quad \text{(b)}
$$

Then

- for $p$ slightly less than $\bar{p}$ the map has a stable fixed point in a neighbourhood of $\bar{x}$;
- for $p$ slightly larger than $\bar{p}$ the map has a stable two-cycle.

This lemma is a particular case of the normal form theorem for the flip bifurcation [Kuznetsov, Theorem 4.4].

It is possible in the Ricker case to find explicit expressions for flip curves.

**Theorem 5.28.** Let $R_0 > e^2$. Then the fixed point of the map (5.22), given explicitly by

$$
x_{1,2} = \frac{\ln R_0 \pm \sqrt{(\ln R_0)^2 - 4 \ln R_0 + 8}}{2},
$$

where $x_1$ corresponds to "+" and $x_2$ corresponds to "-", undergoes a flip bifurcation if $p = p_{1,2}$, respectively, where

$$
p_{1,2} = e^{x_{1,2}} \left( \frac{\ln R_0}{x_{1,2}} - 1 \right).
$$

As a result of this bifurcation a stable two-cycle appears around the corresponding fixed point, for parameter values in regions of $(R_0, p)$ slightly below the curve $p_1(R_0)$ and slightly above $p_2(R_0)$. 
Figure 5.2: The bifurcation diagram of the map (5.22) where the local bifurcations are represented. See explanations in the text.

The proof of the theorem is given in Appendix B. The curves $p_1(R_0)$ and $p_2(R_0)$ corresponding to the flip bifurcation are shown in Figure 5.2.

Now a stability result is as follows.

**Theorem 5.29.** Let $R_0 > 1$.

1. If the map (5.22) has a unique nontrivial fixed point, it is stable in two cases: if $R_0 \leq e^2$ or $R_0 > e^2$ and $p \in [p_1, p_2]$.

2. If the map (5.22) has three nontrivial fixed points, the smallest of them is locally stable if $p \in (p_-, p_2)$.

3. Similarly, if the map (5.22) has three nontrivial fixed points, the largest of them is locally stable if $p \in (p_1, p_+)$.

**Remark.** For $R_0 > e^4$ the map (5.22) has a unique fixed point if $p \notin [p_-, p_+]$. It is stable if $p \in [p_1, p_2]$. Thus in this case a unique fixed point can be stable only if $p_1 < p_- \text{ or } p_2 > p_+$, i.e., the flip bifurcation curves lie outside the region of three fixed points. Remark that intersections of the flip curves $p_{1,2}(R_0)$ with the fold curves $p_{1,2}(R_0)$ respectively, take place for the same value of $R_0$ due to the symmetry of the curves.

The proof of the theorem is given in Appendix B. Now we are able to describe all the local bifurcations of the fixed points which take place in the map (5.22) for positive values of $R_0$ and $p$. The bifurcation diagram is represented in Figure 5.2. There are four bifurcation curves for $R_0 > 1$, fold bifurcation curves $p = p_{\pm}(R_0)$ and flip bifurcation curves $p = p_{1,2}(R_0)$, where $p_{\pm}$ and $p_{1,2}$ are given by (5.23) and (5.28) respectively. The parameter space is divided into seven regions.

- In the region 1 the map has a unique nontrivial fixed point and it is locally stable. This region consists of three subregions. The first one
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Figure 5.3: The surface of fixed points of the 2-Ricker-map in the space \((R_0, p, x)\) with the fold curves \(p_{\pm}\) and flip curves \(p_{1,2}\) shown. \(\textbf{a}:\) The two flip curves correspond to distinct fixed points, so the curves do not intersect. \(\textbf{b}:\) A view from above, the upper part of the surface is removed and one sees the same diagram as in Figure 5.2.

is the domain \(\{R_0 \in (1, e^2]\text{ and } p \in (0, \infty)\}\), the second one is \(\{R_0 \in (e^2, e^4] \text{ and } p \in [p_1, p_2]\}\) and the third one is \(\{R_0 \in (e^4, R_0') \text{ and } p \in [p_1, p_-) \cup (p_+, p_2]\}\), where \(R_0'\) is the value of \(R_0\) which corresponds to the intersection of the flip curves \(p_{1,2}(R_0)\) with the fold curves \(p_{\pm}(R_0)\).

- In the region 2 the map has three nontrivial fixed points: the middle one is unstable while the upper and the lower ones are locally stable.

- The regions 3 and 4 are symmetric with respect to the transformation \(p \mapsto R_0 \frac{p}{p'}\). In these regions the map has a unique nontrivial fixed point which is locally unstable. Moreover, in a neighbourhood of the curves \(p_{1,2}(R_0)\) in these regions there exists a stable two-cycle around the unstable fixed point.

- The regions 5 and 6 are also symmetric. The map has three nontrivial fixed points, two of which are unstable and one is stable: the upper (in the region 5) or the lower one (in the region 6). Moreover, in the region 5 in a neighbourhood of the curve \(p_2(R_0)\) there exists a stable two-cycle around the lower fixed point. Similarly for the region 6.

- In the region 7 the SYC-map has three locally unstable fixed points. In a neighbourhood of \(p_{1,2}(R_0)\) there exist two-cycles around the upper and the lower fixed points, respectively.

We emphasize that for arbitrary large values of the multiplicative factor \(R_0\), there is still a window for \(p\), where the map has a stable equilibrium. One can get a wrong impression looking at Figure 5.2 that flip curves may intersect. It happens only because the curves \(p_1\) and \(p_2\) are projections
of flip curves in the space \((R_0, p, x)\). In other words, the two flip curves correspond to distinct fixed points \(x_1\) and \(x_2\) (given by (5.27)) of the 2-Ricker-map. In Figure 5.3 we show a surface of fixed points of the 2-Ricker-map (5.22) for different values of \(R_0\) and \(p\) which is given by

\[ \nu(x, p) = \ln R_0. \]

We see that this surface "folds" in the space so that there can be one, two or three fixed points for different sets of \((R_0, p)\). Curves, where the normal vector to the surface is parallel with the coordinate plane \((R_0, p)\), are the fold curves \(p_+\) and \(p_-\). The flip curves \(p_1\) and \(p_2\) are also shown so one can see that they do not intersect indeed (Fig. 5.3a). If we look at the surface from above (along the \(x\)-axis) and remove the upper part of the surface (Fig. 5.3b), we see the diagram of Figure 5.2.

The structure similar to this diagram with a cusp point, two fold curves and two flip curves is well-known and rather universal (see, e.g. [Broer et al.] and references in there, [El-Hamouly et al., Carcasses, Mira]), it is also called crossroad area. What is especially nice for the 2-Ricker-map is that the fold and flip curves are given by explicit formulas. As we will see later in this chapter (see the numerical bifurcation diagram in Figure 5.8), the same structure appear also for higher iterates of the map, i.e. we observe cusps of periodic points accompanied by fold and flip curves. Moreover, the cusp points accumulate, so there occur so-called cascades of cusps. We conclude it from the fact that flip curves which accompany cusp points accumulate indeed. (For example, if we fix \(p < e^2\), the 2-Ricker-map is unimodal. So, if we let \(R_0\) grow, we observe well-known Feigenbaum cascades of period-doublings.)

5.7 Some results on global dynamics. \(k = 2\).

We have described local bifurcations of fixed points in the family of 2-Ricker-maps (5.22). Now we are going to consider some global aspects of the dynamics. The main goal of this section is to prove, at least partly, that the bifurcation diagram in Figure 5.7 of the 2-Ricker family is correct. However all the results in this section are formulated for a general 2-SYC-map.

To formulate the results we need various notions from one-dimensional dynamics. To help a non-expert reader we collect some basic notions and results in Appendix A.

First we show that a 2-SYC-map with one critical point can be viewed as an S-unimodal map and hence it has one metric attractor.

In the second subsection we show that a 2-SYC-map can have at most two metric attractors. We prove also that local stability of a unique fixed point
means automatically global stability. These results are valid for a general (even non-parametric) \( H_2 \) 2-SYC-map.

Finally, we describe nonlocal homoclinic bifurcations which influence the number of attractors.

In this and the following sections we denote by \( c \) a unique turning point of the building block map \( f \) of an \( H_2 \) parametric 2-SYC-map (5.8). If \( p_0 \leq \frac{c}{f(c)} \) the SYC-map \( F \) has a unique turning point which is again \( c \) (see Proof of Proposition 5.10 and Lemma 5.16), then \( F(c) \) is the value of maximum. Otherwise, the SYC-map has three turning points: two maxima and a local minimum in between. The values of maxima are the same, we denote them by \( M \). The local minimum is at the point \( c \) (Lemma 5.16) and the value of minimum is \( m \).

5.7.1 One critical point. The map is \( S \)-unimodal.

**Proposition 5.30.** Let an \( H_S \) 2-SYC-map \( x \mapsto F(x) \) have one critical point \( c \). Then the map is topologically conjugated to a unimodal map on an interval \( [0,a] \) with \( F(c) < a < \infty \).

**Proof.** While the SYC-map is defined on \( [0,\infty) \), after the first iteration we land in the interval \([0,F(c)]\). We choose a unimodal map \( \tilde{F} \) on an interval \([0,a]\) with \( F(c) < a < \infty \) such that \( \tilde{F}(x) = F(x) \) for \( x \in [0,F(c)] \) and \( \tilde{F} : (F(c),a) \rightarrow (0,F^2(c)) \) is decreasing. \( \quad \square \)

From Blokh and Lyubich’s theorem (see Appendix A) the corollary below follows.

**Corollary 5.31.** Let an \( H_S \) 2-SYC-map (5.1) have one non-flat critical point. Then it has a unique metric attractor.

For a parametric 2-SYC-map (5.8) we can formulate the following corollary.

**Corollary 5.32.** Consider a parametric \( H_S \) 2-SYC-map (5.8). Let \( R_0 > 1 \), \( c \) is a non-flat critical point of the building-block map \( f \) and \( p_0 < \frac{c}{f(c)} \) or, alternatively, \( R_0 < \left( \frac{c}{f(c)} \right)^2 \). Then the map has a unique metric attractor.

**Proof.** Note that if \( c \) is non-flat and \( p_0 < \frac{c}{f(c)} \), the SYC-map has a unique non-flat critical point \( c \). \( \square \)

If \( p_0 = \frac{c}{f(c)} \), the critical point of the 2-SYC-map can be flat, so the Blokh and Lyubich’s theorem is not applicable, but we can apply another theorem, namely Singer’s theorem 5.46 to prove the proposition below.
To apply Singer’s theorem to a SYC-map we should consider it on a bounded interval. We can easily do it because the range of the SYC-map is bounded by its maximum $F(c)$. So after the first iteration all the values of $x$ lie in the interval $[0, F(c)]$.

**Proposition 5.33.** Let an $H_S$ 2-SYC-map (5.1) have one critical point (flat or nonflat). Then it has at most one periodic attractor. In particular, if for a parametric $H_S$ 2-SYC-map (5.8) $R_0 > 1$ and $p_0 \leq \frac{c}{F(c)}$ or, alternatively, $R_0 < \left( \frac{c}{F(c)} \right)^2$, the map has at most one periodic attractor.

### 5.7.2 Three critical points. Number of attractors.

#### Global stability of a unique fixed point.

**Proposition 5.34.** If an $H_S$ 2-SYC-map (5.1) has three critical points, it has at most two periodic attractors.

**Proof.** We use the Singer’s theorem again.

A 2-SYC-map has three critical points: two maxima and a minimum in between. The values of the maxima are the same. Hence the corresponding critical points belong to the immediate basin of the same periodic attractor.

We consider the SYC-map on a bounded interval $[0, M]$, where $M$ is the value of maximum of $F$. Since the right bound of the interval $M$ coincides with the maximum, this boundary point belongs to the same basin of attraction as the points of maximum.

Further, observe that the left boundary point 0 does not belong to any basin of attraction because it is itself a fixed point.

In conclusion, the map can have two periodic attractors, one that attracts the maximum of the map and one that attracts the minimum.

**Corollary 5.35.** Let an $H_S$ 2-SYC-map have two stable fixed points. Then they are the only metric attractors in the system, moreover, their immediate basins are divided by the unstable fixed point between them.

**Proposition 5.36.** If an $H_S$ 2-SYC-map $x \mapsto F(x)$ has a unique nontrivial fixed point, local stability implies global stability.

**Proof.** For any $H_1$ SYC-map we have that $\lim_{x \to \infty} F(x) = 0$. We choose an $a$ so that $a > M$, where $M$ is the maximum value of $F(x)$, and, in addition, $F(a) < F(M)$ and $F(a) < m$, where $m$ is a unique local minimum. We note that the interval $[0, a]$, is invariant and absorbs all initial conditions. Therefore we can restrict our consideration to this interval.
5.7. SOME RESULTS ON GLOBAL DYNAMICS. $K = 2$.

By Singer’s theorem the immediate basin $I$ of the nontrivial fixed point should contain either a boundary point or a critical point.

Let the basin contain the boundary point $a$ (the boundary point $0$ does not belong to any basin). Then it contains the interval $[F(a), a]$. This interval also absorbs all initial conditions, because $F(a) < \min(m, F(M))$. Hence the fixed point is globally stable.

If the basin $I$ does not contain $a$, we have that $I$ contains a critical point and that $F(\partial I) \subset \partial I$, because $I$ is invariant. Since we have a unique nontrivial fixed point, the boundary $\partial I$ can not contain a fixed point, hence points of the boundary are fixed points of the twice iterated map, i.e., $F^2(x) = x$ for $x \in \partial I$. The function $F(x)$ is decreasing in these points, otherwise $I$ is not invariant. We assumed that $I$ contains a critical point. It is a turning point, i.e. the derivative $F'(x)$ changes the sign in this point. Hence there should be two turning points in $I$. Therefore, the interval $[\min(m, F(M)), M] \subset I$. This interval absorbs all initial conditions, hence the fixed point is globally stable.

\[ \square \]

Corollary 5.37. Let $R_0 < 1$. A parametric $H_2$ 2-SYC-map (5.8) has a unique globally stable nontrivial fixed point if $p_0 \leq \frac{R_0}{\gamma(c)}$ ($p \leq e$ for the Ricker case) or, alternatively, $R_0 \leq \left( \frac{c}{\gamma(c)} \right)^2$ ($R_0 \leq e^2$).

5.7.3 Three critical points. Homoclinic bifurcation.

It is a more subtle question whether attractors, which are not fixed points, can coexist. We are going to give sufficient conditions for existence of two attractors in the system. It has to do with the behaviour of critical points under the action of the map. We consider two cases for the maximum $M$:

\[
\begin{align*}
F(M) > x_{mid} & \quad (a) \\
F_{l}^{-1}(x_{mid}) < F(M) < x_{mid} & \quad (b)
\end{align*}
\]

(5.29)

where $x_{mid}$ is the middle fixed point and $F_{l}^{-1}(x_{mid})$ is its closest preimage to the left (see Figure 5.4). Similarly, for the minimum $m$ we also consider two cases:

\[
\begin{align*}
m > F_{l}^{-1}(x_{mid}) & \quad (a) \\
F_{ll}^{-1}(x_{mid}) < m < F_{l}^{-1}(x_{mid}) & \quad (b)
\end{align*}
\]

(5.30)

where $F_{ll}^{-1}(x_{mid})$ is the left side preimage of $x_{mid}$ (Fig. 5.4). In Figure 5.5 we show four possible combinations: $A$ corresponds to (5.29a) and (5.30a), $B$ to (5.29a) and (5.30b), $C$ to (5.29b) and (5.30a), $D$ to other possibilities.
Figure 5.4: A graph of the SYC-map. The fixed point $x_{mid}$ has four preimages. In the figure the cases (5.29b) and (5.30b) are illustrated.

Figure 5.5: We show four possible combinations of mutual positions of $F(M)$ and $m$: A corresponds to (5.29a) and (5.30a), B to (5.29a) and (5.30b), C to (5.29b) and (5.30a), D to other possibilities.

**Proposition 5.38.** Let an $H_S$ 2-SYC-map $F$ have three fixed points. If the condition (5.29a) is satisfied, the map is $S$-unimodal on $I_r = [x_{mid}, F_r^{-1}(x_{mid})]$, where $F_r^{-1}(x_{mid})$ is the right preimage of $x_{mid}$.

Similarly, if (5.30a) is satisfied, the map is topologically conjugated to an $S$-unimodal map on $I_l = [F_l^{-1}(x_{mid}), x_{mid}]$.

**Proof.** Under the condition (5.29a) $F(I_r) \subseteq I_r$, the map has a unique critical point on this interval and the Schwarzian is negative. So, we get the desired conclusion.

For the case of minimum the proof is similar.

Now, using Blokh & Lyubich’s theorem 5.44, we obtain the following results.
Corollary 5.39. Let an $H_S$ 2-SYC-map 5.8 have three nonflat critical points and three nontrivial fixed points.

i) Let (5.29a) and (5.30a) both be satisfied (Fig. 5.5 A). Then the map has two metric attractors: one belongs to $I_l$ and one to $I_r$.

ii) Let (5.29a) and (5.30b) both be satisfied (Fig. 5.5 B). Then the map has a unique metric attractor, which belongs to $I_r$.

iii) Similarly, if (5.30a) and (5.29b) are both satisfied (Fig. 5.5 C), the map has a unique metric attractor which belongs to $I_l$.

Proof. To show ii) we note that, under (5.30b), the image of the minimum $m$ lies in the interval $I_r$, which is absorbing.

For the left interval the proof is similar. □

If we have both (5.29b) and (5.30b), or if either (5.29) or (5.30) does not hold (Fig. 5.5 D), it is not so easy to determine how many attractors there are. On the parameter plane $(R_0, p)$ boundaries between regions corresponding to the a and b conditions are given by the following implicitly defined functions

$$F(M) = x_{mid}$$  \hspace{1cm} (5.31)

and

$$F(M) = F^{-1}_l(x_{mid})$$  \hspace{1cm} (5.32)

for (5.29) (where $F(M)$ and $x_{mid}$ are functions of $R_0$ and $p$) or

$$m = F^{-1}_l(x_{mid})$$  \hspace{1cm} (5.33)

and

$$m = F^{-1}_{ll}(x_{mid})$$  \hspace{1cm} (5.34)

for (5.30), respectively. These curves correspond to homoclinic bifurcations. More precisely, we mean the following. Consider, for example, the case (5.31). Take a right neighbourhood $N^+$ of $x_{mid}$ (Figure 5.6). Since $F(x) > x$ on $(x_{mid}, x_2)$ and $c_2 < x_2$, where $c_2$ is the right point of maximum and $x_2$ is the upper fixed point, there exists a number $k$ such that $F^k(N^+) \supset M$. The image of $M$ is $x_{mid}$. Therefore we can find an orbit which starts just to the right of $x_{mid}$ (and converging to $x_{mid}$ backwards in time) and which arrives in exactly $x_{mid}$ in finitely many steps. This is a homoclinic orbit. It contains the critical point and we call this orbit degenerate. So, we say that a one-parametric family $x \mapsto F(x, a)$, with $a$ as a parameter, possesses a homoclinic bifurcation if there exists $a_0$ such that the map $x \mapsto F(x, a_0)$ has a degenerate homoclinic orbit.
CHAPTER 5. SYC-MAP

To prove rigorously that these homoclinic orbits do exist for a parametric 2-SYC-map (5.8), we need to provide values of the parameters $p$ and $R_0$ such that one of the conditions (5.31)–(5.34) is satisfied. It is not possible to give explicit expressions, but numerically (for the Ricker case) we can construct the corresponding curves in the bifurcation diagram in the plane $(R_0, p)$ (Fig. 5.7). The curve $M_1$ corresponds to the homoclinic bifurcation (5.31), $M_2$ corresponds to (5.32), $m_1$ to (5.33) and the curve $m_2$ to (5.34). The homoclinic bifurcation is a complicated phenomenon, namely, in every parameter neighbourhood of this bifurcation the map has bifurcation of either fold or flip type [Devaney, p. 126]. Thus, this bifurcation is an accumulation point of simple bifurcations.

Again we find the symmetry of the section 5.3.

Proposition 5.40. Let an $H_2$ parametric SYC-map (5.8) have three fixed points. In the parameter plane $(R_0, p)$ the curves given by (5.31) and (5.33) are symmetric under the reflection $(R_0, p) \mapsto (R_0, \frac{R_0}{p})$. Similarly, (5.32) and (5.34) are also symmetric.

Proof. Let a parametric SYC-map given by $F_1 = f_1 \circ f_0$ have parameters $R_0$ and $p$. Then a symmetric map $F_0 = f_0 \circ f_1$ has parameters $R_0$ and $\frac{R_0}{p}$. We denote the maxima of $F_1$ and $F_0$ as $M_1$ and $M_0$ respectively and the minima $m_1$ and $m_0$. We divide the domain of $f_0$ and $f_1$ into two parts and define invertible functions $i = \{0, 1\}$:

$$f_{il} = f_i : [0, c] \rightarrow [0, M_i] \quad f_{ir} = f_i : [c, \infty) \rightarrow [0, M_i],$$

where $c$ is a unique turning point of the building block map $f$ (recall that $f_i = p_i f$). We notice that $f_j(f_i(c)) = m_j, i \neq j, f_i(c) = M_i$ and hence

$$a) \quad f_{0l}(m_1) = F_0(M_0) \quad c) \quad f_{0r}(M_1) = m_0$$

$$b) \quad f_{1l}(m_0) = F_1(M_1) \quad d) \quad f_{1r}(M_0) = m_1. \quad (5.35)$$
5.8. **BIFURCATION DIAGRAMS OF THE 2-RICKER-MAP.**

We consider, for example, (5.31) and rewrite it as

\[ F_1(M_1) = x^1_{\text{mid}}, \quad (5.36) \]

where \( x^1_{\text{mid}} \) is the middle fixed point of \( F_1 \). For the middle fixed points \( x^0_{\text{mid}} \) of \( F_0 \) and \( x^1_{\text{mid}} \) we have

\[ f_0 \circ (x^1_{\text{mid}}) = x^0_{\text{mid}}. \]

Having (5.35b) we apply \( f_1^{-1} \) to both sides of the equation (5.36):

\[ m_0 = f_1^{-1} (x^1_{\text{mid}}) = f_1^{-1} (f_0 \circ (x^0_{\text{mid}})) = F_1^{-1} (x^0_{\text{mid}}), \]

i.e., we get (5.33). In a similar way we check the equivalence in the opposite direction and also show that (5.32) and (5.34) are symmetric too.

5.8 **Bifurcation diagrams of the 2-Ricker-map.**

In this section we present detailed bifurcation diagrams of the 2-Ricker-map (5.22), i.e., we describe the behaviour of the system in different regions of the parameter plane \((R_0, p)\). We summarize the analytical results we have got in the previous sections and add some numerical observations. In particular, we find cascades of cusps.

First we give a short description of the bifurcation diagram in Figure 5.7 and then explain it in detail.

- There exists a region \( R_0 \leq 1 \) (not shown in the picture) where the map has only the trivial fixed point which is globally stable.
- In the region 1 the map has a unique nontrivial fixed point which is globally stable (Proposition 5.36).
- In the regions 3 and 4 the unique fixed point is unstable and we observe cascades of cusps.
- In the region 2 the SYC-map has two stable fixed points and an unstable one in between. (Corollary 5.35 gives more info.)
- In the regions 5 and 6 one of these fixed points loses its stability and a cascade of period-doublings (-halvings) takes place.
- In the region 7 both fixed points are unstable and there exist two attractors, one “around” each of the fixed points.
Figure 5.7: The bifurcation diagram of the 2-Ricker-map (5.22) on which the curves corresponding to homoclinic bifurcations are represented. For further explanations see the text.

- In the regions 8 and 9 the map has a unique metric attractor which is a fixed point.

- In the regions 10 and 11 the map has again a unique attractor: the stable fixed point from the regions 8 and 9 is now unstable and there is an attractor around it.

- In the regions 13 and 14 the map has a stable fixed point, but another attractor might exist as well.

- In the regions 12, 15 and 16 the map can have two metric attractors which are not fixed points. Cascades of cusps are observed in these regions.

We notice that the regions 4, 6, 9, 11, 14, 16 are symmetric to 3, 5, 8, 10, 13, 15, respectively. Let us consider the region of three fixed points, which is the interior of the wedge between \( p_- \) and \( p_+ \) (\( p_- \) and \( p_+ \) are fold curves, see section 5.6). We note first that the points of intersection of the curves \( p_- \) and \( p_+ \) with the flip-curves \( p_1 \) and \( p_2 \) respectively do not correspond to
a local bifurcation of higher codimension because these bifurcations happen with different fixed points of the SYC-map.

In the region 2 of the bifurcation diagram the SYC-map has two stable fixed points and one unstable between them. The stable fixed points are the only attractors in the system because, as we have shown in section 5.7 (Corollary 5.39 (i)), the SYC-map has two metric attractors. Changing the parameters (but staying within the wedge), we intersect generically either the \( p_1 \) or the \( p_2 \) flip-curve. Hence one of the stable fixed points loses its stability and a two-cycle appears. As long as we move to larger values of \( R_0 \), but do not leave regions 5 or 6, a sequence of period-doubling or, possibly, -halving (see, for example, [Nusse & Yorke]) bifurcations of the two-cycle takes place. If we intersect the second flip-bifurcation curve and enter the region 7, the second fixed point becomes unstable and again a period-doubling cascade occurs. According to Corollary 5.39 in the regions 5–7 the map has two metric attractors.

If we intersect one of the homoclinic bifurcation curves, say, \( m_1 \), entering the regions 8 or 10, we have only one metric attractor left, which is in fact a fixed point in the region 8. Hence in the region 8 (and also in 9) the map has a stable fixed point (which attracts almost all initial values), a rather unexpected phenomenon. In the regions 12, 13 and 15, after the intersection of a second homoclinic curve either \( m_2 \) or \( M_1 \), this is no longer the case, i.e., we can have two attractors.

Numerical experiments, which we have made for the 2-Ricker-map, show however that the map can have two attractors also outside the wedge. More precisely, we have a cascade of cusp bifurcations ([Broer et al., Sections 4.2, 5.1.3] and references in there, see also [Skjolding et al.]). In Figure 5.8 we illustrate this cascade. The bifurcation diagram is produced by using the package CONTENT [Kuznetsov & Levitin]. Black curves correspond to the local bifurcations of the fixed points (cf. Figure 5.7), blue curves correspond to those of period 2 points, green curves — 4-points, red curves — 3-points.

Look at, say, the upper part of the diagram, at the region outside the black wedge. On the black flip curve the unique fixed point becomes unstable and a period-2 point occurs. The second iterate of the map has also a cusp bifurcation (blue one) associated with this point. Therefore we can have two period-2 points which can also become unstable, periodic and other attractors can arise around these points, the attractors can merge by a homoclinic bifurcation, i.e. 2-points repeat the fate of the fixed points of the map.

Above and below the blue wedge we have a period-4 point and similarly a cusp bifurcation (green one) in the fourth iterate of the map, et cetera.

A period-3 point (and other odd-period points) can not occur as a result of period-doubling. Hence we have a fold bifurcation resulting in the appearance
Figure 5.8: A numerical bifurcation diagram of the 2-Ricker-map (5.22). Black curves correspond to the local bifurcations of the fixed points (cf. Figure 5.7), blue curves correspond to that of period 2 points, green curves — 4-points, red curves — 3-points. A cascade of cusps is clearly presented. The bifurcation diagram is produced by using the package CONTENT [Kuznetsov & Levitin].
of such points. But the interesting thing is that even-period points can also appear in such fold bifurcations. Inside the black wedge, on the right of the bifurcation diagram we see that this happens for 4-points.

The precise pattern of cusps in the parameter plane is not clear, but we believe that in some intervals along a line $R_0 = const$ we have Sharkovski order of the cusps. This can be observed in Figure 3 on the back cover.

This figure is made using the package DYNAMICS [Nusse & Yorke: Dynamics]. We show the period of an attractor of the 2-Ricker-map (it is a so-called period plot). We can have bistability but we show only one of the attractors, namely, those which attracts the critical point of minimum $c = 1$ of the 2-Ricker-map. Fortunately, this critical point does not change its position as we change $R_0$ and $p$. The horizontal interval is $R_0 \in [0, 500]$, the vertical interval is $p \in [0, 40]$. Different colours correspond to different periods of attractors: cyan — fixed point (period 1), blue — period 2, red — 3, green — 4, yellow — 5, rose — 6, orange — 7, dark green — 8. Black regions are regions with other behaviour, e.g. higher periods, chaos.

Also in this figure we can see cusps inside the wedge (more precisely, in the regions 12, 15 and 16 of Figure 5.7).

If we follow an appropriate path in the bifurcation diagram, we can observe the Feigenbaum cascades of period-doublings. More interesting behaviour can be observed if we choose a path across flip curves (for example along a line with fixed $p$). If we intersect the same flip curve two times: at the first time we have the period-doubling bifurcation and, for example, a
two-cycle appears from the fixed point, at the second time a period-halving bifurcation takes place and the two-cycle lands onto the fixed point which becomes stable (Figure 5.9). In between, for not so large values of \( p \) (say, \( p = 11 \)), the two-cycle grows at first and then its amplitude decreases again. It forms a loop in a bifurcation diagram with coordinates \((R_0, x_{\text{attr}})\), where \( x_{\text{attr}} \) denotes points of an attractor. For larger values of \( p \), the period-doubling cascade does take place if we move along \( p = \text{const} \), but the cascade stops and evolves back (via period-halving bifurcation) forming so-called "bubbles" in a bifurcation diagram [Nusse & Yorke].

We have not used the word "chaos" yet. However, chaotic dynamics is indeed observed in the 2-Ricker-map. In Figure 1 on the back cover we present a chaos plot of the 2-Ricker map produced with the aid of DYNAMICS [Nusse & Yorke: Dynamics]. To make it nicely symmetric we choose \((p_1, p_0) = (\frac{R_0}{p}, p)\) as axes. If an attractor is chaotic, any two initially close points yield trajectories that diverge and the rate of the divergence is given by the so-called Lyapunov exponent. A necessary condition for an attractor to be chaotic is that a corresponding Lyapunov exponent is positive. In the figure black regions correspond to negative values of the Lyapunov exponent, while blue to yellow regions correspond to positive values of the Lyapunov exponent, moreover, the larger the Lyapunov exponent, the lighter the color of a point.
5.9. DISCUSSION. GENERAL $K$.

Remark. There is one more interesting feature of the bifurcation diagram for the Ricker map $x \mapsto R_0 x e^{-x}$. In Figure 5.10 we see parametric windows in which the map has a stable orbit of successive periods 3, 4, 5, ... . Similar windows we observe in the bifurcation diagram in Figure 2 on the back cover for the 2-Ricker-map (5.22). Figure 2 is a numerical bifurcation diagram, namely, a period plot, on which windows are shown in which the map has a stable orbit of successive periods 2, 3, 4, 5, ... . The horizontal interval is $p_0 \in [0, 12]$, the vertical interval is $p_1 \in [10, 150]$. Different colours correspond to different periods of attractors: cyan — fixed point (period 1), blue — period 2, red — 3, green — 4, yellow — 5, rose — 6, orange — 7, dark green — 8. Black regions are regions with other behaviour, e.g. higher periods, chaos.

Remark. To produce the figures on the back cover, we used not the 2-Ricker-map (5.22) itself, but a topologically conjugated map

$$y \mapsto y + \ln R_0 - e^y (1 + p \exp(-e^y)),$$

which is obtained from the 2-Ricker-map by the transformation $y = \ln x$, $x > 0$. This choice was dictated by numerical needs. Namely, for large values of $R_0$ and $p$ periodic points of the 2-Ricker-map, which lie near zero, are so close to each other that the programme is not able to distinguish between them. The logarithmic scale allows to overcome this problem.

Besides, if we consider a "logarithmic" version of the 1-Ricker-map

$$x \mapsto R_0 x e^{-x},$$

which is given by

$$y \mapsto y + \ln R_0 - e^y,$$

we will find that this expression has a structure similar to the well-know circle map

$$y \mapsto y + \rho + c_1 \sin y \mod 2\pi.$$

5.9 Discussion. General $k$.

In Section 5.2 we have defined a general SYC-map of order $k$, but most attention was paid to maps of order 2. The dynamics of a general SYC-map is beyond the scope of this work, but here we would like to mention some interesting properties of such maps.
Along the symmetry axis all parameters $p_i$ of the parametric SYC-map (5.7) are equal (and equal to a value $p$) and the map is just the $k$-th iterate of the building block map $x \mapsto pf(x)$. We assume that the dynamics of this building block map are known. What are the corresponding dynamics of the SYC-map? For example, let the building block map have a period-$n$ orbit. If $n$ is a multiple of $k$, namely $n = km$, the SYC-map has $k$ period-$m$ orbit. If $k$ is a multiple of $n$, the map has $n$ fixed points. Otherwise, it has again a period-$n$ orbit.

If an attractor in the SYC-map on the symmetry axis is structurally stable, small perturbations do not change its qualitative structure. Therefore we know the behaviour of the system in the neighbourhood of the axis. Besides it can serve as a starting point of a continuation analysis.

But what can we say about bifurcation points? We conjecture that a bifurcation point in the building block map (i.e., on the symmetry axis) is an intersection of corresponding bifurcation surfaces in the parameter space. For example, the first period-doubling of a 1-SYC-map is the cusp point in the corresponding 2-SYC-map (Proposition 5.24). A general question, that we leave out the scope of this article, is: given a bifurcation in the building block map, what does it "generate" as bifurcation on the symmetry axis for the $k$-SYC-map?

We have already mentioned that the multiplicative factor of the SYC-map $R_0$ has a clear biological meaning (it is the expected number of offspring per individual). Therefore, it makes sense to consider bifurcations which happen in the SYC-map when we change $R_0$. In particular, we choose all other parameters so that we are on the symmetry axis.

Let the building block map $x \mapsto pf(x)$ have the first period-doubling for some $p = p_*$ and let it have for $1 < p < p_*$ a (globally) stable (nontrivial) fixed point. Hence any iteration of this map has the same stable fixed point. And thus the $k$-SYC-map has a stable fixed point for all $1 < R_0 < p_*^k$. Therefore the interval of $R_0$ values for which the SYC-map is stable grows very rapidly with $k$. This has a very interesting (and counter-paradigmal) biological interpretation: the introduction of age-structure in the population model allows for a much wider range of basic reproduction ratio values for which the system is stable. We must say that a stable SYC fixed point corresponds to a $k$-years cycle with a single year class present, i.e. the population exhibits cyclic and not steady behaviour. However we consider this behaviour as "stable" comparing with (almost) irregular behaviour which corresponds to high-periods and chaotic attractors which one observes in the building block map for large values of $R_0$.

Another possibility to start continuation analysis is coordinate planes $\{ (R_0, p_i) : p_j = 0, j \neq i \}$ of the parameter space $\{R_0, p_0, ..., p_{k-2}\}$. On these
planes a SYC-map of order $k$ reduces to a 2-SYC-map whose dynamics are known.

The aim of this work was to introduce SYC-maps, a particular case of composition maps. They seem to be an interesting and a logical extension of unimodal maps. An extension in the sense that the dimension of the parameter space increases and also the number of fixed points increases. This allows to observe in these systems such interesting phenomena as homoclinic bifurcations and cascades of cusps.

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Appendices

A Basic notions of one-dimensional dynamics.

Here we present some basic notions of one-dimensional dynamics to make it easier for non-experts to understand the results. Definitions and results presented here are mostly taken from [Thunberg, De Melo & van Strien].

Definition 5.41. The Schwarzian derivative $SF$ of a function $F$ is defined by

$$SF = \frac{F'''}{F'} - \frac{3}{2} \left( \frac{F''}{F'} \right)^2$$

for all $x$ such that $F'(x) \neq 0$.

Definition 5.42. A continuous interval map $F: I = [a, b] \to I$ is unimodal if $F(\partial I) = \partial I$ and there is a unique point of maximum $c$ in the interior of $I$ such that $F$ is strictly increasing on $[a, c)$ and strictly decreasing on $(c, b]$. The map $F$ is S-unimodal if it is unimodal, $C^3$ and $SF(x) < 0$ for all $x \neq c$.

Definition 5.43. If a continuous map is increasing in a left neighbourhood of a point $c$ and decreasing in its right neighbourhood, or vice versa, the point $c$ is called a turning point.
• If for a $C^1$ map $F'(c) = 0$, the point $c$ is called a critical point.

• If for a $C^m$ map $c$ is a critical point such that $F^{(n)}(c) = 0, 1 < n \leq m$, it is called a flat critical point. Otherwise, it is nonflat.

If in a critical point the second derivative of the map is non-zero, this critical point is a turning point.

For $S$-unimodal maps we have the following significant property (which we copy from [Thunberg]).

**Theorem 5.44.** [Blokh & Lyubich] Let $F : I \to I$ be an $S$-unimodal map with nonflat critical point. Then $F$ has a unique metric attractor $\Omega$, such that the $\omega$-limit set $\omega(x)$ is equal to $\Omega$ for Lebesgue almost all $x \in I$. The attractor $\Omega$ is of one of the following types:

1) an attracting periodic orbit;

2) a Cantor set of measure zero;

3) a finite union of intervals with a dense orbit.

In the first two cases, $\Omega = \omega(c)$.

The definition of metric attractor can be found in [Milnor]. We repeat it here. A set $\Gamma$ is called forward invariant with respect to $F$ if $F(\Gamma) = \Gamma$. The set $B(\Gamma) = \{ x | \omega(x) \subset \Gamma \}$ is the basin of attraction of $\Gamma$.

**Definition 5.45.** [Milnor] A forward invariant set $\Omega$ is called a metric attractor if

1) $B(\Omega)$ has positive Lebesgue measure;

2) $\Omega$ is maximal in the sense that if $\Omega'$ is another forward invariant set, strictly contained in $\Omega$, then $B(\Omega) \setminus B(\Omega')$ has positive measure.

In Blokh & Lyubich theorem we read that the first two types of metric attractors attract the critical point $c$. To have this property it is not necessary to deal with a unimodal map. The following theorem says that the most important condition is negativity of the Schwarzian.

**Theorem 5.46.** [Singer] Let $F : I \to I$ be a $C^3$ interval map with negative Schwarzian derivative. Then the immediate basin of any stable periodic orbit contains either a critical point of $F$ or a boundary point of the interval $I$.

The immediate basin of an attracting periodic orbit is the union of the connected components of its basin which contain a point of the periodic orbit. In the case of a stable fixed point the immediate basin is an interval around the point.
APPENDIX B. Proofs of theorems from Section 5.6.

B Proofs of theorems from Section 5.6.

Proof of Theorem 5.25.

Proof. A nontrivial fixed point of the map (5.22) should satisfy the equation

\[ \nu(x, p) = \ln R_0, \]  

(5.37)

We can rewrite this equation in the form

\[ g(x) = p, \]  

(5.38)

with

\[ g(x) = e^x \left( \frac{\ln R_0}{x} - 1 \right). \]  

(5.39)

(i) If \( R_0 > e^4 \) the function \( g \) has two local extrema: a minimum at a point \( x_- \) and a maximum at a point \( x_+ \), where \( x_- \) and \( x_+ \) satisfy

\[ x^2 - \ln R_0 x + \ln R_0 = 0 \]  

(5.40)

and hence they are given by (5.24) (Fig. 5.11).

If \( p_- = g(x_-) < p < g(x_+) = p_+ \) then the equation (5.38) has three simple roots, and, consequently, the map (5.22) has three nontrivial fixed points.

The multiplier of a fixed point \( \bar{x} \) is equal to

\[ 1 - \bar{x} \nu_x(\bar{x}, p) \]  

(5.41)

with

\[ \nu_x(\bar{x}, p) = -\bar{x}g'(\bar{x})e^{-\bar{x}}. \]  

(5.42)

Since the derivative \( g'(\bar{x}) \) at the middle point is positive, the multiplier of this point is always bigger than +1. Thus the first assertion of the theorem is proved.
(ii) For \( p = p_\pm \) and \( R_0 > e^4 \) the equation (5.38) has exactly two roots (not counting multiplicity). If \( p = p_+ \) then the upper root coincides with the local maximum \( x_+ \) of the function \( g \), and if \( p = p_- \) the lower root is \( x_- \). From (5.41) and (5.42) we see that the multipliers of these points are 1, because the derivative \( g'(x) \) is zero.

(iii) If \( R_0 \leq e^4 \) or \( R_0 > e^4 \), but \( p \) lays outside the interval \([p_-, p_+]\), then the equation (5.38) has just one root.

The point \((e^4, e^2)\) is the cusp point in the plane \((R_0, p)\), where the curves \( p = p_+ \) and \( p = p_- \) intersect. It corresponds to the unique fixed point \( x = 2 \). Indeed, the functions \( g(x) \) is monotone in this case and its derivative is given by

\[
g'(x) = \frac{e^x}{x^2} (x - 2)^2.
\]

According to Theorem 5.22, this is a cusp point, since \( h(x) = e^{-x} \),

\[
\frac{xh'(x)}{h(x)} = -x.
\]

and the conditions (5.19) and (5.18) are satisfied \((p \equiv p_0 \text{ and } R_0 = p_0 p_1)\).

The cusp is generic and of spring type. Indeed, a Ricker map is \( HS \) and the condition (5.20) is also satisfied.

\[\square\]

**Proof of Theorem 5.28.**

*Proof.* We can rewrite the condition (5.25a) in the following form:

\[
\nu_x(\bar{x}, p) \bar{x} = 2,
\]

where a fixed point \( \bar{x} \) should satisfy the equation (5.38)–(5.39). From this we find that it satisfies

\[
x^2 - \ln R_0 x + \ln R_0 - 2 = 0. \quad (5.43)
\]

Hence \( \bar{x} = x_{1,2} \), where \( x_{1,2} \) are given by (5.27). To have nonnegative and bounded \( x \) and \( p \) we need \( R_0 > e^2 \). Thus we have proved that the flip bifurcation takes place under the conditions (5.27)–(5.28).

We are going to check the genericity conditions (5.26). Since the 2-Ricker-map has negative Schwarzian, (5.26a) is satisfied.

Now consider the transversality condition (5.26b). Let us fix \( R_0 \) and consider the derivative with respect to \( p \). At a flip point \((\bar{x}, \bar{p}, R_0)\)

\[
F_{xp}(\bar{x}, \bar{p}, R_0) = \bar{x}e^{\bar{x}(\bar{x} - 2)}.
\]

We notice that for \( R_0 > e^2 \) \( x_1 > 2 \) and \( x_2 < 2 \). Therefore, slightly under \( p_1(R_0) \) and slightly above \( p_2(R_0) \) the map has a stable two-cycle according to Lemma 5.27.

\[\square\]
Proof of Theorem 5.29.

Proof. The multiplier of a fixed point is given by (5.41). For the fixed point to be locally stable the absolute value of the multiplier should be less than 1. If the multiplier is 1, the fixed point is stable if it is our cusp point (with $F_{xxx} < 0$) but not if it is a generic fold point. If the multiplier is $-1$ it is stable if $SF < 0$ that is, indeed, the case. Using (5.39), (5.41) and (5.42) we find that this is equivalent to

$$0 < x^2 - \ln R_0 x + \ln R_0 \leq 2$$

(compare with (5.40) and (5.43)) or $x = x_{cusp}$. For $R_0 < e^4$ the left inequality is satisfied, while the right inequality holds if $x_2 \leq x \leq x_1$, where $x_{1,2}$ are given by (5.27). For $R_0 = e^4$ a fixed point $x$ is also stable in this interval because we allow for the cusp. For $R_0 > e^4$ (5.44) is equivalent to $x \in [x_2, x_-) \cup (x_+, x_1]$.

Fixed points of the map are given by (5.38). We consider the function $g$ for different values of $R_0$.

For $R_0 \leq e^4$ the function $g$ is monotone decreasing. Hence we have a unique nontrivial fixed point which is stable in two cases: $R_0 \leq e^2$ or $e^2 < R_0 \leq e^4 \cup p \in [p_1, p_2]$.

For $R_0 > e^4$ the function $g(x)$ is not monotone. If $p \notin [p_-, p_+]$ the SYC-map has still a unique fixed point which is stable if $p \in [p_1, p_2]$.

If the map has three fixed points, the lower $x_l$ and the upper $x_u$ of them can be stable if $x_l \in [x_2, x_-)$ and $x_u \in (x_+, x_1]$, respectively. Since the function $g(x)$ is decreasing on these intervals the conditions are equivalent to the conditions of stability given in assertions (ii) and (iii) of the theorem. \[\square\]