The geometry of conflict sets

De meetkunde van conflict verzamelingen
(met een samenvatting in het Nederlands)

Proefschrift

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Pourquoi donc, reprit le Sirien, citez-vous un certain Aristote en grec ? – C’est, répliqua le savant, qu’il faut bien citer ce qu’on ne comprend point du tout dans la langue qu’on entend le moins.

*Voltaire*, Micromegas

Schüler: Kann Euch nicht eben ganz verstehen.
Mephistopheles: Das wird nächstens schon besser gehen,
Wenn Ihr lernt alles reduzieren
Und gehörig klassifizieren.

*Goethe*, Faust
Preface

Sets in the middle

Conflict sets are sets where two wavefronts coming from different objects meet. To see a conflict set throw not one, but two stones in a pond. The interference pattern you get is a conflict set.

We will study how these are sets are curved, if they are smooth at all. When they are not smooth we will study their singularities. The main theme is that conflict sets are very much like wavefronts themselves.

The study of conflict sets was motivated by some very tangible geometric notions. The first of these is the notion of a Voronoi diagram. A Voronoi diagram is a division of the plane in different regions \( \{ V_i \}_{i=1,...,l} \), such that for \( x \in V_j \) the closest of a number of points \( \{ p_i \}_{i=1,...,l} \) is \( p_j \). Voronoi diagrams arise in many sciences, for instance the \( p_i \) can be roots of plants and the \( V_i \) the regions that each of the plants can take their water from.

The second of these is the idea of a skeletal set describing the form of an object in \( n \)-space. If we have a closed compact hypersurface \( M \) in \( \mathbb{R}^n \), for instance the surface of a dog bone, the distance from some interior point to \( M \) can have non-unique absolute minima: the minimal distance is \( d_0 \) and there are at least two points on \( M \) for which it is attained. Normally, there is just one absolute minimum and thus the locus of points where there are two minima forms a codimension one subset in the interior of \( M \). The set so obtained is commonly called, the medial axis or the central set. The two concepts are related as we can see from figure 1.

For the definition of the conflict set we will replace the points \( p_i \) in the definition of the Voronoi diagram by compact manifolds \( M_i \) and the critical values of the distance function are no longer required to be absolute minima. In this way we will obtain a set in the middle: the conflict set.

Generic differential geometry

Let \( M \) be an embedded manifold in an Riemannian manifold \( X \). In differential geometry one studies the properties of such an embedding that are invariant under isometries of the ambient manifold \( X \).
In this way a lot of results can be obtained. Even more can be said when one restricts attention to certain embeddings that we call “generic”. As the name suggests most embeddings are generic. Genericity is defined using transversality conditions. For instance, a generic line in \( \mathbb{R}^3 \), intersects the \( XY \)-plane in a single point. A line contained in the \( XY \)-plane can be rotated slightly in space. The rotated line intersects the \( XY \)-plane in just one point. 

Another example: most quadruples of points span a positive volume. Again, if the points do not span up a positive volume we can move one just a little bit and obtain four points that do span a positive volume.

What the transversality condition actually is varies from problem to problem but it is a method by which strong results can be obtained under mild conditions.

Here is one such result, concerning the envelope of the normal lines from a curve in the plane. Let \( \gamma : S^1 \to \mathbb{R}^2 \) be an embedding of the circle in the plane. We may think of an ellipse. At every point \( s_0 \) the curvature \( \kappa(s_0) \) and the normal \( \nu(s_0) \) are defined. The focal set or evolute, or caustic, is defined as the set of points in the plane traced out by \( \gamma(s) + \kappa^{-1}(s)\nu(s) \). It is the envelope of the normals to the curve \( \gamma \). Let us mention a well-known theorem.

**Theorem .1.** For a generic embedding \( \gamma \) the focal set \( F \) has the following property: every point \( p \in F \) has a neighborhood in \( p \in U \subset \mathbb{R}^2 \) such that the intersection \( U \cap F \) is diffeomorphic to one of the following algebraic varieties \( x_2 = 0 \), \( x_1x_2 = 0 \) or \( x_3^2 - x_1^2 = 0 \).

In this theorem we have characterized all generic local forms of the focal set. Locally only a few singular situations can arise. In chapter four we obtain a similar characterization of conflict sets in low dimensions.

**Cut locus, Maxwell strata, central set, medial axis, symmetry set**

In generic differential geometry one often studies singular sets which are associated to a distance function from a submanifold and which measure some sort of symmetry. The most well-known of these sets is probably the cut-locus associated to a Riemannian manifold \( X \) and a point \( p \). In a neighborhood \( U \) of \( 0 \in T_pX \) the exponential map

\[
\exp : T_pX \rightarrow X
\]

is a diffeomorphism: small vectors are mapped to geodesics that are globally minimizing. However, what is locally the shortest path need not be the shortest path globally. For each \( v \in T_pX \) with \( ||v|| = 1 \) let \( t = t(v) \) be the largest number such that for all \( 0 \leq s_0 \leq t(v) \) the path

\[
\{ s \in [0, s_0] | \exp(p, sv) \}
\]

is the shortest path from \( p \) to \( \exp(p, s_0v) \). Clearly \( t(v) \) might be infinite. If \( t(v) \) is finite the point \( \exp(p, t(v)v) \) is called the cut point of \( p, v \).

All cut points together form the cut-locus of \( X \) wrt. \( p \). At points of the cut-locus the exponential map \( \exp(p, \cdot) \) is no longer injective or its differential is no longer injective. If \( X \) is compact the cut-locus is a deformation retract of \( X \setminus \{p\} \).

Another such set is related to an embedded manifold \( M \subset X \), where \( X \) is supplied with a Riemannian metric. For \( (p, v) \in NM \), where \( ||v|| \) is small, that a neighborhood of \( p, v \) is mapped diffeomorphically to \( X \). Where the exponential map is no longer a injective or immersive we have a locus called the central set or medial axis. It is what Thom called the “cut-locus d’une variété plongée”, see [Tho72]. If \( M \) is a parabola and \( X \) is the plane it lies in we can see that the central set measures symmetry.

The central set of an ellipse in the plane consists of just one line segment. But the ellipse has two symmetry axes. So how do we incorporate the other axis?
The other axis is contained in the symmetry set. The symmetry set is the closure of those points \( p \in X \) where the distance function \( M \ni q \rightarrow d(p, q) \) has two non-degenerate critical points with the same critical value. When we consider the symmetry set of the ellipse in the plane we will get the desired two line segments.

There is a drawback here. The definitions of cut-locus and central set presuppose that we work in an ambient manifold \( X \) where the unit speed geodesics are defined for all times \( t \). We ask that \( X \) is geodesically complete.

Examples of geodesically complete spaces are compact Riemannian manifolds, say an ellipsoid \( \subset \mathbb{R}^3 \), with three different axes. A geodesic on an ellipsoid can be non-periodic. In fact most geodesics on ellipsoids are non-periodic. Hence, symmetry sets of curves on the ellipsoid might become quite awkward, because not just the absolute minimum but every critical value is considered. Completeness of \( X \) is not enough for the definition of symmetry set to give reasonable results.

To avoid unwanted behavior rather strict conditions have to be imposed on the ambient spaces that we consider. In this thesis these conditions are stated in section III.1.9. Basically \( X \) is assumed to be complete and all points in \( X \) are assumed to have an empty cut-locus.

It is not only in this respect that the symmetry set is different from the central set. Logically the two can exhibit quite different behaviour. Consider a compact manifold without boundary in \( \mathbb{R}^3 \). Suppose the surface is the surface of a generic smooth dog bone. At one of the sides we will find a point where the central set looks like we can see at the left hand side of figure 3. There are \( \binom{4}{2} = 6 \) planes intersecting in 4 lines meeting at a point. Centered at that point of the central set there is a sphere completely contained inside the dog bone that touches the surface of the dog bone in four points. Such a singularity happens generically. It cannot be removed by deforming the dog bone. However the symmetry set looks quite different at such a point. On the symmetry set the six planes continue: there are still critical values but they are no longer absolute minima.

Both the symmetry set and the central set can have what we call endpoints. Endpoints are points where the axes of symmetry of the ellipse stop. Such endpoints are really singular points of a focal set. Though interesting, we will not consider these endpoints here. Instead we will measure symmetry between several manifolds. We take \( l \) manifolds \( M_i \) in our ambient manifold \( X \) and define the conflict set as the closure of those points \( p \in X \) for which

![Figure 2](image-url)
there exist $q_i \in M_i$ such that the distance functions
\[ d_{i,p}: M_i \to \mathbb{R} \quad q_i \mapsto d_i(p, q_i) \]
all have a non-degenerate critical point and at the same level, that is, with the same critical value. Such points $q$ form the conflict set. Pictures of the conflict set are in the figures I.1, II.2 and a singular one in figure III.5.
To develop some intuition for the differences between the conflict set and the symmetry set consider again the example of figure 3. The sphere that touches four patches of surface can come up in three different types of conflict sets. First of all there is the conflict set of four (compact) surfaces in $\mathbb{R}^3$. Such a conflict set generically contains only isolated points. Hence when $l = 4$ and $\dim X = 3$ figure 3 reduces to the intersection point of six planes.
In the setting of three surfaces in 3-space their conflict set is a line. Figure 3 will happen when two of the four points lie on one surface $M_1$ and the other two lie each on a different surface: $M_2$ and $M_3$. The symmetry set of $M_1$ and the conflict set of $M_1$, $M_2$ and $M_3$ intersect. Figure 3 reduces to two intersecting curves.
In the setting of two surfaces in 3-space we can get a sphere touching four pieces of surface when two symmetry sets meet on a conflict set. From the six planes in figure 3 four remain. Hence, there is a correspondence between singularities of central sets, symmetry sets and conflict sets, but there are also differences.

**Symplectic and contact geometry**

The focal set we introduced above is an instance of a much studied class of manifolds: Lagrangian submanifolds of a symplectic manifold. We will use Lagrangian manifolds to study the singularities of conflict sets.
Lagrangian submanifolds arise as one tries to solve first order partial differential equations. The standard example of such a PDE is one that describes equidistants to a submanifold. If $u: X \to \mathbb{R}$ is the function that describes the distance from a point $x$ to a submanifold $M \subset X$ then at each point $x \in X$ we will have that the $\frac{\partial u}{\partial x}$ are unit length normal vectors. Thus we would need that
\[ \| \frac{\partial u}{\partial x} \| = 1 \]
Equation (1) describes a PDE, with initial condition $u|_{M} = 0$.

The solutions of the PDE are well-defined except at points of the caustic or the symmetry set. Hence the function that describes the distance to a submanifold is not really a function: we consider not the function but what would be the graph of $\frac{\partial u}{\partial x}$, this is a Lagrangian submanifold of $T^*X$, or the graph of $u, \frac{\partial u}{\partial x}$, this is a Legendrian submanifold in $J^1(X, \mathbb{R})$. If we project these graphs to $X$ the singular points of the projection form the focal set.

Lagrangian and Legendrian manifolds are usually constructed in a down to earth way. Let $d: X \times X \rightarrow \mathbb{R}$ be a metric on $X$. If $\gamma: M \rightarrow X$ is an embedding of $M$ then the distance from $s \in M$ to $x \in X$ is $F(x, s) = d(\gamma(s), x)$. The distance function has a critical point where

\[ \frac{\partial F}{\partial s} = \frac{\partial d(\gamma(s), x)}{\partial s} = 0 \]

Then where (2) holds the derivative of the distance “function” $u$ above is $\frac{\partial F}{\partial x}$.

In general if we have a family of functions $F(x, s)$ such that the set $\frac{\partial F}{\partial s} = 0$ still behaves reasonably - i.e. it is a manifold - then the “graph” over $X$ of $\frac{\partial F}{\partial x}$ is

\[ \Lambda_F = \{ x, \frac{\partial F}{\partial x} \mid \frac{\partial F}{\partial s} = 0 \}. \]

This “graph” is called a Lagrangian manifold. In chapter three we will elaborate on these remarks. It turns out that for conflict sets such a family of functions exists as well and generically it behaves well.

Outline of the results

The contents and results of this thesis are as follows.

- In chapter one we will define the conflict set of $l$ oriented hypersurfaces in $\mathbb{R}^n$. A remark concerning the smoothness of the conflict set is made. In chapters one and two we will study curvature and torsion properties of conflict sets, in terms of the properties of the base manifolds $M_i$. We will prove a formula for the curvature in the case where $l = 2$, and where $l = n = 3$. We will indicate how to proceed in the case $l = n > 3$. Chapter two concludes with a long computation that through an example proves the curvature formula for $l = n = 3$ in another way.
- The torsion of a conflict set is calculated in chapter two. The fact that in the case of spheres it is generically zero is the subject of a theorem in chapter 1.
- In chapter three we will prove that the singularities of conflict sets generically are of the same nature as the singularities of wave fronts. We also discuss what happens when the singularity of the conflict set is non-generic. The chapter starts with a review of the results from symplectic and contact geometry that we need. Part of the review is a slightly novel treatment of the Gauss map. We also use the language of symplectic geometry to rephrase some of the results of chapter one.
- In chapter four we discuss some notions similar to the conflict set. We introduce the center set, and show that it generically has the structure of a wavefront as well. We also discuss its relation to the center symmetry set introduced by Janeczko. In the last section we relate the notions of conflict set and center set to existing notions such as orthomtic, billiard transformation and pedal curve.
- In chapter five we classify singularities of conflict sets. In low dimensions local models for the conflict set are fabricated using non-versal deformations of sums of the well-known $ADE$-singularities. We determine explicitly in which dimensions there is a finite list
of singularities. The methods employed here are very similar to ones used to classify
singularities of caustics and wavefronts. We thus try to put our results in that perspective,
reviewing some of these things. It turns out that our analysis still gives rise to some
surprises. For instance from the generic point of view the singularities of the conflict set
of $n$ surfaces in $\mathbb{R}^n$ are the same as those of 3 surfaces in $\mathbb{R}^3$.
The reader is notified that a shorter version of chapters one and two is in [vM03]. Chapters
three, four and five are an extension and improvement of [vM]. Chapter 3 was the subject
of talks of the author at conferences in the second half of 2000 in Cambridge and Liverpool.
I Curvature for two surfaces

I.1. Introduction

Let $M_1, \cdots, M_l$ be $l$ manifolds in $\mathbb{R}^n$, all of smoothness $C^j$ with $j \geq 1$ and of dimension $n-1$. Suppose the $M_i$ are orientable. The orientations supply us with $n_1, \ldots, n_l$, unit sections of the conormal bundle $N^* (M_i)$ of the $M_i$ in $\mathbb{R}^n$. We can speak of wavefronts that emanate from the $l$ manifolds in distinct directions as “time” increases or decreases.

The conflict set $M_c$ of $(M_1, n_1), (M_2, n_2), \ldots, (M_l, n_l)$ is defined as the set of points where these wavefronts meet. The conflict set of $M_1, M_2, \ldots, M_l$ is defined as the union of all the conflict sets associated to particular configurations of orientations. The symmetry set of $(M_i, n_i)$ is defined as the set of points traced out by the self intersections of the wavefronts of $M_i$.

Given the second fundamental forms of $M_1$ and $M_2$, can we determine the second fundamental form $\mathbf{II}_c$ of $M_c$?

In the next few theorems and their proofs we will assume that the distance function for each of the $k$ manifolds has only regular points on the conflict set and that - again on the conflict set - the wavefronts travel in different directions, or if they travel in the same direction, they should do so at different speeds. This is to say that the sphere with center $c$ on the conflict set and touching one of the $M_i$'s, touches at one point $p$ only and $c$ is not one of the centers of curvature of $p$. A more precise notion is provided in paragraph I.3. The regular part of the conflict set makes up what one might call the $A^1$ stratum. It consists of those points that do not correspond to ones on the symmetry and/or focal set of the $M_i$.

**Theorem I.1.** Let $M_i, i = 1, \cdots, l \leq n$, be hypersurfaces in $\mathbb{R}^n$ of smoothness $C^j$, $j \geq 1$. Let wavefronts at constant speeds $\lambda_i$ emanate from $M_i$. The conflict set $M_c$ is at least $C^3$ at regular points.

**Theorem I.2.** If $M_1$ and $M_2$ have second fundamental forms $\mathbf{II}_1$ and $\mathbf{II}_2$ then $M_c$ has second fundamental form $(2 \cos \phi)^{-1} \mathbf{P}^T (\mathbf{II}_1^* - \mathbf{II}_2^*) \mathbf{P}$, where $2\phi$ is the angle that the tangent spaces encompass, $\mathbf{II}_1^*$ denotes the second fundamental form at the point where the wavefronts meet, and $\mathbf{P}: T_p \mathbb{R}^n \rightarrow T_p M_c$ is the projection - restricted to the tangent space of the wavefronts of the $M_i$. The two manifolds are required to have smoothness $C^2$.

**Theorem I.3.** Let wavefronts emanate from $M_i$, $i = 1, \cdots, l$, spheres with different radii in $\mathbb{R}^n$, $l \leq n$. Let the convex hull of the centers of these scaled spheres be at least $l - 1$ dimensional. The $A^1$ stratum of $M_c$ is a conic section in an $(n-l+2)$-dimensional affine subspace of $\mathbb{R}^n$.

**Example I.4.** Consider two circles $\alpha_1$ and $\alpha_2$ in the plane, with centers $(0, b_1)$ and $(0, -b_2)$ and radii $r_1$, $r_2$, such that $0 < b_1 < b_2$ and $b_1 + b_2 \neq \pm r_1 \pm r_2$. For each configuration of normals a separate conflict set arises. In concordance with theorem I.3 the conflict set is with all configurations a conic section. See the next chapter for a more meaningful example.

In the papers [Sie99] and [SSG99] similar results concerning curves in $\mathbb{R}^2$ and convex hypersurfaces in $\mathbb{R}^3$ are proved. Our results however are more general and our method of proof is completely different. Similar calculations are in [Ber95] and [BW59]. In section IV.4 we will explain their relevance and discuss some applications of the curvature formula. The main lemmas we use to prove theorem I.2 are reexamined at the end of chapter three. The proofs here are lengthy but elementary and thus may provide some more insight.
1.2. Preliminaries on contact between submanifolds

Let \( M \) and \( N \) be two manifolds in \( \mathbb{R}^n \) both of dimension \( m \). If \( M \cap N \) is not empty we can study the contact between \( M \) and \( N \) at points \( p \) in \( M \cap N \). Let \( \sigma_N \) be locally a submersion for \( N \) and \( \iota_M \) be an immersion for \( M \) near \( p \). The contact map for \( M \) and \( N \) at \( p \) is \( \kappa_{N,M} = \sigma_N \circ \iota_M \). If \( d^i \kappa_{N,M} \) is zero for \( i < k \) then we say that \( M \) and \( N \) have \( k \)-contact at \( p \).

The \( \kappa \)-class of \( \kappa_{M,N} \) is the equivalence class of functions \( f : \mathbb{R}^m \mapsto \mathbb{R}^{n-m} \) that are contact equivalent to \( \kappa_{M,N} \). If \( K \) and \( L \) are also \( m \)-manifolds in \( \mathbb{R}^n \) and they have such a function as contact map at \( q \) then there is a diffeomorphism \( \phi \) carrying a neighborhood of \( p \) to a neighborhood of \( q \) and such that \( \phi(M) = K \) and \( \phi(N) = L \).

It is also possible to consider contacts between anequidimensional manifolds. Let \( M_1 \) have dimension \( m_1 \) and \( M_2 \) dimension \( m_2 \), and \( m_1 < m_2 \). Put

\[
\kappa_{M_1,M_2} = \sigma_{M_1} \circ \iota_{M_2} : \mathbb{R}^{m_2} \mapsto \mathbb{R}^{n-m_1}
\]

To compare two contacts one considers the \( \kappa \)-class of \( \kappa_{M_1,M_2} \). A result of Montaldi says that \( \kappa_{M_1,M_2} \) is a trivial unfolding of \( \kappa_{M_2,M_1} \), see [Mon83].

If \( M_1 \) and \( M_2 \), with \( \dim M_1 \leq \dim M_2 \), have at least 2-contact, then the tangent space of \( M_1 \) at \( p \) will be contained in \( T_p(M_2) \). In this case we also employ the notion of \( k + 1 \)-contact in a certain direction \( V \) of \( T_p(M_1) \). \( V \) will in general be a subspace of the tangent space.

**Definition I.5.** \( M_1 \) and \( M_2 \) have \((k+1)\)-contact, with \( k > 0 \) in the direction \( V \subseteq (T_pM_1 \cap T_pM_2) \) if

1. \( M_1 \) and \( M_2 \) have \( k \)-contact at the point \( p \) and,
2. there is a third manifold \( L_1 \) with \( T_pL_1 = V \neq (T_pM_1 \cup T_pM_2) \) and such that \( L_1 \) has \((k+1)\)-contact with both \( M_1 \) and \( M_2 \).

Our main interest will be in two manifolds that have 3-contact at \( p \) in a certain direction. Here the second derivative of the contact map \( d^2 \kappa_{M_1,M_2} \) and its pushforward along \( \iota_{M_1} \) is a bilinear form - possibly vector valued - on the tangent space of \( \mathbb{R}^n \) at \( p \) with values in a euclidean space. We say that a vector \( v \in T_p\mathbb{R}^n \) is in the kernel of \( \iota^* d^2 \kappa_{M_1,M_2} \) if

\[
v^T \iota^* d^2 \kappa_{M_1,M_2} v = 0.
\]
Example I.6. Let a hypersurface $M$ in $\mathbb{R}^{n+1}$ be given by an immersion $\iota_M = (t, f(t))$, with $f: \mathbb{R}^n, 0 \mapsto \mathbb{R}, 0$. Let $N$ be another hypersurface with immersion $\iota_N = (t, g(t))$ and $g: \mathbb{R}^n, 0 \mapsto \mathbb{R}, 0$. The hypersurfaces have $k$-contact iff. the first $k - 1$ derivatives of $f$ and $g$ agree in 0.

Lemma I.7. Two manifolds $M_1$ and $M_2$ have 3-contact at the point $p$ in the directions $W \subset (T_pM_1 \cap T_pM_2)$ iff. $v^T v^* d^2 f_i v = 0, \forall v \in W$ and for all components $f_i$ of the contact map $\kappa_{M_1, M_2}$

Proof. "\(\Rightarrow\)" Let $L$ be the manifold that has 3-contact with both $M_1$ and $M_2$. Let $\dim L = l \leq m_1 = \dim M_1$ and $m_1 \leq m_2 = \dim M_2$. We can assume that $L = \mathbb{R}^l \times \{0\}$. Because, at $p = 0$ the tangent space of $L$ is contained in the tangent space of $M_i$ we can write the immersions of $M_1$ and $M_2$ as

$$\iota_{M_1} = (s_1, \ldots, s_{m_1}, f_{m_1+1}(s_1, \ldots, s_{m_1}), \ldots, f_n(s_1, \ldots, s_{m_1}))$$

and

$$\iota_{M_2} = (s_1, \ldots, s_{m_2}, g_{m_2+1}(s_1, \ldots, s_{m_2}), \ldots, g_n(s_1, \ldots, s_{m_2}))$$

Because

$$d^2 \kappa_{L, M_2} = 0$$

and

$$d^2 \kappa_{M_1, L} = 0$$

are zero by hypothesis it follows that

$$d^2 \kappa_{M_1, M_2} = 0$$

in the point $p = 0$. Here $d_i$ denotes the derivative with respect to $(x_1, \ldots, x_l)$. This is exactly what we needed to prove.

"\(\Leftarrow\)" Choose an orthonormal basis for $T_pM_1$ so that $W$ is spanned by $e_1, \ldots, e_k$, the first $k$ basis vectors. $L$ is then the manifold that is the image of

$$\iota_{M_1}(x_1, \ldots, x_k, 0, \ldots, 0). \quad \square$$

From the proof it is clear that a similar statement holds for higher order contacts.

1.3. Proof of theorem I.1

If the speeds of the wavefronts are all equal we can think of spheres centered at the conflict set having at least 2-contact with each of the $M_i$. Note that there may be more than one sphere with center on the conflict set, i.e. the conflict set of oriented hypersurfaces may have self-intersections.

When the speeds of the wavefronts are different, that is $\lambda_i$, each of the spheres centered on the conflict set decomposes into $l$ different spheres. The ratio of the radii of these spheres will be $\lambda_1 : \ldots : \lambda_l$.

The spheres in both cases will be called the kissing spheres, the terminology coming from the two-dimensional case. The points where the kissing spheres $(\alpha_j)_i$ of a point $p$ on the conflict set $M_c$ and $M_i$ touches $M_i$ will be called the basepoints $p_j$. At each basepoint we have a singularity type of the contact between the kissing sphere and the corresponding $M_i$. This is the same as the singularity type of the distance function $d(p, \cdot)$ on $M_i$. The following definition ensures that we are away from focal sets and also away from selfintersections.

Definition I.8. A point $p$ on the conflict set $M_c$ is called regular if
For each $i$ the distance function $s \to d(p, s)$ $d: M_i \to \mathbb{R}$ is smooth and has a non-degenerate critical point at the basepoint $s = p_i$.

The vectors
\[
n_i^* = \left( \lambda_i, \frac{\partial d}{\partial x} \big|_{x=p} \right)
\]
are linearly independent.

Even in the generic case - we won't specify what that means until chapter 3 - we expect other than regular points on the conflict set. The definition of regularity assures that the conflict set is an immersed submanifold at $p$. It does not assure that the conflict set is an embedded submanifold. A weaker version of the theorem I.1 is in fact a known result, see [JB85]. A short version of this argument goes as follows.

The distance functions induce a mapping of
\[
\mathbb{R}^n \times \prod_{i=1}^l M_i \mapsto \prod_{i=1}^l J^1(M_i, \mathbb{R})
\]

The projection of the $A_i^j$ stratum on to the first factor is an immersed $C^{j-1}$ manifold. This stratum represents exactly those points $q$ that are the center of a sphere having 2-contact with all manifolds $M_i$.

However, this proves only that the conflict set is $C^{j-1}$.

We have conormal bundles $N^*(M_i)$. Denote by $*_{i}$ a map $N^*(M_i) \to \mathbb{R}^{n+1}$ defined by
\[
*_{i}: (p, \xi) \mapsto (p + \xi \vec{n}(p), \lambda_i \xi)
\]
The images of these maps in $\mathbb{R}^{n+1}$ are $n$-manifolds, possibly singular. In the literature one sometimes uses the term big wavefront for the image of $N^*(M_i)$ by $*_{i}$. To check whether the image is a manifold around a point $*_{i}(p, \xi)$ we have to check two things. Firstly the map should be injective and secondly the derivative should have rank $n$. The first condition fails when there is more than one basepoint. Hence we have to be away from the symmetry set of $M_i$. The second condition fails when the first $n$ coordinates represent the point on the focal surface corresponding to the basepoint or one of the basepoints, see [Mil63]. This is exactly the definition of a regular point.

Now the intersection of the $k$ images in $\mathbb{R}^{n+1}$ is a transversal intersection at all but the above mentioned points, because $\vec{n}_i \lambda_i \neq \vec{n}_j \lambda_j$. Furthermore it is $C^{j-1}$. The projection onto the first $n$ coordinates has no critical points because the vector that is the direction of projection
\[
(0, \cdots, 0, 1) \in \mathbb{R}^{n+1}
\]
ever lies in the tangent space of the big wavefront. The projection is also the conflict set and thus we have established that the conflict set is $C^{j-1}$. But in fact we shall see that the big wavefront is a $C^j$ manifold. And thus the conflict set is $C^j$.

A nice proof of the fact that the big wave front of a hypersurface $M_1$ at points where the distance function has regular values has the same smoothness as $M_1$ was supplied by J. J. Duistermaat.

Let $p$ be a point on the original hypersurface $M$. Let $q$ be a point on the wavefront $W$ at distance $d$. If $q$ is a regular point of the distance function on $W$ with $p$ being the basepoint of $q$ then the conormal bundle $N^*W$ has the same smoothness as $N^*M$ around these points. They both have smoothness $C^{j-1}$.

But the conormal bundle of a manifold $W$ has smoothness $C^{j-1}$ iff. the the manifold itself has
smoothness $C^j$. Thus the wavefronts have equal smoothness at regular points of the distance function. As the big wavefront is a union of these wavefronts, smoothly parameterized, the big wave front is also $C^j$.

I.4. Proof of theorem I.2

The proof of the theorem will depend on some lemmas. Basically we will construct for each basepoint a Meusnier sphere to the $M_i$ it lies on. The Meusnier spheres $N_i$ will have 3-contacts in appropriate directions $V_i$. As a consequence the intersections of the images of $N^*(N_i)$ and those of $N^*(M_i)$ in $\mathbb{R}^{n+1}$ will also have a 3-contact in some direction $V_c$. With this we can find the second fundamental form of the conflict set.

**Lemma I.9.** Let $M$ and $N$ have 3-contacts in the direction $V \subset T_{p}M$. Then $M^* = *_{M}(M)$ and $N^* = *_{N}(N)$ have 3 contact in the directions of the linear span of $(n,1), n \in N_pM$ and $\{(v,0) \mid v \in V\}$

**Proof.** It is enough to consider the Monge forms - that is : the second order Taylor approximations of the immersions and submersions of $M$ and $N$. Thus choose coordinates such that

$$\iota_M(t_1, \ldots , t_{n-1}) = (t_1, \ldots , t_{n-1}, (t_1, \ldots , t_{n-1})^T A(t_1, \ldots , t_{n-1}))$$

where $A$ is a symmetric matrix.

In the same fashion presume that

$$\sigma_N(x_1, \ldots , x_n) = x_n - (x_1, \ldots , x_{n-1})^T B(x_1, \ldots , x_{n-1})$$

The contact map between the manifolds is then

$$(x_1, \ldots , x_{n-1})^T(A - B)(x_1, \ldots , x_{n-1})$$

For vectors in $v \in V$ we will have $v^T (A - B)v = 0$. We shall write $(t_1, \ldots , t_n) = (t', t_n)$ and $(s_1, \ldots , s_n) = (s', s''_n)$. The normal $n_M$ to $M$ in a neighbourhood of 0 is:

$$n_M = (- \sum_k A_{k1} t_1, \ldots , - \sum_k A_{k,n-1} t_{n-1}, 1) = (-A t', 1)$$

Although this normal does not have unit length we can use it to write $*_{M}$ in coordinates: $*_{M} = (\iota_M + t_n n_M, t_n)$.

We make a transformation on the first $n$ coordinates of $\mathbb{R}^{n+1}$ by

$$H : \begin{cases} s_i = (*_{M})_i & i = 1, \ldots , n-1 \\ s_n = t_n \end{cases}$$

This transformation is not singular in some neighborhood of 0 for $\frac{\partial H}{\partial t}(0) = I$. Its inverse can be calculated when $t$ is small:

$$t' = \left(I + s_n A + \frac{s_n^2}{2} A^2 + \ldots \right) s'$$

It is also convenient that the transformation $H$ is the identity map on the tangent space of $M$, $N$, $M^*$ and $N^*$.

We are interested in the Monge form of $*_{M}$ and $*_{N}$ in these new coordinates. The map $*_{M}$ has the following particularly simple form:

$$*_{M} = (s_1, \ldots , s_{n-1}, (s_1, \ldots , s_{n-1})^T A(s_1, \ldots , s_{n-1}) + s_n, s_n),$$
We will compute a Monge form for \( N^* \). The first \( n - 1 \) components of \( *_N \) are
\[
(I - t_n B) t' = s' + s_n (A - B) s' + o(t^2)
\]
The other components are simply \( s'^T Bs' + s_n \) and \( s_n \). The submersion for \( M^* \) is
\[
\sigma_{M^*} = x_n - x_{n+1} - (x_1, \ldots, x_{n-1})^T A(x_1, \ldots, x_{n-1})
\]
and as a consequence
\[
\kappa_{M^*, N^*} = \sigma_{M^*} \circ *_N = s'^T Bs' - s'^T As'
\]
whereby the lemma is proved.

At \( p = 0 \) we have 2 pairs of hypersurfaces: \( M_1, N_1 \) and \( M_2, N_2 \). Each element of a pair
intersects transversally with elements of the other pair and the elements in one pair have
identical tangent spaces. We have a third pair of manifolds: \( M_c = M_1 \cap M_2 \) and \( N_c = N_1 \cap N_2 \).

**Lemma I.10.** If \( M_1 \) and \( N_1 \) have 3-contact in the directions \( V_a \) and \( M_2 \) and \( N_2 \) have 3-contact in the directions \( V_b \) then \( M_c \) and \( N_c \) have 3-contact in the directions \( V_c = V_a \cap V_b \).

**Proof.** As in the proof of the previous lemma we will only consider Monge forms. Of
course we want to calculate the contact map. The submersion for \( M_c \) is quickly found:
\[
\sigma_{M_c} = (\sigma_{M_1}, \sigma_{M_2})
\]
To find an immersion for \( N_c \) first notice that the immersion for \( N_1 \) and \( N_2 \) can be written
in the form:
\[
\begin{align}
\iota_{N_1} &= (t_1, \ldots, t_{n-2}, R(\phi)(t_{n-1}, t'^T A t')) \\
\iota_{N_2} &= (t_1, \ldots, t_{n-2}, R(\psi)(t_{n-1}, t'^T B t'))
\end{align}
\]
Here we wrote \( R(\phi) \) for a rotation by an angle \( \phi \). The points of \( N_c \) are the points where
\[
\iota_{N_1}(t') = \iota_{N_2}(s')
\]
from which we conclude that
\[
s_i = t_i, \quad i = 1, \ldots, n - 2
\]
We have \( s_{n-1} = s_{n-1}(s_1, \ldots, s_{n-2}) \) and \( t_{n-1} = t_{n-1}(t_1, \ldots, t_{n-2}) \). With these relations
we obtain two different immersions for \( \iota_{M_c} \), number one is derived from \( \iota_{N_1} \) and \( t_{n-1} = t_{n-1}(t_1, \ldots, t_{n-2}) \), number two from \( \iota_{N_2} \) and \( s_{n-1} = s_{n-1}(s_1, \ldots, s_{n-2}) \). We combine this
with the above obtained to obtain a contact map.
\[
\kappa_{M_c, N_c} = (\sigma_{M_1} \circ \iota_{N_1}|_{t_{n-1}=t_{n-1}(t_1, \ldots, t_{n-2})}, \sigma_{M_2} \circ \iota_{N_2}|_{t_{n-1}=s_{n-1}(t_1, \ldots, t_{n-2})})
\]
Now we want to calculate the second derivative of this map. We will start with the first
component.
\[
\begin{align}
\frac{\partial \kappa_{M_1, N_1}|_{t_{n-1}=t_{n-1}(t_1, \ldots, t_{n-2})}}{\partial (t_1, \ldots, t_{n-2})} &= \frac{\partial \kappa_{M_1, N_1}}{\partial (t_1, \ldots, t_{n-2})} + \frac{\partial \kappa_{M_1, N_1}}{\partial t_{n-1}} \frac{\partial t_{n-1}}{\partial (t_1, \ldots, t_{n-2})} \\
\frac{\partial^2 \kappa_{M_1, N_1}|_{t_{n-1}=t_{n-1}(t_1, \ldots, t_{n-2})}}{\partial (t_1, \ldots, t_{n-2})^2} &= \frac{\partial^2 \kappa_{M_1, N_1}}{\partial (t_1, \ldots, t_{n-2})^2} + \frac{\partial \kappa_{M_1, N_1}}{\partial t_{n-1}} \frac{\partial^2 t_{n-1}}{\partial (t_1, \ldots, t_{n-2})^2} + \frac{\partial^2 \kappa_{M_1, N_1}}{\partial t_{n-1}^2} \frac{\partial t_{n-1}}{\partial (t_1, \ldots, t_{n-2})}
\end{align}
\]
Because $M_1$ and $N_1$ have 3-contact in some direction we will surely have

$$\frac{\partial \kappa_{M_1,N_1}}{\partial t_n} = 0$$

and thus we are to evaluate the derivative

(I.5) \[ \frac{\partial t_{n-1}}{\partial (t_1, \ldots, t_{n-2})} \]

in $p = 0$. As the surface $x_{n-1} = x_n = 0$ is the tangent space of $M_c$ in $0$ we can see that the derivative (I.5) evaluates to 0 in $p = 0$. If we write for a $n - 1 \times n - 1$ matrix $C$

$$C = \begin{pmatrix} C_{(n-2,n-2)} & C_{(n-1)} \\ C_{(n-1)} & C_{n-1,n-1} \end{pmatrix}$$

then the second derivative of the contact map of $M_c$ and $N_c$ in $0$ equals

$$\left( d^2 \kappa_{M_1,N_1(n-2,n-2)}, d^2 \kappa_{M_2,N_2(n-2,n-2)} \right)$$

This proves the lemma.

□

**Lemma I.11.** Let $M$ and $N$ be manifolds $\mathbb{R}^{n+k}$ with a 3-contact in the direction $V \subset T_p \mathbb{R}^{n+k}$. Let $\text{pr} : \mathbb{R}^{n+k} \to \mathbb{R}^n$, such that $T_p M$ does not lie in the direction of the projection: $\dim V = \dim \text{pr}(V)$ then $\text{pr}(M)$ and $\text{pr}(N)$ have 3 contact in the direction $\text{pr}(V)$ at $\text{pr}(p)$.

**Proof.** The proof consist of a simple verification with the contact map. □

After these lemmas we are ready to complete the proof I.2. Suppose we are at a point $p$ in $M_c$.

There are two base points $p_a$ and $p_b$ on $M_1$ and $M_2$ respectively. Propagate the wave fronts until the three points $p_a$, $p_b$ and $p_c$ fall together, say $p$. Now take an arbitrary direction in the tangent space of $M_c$, say $v_c$. The direction $v_c$ projects to directions $v_1$ and $v_2$ in $T_p M_1$ and $T_p M_2$ respectively.

We will proceed to construct the big wavefront of $M_1$ and $M_2$: $M_1^h$ and $M_2^h$. Because $p$ is a regular point for the distance function at $(p, 0) \in \mathbb{R}^{n+1}$ they intersect transversally in a manifold $M_c^h$. The manifold $M_c^h$ projects down to $M_c$ in $\mathbb{R}^n$.

There are Meusnier spheres to $M_1$ and $M_2$ in the directions $v_1$ and $v_2$: $N_1$ and $N_2$. Their big wavefronts can also be constructed: $N_1^h$ and $N_2^h$.

We apply lemma I.9 to obtain that $N_1^h$ and $M_1^h$ have 3-contacts in the direction $V_1 = \text{sp} \left( \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} n_1 \\ 1 \end{pmatrix} \right)$.

In a similar vein we define $V_2$.

We define $N_1^h$ as the transversal intersection of $N_1^h$ and $N_2^h$ at $(p, 0)$. By lemma I.10 $M_1^h$ and $N_1^h$ have 3-contact in the direction $V_c = V_1 \cap V_2$. The directions in $V_c$ can be projected down to $\mathbb{R}^n$ by a projection $\text{pr}$.

We claim that

(I.6) \[ \text{pr} (V_1 \cap V_2) = v_c. \]

To prove equation I.6 we first try to solve the equation

$$\lambda_1 \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} n_1 \\ 1 \end{pmatrix} = \lambda_3 \begin{pmatrix} v_2 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} n_2 \\ 1 \end{pmatrix}$$
If \( n_1 = n_2 \) then also \( v_1 = v_2 \). In this case \( \lambda_2 = \lambda_4 = 0 \) and equation I.6 will hold. Thus we will assume \( \lambda_2 = \lambda_4 \neq 0 \) and we can write (with different \( \lambda_i \))

\[
(I.7) \quad \lambda_1 \left( \begin{array}{c} v_1 \\ 0 \\
1
\end{array} \right) + \left( \begin{array}{c} n_1 \\
1
\end{array} \right) = \lambda_2 \left( \begin{array}{c} v_2 \\ 0 \\
1
\end{array} \right) + \left( \begin{array}{c} n_2 \\
1
\end{array} \right)
\]

Because \( n_1 \neq n_2 \) we can write \( n_c = n_1 - n_2 \). Now \( v_1 \) and \( v_2 \) are projections of \( v_c \) onto the tangent planes of \( T_pM_c \). The tangent planes of \( M_1 \) and \( M_2 \) make an equal angle with \( M_c \). Therefore we can solve I.7 if we put \( \lambda_1 = \lambda_2 \). The line that is thus in \( V_c \) is of the form

\[
v_c^h = \left( \begin{array}{c} v_1 \\ 0 \\
1
\end{array} \right) + \lambda \left( \begin{array}{c} n_1 \\
1
\end{array} \right) = \left( \begin{array}{c} v_2 \\ 0 \\
1
\end{array} \right) + \lambda \left( \begin{array}{c} n_2 \\
1
\end{array} \right)
\]

Equation I.6 is proved.

We also know that \( \text{pr}(v_c^h) \) makes an equal angle with \( v_1 \) and \( v_2 \) so that we will have

\[
v_c^h = \left( \begin{array}{c} v_1 + v_2 \\ \mu
\end{array} \right)
\]

Projecting \( N_c^c \) down to \( N_c \) we see that this projection is regular around \( p \) and that, by lemma I.11, the manifold \( N_c \) will have 3-contact with \( M_c \) at \( p \) in the direction \( v_c \). Thus the curvature of \( M_c \) in the direction \( v_c \) is that of \( N_c \). Indeed, intuitively, this is the direction in which a minimal change in chord length is achieved.

We now see that it is enough to calculate the curvature of the spheres \( N_1 \) and \( N_2 \) in the directions \( v_1 \) and \( v_2 \) and then to compute the curvature of the conflict set of the spheres in the direction \( v_c \). In particular, for two wavefronts starting from a plane curve, it will be enough to calculate the curvature of the conflict set of two curvature circles. Example I.4 is characteristic. The conflict line is parameterized by \( (\sqrt{b_1 b_2} \sinh t, 2^{-1}(b_1 - b_2)(1 - \cosh t)) \).

Its curvature is

\[
(I.8) \quad \frac{1}{2} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \cos \phi
\]

where \( T_1 \) and \( T_2 \) are the distances from the conflict set to the curvature centers, and \( \phi \) is half on the angle that \( n_1 \) and \( n_2 \) thus their inverses are the curvatures of the circles at the point where the wavefronts meet.

**Remark I.12.** For a further discussion of (I.8) and its applications see section IV.4.

**Remark I.13.** Sometimes it is more convenient to choose \( n_1 \) and \( n_2 \) so that \( n_c = n_1 + n_2 \). In this case the minus sign in equation I.8 becomes a plus sign. Thus, in equation I.8 the distances \( T_1 \) and \( T_2 \) are measured with respect to the direction of the normals \( n_1 \) and \( n_2 \). If one chooses the normals to the wavefronts as in the setup then the formula holds but when one chooses the distances differently the mentioned sign change occurs. In the sequel we will stick to the original setup where the sign change does not occur.

**Example I.14.** In figure I.2 one sees two circles with their conflict set. The conflict set is an ellipse. The normals at the left side are the normals that give this ellipse as conflict set. But, chosen thus \( T_1 \) and \( T_2 \) will have different signs.

The normals as chosen in the picture on the right do not correspond to the conflict set as shown. However if we use these normals to compute the curvature of the conflict set as shown in the picture we need to change the minus sign in equation I.8 into a plus sign.

With two spheres in \( \mathbb{R}^n \) the conflict set is a hyperboloid or ellipsoid. \( v_1, v_2, v_c \) and \( n_c \) all lie in one 2-dimensional plane. Thus the above formula also applies to each direction in \( T_pM_c \). When \( v_c \in T_pM_1 \cap T_pM_2 \) the angle \( \gamma \) will be 0. In that case we can simply add the
curvatures of the circles we obtain. The spheres have radius $T_1$ and $T_2$. The corresponding circles have radius $T_1 \cos \phi$ and $T_2 \cos \phi$ where $\phi$ is the angle between $n_1$ and $n_c$. In the other case $v_c \notin T_p M_1 \cap T_p M_2$ and we choose $v_c$ to lie orthogonally to this intersection. Then $v_1$, $v_2$ and $v_c$ lie in the plane spanned by $n_1$, $n_2$ and $n_c$. The angle $v_c$ and $v_1$ make in this case is also $\phi$. Thus we have formulas for the curvature, in the two cases an illustration of which is in figure I.3.

(I.9) $$\kappa_c = \frac{1}{2} \left( \frac{\kappa_1}{\cos \phi} - \frac{\kappa_2}{\cos \phi} \right)$$

(I.10) $$\kappa_c = \frac{1}{2} (\kappa_1 - \kappa_2) \cos \phi$$

The final step in the proof consists of linking these two formulas to the elegant matrix representation. Choose an orthonormal basis $\{e_1, \ldots, e_{n-2}\}$ for $T_p M_1 \cap T_p M_2$ and choose vectors $e_a$, $e_b$ and $e_c$ that complete to orthonormal bases for $T_p M_1$, $T_p M_2$ and $T_p M_c$ respectively. $e_a$, $e_b$ and $e_c$ correspond with each other.

The projection of $T_p \mathbb{R}^n$ onto $T_p M_c$ restricted to $T_p M_1$ is an isomorphism. It is the identity on $\text{sp}\{e_1, \ldots, e_{n-1}\}$ and $e_a$ is mapped to $e_c \cos \phi$. Call the matrix representation of this $P$.

The function that assigns a curvature to each direction of the tangent space of a $C^2$ hypersurface is in fact a symmetric bilinear form on the tangent space. It is the inverse of the first fundamental form multiplied by the second fundamental form. Thus the representation for this function is a matrix. One such matrix for $T_p M_c$ that gives the correct curvatures in
the directions $e_1, \ldots, e_{n-2}$ and $e_c$ would be

$$\frac{1}{2 \cos \phi} P^T (A - B) P$$

where $A$ and $B$ are the matrices of the curvature functions on the tangent spaces $T_p M_1$ and $T_p M_2$. To prove that this is in fact the curvature function for $T_p M_c$ all we need to do is write the equations for the curvature in other directions than $e_1, \ldots, e_{n-2}$ and $e_c$. We will omit this calculation.

We now want to pass on from the curvature functions to second fundamental forms. For $M_1$ and $M_2$ we can take immersions as the ones in equations I.2, replacing $\psi$ by $-\phi$. We now see that their first fundamental forms in intrinsic coordinates at 0 are simply identity matrices. The first fundamental form of the conflict set is the identity matrix. We have proven that at 0 the second fundamental form of the conflict set $\mathbf{I}_c$ is:

$$\mathbf{I}_c = \frac{1}{2 \cos \phi} P^T (\mathbf{I}_1 - \mathbf{I}_2) P$$

I.5. Proof of theorem I.3

The special case of $l$ spheres in $\mathbb{R}^n$ the conflict set has an appealing property: each of its components is a conic section. So let us have $l$ spheres in $\mathbb{R}^n$, each having center $p_i$, $i = 1, \ldots, l$. Denote their radii by $r_i$. Each component is given by $l - 1$ equations:

(I.11) $\|x - p_i\| = \|x - p_1\| + d_i$, $i = 2, \ldots, l$

The equations (I.11) describe conic sections. Each $d_i$ will be written $d_{1,i} = d_i = a_i - a_1$ where $a_i = \pm r_i$. Thus we have in total a maximum of $2^l$ components of the complete conflict set. We will consider a fixed component, that is we will fix $a_i$ for the rest of the proof. Other equations are

$$\|x - p_j\| = \|x - p_i\| + d_{i,j}$$

with $d_{i,j} = d_{1,j} - d_{1,i}$. Now we square both sides of these equations.

$$\langle x, x \rangle - 2 \langle p_j, x \rangle + \langle p_j, p_j \rangle = \langle x, x \rangle - 2 \langle p_i, x \rangle + \langle p_i, p_i \rangle + 2 d_{i,j} \|x - p_i\| + d_{i,j}^2, \quad i, j = 1, \ldots, l$$

and we obtain

$$2 d_{i,j} \|x - p_i\| = 2 \langle x, p_i - p_j \rangle + \|p_j\|^2 - \|p_i\|^2 - d_{i,j}^2, \quad i, j = 1, \ldots, l$$

When $d_{i,j} = \pm \|p_i - p_j\|$ the solution to the equation I.11 is a line. So the component of the conflict set we consider lies in a line and the theorem is trivially true. We can safely exclude this case and all the others where

$$d_{i,j} = \pm \|p_i - p_j\|$$

If they exist choose mutually disjoint subsets $J_1, \ldots, J_k$ of $\{1, \ldots, n\}$ so that

- $a_{i_1} = a_{i_2}$ for all $i_1, i_2 \in J_i$
- the cardinalities $j_i = |J_i|$ are ordered $j_1 \geq j_2 \geq \cdots \geq j_k$.
- $j_i \geq 2$
The complement of the union of these subsets will be denoted by $J_0$:

$$\bigcup_{i=0}^{k} J_i = \{1, \ldots, l\}$$

The number of elements $j_0$ in $J_0$ is possibly 0. If all $a_i$ are different then $j_0 = n$, and $k = 0$. It is now convenient to put

$$P(i, j) = 2d_{i,j} \|x - p_i\| = 2\langle x, p_i - p_j \rangle + \|p_j\|^2 - \|p_i\|^2 - d_{i,j}^2$$

We finally relabel the points such that

$$J_0 = \{1, \ldots, j_0\}, J_1 = \{j_0 + 1, \ldots, j_0 + j_1\}$$

and so on. Let us first assume that $k \geq 1$ and $j_0 \geq 3$.

From each of the $J_m$ we have $j_m - 1$ linear equations

(I.12) \[ P(j_0 + \cdots + j_m, i) = 0 \quad i \in J_m \setminus \{j_0 + \cdots + j_m\} \]

We also have $j_0 - 2$ linear equations

(I.13) \[ \frac{P(1,2)}{d_{1,2}} = \frac{P(1, i)}{d_{1,i}} \quad i = 3, \ldots, j_0 \]

There is a third set of $m$ linear equations that read

(I.14) \[ \frac{P(1,2)}{d_{1,2}} = \frac{P(1, i)}{d_{1,i}} \quad i = j_0 + j_1, j_0 + j_1 + j_2, \ldots, j_0 + j_1 + \cdots + j_m \]

In total we have $l - 2$ linear equations. These equations all have the form

(I.15) \[ \langle v_i, x \rangle = w_i \]

From I.12, I.13 and I.14 there are three sets of $v_i$. The first set is

(I.16) \[ p_{j_0 + j_1 + \cdots + j_m} - p_i \quad i \in J_m \setminus \{j_0 + \cdots + j_m\} \]

The second set is

(I.17) \[ \frac{p_1 - p_2}{d_{1,2}} - \frac{p_1 - p_i}{d_{1,i}} \quad i = 3, \ldots, j_0 \]

The third set is

(I.18) \[ \frac{p_1 - p_2}{d_{1,2}} - \frac{p_1 - p_i}{d_{1,i}} \quad i = j_0 + j_1, j_0 + j_1 + j_2, \ldots, j_0 + j_1 + \cdots + j_m \]

Each of the equations determines an affine hyperplane in $\mathbb{R}^n$. The hypersurfaces will intersect transversally if the vectors $v_i$ are linearly independent. The $l - 2$ vectors are linearly independent iff. there are 2 more vectors such that these $l$ will be independent. Two vectors that will do are $p_1$ and $p_2$ because by hypothesis the $p_i$ span up a $l - 1$ dimensional simplex. If $k = 0$ then all the $a_i$ are different and the vectors are given by equations as in I.13 and I.17. If $k > 0$ and $j_0 = 2$ the $v_i$ are given by equations I.12, I.14 and I.16, I.18. If $k > 0$ and $j_0 = 1$ $j_1$ will equal 2. The case $j_0 = 1$ is similar. When $l = 2$ the theorem is trivial. Saying that the conflict set is a conic section is equivalent to saying that the conflict set is the zero-set of some quadratic equations. The quadratic equations are given by the $P(i, j)$. This completes the proof of theorem I.3.
Remark I.15. Theorem I.3 should also hold with other conditions. There are examples of 3 spheres in $\mathbb{R}^n$, with their centers lying on a line, such that a component of the conflict set lies in a hyperplane and that it is a conic section in this plane. Caution is needed though because there are also examples of four spheres in $\mathbb{R}^n$ - i.e. two concentric spheres mirrored through some hyperplane - where the theorem is not true.

Remark I.16. If $l = n = 3$ we have a special case, that was studied a lot classically. The 1 parameter family of spheres with center on the conflict set that touch the three given spheres has a special surface, called a Dupin cyclide, as an envelope. By an appropriate inversion the conflict set - which is on the symmetry set of the cyclide - is mapped onto a circle. Under this inversion the cyclide is mapped to a torus. Thus the conflict set itself is a conic section in a hyperplane in $\mathbb{R}^3$, see [Cox52]. For cyclides one might consult several works of T.E. Cecil.
II Three and more surfaces

II.1. Introduction

In this chapter we will elaborate on the results of the previous chapter. We will obtain a formula for the curvature and the torsion of the conflict set of three surfaces. The relevant formulas are in theorems II.6 and II.10. The last section of this chapter - which can be safely skipped - contains an illustration of the formulas for curvature and torsion and of theorem I.3.

II.2. The tangent space to the conflict set

We would like to give a description of the tangent space of the conflict set of \((M_1, n_1), (M_2, n_2), \ldots, (M_l, n_l)\) at regular points, where the \(M_i\) are hypersurfaces in \(\mathbb{R}^n\).

So at regular points we have - as in the previous chapter - a set of basepoints \(p_i\) on each of the \(M_i\). The tangent spaces \(T_p M_i\) define \(l\) affine hyperplanes \(W_i\) in \(\mathbb{R}^n\) that intersect transversally. The conflict set of the \(W_i\) has the same tangent space as the conflict set of the \(M_i\). So we might as well assume that the \(M_i\) are affine hyperplanes, with a point \(p\) in their intersection.

Here the tangent space will be \(n - l + 1\) dimensional. We will split it into two components. First of all a part of it will correspond to the intersection of the \(M_i\). Let \(V_c = \bigcap_{i=1}^l T_p M_i\), then \(V_c \subset T_p M_c\). This part of the tangent space is readily calculated and so we can study the projection

\[
\pi: T_p \mathbb{R}^n \rightarrow T_p \mathbb{R}^n / V_c.
\]

The \(\pi(n_i)\) form a basis for \(T_p \mathbb{R}^n / V_c\) and we also have that \(\pi(n_i) \perp \pi(T_p M_i)\). Denote \(M_c^\pi\) the conflict set of the \(\pi(M_i)\). This will be just a line. And this line corresponds to the second part of the tangent space. We thus have

\[
V_c \oplus \text{“line”} = T_p M_c
\]

**Proposition II.1.** \(\pi(T_p M_c) = T_p M_c^\pi\) and \(V_c \oplus T_p M_c^\pi = T_p M_c\)

**Proof.** Consider a vector \(x\) that is in \(T_p M_c\). It has a lift to \(\mathbb{R}^{n+1}\) where it has to be in the intersection of the tangent spaces to the big fronts. Thus the vectors \(x\) that are in \(T_p M_c\) all have

\[
\langle x, n_i \rangle = \langle x, n_j \rangle
\]

From here it follows that

\[
\langle \pi(x), \pi(n_i) \rangle = \langle \pi(x), \pi(n_j) \rangle
\]

On the other hand if \(\pi(x) \in T_p M_c^\pi\) then either \(x \in V_c\) or we may assume that \(x \perp V_c\), in which case the above two equations are equivalent. \(\square\)

For this second part of the tangent space it is thus enough to look at the case where \(l = n\) and \(V_c = \{0\}\). In this case we have a formula. In order to write this formula we introduce the following notations. If \(i, j, k\) are integers then we write

\[
q(i, j, k) = 1 + ((i + j - 1) \mod k)
\]

This function is a “circulator” over a finite index set \(\{1, \cdots, k\}\). Also, if we have \(n - 1\) vectors in \(\mathbb{R}^n\) we define a cross product, that generalizes the cross product of 2 vectors in
The cross-product of \( n-1 \) vectors \( v_i \) in \( \mathbb{R}^n \) is defined by the requirement that for each vector \( w \in \mathbb{R}^n \) we have
\[
\langle \times_{i=1}^{n-1} v_i, w \rangle = \det(v_1, \ldots, v_{n-1},w)
\]

**Proposition II.2.** If \( l = n \) then the vector
\[
t_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} n_q(i,j,l)
\]
defines the tangent space to the conflict set.

**Proof.** All we need to verify is that \( \langle t_c, n_i \rangle = \langle t_c, n_j \rangle \).

We summarize our discussion in a theorem:

**Theorem II.3.** The tangent space to the conflict set of \((M_1, n_1), (M_2, n_2), \ldots, (M_l, n_l)\), all hypersurfaces in \( \mathbb{R}^n \) is spanned by
\[
V_c \oplus \left( \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} \pi(n_q(i,j,l)) \right)
\]

provided that we are at regular points of the conflict set.

**II.3. A curvature formula**

We next want to calculate the curvature of the conflict set of \((M_1, n_1), (M_2, n_2), \ldots, (M_l, n_l)\) at regular points in terms of the curvature at the basepoints. We calculate the curvature for the case \( l = n = 3 \). The result is in theorem II.6.

The curvature formula will be obtained by differentiation. As before we write:

(II.1) \[
t_c = n_1 \times n_2 + n_2 \times n_3 + n_3 \times n_1
\]

We will also write
\[
e = \frac{t_c}{\|t_c\|}
\]

We will denote \( t_i \) the unit vector of the projection of \( t_c \) to \( T_p M_i \). For the curvature of the conflict set we need to calculate, according to the Frenet-Serret equation, the derivative \( \nabla_v e \):
\[
\nabla_v e = \nabla_e \frac{t_c}{\|t_c\|} = \frac{\|t_c\|^2 \nabla_v t_c - \langle t_c, \nabla_v t_c \rangle t_c}{\|t_c\|^3}
\]

By \( \nabla_v \) we mean the directional derivative
\[
\nabla_v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}
\]

It is clear that
\[
\langle \nabla_v e, t_c \rangle = 0
\]

We can choose coordinates so that \( e \) is \((0,0,1)\). In particular this means that \( \nabla_v e \) has a zero component in the last coordinate, the “\( z \)” coordinate. The vector \( t_c \) has only non-zero components in all but this last coordinate. Now denote \((\cdot)_i\) the \( i \)-th component of a vector.

We have just concluded that:
\[
(\nabla_v e)_3 = 0
\]

and that thus:
\[
\kappa_c^2 = \frac{(\nabla_v t_c)_1^2 + (\nabla_v t_c)_2^2}{\|t_c\|^2}
\]
II.3. A CURVATURE FORMULA

\[ n_1 - n_2 - n_3 - t_1 - t_2 - t_c \]

**Figure II.1.** The geometry of the three tangent planes

**Remark II.4.** We will calculate the square of the curvature. In this way we avoid “sign” problems. There seems to be no natural choice of frame for the conflict set. For instance when we interchange \( n_1 \) and \( n_2 \) in (II.1) \( t_c \) changes sign. For the first normal to the conflict set things get even more complicated.

We compute \( \nabla e t_c \) through a further choice of coordinates namely:

\[
\begin{align*}
(\text{II.2}) & \quad e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
& \quad n_i = \begin{pmatrix} \sin \alpha \sin \beta_i \\ \sin \alpha \cos \beta_i \\ \cos \alpha \end{pmatrix} \\
& \quad t_i = \begin{pmatrix} -\cos \alpha \sin \beta_i \\ -\cos \alpha \cos \beta_i \\ \sin \alpha \end{pmatrix}
\end{align*}
\]

Furthermore

\[
(\text{II.3}) \quad \nabla_e n_i = \kappa_i t_i \sin \alpha
\]

where \( \kappa_i \) is the normal curvature of the wavefront of \( M_i \) at the point of the conflict set that we are considering.

**Remark II.5.** Suppose a curve \( \gamma \) is contained in a hypersurface \( M \subset \mathbb{R}^3 \). This curve has a so-called normal curvature \( \kappa_n \) and a so-called geodesic curvature \( \kappa_g \). The geodesic curvature depends only on the interior geometry of \( M \) and the normal curvature depends only on the exterior geometry of \( M \). The curvature of \( \gamma \) as a space curve is \( \kappa \). Between \( \kappa \) and the aforementioned curvatures we have the following relation:

\[
\kappa^2 = \kappa_g^2 + \kappa_n^2
\]

Thus equation (II.3) merely says that the derivative only depends on the exterior geometry.
With these choices

\[ \|t_c\|^2 = \sin^4 \alpha (\sin(\beta_1 - \beta_2) + \sin(\beta_2 - \beta_3) + \sin(\beta_3 - \beta_1))^2 \]

(II.4)

\[ = 16 \sin^4 \alpha \sin^2 \left( \frac{\beta_1 - \beta_2}{2} \right) \sin^2 \left( \frac{\beta_2 - \beta_3}{2} \right) \sin^2 \left( \frac{\beta_3 - \beta_1}{2} \right) \]

and

\[ (\nabla e t_c)_1 = \sin^3 \alpha (\kappa_1 (\cos \beta_3 - \cos \beta_2) + \kappa_2 (\cos \beta_1 - \cos \beta_3) + \kappa_3 (\cos \beta_2 - \cos \beta_1)) \]

Similarly

\[ (\nabla e t_c)_2 = \sin^3 \alpha (\kappa_1 (\sin \beta_3 - \sin \beta_2) + \kappa_2 (\sin \beta_1 - \sin \beta_3) + \kappa_3 (\sin \beta_2 - \sin \beta_1)) \]

This is because

\[ \nabla e t_c = (\nabla e n_1) \times (n_2 - n_3) + (\nabla e n_2) \times (n_3 - n_1) + (\nabla e n_3) \times (n_1 - n_2) \]

\[ = \kappa_1 \sin \alpha t_1 \times (n_2 - n_3) + \kappa_2 \sin \alpha t_2 \times (n_3 - n_1) + \kappa_3 \sin \alpha t_3 \times (n_1 - n_2) \]

Now we have

\[ (\nabla e t_c)^2 + (\nabla e t_c)^2 = \sin^6 \alpha \left( (\kappa_1 - \kappa_2)^2 + (\kappa_2 - \kappa_3)^2 + (\kappa_3 - \kappa_1)^2 \right) \]

\[ + 2 (\cos \beta_1 \cos \beta_2 + \sin \beta_1 \sin \beta_2)(\kappa_2 - \kappa_3)(\kappa_3 - \kappa_1) \]

\[ + 2 (\cos \beta_2 \cos \beta_3 + \sin \beta_2 \sin \beta_3)(\kappa_3 - \kappa_1)(\kappa_1 - \kappa_2) \]

\[ + 2 (\cos \beta_3 \cos \beta_1 + \sin \beta_3 \sin \beta_1)(\kappa_1 - \kappa_2)(\kappa_2 - \kappa_3) \]

\[ = \sin^6 \alpha \left( (\kappa_1 - \kappa_2)^2 + (\kappa_2 - \kappa_3)^2 + (\kappa_3 - \kappa_1)^2 \right) \]

\[ + 2 \cos(\beta_1 - \beta_2)(\kappa_2 - \kappa_3)(\kappa_3 - \kappa_1) \]

\[ + 2 \cos(\beta_2 - \beta_3)(\kappa_3 - \kappa_1)(\kappa_1 - \kappa_2) \]

\[ + 2 \cos(\beta_3 - \beta_1)(\kappa_1 - \kappa_2)(\kappa_2 - \kappa_3) \]

This is finally equal to

\[ \text{(II.5)} \quad = -4 \sin^6 \alpha \sum_{i=1}^{3} \sin^2 \left( \frac{\beta_q(i,1,3) - \beta_q(i,2,3)}{2} \right) (\kappa_q(i,2,3) - \kappa_i)(\kappa_i - \kappa_q(i,1,3)) \]

We have proven the following theorem.

**THEOREM II.6.** The squared curvature of the conflict set of 3 hypersurfaces is at regular points given by the formula:

\[ \kappa_c^2 = -\frac{\sin^2 \alpha}{4} \sum_{i=1}^{3} \frac{(\kappa_i - \kappa_q(i,1,3))(\kappa_q(i,1,3) - \kappa_q(i,2,3))}{\sin^2 \left( \frac{\beta_q(i,1,3)}{2} \right) \sin^2 \left( \frac{\beta_q(i,2,3)}{2} \right)} \]

(II.6)

We can also find a more coordinate free form of formula (II.6). We reformulate theorem II.6:

\[ \kappa_c^2 = \sin^6 (\alpha) \sum_{i=1}^{3} \frac{(\kappa_i - \kappa_q(i,1,3))(\kappa_q(i,1,3) - \kappa_q(i,2,3))}{(1 - \langle n_i, n_q(i,1,3) \rangle)(\langle n_q(i,1,3), n_q(i,2,3) \rangle - 1)} \]

(II.7)

We used that:

\[ 1 - \langle n_i, n_j \rangle = 2 \sin^2 (\alpha) \sin^2 \left( \frac{\beta_i - \beta_j}{2} \right), \]

an identity that follows from (II.2). We also have the following:

**PROPOSITION II.7.** It holds \( \kappa_c = 0 \) iff. \( \kappa_i = \kappa_j \) for \( 1 \leq i, j \leq 3 \).
II.4. Higher derivatives: Torsion

Theorem I.3 says that the conflict set with the three spheres has zero torsion. We are interested in computing in general the torsion of the conflict set when \( l = n = 3 \). To compute the torsion we need the Frenet-Serret equations, see [Spi], vol. 2. The Frenet-Serret equations can be applied once we know the first and second normal to the conflict set.

We contend that the first normal to the conflict set is

\[
\mathbf{n}_c = \kappa_1 \mathbf{e} \times (n_2 - n_3) + \kappa_2 \mathbf{e} \times (n_3 - n_1) + \kappa_3 \mathbf{e} \times (n_1 - n_2)
\]

This is proven by using the polar coordinates and comparing each term in the cyclic sum to the corresponding terms in

\[
\text{pr}_e (\mathbf{t}_c / \| \mathbf{t}_c \|)
\]

We can also use a lemma:

**Lemma II.8.** If \( \langle u, v \rangle = 0 \) then

\[
\text{pr}_u (w \times v) = \frac{\langle w, u \rangle}{\| u \|^2} u \times v \quad \text{and} \quad u \times (u \times v) = -\| u \|^2 v
\]

**Proof.** Set \( u = (0, 0, 1) \). □

We can apply this lemma to verify the assertion of equation (II.8). Let \( v = n_2 - n_3 \), \( u = \mathbf{e}_l \) and \( w = t_1 \). Then

\[
\text{pr}_e (\kappa_1 t_1 \times (n_2 - n_3)) = \kappa_1 \sin \alpha \mathbf{e} \times (n_2 - n_3)
\]

which in view of (II.1) proves (II.8).

To find the third vector in the Frenet frame - up to sign - we take the outer product:

\[
\mathbf{b}_c = \mathbf{n}_c \times \mathbf{e}
\]

Using \( \mathbf{e} \times (\mathbf{e} \times (n_i - n_{q(i,1,3)})) = n_{q(i,1,3)} - n_i \) we can write:

\[
\mathbf{b}_c = \kappa_1 (n_2 - n_3) + \kappa_2 (n_3 - n_1) + \kappa_3 (n_1 - n_2)
\]

This further entails that \( \| \mathbf{b}_c \| = \| \mathbf{n}_c \| \). Recalling the Frenet-Serret equations ( see [Spi] ) the derivative we are interested in is

\[
\nabla_e \frac{\mathbf{b}_c}{\| \mathbf{b}_c \|} = -\tau_c \frac{\mathbf{n}_c}{\| \mathbf{n}_c \|}
\]

We can now compute this derivative and the torsion

\[
\nabla_e \frac{\mathbf{b}_c}{\| \mathbf{b}_c \|} = \frac{\| \mathbf{b}_c \|^2 \nabla_e \mathbf{b}_c - \langle \mathbf{b}_c, \nabla_e \mathbf{b}_c \rangle \mathbf{b}_c}{\| \mathbf{b}_c \|^3} = -\tau_c \frac{\mathbf{n}_c}{\| \mathbf{n}_c \|}
\]

This leads to

\[
\tau_c = -\frac{\langle \nabla_e \mathbf{b}_c, \mathbf{n}_c \rangle}{\| \mathbf{b}_c \|^2}
\]
As a first step we will derive some expression for $\|b_c\|^2$.

$$
\|b_c\|^2 = \sum_{i=1}^{3} (\kappa_i - \kappa_{q(i,1,3)})^2 + 2 \cos(\beta_i - \beta_{q(i,1,3)})(\kappa_{q(i,1,3)} - \kappa_{q(i,2,3)})(\kappa_{q(i,2,3)} - \kappa_i)
$$

$$
= -4 \sin^2 \alpha \sum_{i=1}^{3} \sin^2 \left( \frac{\beta_i - \beta_{q(i,1,3)}}{2} \right) (\kappa_{q(i,1,3)} - \kappa_{q(i,2,3)})(\kappa_{q(i,2,3)} - \kappa_i)
$$

$$
= \frac{k_c^2 \|t_c\|^2}{\sin^4 \alpha}
$$

The derivative of interest is thus

$$
\nabla_{e} b_c = \sin \alpha (\kappa_1 (\kappa_2 t_2 - \kappa_3 t_3) + \kappa_2 (\kappa_3 t_3 - \kappa_1 t_1) + \kappa_3 (\kappa_1 t_1 - \kappa_2 t_2))
$$

$$
+ \nabla_e \kappa_1 (n_2 - n_3) + \nabla_e \kappa_2 (n_3 - n_1) + \nabla_e \kappa_3 (n_1 - n_2))
$$

Now in this derivative we encounter terms of the form $\nabla_{e} \kappa_i$. Such terms embody two variations. One is the change of curvature on the wavefront. The other one is the distancing between the point on the conflict set and the corresponding basepoint. Heuristically we feel that the last influence should have no effect on the conflict set. The curvature of the conflict set should only depend on the relative position of the base manifolds.

Note that by only taking into account the distance variation we have:

$$
\nabla_{e} \kappa_i = \nabla_{e} \frac{1}{T_i} \approx - \frac{\nabla_e T_i}{T_i^2} = -\kappa_i^2 \nabla_e T_i = -\kappa_i^2 \cos \alpha
$$

Here we have used the notation $T_i$ for the distance between the point under consideration on the conflict set and the center of the Meusnier sphere of the surface $M_i$ at the basepoint on $M_t$. Because we are at regular points this distance is never zero. Taking both variations into account we thus have:

$$
\nabla_{e} \kappa_i = -\kappa_i^2 \cos \alpha + \sin \alpha \nabla_{t_i} \kappa_i'
$$

Here $\kappa_i'$ is the normal curvature of the traced out curve on the wavefront. There seems to be no obvious expression for its derivative.

REMARK II.9. The meaning of the term $D_{t_i} \kappa_i'$ is as follows. It is the derivative of the normal curvature along the basecurve on the equidistant at $p$. The relation between normal curvature and normal curvature on an equidistant is clear. At distance $d$ the normal curvature is

$$
\frac{\kappa}{1 + d \kappa}
$$

Where their derivatives are concerned this is much less the case. Let us start by clarifying how the 3-jet of a hypersurface determines the derivative of the normal curvature.

In case of an immersion that up to a quadratic form looks like

$$
(s,t) \mapsto (s,t, \frac{1}{6}(As^3 + 3Bs^2 t + 3Cs^2 t + Dt^3))
$$

and a curve with tangent $(\cos \mu, \sin \mu)$ the derivative is

$$
A \cos^3 \mu + 3B \cos^2 \mu \sin \mu + 3C \cos \mu \sin^2 \mu + D \sin^3 \mu.
$$

This means that - as was to be expected - if two surfaces have 2 contact and 4 contact in the direction $\vec{v}$ curves in that direction on the surfaces have equal derivative of the normal curvature.

The relation between the derivative of the normal curvature at the base manifold and at its
equidistant is most conveniently found by just using curves in $\mathbb{R}^2$. If we want a curve with prescribed derivative of the curvature $\alpha$ at 0 we can take

$$\gamma: t \mapsto (t, \frac{\kappa}{2}t^2 + \frac{\alpha}{6}t^3)$$

At distance $d$ its derivative of the normal curvature is

$$\frac{\alpha}{(1 + d\kappa)^3}$$

We write the derivative of $b_c$ as a sum of three components:

$$\nabla_e b_c = \sin \alpha (\kappa_1 (\kappa_2 t_2 - \kappa_3 t_3) + \kappa_2 (\kappa_3 t_3 - \kappa_1 t_1) + \kappa_3 (\kappa_1 t_1 - \kappa_2 t_2))$$

$$- \cos \alpha (\kappa_1^2 (n_2 - n_3) + \kappa_2^2 (n_3 - n_1) + \kappa_3^2 (n_1 - n_2))$$

$$+ \sin \alpha (\nabla t_1 \kappa_1' (n_2 - n_3) + \nabla t_2 \kappa_2' (n_3 - n_1) + \nabla t_3 \kappa_3' (n_1 - n_2))$$

$$I_1 + I_2 + II$$

With some perseverance we obtain $\langle I_1 + I_2, n_c \rangle = 0$. We thus have

$$\langle \nabla_e b_c, n_c \rangle = \langle II, n_c \rangle = \sin \alpha \|t_c\| (\kappa_1 (\nabla t_2 \kappa_2' - \nabla t_3 \kappa_3') + \kappa_2 (\nabla t_3 \kappa_3' - \nabla t_1 \kappa_1') + \kappa_3 (\nabla t_1 \kappa_1' - \nabla t_2 \kappa_2'))$$

Consequently

**Theorem II.10.** With notations as above the torsion of the conflict set is given by:

$$\tau_c = \pm \frac{\sin^5 \alpha}{\|t_c\| \kappa_c^2} (\kappa_1 (\nabla t_2 \kappa_2' - \nabla t_3 \kappa_3') + \kappa_2 (\nabla t_3 \kappa_3' - \nabla t_1 \kappa_1') + \kappa_3 (\nabla t_1 \kappa_1' - \nabla t_2 \kappa_2'))$$

**Remark II.11.** Compare this formula to a classical formula for the torsion of a space curve with nonzero curvature $\kappa$ and a unit parameterization $c(t)$ (see [Spi] )

$$\tau = \frac{1}{\kappa^2} \langle \frac{dc}{dt} \times \frac{d^2 c}{dt^2}, \frac{d^3 c}{dt^3} \rangle$$

We see that the term $\frac{1}{\kappa_c^2}$ is natural in this respect.

**II.5. Higher dimensional analogues**

The same differentiation techniques can be used to determine the curvature of $l$ surfaces in $\mathbb{R}^l$. The trouble is that it is less clear what angles and geometrical data other than just the normals we should take in order to get useful formulae.

The tangent to the conflict set is

$$t_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} n_{q(i,j,l)}$$

If we differentiate $t_c$ wrt. arclength we obtain

$$\nabla_e t_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} (\nabla_e n_i) \times_{j=1}^{l-2} (n_{q(i,j,l)} - n_{q(i,j+1,l)})$$

Here we have written again:

$$\epsilon = \frac{t_c}{\|t_c\|},$$

and we have used the formula

$$\nabla_e n_i = \kappa_i t_i \sin \alpha$$
We know that the first normal in the Frenet frame (up to sign and length) to the conflict curve is
\[ n_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \kappa_i e \times \sum_{j=1}^{l-2} (n_{q(i,j,l)} - n_{q(i+1,j,l)}) \]
This means that we might also calculate the curvature in the following way:
\[ \kappa_c = \frac{\langle \nabla e t_c, n_c \rangle}{\| n_c \| \| t_c \|} \]
It is also remarked that proposition II.7 generalizes to higher \( l = n \).

**II.6. An example: three disjoint spheres**

Conflict sets are very difficult to calculate, both algebraically and numerically. Here we will present an example with three surfaces in \( \mathbb{R}^3 \) in order to have a non-trivial example of the situation encountered in theorem II.6, in which we can actually compute explicitly a parameterization of the conflict set. Our example is so typical that the calculations we do to compute the curvature almost provide a new proof of (II.6).

The example we are talking about is the example of three disjoint spheres in \( \mathbb{R}^3 \) with outward pointing normals. Their conflict set can be explicitly calculated if we make one further assumption: we will demand that the convex hull of their centers is a triangle with acute angles only.

**Lemma II.12.** Let \( p_1, p_2, p_3 \) be three points in \( \mathbb{R}^3 \), such that their convex hull is a triangle with acute angles. These points are in general position wrt. to linear subspaces of \( \mathbb{R}^3 \). Coordinates can be so chosen that mutual distances are preserved and \( p_1 = (b_1, 0, 0) \), \( p_2 = (0, b_2, 0) \), \( p_3 = (0, 0, b_3) \).

**Proof.** Through each pair of points there passes a sphere, so that the two points are poles of the sphere. The three spheres have one point of intersection because the three points are in general position. This point of intersection is chosen to be the origin \( O \). The lines through \( O \) and \( p_i \) will be the three coordinate axes. □

If the centers of the spheres do not span up a triangle with acute angles such a coordinate representation is not possible. In that case one could revert to the following more general representation. If we have \( n \) points in \( \mathbb{R}^n \) that lie in general position this is a \( \frac{n(n-1)}{2} \) dimensional space. Coordinates for it can be written
\[
\begin{pmatrix}
    b_{11} & b_{12} & \cdots & \cdots & 1 \\
    b_{21} & b_{22} & \cdots & 1 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    b_{n-1,1} & 1 & \cdots & 0 & 0 \\
    1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
Proof is by induction. For \( n = 2 \) the statement is clear.

**II.6.1. A parameterization for the conflict set.** We will continue with the diagonal representation of 3 points. We choose the three points as centers of mutually disjoint spheres with radii \( r_i \). The mutual disjointness can be achieved with a condition on the radii:
\[ r_i + r_j < \sqrt{b_i^2 + b_j^2} \]
We know that the conflict set of these spheres with outward normals lies in a plane. For the sake of brevity we will write this conflict set as $M_{p,r}$. The equations for the lift of $M_{p,r}$ to $\mathbb{R}^4$ are

$$x_0 = \|x - p_i\| - r_i \quad 1 \leq i \leq 3$$

They can be rewritten in algebraic form:

$$(x_0 + r_i)^2 = \|x - p_i\|^2$$

It seems that these equations will add another algebraic component to the conflict set. They do but this component will be just another conflict set, namely the one with the orientations reversed.

We will frequently use $\bar{x} = (x_0, x)$ for a point in $\mathbb{R}^{n+1}$ whose projection to $\mathbb{R}^n$ is $x$. In this section $n = 3$. It is also useful to introduce a quadratic form of sign $(1,3)$ on $\mathbb{R}^4$ namely

$$\|x_0, x\|-1 = x_0^2 - \|x\|^2$$

and the corresponding form

$$\langle(x_0, x), (y_0, y)\rangle-1 = x_0y_0 - \langle x, y \rangle$$

The unique plane through the points $p_1, p_2, p_3$ is given by

$$(\text{II.9}) \sum_{i=1}^3 x_i b_i = 1$$

**Lemma II.13.** If $q$ lies on $M_{p,r}$ then its reflection through the above plane also lies on the conflict set.

**Proof.** The reflection through the plane (II.9) is an isometry that maps spheres centered at $p_i$ to themselves. The reflection thus leaves the conflict set $M_{p,r}$ also invariant. $\square$

The intersection of this plane with the conflict set is a point $\alpha$. The point lifts to a point $\bar{\alpha} = (\alpha_0, \alpha)$ in $\mathbb{R}^4$. Here $\alpha_0$ is the “time” at which this component of starts appearing. So we have for $\bar{\alpha}$ that

$$\|(\alpha_0 + r_i, \alpha - p_i)\|-1 = 0 \quad i = 1, \cdots, 3 \quad \text{and} \quad \sum_{i=1}^n \frac{\alpha_i}{b_i} = 1$$

From the equations we can find two solutions for $\bar{\alpha}$. The solutions take the form

$$\alpha_i = b_i \frac{f_i \pm g_i \sqrt{D}}{2N_i} \quad i = 1, \cdots, 3$$

Here $D$ is a discriminant.

$$D = (b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2) \left( b_1^2 + b_2^2 - (r_1 - r_2)^2 \right)$$

$$\times \left( b_1^2 + b_3^2 - (r_1 - r_3)^2 \right) \left( b_2^2 + b_3^2 - (r_2 - r_3)^2 \right)$$

The equation $D = 0$ corresponds exactly to the degeneracy condition in the previous chapter, in the last section. If $D = 0$ then we will have for instance that

$$b_1^2 + b_2^2 = \|p_1 - p_2\|^2 = (r_1 - r_2)^2$$

so that

$$\pm\|p_1 - p_2\| = r_1 - r_2$$
In this case the conflict set is reduced to a line, but by our previous demand this is impossible. The condition corresponds exactly to the two cones in 4-space having 2-contact. For \( i = 1, \cdots, 3 \) the nominator \( N_i \) is

\[
(b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2) \left( b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2 - b_3^2(r_1 - r_2)^2 - b_2^2(r_1 - r_3)^2 - b_1^2(r_2 - r_3)^2 \right)
\]

The most obvious geometrical significance of the term

\[
F = b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2
\]

is that the surface of the triangle spanned by the \( p_i \) is

\[
\frac{1}{2} \sqrt{b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2}
\]

This can be proven using Archimedes formula.

The other term is

\[
E = b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2 - b_3^2(r_1 - r_2)^2 - b_2^2(r_1 - r_3)^2 - b_1^2(r_2 - r_3)^2.
\]

In fact this term introduces no new geometry, it is in the ideal generated by

\[
(b_1^2 + b_2^2 - (r_1 - r_2)^2, b_2^2 + b_3^2 - (r_2 - r_3)^2, b_3^2 + b_1^2 - (r_3 - r_1)^2)
\]

This can be checked by computing a standard basis. Or - a little less obvious - one can remark that if we write \( d_{i,j} = b_i^2 + b_j^2 - (r_i - r_j)^2 \) that

\[
E = -\frac{d_{1,2}^2 + d_{2,3}^2 + d_{3,1}^2}{4} + \frac{d_{1,2}d_{2,3} + d_{2,3}d_{3,1} + d_{3,1}d_{1,2}}{2}
\]

The \( f_i \) are

\[
f_1 = b_1(b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2)(b_2^2 + b_3^2 - (r_2 - r_3)^2)(b_1^2 + (r_1 - r_2)(r_3 - r_1))
\]

\[
f_2 = b_2(b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2)(b_3^2 + b_1^2 - (r_3 - r_1)^2)(b_1^2 + (r_2 - r_3)(r_1 - r_2))
\]

\[
f_3 = b_3(b_1^2b_2^2 + b_1^2b_3^2 + b_2^2b_3^2)(b_1^2 + b_2^2 - (r_1 - r_2)^2)(b_3^2 + (r_2 - r_3)(r_3 - r_1))
\]

The \( g_i \) are

\[
g_1 = r_1(b_2^2 + b_3^2) - r_2b_3^2 - r_3b_2^2
\]

\[
g_2 = r_2(b_3^2 + b_1^2) - r_3b_1^2 - r_1b_3^2
\]

\[
g_3 = r_3(b_1^2 + b_2^2) - r_1b_2^2 - r_2b_1^2
\]

We remark that all \( f_i \) and \( g_i \) are in the ideal generated by \( (d_{1,2}, d_{2,3}, d_{3,1}) \) and that also \( b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2 \) is in this ideal. In fact \( f_i \) can be expressed in terms of the generators of the ideal. We have

\[
f_1 = \frac{d_{1,2}d_{2,3} + d_{2,3}d_{3,1} - d_{3,1}^2}{2}
\]

\[
f_2 = \frac{d_{2,3}d_{3,1} + d_{3,1}d_{1,2} - d_{1,2}^2}{2}
\]

\[
f_3 = \frac{d_{3,1}d_{1,2} + d_{1,2}d_{2,3} - d_{2,3}^2}{2}
\]

The \( g_i \) can be expressed as multiples of the differences \( r_1 - r_{q(i,1,3)} \) and the \( d_{i,j} \).

\[
\alpha_0 = \frac{\sqrt{Fd_{1,2}d_{2,3}d_{3,1}} + \sum_{i=1}^{3} (r_i + r_{q(i,1,3)})(b_{q(i,2,3)}^2(r_i - r_{q(i,1,3)})^2 - b_{q(i,1,3)}^2b_{q(i,1,3)}^2)}{2E}
\]
If again, \( d_{i,j} \neq 0 \) then each of the two components calculated can be parameterized with a parameterization
\[
\mathbf{\bar{s}}(t) = (\mathbf{\sigma}_0, \mathbf{\alpha}) = (\alpha_0, \alpha) + (\beta_0, \beta)(\cosh(t) - 1) + (\gamma_0, \gamma) \sinh(t)
\]
The \( \gamma \) vector will be orthogonal to the \( \beta \) vector, and the \( \beta \) vector will lie in the plane defined by the points \( p_1, p_2, p_3 \). We thus have the relations:
\[
\langle \beta, \gamma \rangle = 0
\]
(II.10)
\[
\sum_{i=1}^{n} \frac{\beta_i}{b_i} = 0
\]
Furthermore as at \( t = 0 \) the spheres will be disjoint we will have to assume that \( \gamma_0 = 0 \). Once we have found the solutions for \( \mathbf{\bar{s}} \) we can use these to find the parameterization. Namely, writing \( u \) for \( \cosh(t) - 1 \) and \( v \) for \( \sinh(t) \),
\[
\| (\alpha_0 + \beta_0 u + \gamma_0 v + r_i, \alpha + \beta u + \gamma v - p_i) \|_{-1} - \| (\alpha_0 + r_i, \alpha - p_i) \|_{-1} = 0
\]
and we get using the \( \langle \cdot, \cdot \rangle_{-1} \) inner product.
(II.11)
\[
\langle (2\alpha_0 + 2r_i + \beta_0 u + \gamma_0 v, \alpha - p_i + \beta u + \gamma v), (\beta_0 u + \gamma_0 v, \beta u + \gamma v) \rangle_{-1} = 0
\]
Now instead of \( v \) we can also use \( -v \) in this equation. Adding the two we get
\[
2\langle (\alpha_0 + r_i, \alpha - p_i), (\beta_0 u, \beta u) \rangle_{-1} + \| (\gamma_0 v, \gamma v) \|_{-1} = 0
\]
What we now do is to insert several values of \( u, v \) in order to obtain a simpler system of equations, one that separates the questions of finding \( \beta \) and \( \gamma \). We can use the values:
\[
u = 1, \ v = \sqrt{3} \quad u = 2, \ v = \sqrt{8}
\]
The
(II.12)
\[
2u\langle (\alpha_0 + r_i, \alpha - p_i), (\beta_0, \beta) \rangle_{-1} + u^2\|\beta\|_{-1} + v^2\|\gamma\|_{-1} = 0
\]
This is only a rank 2 system of equations, because \( v^2 = u^2 + 2u \). We thus conclude a set of equations:
(II.13)
\[
\langle (\alpha_0 + r_i, \alpha - p_i), (\beta_0, \beta) \rangle_{-1} - \| (\beta_0, \beta) \|_{-1} = 0 \quad i = 1, \ldots, 3
\]
We now take the identity (II.10) and the first of these sets to find the \( \mathbf{\bar{\beta}} \) vector. From the same source we obtain the
\[
\| (\beta_0, \beta) \|_{-1} + \| (\gamma_0, \gamma) \|_{-1} = 0
\]
This does not determine \( \gamma \) but we have that \( \gamma \) is a multiple of \( (b_1^{-1}, b_2^{-1}, b_3^{-1}) \) and that from (II.11 ) we can also conclude - now by subtracting the equation for \( v \) from the one for \( -v \):
\[
\langle (\alpha_0 + r_i + \beta_0 u, \alpha - p_i + \beta u), (\gamma_0, \gamma) \rangle_{-1} = 0
\]
This gives rise to
\[
\langle (\gamma_0, \gamma), (\beta_0, \beta) \rangle_{-1} = 0
\]
So that \( \gamma_0 = 0 \) - which was to be expected.
One might remark that a solution for \( \mathbf{\bar{\beta}} \) is constructed from \( \mathbf{\bar{\alpha}} \), of which there are two. Also the equations for \( \mathbf{\bar{\beta}} \) are quadratic so that you would expect four solutions for \( \mathbf{\bar{\beta}} \). In fact this is not a problem. For each solution \( \mathbf{\bar{\alpha}} \) we have only one non-zero \( \mathbf{\bar{\beta}} \) and the solutions for the
two $\vec{\alpha}$ differ only by a sign change. This agrees with the geometry.

Our results are that

\[(\text{II.14}) \quad \|\vec{\beta}\|^2 = \frac{d_{1,2}d_{2,3}d_{3,1}(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2 - E)}{4E}\]

and that

\[(\text{II.15}) \quad \|\vec{\beta}\|_1 = \frac{d_{1,2}d_{2,3}d_{3,1}}{4E}\]

Furthermore

\[
\beta_0 = \frac{\sqrt{F}d_{1,2}d_{2,3}d_{3,1}}{2E} \\
\beta_1 = -\frac{b_1(r_1(b_2^2 + b_3^2) - b_3^2r_2 - b_2^2r_3)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}} = -\frac{b_1g_1\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{F}} \\
\beta_2 = -\frac{b_2(r_2(b_3^2 + b_1^2) - b_1^2r_1 - b_3^2r_3)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}} \\
\beta_3 = -\frac{b_3(r_3(b_1^2 + b_2^2) - b_1^2r_2 - b_2^2r_1)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}}
\]

Note that the directions of the $\beta$ and $\gamma$ vectors can also be found from the proof of theorem I.3. It seems useful to check whether the two results obtained from different reasonings do indeed coincide, as they should.

According to the proof of theorem I.3 the normal vector to the plane in which the conflict set lies in is - if defined:

\[(\text{II.16}) \quad \frac{p_1 - p_2}{r_2 - r_1} = \frac{p_1 - p_3}{r_3 - r_1}\]

All cyclic permutations of (II.16) are also allowed, but these are just collinear. Taking the outer product of the vector (II.16) and the vector \((\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3})\) we obtain the direction of the $\beta$ vector.

In figure II.2 the two components of the conflict set of three spheres are depicted.

**II.6.2. The curvature.** We can compute the curvature of the conflict set of the three spheres by using the parameterization. Once this has been done we compare the outcome with the outcome of (II.6). These should be the same.

The curvature of this curve will only depend on the $\beta$ and the $\gamma$ coefficients. In fact it will only depend on their euclidean norms. If a curve in $\mathbb{R}^2$ has parameterization \((B(\cosh(t) - 1), C\sinh(t))\) its curvature is

\[
\kappa(t) = \frac{BC}{(B^2\sinh^2(t) + C^2\cosh^2(t))^{\frac{3}{2}}}
\]

Here we will have that

\[
B = \|\vec{\beta}\| \quad C = \|\vec{\gamma}\|
\]

because the conflict set is in fact a plane curve.

As before it is more convenient to consider the square of the curvature.

\[
\kappa_c^2(t) = \frac{\|\vec{\beta}\|^2\|\vec{\gamma}\|^2}{(\|\vec{\beta}\|^2\sinh^2(t) + \|\vec{\gamma}\|^2\cosh^2(t))^{\frac{3}{2}}} = \frac{\|\vec{\beta}\|^2\|\vec{\gamma}\|^2}{\|\sigma'(t)\|^6}
\]
So that for the curvature we only need to determine the euclidean lengths of the $\beta$ and $\gamma$ vectors in $\mathbb{R}^3$. However this expression can still be considerably simplified. In the above we have seen that

$$\|(\beta_0, \beta)\|_1 + \|(\gamma_0, \gamma)\|_1 = 0$$

and that we have for geometrical reasons:

$$\gamma_0 = 0$$

(This is because time will only increase or decrease on one component.)

We have obtained the formula for the curvature in terms of the $b_i$ and the $r_i$ but for the formula to be useful we would like to obtain this formula with other parameters. The first of these are the $T_i$, the distances from the point on the conflict set to the three centers of curvatures.

$$T_i^2 = \|\alpha + u\beta + v\gamma - p_i\|^2 = (\alpha_0 + r_i + u\beta_0)^2$$

Now we note that we also have $T_i - T_j = r_i - r_j$.

We strive for a representation in terms of these distances and some angles. Denote $t_c$ the tangent to the conflict set and for further simplicity the parametrization of the conflict set in $\mathbb{R}^3$ will be written:

$$\sigma(t) = \alpha + \beta u + \gamma v$$

We also denote $n_i$ the normal from the “center of curvature” to the conflict set, thus:

$$n_i = \frac{\sigma(t) - p_i}{T_i}$$

It is clear that

$$\sigma'(t) = \beta v + \gamma u + \gamma$$

Both vectors $t_c$ and $\sigma'(t)$ lie along the conflict set.

We now calculate the inner product $\langle \sigma'(t), n_i \rangle$.
Lemma II.14. \( \langle \sigma'(t), n_i \rangle = \|\sigma'(t)\| \cos \alpha = v \beta_0 \)

Proof.

\[
\langle \sigma'(t), n_i \rangle = \|\sigma'(t)\| \cos \alpha \\
= \langle \frac{\sigma - p_i}{T_i}, \beta v + \gamma u + \gamma \rangle \\
= \frac{1}{T_i} \langle \alpha + \beta u + \gamma v - p_i, \beta v + \gamma u + \gamma \rangle \\
= \frac{1}{T_i} (\langle \alpha - p_i, \beta v \rangle + uv\|\beta\|^2 + uv\|\gamma\|^2 + v\|\gamma\|^2)
\]

Now we use the identity (II.17) which is in fact:

\[
\beta_0^2 = \|\beta\|^2 + \|\gamma\|^2
\]

Consequently:

\[
= \frac{1}{T_i} (\langle \alpha - p_i, \beta v \rangle + uv\beta_0^2 + v\|\gamma\|^2) \\
= \frac{1}{T_i} ((\alpha_0 + r_i)\beta_0 v + uv\beta_0^2) \\
= \frac{1}{T_i} (\alpha_0 + u\beta_0 + r_i) v \beta_0 = v \beta_0
\]

In this last calculation we used that

\[
\langle \alpha - p_i, v \beta \rangle + v \|\gamma\|^2 = \langle \alpha_0 + r_i, v \beta_0 \rangle
\]

which is a straight consequence of (II.13) and (II.17).

We try to find the sine of the angle \( \alpha \). We note that

\[
\|\sigma'(t)\|^2 = \|\beta\|^2 v^2 + \|\gamma\|^2 (u + 1)^2 = \beta_0^2 v^2 + \|\gamma\|^2
\]

We combine the above with the lemma to find

\[
\|\sigma'(t)\|^2 \sin^2 \alpha = \|\gamma\|^2
\]

We can use this to further simplify the expression for \( \kappa_c \).

(II.19) \[ \kappa_c^2 = \sin^2 \alpha \frac{\|\beta\|^2}{\|\sigma'(t)\|^4} \]

We now want to check whether formula (II.19) agrees with the previous calculations. First of all we check our description of the tangent spaces. The lines from the \( p_i \) to \( \sigma \) meet the spheres around the \( p_i \) in points \( q_i \).

\[
q_i = \frac{r_i}{T_i} (\sigma - p_i) + p_i = \sigma - (\alpha_0 + \beta_0 u) n_i
\]

They span up a simplex with the point \( \sigma \).

Lemma II.15. The median starting from the vertex \( \sigma \) is the tangent line to the conflict set.

Proof. From an analysis of the tangent space to the conflict set in \( \mathbb{R}^n \) of \( k \) hypersurfaces we know that \( \sigma'(t) \) lies along \( n_1 \times n_2 + n_2 \times n_3 + n_3 \times n_1 \). This last vector should be orthogonal to \( q_i - q_j = (\alpha_0 + \beta_0 u)(n_j - n_i) \). Indeed, the outer product is zero.
This median is in itself again a conflict set of the tangent planes to the spheres at the points \( q_i \). The center of the circumradius of the points \((q_1, q_2, q_3)\) is the intersection of this median with the plane through \((q_1, q_2, q_3)\). We would like to calculate this radius. The conflict set projects on each sphere. The projection is in each case a circle. For a fixed \( \sigma \) the tangent line is

\[
\sigma'(t) - \langle \sigma'(t), n_i \rangle n_i
\]

One can easily prove (using a cosine formula) that the distance from \( q_i \) to \( q_j \) is

\[
\sqrt{d_{i,j}^2 - \alpha_0^2 - \beta_0^2 u^2}
\]

where \( d = \alpha_0 + \beta_0 u \).

Somewhat more work is involved in comparing (II.19) and (II.6). We recall that for four vectors in \( \mathbb{R}^3 \) we have the following identity

\[
\langle a \times b, c \times d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle
\]

With this we can calculate the cosine of the angles \( \beta_i - \beta_j \) that are the angles between the planes spanned by \( \sigma'(t), n_i \) and \( \sigma'(t), n_j \).

\[
\langle \sigma'(t) \times n_i, \sigma'(t) \times n_j \rangle = \| \sigma'(t) \|^2 \left( \frac{\sigma - p_i}{T_i}, \frac{\sigma - p_j}{T_j} \right) - \| \sigma'(t) \|^2 \cos^2 \alpha
\]

On the other hand using (II.2) we have:

\[
\langle \sigma'(t) \times n_i, \sigma'(t) \times n_j \rangle = \| \sigma'(t) \|^2 \sin^2 \alpha \cos(\beta_i - \beta_j)
\]

We combine the two to obtain

\[
\cos(\beta_i - \beta_j) = \frac{\langle n_i, n_j \rangle - \cos^2 \alpha}{\sin^2 \alpha}
\]

And \( 1 - \cos(\beta_{q(i,1,3)} - \beta_{q(i,2,3)}) \) is

\[
(II.20) \quad \frac{1 - \langle n_{q(i,1,3)}, n_{q(i,2,3)} \rangle}{\sin^2 \alpha}
\]

We also write

\[
\sin^2 \left( \frac{\beta_1 - \beta_2}{2} \right) = \frac{1 - \cos(\beta_1 - \beta_2)}{2}
\]

\[
= \frac{1 - \langle n_1, n_2 \rangle}{2 \sin^2 \alpha} = \frac{T_1 T_2 - \langle \sigma - p_1, \sigma - p_2 \rangle}{2 T_1 T_2 \sin^2 \alpha}
\]

\[
= \frac{2 T_1 T_2 - \| \sigma - p_1 \|^2 - \| \sigma - p_2 \|^2 + b_1^2 + b_2^2}{4 T_1 T_2 \sin^2 \alpha}
\]

\[
= \frac{d_{1,2}}{4 T_1 T_2 \sin^2 \alpha}
\]

Now we switch back to the formulas in paragraph II.3

\[
\| t_c \|^2 = \frac{1}{4 \sin^2 \alpha} \frac{d_{1,2} d_{2,3} d_{3,1}}{T_1^2 T_2^2 T_3^2}
\]
And we also recalculate
\[(\nabla e^Tc)^2_1 + (\nabla e^Tc)^2_2 = \sin^4 \alpha \frac{b_1^2(r_2 - r_3)^2 + b_2^2(r_3 - r_1)^2 + b_3^2(r_1 - r_2)^2}{T_1^2T_2^2T_3^2}\]

Using the previous identities for \(\|\beta\|^2\) and \(\|\gamma\|^2\) in equations (II.14) and (II.15) it follows that the curvature formula (II.19) agrees with the result obtained in (II.6).
III The conflict set as a wavefront

III.1. Some symplectic and contact geometry

The equidistants and focal sets we have encountered in the previous chapter can be studied from a slightly more abstract point of view, that of symplectic geometry. It turns out that this is a very fruitful approach. In this chapter we start to apply this theory to the geometry of conflict sets. A standard introduction to symplectic geometry is [Arn86]. See also [Dui96].

III.1.1. Lagrangian manifolds and symplectic manifolds. A symplectic manifold is a manifold with a closed non-degenerate 2-form \( \omega \) on it. Cotangent bundles \( T^*X \) are symplectic manifolds. On them there is defined a canonical 1-form \( \sigma \) such that \( \omega = d\sigma \). If \( X = \mathbb{R}^n \) the canonical 1-form is \( \sigma = \sum \xi_i \, dx_i \) and \( \omega = \sum d\xi_i \wedge dx_i \).

Definition III.1. A submanifold \( L \) of \( T^*X \) is called \( \text{Lagrangian} \) if the restriction of \( \omega \) to \( TL \) is 0 and \( \dim L = n \). If in addition the restriction of \( \sigma = \sum \xi_i \, dx_i \) to \( TL \) is zero then \( L \) is a conic Lagrangian manifold.

From a conic Lagrangian manifold we can always remove the zero section of \( T^*X \to X \), the resulting manifold will still be a conic Lagrangian manifold. For local considerations we can put \( X = \mathbb{R}^n \). There are several ways to construct Lagrangian submanifolds of \( T^*\mathbb{R}^n \).

- **Generating functions**: Divide the index set \( \{1, \cdots, n\} \) into two disjoint parts I and J. Let \( S: \mathbb{R}^n \to \mathbb{R} \) be a real-valued function: \( S = S(x_I, \xi_J) \). Then \( \{ (x, \xi) \in T^*\mathbb{R}^n \mid x_J = \partial S/\partial \xi_J \quad \xi_I = -\partial S/\partial x_I \} \) is lagrange.

- **Phase functions**: Let \( F(x, s): \mathbb{R}^{n+k} \to \mathbb{R} \) be such that a \( d_s F \) has a regular value at 0. Then \( (d_s F)^{-1}(0) \) is an \( n \)-dimensional submanifold of \( \mathbb{R}^{n+k} \) whose image under the mapping \( (x, d_x F) \) is an immersed Lagrangian submanifold of \( T^*\mathbb{R}^n \).

All Lagrangian submanifolds of \( T^*\mathbb{R}^n \) can be constructed in both ways, at least locally. The two representations are connected by the following. If \( L \) is represented around \( (x_0, \xi_0) \in T^*\mathbb{R}^n \) by \( S(x_I, \xi_J) \) then there is a phase function \( F(x, s) = S(x_I, s) + \langle s, x_J \rangle \) that also represents \( L \).

III.1.2. Legendrian manifolds and contact manifolds. A contact manifold is a manifold \( \mathcal{M} \) with a contact structure. The contact structure is defined using a 1-form \( \sigma \), appropriately called the contact form.

The contact form \( \sigma \) is to satisfy two demands. The derivative \( d\sigma \) is not degenerated on the hypersurfaces in the tangent space of \( \mathcal{M} \) that result from \( \sigma = 0 \). From this condition it already follows that the manifold has to be odd-dimensional. Indeed, non-degenerate 2-forms only exist on even-dimensional spaces. So the tangent space to the contact-manifold has odd dimension, say \( 2n + 1 \).

The second condition is that the 1-form is maximally non-integrable. In earthly terms this is equivalent to saying that \( \sigma \wedge (d\sigma)^n \) is a volume form.

It will also be equivalent to an integral manifold of \( \sigma \) having maximal dimension \( n - 1 \). The contact structure on \( \mathcal{M} \) is the hyperplane subbundle \( \ker \sigma \) of \( T\mathcal{M} \).

Definition III.2. Submanifolds of contact manifolds that are integral manifolds of maximal dimension of \( \ker \sigma \) are called \( \text{Legendrian submanifolds} \).

Let the contact manifold be fibered such that each fiber is a Legendrian submanifold of the contact manifold. We will only consider such \( \text{fibered contact manifolds} \).
The most important example of such a contact manifold is the following. Let $N$ be any manifold. At a point $q$ of $N$ consider the set of all hyperplanes through the origin in the tangent space $T_q N$. This is a projective space $\mathbb{P}T^*_q N$. To construct a field of hyperplanes on the manifold $\mathbb{P}T^*_N$ consider a point $(x_1, ..., x_n; \xi_1; ..., \xi_n)$ on it. This point defines a hyperplane in the tangent space of the base manifold $N$. This plane lifts to an $n - 1$ dimensional plane in the $2n - 1$ dimensional tangent space to the total space $\mathbb{P}T^*_N$ in \((x_1, ..., x_n, \xi_1; ..., \xi_n)\). Join this plane with the vertical directions and one obtains a field of hyperplanes in $T(\mathbb{P}T^*_N)$. This field of hyperplanes defines a contact structure.

It is verified that each fiber of the fibration $\mathbb{P}T^*_N \to N$ is an integral manifold of the contact structure. Indeed, all tangent directions to fibers are vertical directions. They always lie in the hyperplanes of the contact structure.

We have thus made $\mathbb{P}T^*_N$ into a fibered contact manifold. Two other examples of contact manifolds are important.

- $J^1(N)$: Every germ is defined by its function value $z$ and derivative $y$ in a point $x$. The 1-form $\sigma = dz - y \, dx$ defines a 1-form on $J^1(N)$. This 1-form defines a contact structure because $\sigma \wedge (d \sigma)^n$ defines a volume form.
- $ST^* N$: If we coorient each contact element in $\mathbb{P}T^*_N$ we obtain a double covering of that space. This is $ST^* N$.

We have the following generalization of the Darboux lemma

**Lemma III.3.** Every contact manifold is locally contactomorphic to $J^1(\mathbb{R}^n)$, for some $n$.

As an example we take $\mathbb{P}T^* \mathbb{R}^n$. Around a point in $\mathbb{P}T^* \mathbb{R}^n$ coordinates can be written

\[(q_1, \cdots, q_n, p_1; \cdots, p_{n-1}; 1).\]

Thus there is a local contactomorphism to $J^1(\mathbb{R}^{n-1})$. In these coordinates the contact form will be $d q_n - \sum_{i=1}^{n-1} p_i \, d q_i$.

As with Lagrangian manifolds Legendrian manifolds can also be constructed using both generating functions and phase functions. Before getting to this point we have to explain both symplectization and contactization.

### III.1.3. Symplectization

Symplectization is a canonical construction that associates to a contact manifold $V$ a symplectic manifold and to a Legendrian submanifold $L$ a Lagrangian submanifold $\Lambda$.

The contact form that determines the field of hyperplanes in the tangent space is determined up to a multiple. Instead of $d \, z - p \, dq$ we might as well write $\lambda \, dz - \lambda p \, dq$. Here $\lambda$ is some independent variable $\neq 0$.

The form $(d(\lambda p \, dq))^{n+1}$ is indeed a volume form so that $\mathbb{R} \times V$ is a symplectic manifold. If $L$ is a Legendrian submanifold of $V$ then $\Lambda = \mathbb{R} \times L$ is lagrange in $\mathbb{R} \times V$. Infact, $\Lambda$ will not just be lagrange, $\Lambda$ will be conic lagrange.

**Example III.4.** Let $L$ be a Legendrian submanifold of $\mathbb{P}T^* \mathbb{R}^n$. The symplectization of $\mathbb{P}T^* \mathbb{R}^n$ will be $T^* \mathbb{R}^n$. The $n - 1$ dimensional Legendrian manifold is send to a conic Lagrangian manifold.

### III.1.4. Contactization

Contactization is a canonical way of constructing a contact manifold out of a symplectic one. First we notice that on the symplectic manifold there is defined the canonical 1-form $\sigma$. We take the direct product of $\mathbb{R}$ with the contact manifold. This is an additional coordinate $z$. Lagrangian submanifolds $\Lambda$ carry over to Legendrian
ones by an integration

\[ z = \int \sigma = \int \xi \, d\mathbf{x} \]

This is how we go from \( T^*\mathbb{R}^n \) to \( J^1(\mathbb{R}^n, \mathbb{R}) \).

**III.1.5. Constructions with phase functions.** In section III.1.1 we have seen how to use phase functions to construct Lagrangian submanifolds of \( T^*\mathbb{R}^n \). There are essentially two ways of constructing Legendrian submanifolds from phase functions.

The first construction is connected with contactization. Take a Lagrangian \( \Lambda \) submanifold of \( T^*\mathbb{R}^n \). Let \( \Lambda \) be given by a phase function \( F(x, s) \). If \( d_s F \) has a regular value at 0 then \((d_sF)^{-1}(0) \ni (x, s) \mapsto (x, d_x F(x, s))\) has a Lagrangian manifold in \( T^*\mathbb{R}^n \) as an image \( \Lambda_F \).

The contactization process now gives a Legendrian submanifold in the contactization of \( T^*\mathbb{R}^n \), that is \( J^1(\mathbb{R}^n) \). This submanifold is given by the image of:

\[(d_s F)^{-1}(0) \ni (x, s) \mapsto (x, F(x, s), d_x F(x, s)) \]

In this way we have constructed a Legendrian submanifold starting from what is a phase function in the Lagrangian sense.

The second construction is connected with symplectization.

**Definition III.5.** If we have a phase function such that \( F, d_s F(x, s) \) has a regular value in 0 then it is called non-degenerate.

**Remark III.6.** Except in isolated cases (e.g. in subsection IV.2.3) this is what we will mean by a non-degenerate phase function.

The image \( L_F \subset \mathbb{PT}^*\mathbb{R}^n \) of

\[(F(x, s), d_s F)^{-1}(0) \ni (x, s) \mapsto (x, d_x F(x, s)) \]

is \( n - 1 \) dimensional and isotropic. We map

\[(\tau, q, p) \mapsto (q, \tau p) \]

\[\mathbb{R} - \{0\} \times L_F \to T^*\mathbb{R}^n\]

or

\[\mathbb{R}_{>0} \times L_F \to T^*\mathbb{R}^n\]
the image of which is a conic Lagrangian manifold and can be interpreted in the first case as a submanifold of $\mathbb{P}T^*\mathbb{R}^n$ and in the second case as a submanifold of $ST^*\mathbb{R}^n$.

There is another way of looking at this. We can regard $F(x, s)$ as a family of (oriented) hyper-surfaces in $T\mathbb{R}^{n+k}$: for each $(x_0, s_0) \in \mathbb{R}^{n+k}$ we have the hyperplane $d_x, s F(x_0, s_0)(\delta x, \delta s) = 0$ in the tangent space $T_{(x_0, s_0)}\mathbb{R}^{n+k}$. The ones that project as hyperplanes in $T\mathbb{R}^n$ are those that have $\frac{\partial F}{\partial s}(x_0, s_0) = 0$. From the requirement that $(F, d_s F)$ has a regular value at 0 it follows that we can always write locally

$$F(x, s) = x_n - \tilde{F}(x_1, \ldots, x_{n-1}, s)$$

so that indeed a Legendrian submanifold is constructed, either in $\mathbb{P}T^*\mathbb{R}^n$ or in $ST^*\mathbb{R}^n$.

### III.1.6. Note on terminology

Now that the notions “conic Lagrange”, “Legendre” and “Lagrange” have been clearly established we will for the sake of readability start using phrases such as: “... hence $C$ is conic lagrange in $X$ ... ”, when we mean to say that there exists a conic Lagrangian manifold $L \subset T^*X$ such that the projection of $L$ to $X$ is $C$: $\pi_X L = C$ In this sort of phrases it does not matter very much whether $C$ is conic Lagrange or Legendre, hence we will use both terminologies. What is meant will always be clear from the context.

### III.1.7. Canonical relations

From one conic Lagrangian manifold one constructs a new one by means of “a section and a projection”. This idea is made precise by the notion of a canonical relation. For canonical relations we refer to the book [Hör85], pages 289 et seq., in particular theorem 21.2.14.

**Definition** III.7. A canonical relation between two symplectic manifolds $(S_1, \omega_1)$ and $(S_2, \omega_2)$ is a Lagrangian submanifold of $(S_1 \times S_2, \pi_1^* \omega_1 - \pi_2^* \omega_2)$.

By $\pi_1$ and $\pi_2$ we mean the projections $\pi_1: S_1 \times S_2 \to S_1$ and $\pi_2: S_1 \times S_2 \to S_2$.

Here are a few ways to construct canonical relations:

- A Lagrangian submanifold $N$ of $(S_1 \times S_2, \omega_1 + \omega_2)$ gives rise to a canonical relation between $S_1$ and $S_2$ by the accent mapping $\hat{\cdot}$:

  $$(N)' = \{x_1, \xi_1, x_2, \xi_2 \in S_1 \times S_2 \mid (x_1, \xi_1, x_2, -\xi_2) \in N\}$$

- If $S_2$ is a point a canonical relation between $S_1$ and $S_2$ is any Lagrangian manifold of $S_1$.
- If $f: S_1 \to S_2$ is a symplectomorphism, then the graph of $f$ is a canonical relation between $S_1$ and $S_2$.

The composition of two maps can be done via their graph. If we have $f: S_1 \to S_2$ and $g: S_2 \to S_3$ symplectomorphisms then the graph of their composition is

$$\pi_{S_1 \times S_3}(\text{gr}(f) \times \text{gr}(g) \cap S_1 \times \Delta(S_2 \times S_2) \times S_3) \subset S_1 \times S_3$$

This is also how canonical relations are composed.

**Theorem** III.8. Let $S_i, i = 1 \cdots 3$ be three symplectic manifolds. Let $G_1$ be a canonical relation between $S_1$ and $S_2$ and $G_2$ one between $S_2$ and $S_3$. If $G_1 \times G_2$ intersects $S_1 \times \Delta(S_2) \times S_3$ transversally then the image $G_3$ under the projection $S_1 \times S_2 \times S_2 \times S_3 \to S_1 \times S_3$ is a canonical relation between $S_1$ and $S_3$. We call this the composition $G_1 \circ G_2$ of $G_1$ and $G_2$.

In our applications we will usually apply this theorem there where $S_3$ is a point. So we use the next proposition, that rephrases the demand in the theorem of Hörmander.
Proposition III.9. Let $G_1$ be a canonical relation between $S_1$ and $S_2$ whose projection to $S_2$ is an immersion and let $G_2$ be a canonical relation between $S_2$ and a point. Then the composition $G_1 \circ G_2$ is a canonical relation, and thus a Lagrangian manifold in $S_1$, if $\pi_2(G_1) \pitchfork G_2$.

Proof. We need that

\[(III.3) \quad G_1 \times G_2 \pitchfork S_1 \times \Delta(S_2)\]

This intersection is contained in the graph of the projection $\pi_2 : G_1 \to G_2$. We have that (III.3) holds iff.

$$\text{gr}(\pi_2) \pitchfork G_1 \times G_2$$

this in turn is true iff.

$$\pi_2(G_1) \pitchfork G_2$$

\[\square\]

III.1.8. The Gauss map. The space of oriented lines in $\mathbb{R}^n$ can be realized as the symplectic manifold $T^*S^{n-1}$. Namely if $\ell$ is a directed line in $\mathbb{R}^n$ then this line has a direction $v \in S^{n-1}$.

The direction $v$ determines a hyperplane through the origin:

$$H_\ell = \{ x \in \mathbb{R}^n \mid \langle v, x \rangle = 0 \} \subset \mathbb{R}^n$$

The hyperplane $H_\ell$ can be identified with the tangent plane $T_v S^{n-1}$, to be identified with $T_v^* S^{n-1}$ through the Legendre mapping. The intersection point of $H_\ell$ and $\ell$ determines thus a point in $T_v^* S^{n-1}$. The normal to an oriented $(M, \vec{n})$ hypersurface in $\mathbb{R}^n$ is a directed line.

![Figure III.2. Construction of space of directed lines](image)

We thus have a map $N^*M \mapsto T^*S^{n-1}$. As a map $\vec{n} : M \to S^{n-1}$ it is known as the Gauss map. The image of this map is Lagrangian in $T^*S^{n-1}$, see [Arn90].
Theorem III.10. There is a canonical relation $G$ between $T^*S^{n-1}$ and $T^*\mathbb{R}^n \setminus \{0\}$ such that its composition with any conic Lagrangian manifold in $T^*\mathbb{R}^n \setminus \{0\}$ yields a Lagrangian submanifold of $T^*S^{n-1}$. In particular, the composition of $G$ with the conormal bundle $N^*M$ of a submanifold $M \subset \mathbb{R}^n$ yields a Lagrangian submanifold of $T^*S^{n-1}$ that coincides with the image of the Gauss map.

Proof. We write $(v, \mu)$ for coordinates on $T^*S^{n-1}$. They are really coordinates on $T^*\mathbb{R}^n$ but we will always have $\langle v, \mu \rangle = 0$ and $\|v\| = 1$ so that they can be used as coordinates on $T^*S^{n-1}$. The canonical symplectic form is $d\ v \wedge d\mu$.

Consider the following subset $G$ of $T^*S^{n-1} \times (T^*\mathbb{R}^n \setminus 0)$

$$\{(v, \mu, x, \xi) \mid v = \frac{\xi}{\|\xi\|}, \ \mu = x - \frac{\langle x, \xi \rangle \xi}{\|\xi\|^2}, \ \|\xi\| = C\}$$

(Here $C > 0$ is some constant.)

The subset $G$ mimicks exactly the geometric construction that associates to a point on a conormal $N^*M$ a directed line in $T^*S^{n-1}$. Our proof will consist of two steps:

- that $G$ is a canonical relation from $T^*\mathbb{R}^n \setminus 0$ to $T^*S^{n-1}$
- that conic Lagrangian manifolds are exactly those that can be pulled back to $T^*S^{n-1}$ by $G$.

Step 1. Remark that instead of $\omega_2 = \sum_{i=1}^n dx_i \wedge d\xi_i$ we can take any multiple $\lambda \omega_2$ of $\omega_2$ as a symplectic form, because $\omega_2 = 0$ on a tangent space iff. a nonzero multiple of it is zero. Both forms give the same structure in the tangent space.

Accordingly if we prove that $\lambda \omega_2 = d\ v \wedge d\mu$ on every $T_pG$ we have shown that $G$ is a canonical relation, between $T^*S^{n-1}, d\ v \wedge d\mu$ and $T^*\mathbb{R}^n \setminus \{0\}, \omega_2$.

We are to prove that for two tangent vectors $(\delta v, \delta \mu, \delta x, \delta \xi)$ and $(\delta v', \delta \mu', \delta x', \delta \xi')$ at $p = (v, \mu, x, \xi)$ we have

$$\lambda \sum_{i=1}^n \delta x_i \delta \xi_i' - \delta x_i' \delta \xi_i = \sum_{i=1}^n \delta v_i \delta \mu_i' - \delta v_i' \delta \mu_i$$

We first get rid of the $v$ coordinates.

$$v = \frac{\xi}{\|\xi\|} \Rightarrow \delta v_i = \frac{\delta \xi_i}{C}$$

So that we are left with:

$$\sum_{i=1}^n \left( \lambda \delta x_i + \frac{\delta \mu_i}{C} \right) \delta \xi_i' = \sum_{i=1}^n \left( \lambda \delta x_i' + \frac{\delta \mu_i'}{C} \right) \delta \xi_i$$

The next candidates for removal are the $\mu$ coordinates:

$$\delta \mu_i = \delta x_i - \frac{\langle x, \xi \rangle}{C^2} \delta \xi_i - \frac{\xi_i}{C^2} \left( \sum_{j=1}^n \xi_j \delta x_j + x_j \delta \xi_j \right)$$

(III.5) \[ \begin{array}{c} \delta \mu_i = \delta x_i - \frac{\langle x, \xi \rangle}{C^2} \delta \xi_i - \frac{\xi_i}{C^2} \left( \sum_{j=1}^n \xi_j \delta x_j + x_j \delta \xi_j \right) \\ \text{The terms from III inserted in each side of (III.4) annul because they are multiples of } \sum \xi_i \delta \xi_i. \text{ The terms from II are identical at both sides of (III.4). Finally choose } \lambda = \frac{1}{C}. \end{array} \]

Step 2: $G$ is a canonical relation. Its projection to $T^*\mathbb{R}^n$ is $\|\xi\| = C$. The projection is an immersion. We can apply proposition III.9: if $L \subset T^*\mathbb{R}^n$ intersects $\|\xi\| = C$ transversely we have that via $G$ we can define in $T^*S^{n-1}$ an “image of the Gauss map”. But $L \cap \{\|\xi\| = C\}$ amounts to saying that $L$ is conic lagrange. \(\square\)
III.1. SOME SYMPLECTIC AND CONTACT GEOMETRY

Remark III.11. Though probably known to experts we have not found this Gauss canonical relation for arbitrary conic Lagrangian manifolds in the literature.

III.1.9. Wavefront from a submanifold. In the sequel we will encounter a distance function on a $\mathcal{C}^{\infty}$ manifold $X$. If $X = \mathbb{R}^n$ and we use the standard euclidean metric it is clear what is meant. In other cases where we define a distance function by means of a Hamiltonian we need to impose rather strong conditions on $X$ and the Hamiltonian $H$.

Definition III.12. A \textit{homogeneous Hamiltonian} is a function on the slit cotangent bundle $T^*X \setminus \{0\}$ that is homogeneous of degree 2 in the coordinates of the fiber of $T^*X \setminus \{0\} \rightarrow X$. In other words, it is the continuous assignment to each fiber of a function homogeneous of degree 2.

Example III.13. • Any finsler metric on $X$ gives rise to a homogeneous Hamiltonian.
• If $g_{ij}(x)$ is the matrix of a Riemannian metric on $X$ a homogeneous Hamiltonian is

$$H: (x, \xi) \mapsto \sum_{i,j} g^{ij} \xi_i \xi_j$$

Throughout this chapter and the next we will assume

• 1 that all our Hamiltonians are $\mathcal{C}^{\infty}$. In physics this will not always be the case. For instance, on the border of two different media there is refraction. Here the Hamiltonian will not be $\mathcal{C}^{\infty}$.
• 2 that the quadratic form

$$\frac{\partial^2 H}{\partial \xi^2}$$

is non-degenerated, for all $x, \xi$.

To contrast our approach with others we explicitly remark that our quadratic Hamiltonians include pseudo-Riemannian metrics.

Definition III.14. For any function $K: T^*X \rightarrow \mathbb{R}$ the Hamiltonian vectorfield $vf(K)$ is defined by

$$(III.6) \quad \omega(vf(K), \cdot) = -dK$$

The Hamiltonian vectorfield $vf(K)$ has an induced exponential mapping

$$\exp_K: T^*X \times \mathbb{R} \rightarrow T^*X$$

The map $\exp_K(x, \xi, t)$ is called the \textit{Hamiltonian flow}.

Take any conic Lagrangian submanifold $L$ of $T^*X \setminus \{0\}$. Because $H$ is homogeneous in the fiber coordinates the intersection $L \cap H^{-1}(1)$ is transversal. For a fixed $t = t_0$ we can flow out $L \cap H^{-1}(1)$ with the Hamiltonian vectorfield of $H$. The image is

$$\exp_H(L \cap H^{-1}(1), t_0)$$

is isotropic and of dimension $n - 1$.

Remark III.15. Note that it does not matter much whether we flow out an isotropic manifold by the Hamiltonian vectorfield of $H$ or by $vf(H^p)$. From equation (III.6) we can deduce that the one vector $vf(H^p)$ is a multiple of the other $vf(H)$ when restricted to $H = 1$. 
If we also “multiply the fibers” of \( \exp_H(L \cap H^{-1}(1), t_0) \)
\[ \mathbb{R}_{>0} \times T^*X \to T^*X \quad (\lambda, x, \xi) \to (x, \lambda \xi) \]
we obtain a map
\[ L \mapsto L_t = \exp_{\mathbb{R}}(L, t) \]
that sends conic Lagrangian submanifolds to conic Lagrangian submanifolds. In particular, we may apply \( \exp_{\mathbb{R}} \) to the fiber \( T^*_{x_0}X \) over \( x_0 \) of \( T^*X \setminus 0 \). Denote
\[ S(x_0, t) = \exp_{\mathbb{R}}(T^*_{x_0}X, t) \]
With the assumptions we make \( S(x_0, t) \) is a \( C^\infty \) manifold. Note that the manifold \( S(x_0, t) \) is not necessarily symmetric or connected. The manifolds \( S(x_0, t) \) are smooth. Their projection to \( X \) is not always smooth.
We will now make an additional assumptions about the pair \( X, H \).

\* 3 The integral curves of \( \text{vf}(H) \) are defined for all \( t \in \mathbb{R} \).

In case \( H \) is a Finsler metric the integral curves of \( \text{vf}(H) \) are geodesics. Thus, the assumption amounts to saying that \( X \) is complete, see [BCS00]. This assertion is called the Hopf-Rinow theorem.

Consider the integral
\[ \text{III.7} \quad A = A(x_0, x_1) = \int_{\Gamma} \xi \, dx \]
where \( \Gamma \) is a curve in \( T^*X \cap \{ H = 1 \} \) that makes this action stationary and whose projection to the \( X \) space goes from \( x_0 \) to \( x_1 \). The solution curves to this problem are locally uniquely defined due to the two assumptions above. Thus the integral \( A \) depends only on \( x_0 \) and \( x_1 \) - and on the energy-level \( H = 1 \) chosen.

The manifolds \( S(x_0, t) \) are level sets of the function \( x \mapsto \pi_X(A(x_0, x)) \). This makes \( A(x_0, x_1) \) the perfect candidate for a distance or time function, as will be shown in the next section.

The function \( A(x_0, x_1) \) is \( C^\infty \) on \( X \times X \setminus \Delta \). We will also call it the \textbf{work function}.

It is the analogue of \( \| x_0 - x_1 \| \) in the euclidean case.

Our last and final assumption is:

\* 4 For any pair \( x_0 \) and \( x_1 \) there is at most one curve \( \Gamma \) as above.

Now we have assembled all we needed for our analysis. The \( S(x_0, t) \) are defined for all \( x_0 \) and \( t \) by the third assumption and their projection to \( X \) is smooth by the fourth assumption. Sadly there is a catch here: these demands are very strong. When \( X = \mathbb{R}^n \) and we use the standard Riemannian metric they are trivially satisfied. There is one other well-known case where these assumptions are satisfied. We cite the Cartan-Hadamard theorem.

\textsc{Theorem III.16.} If \( X \) is a simply connected complete Riemannian manifold with all sectional curvatures negative or 0 then any two points can be joined with a unique geodesic. The exponential map \( \exp \colon T_pM \to X \) is a diffeomorphism for every \( p \in X \).

The Cartan-Hadamard theorem also holds in the finsler setting. It singles out an exceptional situation because already when \( X = S^n \) our assumptions are no longer satisfied because closed geodesics abound there. This is clearly not what we want.

We are thus led to contemplating whether we can do a little more. For instance we could impose our conditions in some open subset of \( X \times \mathbb{R} \). So if \( X = S^n \) then we could leave out say the north-pole \( \{0\} \) and consider for each \( x \in S^n \setminus \{0\} \) only those times for which the north pole is not attained. However \( S^n \setminus \{0\} \) can be spread out and in that case it will look just like \( \mathbb{R}^n \) and such additional time and space restrictions will certainly not ease notation.
Instead we note that if we have on $\mathbb{R}^n$ a quadratic Hamiltonian independent of $x$ then all our demands are satisfied. Such quadratic Hamiltonians are often called translation invariant. In the case where $H = \sum g^{ij}(x)\xi_i\xi_j$ it is possible to choose local coordinates such that

\begin{equation}
\frac{\partial g^{ij}}{\partial x_k}(0) = 0
\end{equation}

Thus in some sense $H$ does not depend on $x$ here. For a proof of this assertion, see [Hör85], part III, appendix C. We conclude that nearly all quadratic Hamiltonians are near to translation invariant ones.

We will now turn our attention to wavefronts not just from a point, but from a submanifold $M$ of $X$ with its accompanying Hamiltonian $H$ the wavefront of $M$ at time $t$.

**Definition III.17.** For $\gamma \in \text{Emb}(M, X)$ define

$$WF(t, M) = \bigcup_{s \in M} S(\gamma(s), t) \subset M \times T^*X$$

The singular values of the projection of $WF(t, M)$ to $T^*X$ are called the wavefront $\Sigma(t, M)$ of $M$ at time $t$.

The wavefront $\Sigma(t, M) \subset T^*X$ is thus defined as an envelope, and not as a flow-out. The following lemma relates the flow-out and the envelope.

**Lemma III.18 (Huygens principle).**

$$\Sigma(t, M) = \exp_{H}^{\mathbb{R}}(N^*M)$$

If $M \subset X$ is cooriented and a hypersurface and $H$ is positive definite the manifold $N^*M \cap \{H = 1\}$ has two components, each corresponding to one sign of the orientation. We could so speak of $\Sigma^+(t, M)$ and $\Sigma^-(t, M)$, assuming $t$ is positive.

**Example III.19.** Take $X = \mathbb{R}^2$ and $H = \xi_1^2 - \xi_2^2$ then the “distance” from $p_0 = (x_0, y_0)$ to $p_1 = (x_1, y_1)$ is $\sqrt{|(x_0 - x_1)^2 - (y_0 - y_1)^2|}$, if we choose energy levels $\pm 1$. Indeed, some points cannot even be connected, for instance the origin and $(1, 1)$. However, our four assumptions are satisfied. In figures III.3 and III.4 we see the envelopes for different values of the energy level. Below are all the wavefronts from different points. The envelopes are above.

![Figure III.3. Fronts for $\xi_1^2 - \xi_2^2 = -1$](image-url)
Several articles among which \[\text{IPS00}\] deal with pseudo-Riemannian metrics to investigate questions in generic differential geometry. With our definition of distance we do not need to distinguish between the case of pseudo-Riemannian geometry and the case of Riemannian geometry.

We repeat that the combination of the work function \(A\) with an embedding \(\gamma\) allows us to produce a globally defined non-degenerate phase function \(F(x, s) = A(x, \gamma(s))\) for \(\Sigma(t, M)\). This will be essential in the sequel.

In the literature one finds the assertion that the existence of a global non-degenerate phase function implies some conditions on the cohomology of the ambient manifold, see for instance \[\text{Zak84}\], p. 2733. In this way the conclusion of the Cartan-Hadamard theorem is not so surprising.

### III.1.10. Quadratic Hamiltonians and the Legendre transform.

We have already seen that the integral curves of the Hamiltonian vectorfield \(vf(H)\) correspond to geodesics. We would like to explain in some detail the relation between the Hamiltonian viewpoint and the more traditional geometric viewpoint. For the geometry we refer to \[\text{BCS00}\]. The main change that we make is that we do not require that our “metrics” are positive definite.

The Hamiltonian dynamics take place in the cotangent space and the differential geometry in the tangent space. There is a pairing between the two:

\[
T^*_x X \times T_x X \rightarrow \mathbb{R} \quad (\xi, v) \mapsto \xi(v)
\]

For a fixed \(v \in T_x X\) we can look for critical values of the function

\[
\xi(v) - H(x, \xi)
\]

These critical values are attained there where

\[
(\text{III.9}) \quad v = \frac{\partial H}{\partial \xi}
\]
Because
\[ \det(\frac{\partial^2 H}{\partial \xi^2}) \neq 0 \]
we know that (III.9) defines a local diffeomorphism from \( T^*X \) to \( TX \). It is the **Legendre transform**. When \( v \) is the Legendre transform of \( \xi \) we write \( v = \xi^b \).

The critical value is uniquely determined by the Legendre transform and we can introduce the Lagrangian \( \mathcal{L} \):
\[
\mathcal{L}(x, v) = \max_{\xi \in T_x^*X} (\xi(v) - H(x, \xi))
\]

If \( v = \xi^b \) then with (III.9) we have:
\[
\mathcal{L}(x, v) = H(x, \xi)
\]

If we have a finsler metric \( F \) on \( TX \) we can put
\[
\mathcal{L}(x, v) = \frac{1}{2} F^2(x, v)
\]

**Example III.20.** The case with Riemannian metrics permits more explicit comparisons between \( H \) and \( \mathcal{L} \). As usual denote \( g_{ij} \) a Riemannian metric. We have \( H = \frac{1}{2} (g^{ij} \xi, \xi) \). And the Lagrangian is \( \mathcal{L}(x, v) = \frac{1}{2} (g_{ij} v_i, v_j) \). The Legendre transform is
\[
v = g^{ij} \xi
\]

The inverse of the Legendre transform can be used to obtain from the canonical 1-form on \( T(T^*X) \) a form on \( T(TX) \)
\[
\xi \, dx \Rightarrow \xi^b \, dx
\]
When restricted to the surface \( \mathcal{L}(x, v) = 1 \) this form is sometimes known as the Hilbert form. We denote it \( \alpha^b \). Contrary to what is done in [BCS00] the form we introduce here does not have the property that it is invariant under rescaling of the \( v \) coordinate, we do not care because we fix the manifold over which we work.
\[
\alpha^b(x, \lambda v) = \lambda \alpha^b
\]
The Hilbert form can be used just as the integral \( \int \xi \, dx \) to obtain extrema of path length. With Riemannian metrics we carry out the following calculation.
\[
\int_{\Gamma} \xi \, dx = \int_{\Gamma'} g_{ij} v_i \, dx
\]
Here \( \Gamma \) is the curve from (III.7) and \( \Gamma' \) is the image of \( \Gamma \) under the Legendre transform. Denote the length of a curve \( \gamma: [a, b] \rightarrow X \) as
\[
L^b_a = \int_a^b \sqrt{g_{ij} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt}} \, dt
\]
whilst the energy is
\[
E^b_a = \int_a^b g_{ij} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} \, dt
\]
It follows that the extremum of the integral of \( \xi \, dx \) is both equal to the length and the energy of the curve, because \( \pi_X(\Gamma) \) is unit parameterized.
III.1.11. The big wavefront. There is one special kind of wave front that we still need to investigate. This is the big wavefront, sometimes also called the graph of the time function.

To define it take a submanifold \( M \subset X \), where \( X \) satisfies all four assumptions stated in section III.1.9. The union of all wavefronts \( \Sigma(t, M) \) as a subset of \( \mathbb{R} \times T^*X \) is Legendrian submanifold of

\[
J^1(X, \mathbb{R}) = \mathbb{R} \times T^*X
\]

with contact form \( d\xi_0 - \sum \xi d\tau \). We can view \( J^1(X, \mathbb{R}) \) as a coordinate patch of \( PT^*(X \times \mathbb{R}) \). We can apply the symplectization and obtain from the union of the \( \Sigma(t, M) \) a conic Lagrangian manifold in \( T^*(X \times \mathbb{R}) \). This is the big wavefront \( N^*M_h^* \).

Let us show another way to define the big wavefront. Denote \( 0_{\mathbb{R}} \) the zero section of \( T^*\mathbb{R} \). We observe that \( 0_{\mathbb{R}} \times N^*M \subset T^*(X \times \mathbb{R}) \setminus 0 \) is a conic Lagrangian manifold in \( T^*(X \times \mathbb{R}) \). Again, in particular \( 0_{\mathbb{R}} \times T^*_x X \) is a conic Lagrangian manifold.

**Lemma III.21.** Let \( H: T^*X \rightarrow \mathbb{R} \) be a Hamiltonian, positively homogeneous of degree 1. Let \( \phi^t: T^*X \times \mathbb{R} \rightarrow T^*X \) be the corresponding flow. The map \( \Psi: T^*(X \times \mathbb{R}) \rightarrow T^*(X \times \mathbb{R}) \) given by

\[
(x, \xi, t, \tau) \xrightarrow{\Psi} (\phi^t(x, \xi), t, \tau - H(x, \xi))
\]

preserves the symplectic form.

**Proof.** The map \( \Psi \) is symplectic iff.

\[
\omega(d\cdot, d\cdot') = \omega(\cdot, \cdot')
\]

We write \( p = (x, \xi) \), and thus

\[
d_{p,t,\tau} \phi^t(\delta p, \delta t, \delta \tau) = d_p \phi^t \delta p + vf(H) \delta t
\]

Writing this out we thus need to prove that

\[
\omega((d_p \phi^t \delta p + vf(H) \delta t, \delta t', \delta \tau - \nabla H \delta p), (d_p \phi^t \delta p' + vf(H) \delta t', \delta \tau' - \nabla H \delta p'))
\]

\[
= \omega(d_p \phi^t \delta p + vf(H) \delta t, d_p \phi^t \delta p' + vf(H) \delta t')
\]

\[
+ \omega((\delta t, \delta \tau - \nabla H \delta p)(\delta t', \delta \tau' - \nabla H \delta p'))
\]

\[
= I + II
\]

equals \( \omega((\delta p, \delta t, \delta \tau), (\delta p', \delta t', \delta \tau')) \). We calculate part I.

\[
I = \omega(d_p \phi^t \delta t, d_p' \phi^t \delta t') + \omega(vfH \delta t, d_p' \phi^t \delta t') + \omega(d_p \phi^t \delta t, vf(H) \delta t', d_p' \phi^t \delta t') + \delta t \delta t' \omega(vf(H), vf(H))
\]

\[
= \omega(\delta p, \delta p') + \omega(vf(H) \delta t, d_p, \phi^t \delta p') - \omega(vf(H) \delta t, d_p, \phi^t \delta p)
\]

Next we calculate part II. Because

\[
II = \omega((\delta t, \delta \tau)(\delta t', \delta \tau'))
\]

the proof is complete. \( \square \)

**Remark III.22.** In our case we have a quadratic Hamiltonian but on the intersection

\[
0_{\mathbb{R}} \times T^*_x X \cap H = 1
\]
III.2. The conflict set via a canonical relation

In this section we apply the notion of a canonical relation to conflict sets in order to prove that the conflict set has a Legendre singularity if a certain transversality condition is satisfied.

III.2.12. Definition of conflict set. As in the previous section we have the following setup: Let \( M_i, i = 1, 2 \) be two smooth manifolds of dimension \( n - 1 \), equipped with an orientation \( n_i \). Suppose that by \( \gamma_i: M_i \to X, i = 1, 2 \), the manifolds are smoothly embedded in a smooth manifold \( X \) of dimension \( n \). Also let \( H_i: X \to \mathbb{R}, i = 1, 2 \) be two homogeneous Hamiltonians.

**Definition III.23.** The conflict set of two submanifolds \( M_1, M_2 \subset X \) relative to two quadratic Hamiltonians \( H_1, H_2: T^*X \to \mathbb{R} \) is the set

\[
C = \{ x \in X \mid \exists t \in \mathbb{R} \quad x \in \pi_X(\Sigma(t, M_1)) \quad x \in \pi_X(\Sigma(t, M_2)) \}
\]

The set \( C \) is the projection of the intersection of big wavefronts, in the way we have encountered it in chapter 1.

Just as with the wavefronts there is a phase function defined on the whole of \( X \) for the conflict set \( C \) of two submanifolds of \( X \). It is given by

\[
F(x, s_1, s_2): X \times M_1 \times M_2 \to \mathbb{R}
\]

\[
F(x, s_1, s_2) = A_1(\gamma_1(s_1), x) - A_2(\gamma_2(s_2), x)
\]

Here \( A(\gamma_i(s_i), x) \) is the work function for the Hamiltonian \( H_i \).

III.2.13. A transversality condition. In the theorem below we will employ the notation:

\[
(\text{III.11}) \quad T^*_\Delta(X^i) \setminus 0 = \{(x_1, \ldots, x_i, \xi_1, \ldots, \xi_i) \in T^*(X^i) \setminus 0 \mid x_i = x_j \; i \neq j\}
\]

**Theorem III.24.** If the flow out of \((N^*M_1 \times N^*M_2)^{\prime} \cap ((H_1 - H_2)^{-1}(0)) \) by \( H_1 - H_2 \) intersects \( T^*_\Delta(X \times X) \) transversally then the conflict set is a conical Lagrangian submanifold of \( T^*X \).

**Proof.** In the setting of theorem (III.8) we choose \( S_1 = T^*\Delta, S_2 = T^*(X \times X) \) and \( S_3 \) a point. The canonical relation in \( S_1 \times S_2 \) we choose is

\[
G_1 = \{(x_1, \xi_1, x_2, x_3, \xi_2, \xi_3 \mid x_1 = x_2 = x_3 \quad \xi_1 = \xi_2 + \xi_3\}
\]

We will now choose an appropriate Legendrian submanifold in \( S_2 \) so that we can apply proposition III.9.

The manifolds \( N^*M_1 \) and \( N^*M_2 \) are two conical Lagrangian submanifolds in \( T^*X \). Apply
the accent mapping ′ to the product \( N^*M_1 \times N^*M_2 \). The non-degeneracy of the Hamiltonians implies that \( \{ H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0 \} \) is a hypersurface in \( T^*(X \times X) \). If we flow out the intersection

\[
(N^*M_1 \times N^*M_2) \cap \{ H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0 \}
\]

by the Hamiltonian vectorfield of \( H_1 - H_2 \) we stay inside the hypersurface

\[
\{ H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0 \}.
\]

Call the flow-out \( N \). The conclusion of the theorem follows if we take the composition \( G_1 \circ N \).

Thus we have a very general criterion under which the conflict set resulting from two homogeneous Hamiltonians is Legendrian.

**III.2.14. Examples, I.** Stated as in theorem III.24 the criterion is not very computable. To obtain a computable criterion we first have to restrict our attention to a computable situation. One of the few situations in which we can calculate wavefronts is in \( \mathbb{R}^n \). To (locally) find equations for a wavefront we take as a phase function \( F \) the squared distance from \( x \) to \( \gamma(s) \). A phase function for the conflict set is

\[
F(x, s_1, s_2) = \| x - \gamma_1(s_1) \|^2 - \| x - \gamma_2(s_2) \|^2
= F_1(x, s_1) - F_2(x, s_2)
\]

so that the equations for a conflict set are

\[
F(x, s_1, s_2) = F_2(x, s_2) - F_1(x, s_1) = 0 \quad \partial F_1 \partial s_1 = 0 \quad \partial F_2 \partial s_2 = 0
\]

Now the demand that \( F \) is non-degenerate phase function wrt. both \( s_1 \) and \( s_2 \), so that the matrix

\[
(III.12)
\]

\[
\left( \begin{array}{ccc}
\frac{\partial F}{\partial x} & \frac{\partial F_1}{\partial s_1} & -\frac{\partial F_2}{\partial s_2} \\
\frac{\partial^2 F_1}{\partial s_1 \partial x} & \frac{\partial^2 F_1}{\partial s_2^2} & 0 \\
-\frac{\partial^2 F_2}{\partial s_2 \partial x} & 0 & -\frac{\partial^2 F_2}{\partial s_2^2}
\end{array} \right)
\]

has maximal rank.

Because we can write

\[
F(x, s_1, s_2) = 2\langle x, \gamma_2(s_2) - \gamma_1(s_1) \rangle + \| \gamma_1(s_1) \|^2 - \| \gamma_2(s_2) \|^2
\]

the matrix \((III.12)\) can be written at points of the conflict set:

\[
K = \left( \begin{array}{ccc}
\gamma_2(s_2) - \gamma_1(s_1) & 0 & 0 \\
-\dot{\gamma}_1(s_1) & \langle \gamma_1(s_1), \dot{\gamma}_1(s_1) \rangle - \langle x - \gamma_1(s_1), \dot{\gamma}_1 \rangle & 0 \\
-\dot{\gamma}_2(s_2) & 0 & \langle \gamma_2(s_2), \dot{\gamma}_2(s_2) \rangle - 2\langle x - \gamma_2(s_2), \dot{\gamma}_2 \rangle
\end{array} \right)
\]

The rank \( \text{rk} K \) of \( K \) is bounded from below by

\[
1 + \text{rk} \left( \frac{\partial^2 F_1}{\partial s_1^2} \right) + \text{rk} \left( \frac{\partial^2 F_2}{\partial s_2^2} \right)
\]

A few remarks are in order:

- The rank \( \text{rk} K \) has to be at least \( 1 + \dim M_1 + \dim M_2 \) for the conflict set to be Legendrian at \( x \).
The first $n$ columns of the matrix are $K$ linearly independent: if $\gamma_1(s_1) \neq \gamma_2(s_2)$ the vector $\gamma_2(s_2) - \gamma_1(s_1)$ cannot lie in the tangent plane of $M_1$ at $\gamma_1(s_1)$. Suppose this were the case. Then

$$\langle x - \gamma_1(s_1), \gamma_2(s_2) - \gamma_1(s_1) \rangle = 0$$

for some $x$ on the conflict set. Because $x$ lies on the conflict set we would also have

$$\langle x - \gamma_1(s_1), x - \gamma_1(s_1) \rangle = \langle x - \gamma_2(s_2), x - \gamma_2(s_2) \rangle$$

Adding (III.13) and (III.14) we get that

$$\langle x - \gamma_1(s_1), x - \gamma_2(s_2) \rangle = \langle x - \gamma_2(s_2), x - \gamma_2(s_2) \rangle$$

and thus, $\gamma_1(s_1) = \gamma_2(s_2)$ which is impossible. On the other hand $\gamma_1(s_1) = \gamma_2(s_2)$ can be avoided because the conflict set does not change when we move both manifolds $M_1$ and $M_2$ by the same distance. We can use this argument only locally but then we are only looking locally.

The square matrices

$$d_s^2 F_i = \langle \gamma_i(s_i), \gamma_i(s_i) \rangle - \langle x - \gamma_i(s_i), \gamma_i \rangle$$

have maximal rank if the wavefront of $M_i$ at $x$ is smooth. Note that the wavefront might well be $C^1$ at points where $d_s^2 F_i$ does not have maximal rank. In fact the equation

$$\det(d_s^2 F_i(x, s_i)) = 0$$

defines the focal surfaces of $M_i$. The relationship between the singularities of the map $F_i(x, s_i)$ and the geometry of $M_i$ is investigated in [Por01].

In chapter V we will apply the singularity theory of Lagrangian and Legendrian mappings to the problem of conflict sets. There it is always good to keep the matrix $K$ in mind.

**III.2.15. Generalization to three surfaces and more.** The previous construction worked to obtain the conflict set of two submanifolds $M_1, M_2$ of an ambient manifold $X$. It also generalizes to the case where we have $l \leq n$ submanifolds of $X$.

We will use the big wavefronts from section III.1.11. Let

$$P = \prod_{i=1}^l N^* M_i^h \subset (T^*(X \times \mathbb{R}))^l$$

**Theorem III.25.** The conflict set of the $M_i$ is Legendrian when

$$P \pitchfork T^*_\Delta ((X \times \mathbb{R})^l)$$

**Proof.** We consider a canonical relation in $T^* X \times T^* \mathbb{R} \times (T^* X \times T^* \mathbb{R})^l$.

$$G_1 = \{(\bar{y}, \bar{\eta}, \bar{x}_1, \bar{\xi}_1, \cdots, \bar{x}_l, \bar{\xi}_l) \mid \bar{y} = \bar{x}_i i = 1, \cdots, l \bar{\eta} = \sum_{i=1}^l \bar{\xi}_i\}$$

To check that $G_1$ is a conic canonical relation we need to show that it is a conic Lagrangian manifold wrt. to the form $\bar{\eta}d\bar{y} - \sum_{i=1}^l \bar{\xi}_i \bar{x}_i$. This is the case.

The manifold $P$ can be pulled back to $T^* X \times T^* \mathbb{R}$ if the conditions in theorem III.8 are fulfilled. I.e. we are to have a transversal intersection

$$G_1 \times P \pitchfork T^* X \times T^* \mathbb{R} \times \Delta \left((T^* X \times T^* \mathbb{R})^l\right)$$

This intersection is transversal iff.

$$P \pitchfork T^*_\Delta ((X \times \mathbb{R})^l)$$
Let $L^h$ be the conical Lagrangian manifold that is the pull back of $P$ by $G_1$. We proceed to pull $L^h$ back to $T^*X$. The canonical relation we use for this is

$$G_2 = \{ (x, \xi, \bar{y}, \bar{\eta}) \mid x = \pi_X \bar{y}, \pi^*_X \xi = \eta, \eta_0 = 0 \}$$

Again we have that upon applying proposition III.9

$$G_2 \times L^h \triangleleft T^*X \times \Delta (T^*(X \times \mathbb{R})) \leftrightarrow W \cap L^h$$

where

$$W = \{ (\bar{y}, \bar{\eta}) \mid \eta_0 = 0 \}$$

The proof will be complete with the following lemma.

**Lemma III.26.** $P \cap T^*_\Delta ((X \times \mathbb{R})^l) \Rightarrow W \cap L^h$

**Proof.** Suppose that we did not have $W \cap L^h$. Because $W$ is a hypersurface that would mean that at some point $p$ in $L^h$ the tangent space $TL^h$ would be contained in $TW$. So it would hold

$$\langle (0, 0, 0, \delta \eta_0, w) \rangle = 0, \forall w \in TL^h$$

Let $J$ denote the usual complexification mapping. In local coordinates we can write:

$$\omega(v, w) = \langle v, J(w) \rangle$$

As $J$ maps $\delta y_0$ to $\delta \eta_0$ the equation (III.18) becomes

$$\omega((0, 0, 0, \delta y_0, 0), w) = 0, \forall w \in T_pL^h$$

with $\omega$ being the canonical symplectic structure. But $L^h$ is Lagrangian, so we would have that this vector $(0, 0, \delta y_0, 0) \in TL^h$. But that is clearly impossible.

It is verified that in the case $k = 2$ the above construct is the same as the one with the accent mapping. We have that $\bar{x}_{i,0} = \bar{x}_{j,0}$. The construction with $W$ implies that we have a fixed energy level, so that $\bar{x}_{i,0}$ is the “time traveled”, i.e. $\bar{x}_{i,0} = A_i(x, \gamma_i(s_i))$. When $k = 2$ there is thus just one equation:

$$A_1(\gamma_1(s_1), x) - A_2(\gamma_2(s_2), x) = 0$$

Here we have the same phase function as the one we got with the accent mapping. As a consequence the constructions are identical.

**III.2.16. Examples, II.** Let $F(x, s)$ be a non-degenerate phase function for a Legendrian submanifold of $\mathbb{P}T^*X$. Suppose we want to construct the corresponding conical Lagrangian manifold with a phase function. We can pick $\lambda F(x, s)$. Indeed it holds - provided that $\lambda \neq 0$ - that

$$F = 0, \quad d_sF = 0 \Leftrightarrow d_{\lambda,s}(\lambda F) = 0.$$
of the ambient manifold $X$ and the fixed energy level $H = 1$ chosen. For a phase function we could also have used the less symmetric

$$F = (F_1 - F_2) + \lambda_1(F_2 - F_3) + \cdots + \lambda_{l-2}(F_{l-1} - F_l)$$

It is instructive to calculate the matrix of derivatives

(III.20) $d_{x,\lambda_1,\cdots,\lambda_{l-1},s_1,\cdots,s_l} \left( d_{\lambda_1,\cdots,\lambda_{l-1},s_1,\cdots,s_l} F \right)$

there where

$$d_{\lambda_1,\cdots,\lambda_{l-1},s_1,\cdots,s_l} F = 0$$

or equivalently

(III.21) $F_i = F_{i+1}, \frac{\partial F_i}{\partial s_i} = 0, \ i = 1, \cdots, l$

Thus the matrix (III.20) evaluated at the points of the conflict set defined by (III.21) looks like

(III.22) $K = \begin{pmatrix}
\gamma_2 - \gamma_1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
\gamma_l - \gamma_{l-1} & \lambda_1 \frac{d^2}{s_1} F_1 & 0 & \cdots & 0 \\
-\lambda_1 \gamma_1 & 0 & (\lambda_2 - \lambda_1) \frac{d^2}{s_2} F_2 & \cdots & 0 \\
-(\lambda_2 - \lambda_1) \dot{\gamma}_2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\lambda_{l-1} \dot{\gamma}_l & 0 & 0 & \cdots & -\lambda_{l-1} \frac{d^2}{s_l} F_l
\end{pmatrix}$

If the matrix has maximal rank, that is $l - 1 + \sum \dim(M_i)$, then the conflict set is Legendrian. We can see that the $\lambda_i$ do not matter a great deal, we can simply eliminate them from the matrix (III.22).

Let us analyze this matrix again in the way such matrices are analyzed in [Por01].

- The first column block contains the derivatives wrt. the $x$ variable. These are $n$ independent vectors, so that if the lower right block of the matrix III.22 has rank $l - 1 - n + l(n - 1)$ then the conflict set is Legendrian. In particular it is Legendrian when it is smooth.
- We can see that when there is one umbilic - i.e. $\frac{d^2}{s_1} F = 0$ - and the other wavefronts are smooth that we then have a Legendrian singularity at the conflict set. This is in concordance with the remarks above about non-Legendrian singularities. The matrix $K$ looks like

$$\begin{pmatrix}
\gamma_2 - \gamma_1 & 0 & 0 \\
-\dot{\gamma}_1 & 0 & 0 \\
\dot{\gamma}_2 & 0 & -\frac{d^2}{s_2} F_2
\end{pmatrix}$$

We might as well forget the middle column. Then it is clear from the lemma that the rank of this $K$ is $n + \rk \frac{d^2}{s_2} F_2$, when $\gamma_2(s_2)$ and $\gamma_1(s_1)$ do not fall together - something which can be avoided. Because the second wavefront was regular at the conflict set we can conclude here that the conflict set is Legendrian.
- On the other hand if $\rk \frac{d^2}{s_2} F_2 < n - 1$ the conflict set will have a non-Legendrian singularity.
Another example is where $l = n = 3$. This is the simplest case with three hypersurfaces. Here the matrix $K$ looks like:

$$
\begin{pmatrix}
\gamma_2 - \gamma_1 & 0 & 0 & 0 & 0 \\
\gamma_3 - \gamma_2 & 0 & 0 & 0 & 0 \\
-\dot{\gamma}_1 & 0 & d_{s_1}^2 F_1 & 0 & 0 \\
-\dot{\gamma}_2 & 0 & 0 & d_{s_2}^2 F_2 & 0 \\
\dot{\gamma}_3 & 0 & 0 & 0 & d_{s_3}^2 F_3
\end{pmatrix}
$$

Disregarding the column with zeroes this is a $8 \times 9$ matrix. Let us first consider the first three columns. They are independent. Now if two of the wavefronts are smooth at the conflict set we can have the third one singular and obtain a singular space curve.

The relation between caustics and wavefronts is that singular points of the wavefronts lie on the caustic. For the conflict set and the caustics of the base manifolds we have something similar. If a point on the conflict set is singular then not all the $d_{s_i}^2 F_i$ can have maximal rank. This means that singular points of the conflict set lie on the caustic of one the manifolds. The singular points of the conflict sets lie on a codimension $n - l$ set on each caustic.

**Example III.27** (Birth and death of a component). Conflict sets are hard to calculate, both numerically and algebraically. Thus we have to restrict our attention to a very simple situation. Already this simple situation leads to some interesting conclusions.

Take

$$\gamma_1(s) = \left( s, \frac{1}{2} s^2 \right) \quad \text{and} \quad \gamma_2(t) = A(\phi) (\gamma_1(t) - b)$$

thus $\gamma_2$ is just an affine transform of $\gamma_1$. We know that non-Legendrian points occur when the rank condition fails. The rank condition fails when the conflict set, the caustic of $\gamma_1$ and the caustic of $\gamma_2$ meet. These are three curves and for them to meet we see from the matrix that we need a one parameter family. Different families can be fabricated: we could choose variations in $\gamma_1$ or $\gamma_2$, or we could vary $H_1$ and $H_2$, that is we vary the metrics associated to $\gamma_1$ and $\gamma_2$. We will take

$$F_z = (1 + z) \| x - \gamma_1(s) \| - (1 - z) \| x - \gamma_2(t) \|$$

The quickest way to calculate the conflict set in this case is by repeated use of resultants. For $z = 0$ we get figure III.5. In the middle of figure III.5 we see a separate component. We will show how it comes to existence through the non-Legendrian point, and why it has exactly four cusps. When $z = 8/10$ the picture looks very different, see figure III.6. Here the component is not born yet. For the region inside the two caustics another four cusps need to be born. But this is about to happen, just below and just above the cuspidal point of the caustic of $\gamma_1$ there are two curve segments of the conflict set. When they have shifted through this cuspidal points the four cusps will be there. Just as with wavefronts cusps are born in pairs. This is what has happened in figure III.7. One now sees the cusps on the right side moving towards the intersection of the caustic of $\gamma_1$ and $\gamma_2$. They will meet there at the same time. At this point the separate component is born. Over there the two Legendrian manifolds corresponding to the wavefronts of $\gamma_1$ and $\gamma_2$ become tangent to each other. It can be traced that it dies indeed at the other point where the caustics of $\gamma_1$ and $\gamma_2$ intersect.

The transition described here is not entirely new in the literature. As a purely local phenomenon it is described in [BG86]. The $A_2^2$ case on p. 195 is the same as we have here. We will return to the $A_2^2$ case in the next chapter, see figure V.4.
III.2. THE CONFLICT SET VIA A CANONICAL RELATION

Figure III.5. $z = 0$: The curves $\gamma_1$ and $\gamma_2$ (thick black line), together with their evolutes (thin black line) and the conflict set (thick grey line).

Figure III.6. $z = 8/10$: The curves $\gamma_1$ and $\gamma_2$, together with their evolutes and the conflict set.

Figure III.7. $z = 5/10$: The curves $\gamma_1$ and $\gamma_2$, together with their evolutes and the conflict set.
Remark III.28. In the calculations carried out in the previous example resultants were much faster than elimination of the $s$-variables by Groebner bases. This has been our experience with a lot of examples. With two surfaces in $\mathbb{R}^3$ we have not seen the end of Groebner basis elimination or resultant methods even with fairly fast computers. The fastest of these systems [GPS01] would run for days without ending.

III.2.17. Further section and projection. All considerations in this subsection will be purely local, hence take $X = \mathbb{R}^n$ for simplicity. Consider a Legendrian submanifold $L$ of $T^*\mathbb{R}^n \setminus 0$. At each $(x_0, \xi_0) \in L$ we can consider the dimension of $T_{x_0,\xi_0}L \cap T_{x_0}\pi^{-1}(x_0)$. This fiber dimension is an upper semi continuous function on $L$. Because the fiber dimension is integer valued there is some open neighborhood $U$ of $(x_0, \xi_0)$ such that on a dense subset of $U \cap L$ the fiber dimension takes on a single value. This number is the type of the Legendrian submanifold at $(x_0, \xi_0)$. If the type is 1 the Legendrian submanifold we have is of general type.

The Legendrian manifolds we encounter if $l > 2$ are not of general type. Their fiber dimension is at regular points $l - 1$. We wish to show how we can apply the section and projection procedure to realize the conflict set as a Legendrian manifold of general type, or more generally how to realize a Legendrian manifold of higher type, as a Legendrian manifold of general type.

Thus let $L$ be a Legendrian manifold of type $l - 1 > 1$ in $T^*\mathbb{R}^n$. For a vector $v$ in $T_0\mathbb{R}^n$ we consider in $T^*\mathbb{R}^n$ the hypersurface

$$W_v = \{(x, \xi) \in T^*\mathbb{R}^n \mid \xi(v) = 0\}$$

By Sard’s theorem for almost all $v$ we have

(III.23) \[ W_v \cap L \]

We have seen in the above (equation (III.17)) that (III.23) implies that we can project immersively $L$ along the direction $v$. The projection of $L$ along $v$ to $T^*\mathbb{R}^{n-1}$ gives rise to a Legendrian manifold in $\pi_v(L)$, it will be of type $l - 2$. This process can be repeated until we have a Legendrian manifold of general type. We summarize.

Theorem III.29. A conic Lagrangian manifold of type $l - 1 > 1$ near $(0, \xi)$ in $T^*\mathbb{R}^n$ can be sectioned and then projected immersively to a Legendrian manifold of general type in $T^*\mathbb{R}^{n-l+2}$. 

Figure III.8. Left $z = 2/10$ and right side $z = 35/100$
III.3. Surjectivity of the jet mapping

In this section we will prove that the condition which assures that the conflict set has a Legendre singularity is generically satisfied under perturbations of the base manifolds. There are two ways of going about proving this. In this section we prove the genericity of the maximal rank criterion (III.22) for phase functions. In the next chapter we prove the genericity of the criterion (III.16).

III.3.18. Maximal rank criterion. Recall from [Hir94] that a residual set in a topological space is a countable intersection of open and dense subsets. The space of $C^\infty$ mappings from $M_i$ to $X$ equipped with the Whitney topology is a Baire space, meaning that countable intersections of open and dense sets are still dense. Hence residual sets lie dense.

For our purposes it is convenient to introduce the space $\text{Emb}(M, X)$. It is the space of closed embeddings of the hypersurface $M$ in $X$. Again from [Hir94] we know that the space of closed embeddings is open in the space $C^\infty(M, X)$.

**Theorem III.30.** Let $M_i$ be $l$ closed compact hypersurfaces $X$. For a residual subset of embeddings in $\bigoplus_{i=1}^{l} \text{Emb}(M_i, X)$ with the Whitney topology the conflict set only has Legendrian singularities.

In the above we have made four assumptions on our Hamiltonian such that for each of the $A_i(x, y)$

$$\frac{\partial A_i}{\partial y} \neq 0 \quad \text{(III.24)}$$

Moreover, because the Hamiltonians are non-degenerate we will also have

$$\det\left(\frac{\partial^2 A_i}{\partial y^2}\right) \neq 0$$

Let each of the $\gamma_i$ depend on an additional parameter $e_i \in \mathbb{R}^n$. So every embedding is replaced by a family of embeddings: $\gamma_i = \gamma_i(s_i, e_i)$.

In the euclidean case we take translations of the $\gamma_i$, there $\gamma_i$ has the form $\gamma_i(s_i, e) = \gamma_i(s_i, 0) + e$. In the general case we assume that near any point coordinates can be chosen such that for $s_i$ near $s_{i,0}$ and small $e$ such that we can write in these coordinates:

$$\gamma_i(s_i, e) = \gamma(s_i) + \frac{\partial \gamma_i}{\partial e} e + O(\|e\|^2)$$

and

$$\frac{\partial \gamma_i}{\partial e}(s_{i,0}, 0) = I_n$$

Let us cite lemma 3.2. in [Mat70b].

**Lemma III.31 (Lemma V.3.2.).** Let $U$ be a submanifold of a manifold $W$. Let $\mathcal{F}$ be a topological space and $j: \mathcal{F} \to C^\infty(V, W)$ a mapping, where $V$ is a manifold. Suppose that for each $f \in \mathcal{F}$ there exists a continuous mapping $\phi: (E, e_0) \to (\mathcal{F}, f)$, where $E$ is a manifold and $e_0 \in E$, such that the mapping $\Phi: E \times V \to W$ (defined by $\Phi(e, v) = j\phi(e)(v)$) is $C^\infty$ and transversal to $U$. Then

$$\{f \in \mathcal{F} | j(f) \text{ is transversal to } U\}$$

is dense in $\mathcal{F}$. 
We want to apply the lemma to our present situation, where we need to prove theorem III.30. Take
\[ \mathfrak{F} = \bigoplus_{i=1}^l \text{Emb}(M_i, X) \]
We need to prove that the matrix \( K \) from (III.22) generically has maximal rank. Hence the graph of
\[ x, s_1, \cdots, s_l \to F_1 - F_2, \cdots, F_{l-1} - F_l, \frac{\partial F_1}{\partial s_1}, \cdots, \frac{\partial F_l}{\partial s_l} \]
is to intersect its zero-level transversally. We have to fill in that \( U \) is the zero-level. So \( U \) is \( \{x, s_1, \cdots, s_l, 0, 0, \cdots, 0\} \) in \( X \times M_1 \times \cdots \times M_l \times \mathbb{R}^{l-1} \times (\mathbb{R}^n)^l \)
The simplest way to get a map \( \Phi: E \times V \to W \) transverse to \( U \subset W \) is to make \( \Phi \) submersive. Let 
\[ V = M_1 \times \cdots \times M_l \times X \]
and let \( E \) be \((\mathbb{R}^n)^l\)
The map \( \Phi \) will be the graph of (III.25), where the \( F_i \) are the distance functions
\[ F_i = A_i(x, \gamma(s_i, e_i)) \]
If the map \( \Phi \) is submersive then for a dense subset of \( \bigoplus_{i=1}^l \text{Emb}(M_i, X) \) the maximal rank criterion is fulfilled and the conflict set is generically Legendre.
To show that the map \( \Phi \) is submersive we need to show that the matrix of derivatives
\[ d_{x,e,s} \begin{pmatrix} F_1 - F_2, \cdots, F_{l-1} - F_l, d_{s_1} F_1, \cdots, d_{s_l} F_l \end{pmatrix} \]
has rank \( l - 1 + l(n-1) \). (We have written \( s = (s_1, \cdots, s_l) \) and \( e = (e_1, \cdots, e_l) \)) It will be enough to show that
\[ d_{x,e} \begin{pmatrix} F_1 - F_2, \cdots, F_{l-1} - F_l, d_{s_1} (F_1), \cdots, d_{s_l} (F_l) \end{pmatrix} \]
has the required rank \( l - 1 + l(n-1) \). Thus we are to calculate a number of partial derivatives
\[ d_{x,e_i} (F_j - F_{j+1}) \text{ and } d_{x,e_i} (d_{s_j} (F_j)) \]
because of these partial derivatives the matrix in (III.26) is composed. We find:
\[ d_x (F_j - F_{j+1}) = \frac{\partial A_j}{\partial x} - \frac{\partial A_{j+1}}{\partial x} \]
Here
\[ \frac{\partial A_j}{\partial x} \]
is the normal to the front from \( M_j \).
\[ d_{e_i} (F_j - F_{j+1}) = \frac{\partial A_j}{\partial y} \frac{\partial \gamma_j(s_j, e_j)}{\partial e_i} - \frac{\partial A_{j+1}}{\partial y} \frac{\partial \gamma_{j+1}(s_{j+1}, e_{j+1})}{\partial e_i} \]
Above we chose our families so that in local coordinates
\[ \frac{\partial \gamma_i(s_i, e_i)}{\partial e_i} \]
is the identity matrix. Hence
\[ d_{e_j} (F_j - F_{j+1}) = \frac{\partial A_j}{\partial y} = d_y A_j \quad d_{e_{j+1}} (F_j - F_{j+1}) = - \frac{\partial A_{j+1}}{\partial y} = -d_y A_{j+1} \]
The next partial derivatives to calculate are
\[ d_x (d_{s_j}(F_j)) = d_x \left( \frac{\partial A_j}{\partial y} \frac{\partial \gamma_j (s_j,e_j)}{\partial s_j} \right) = \frac{\partial^2 A_j}{\partial x \partial y} \frac{\partial \gamma_j (s_j, e_j)}{\partial s_j} \]
and
\[ d_{e_j} (d_{s_j}(F_j)) = d_{e_j} \left( \frac{\partial A_j}{\partial y} \frac{\partial \gamma_j (s_j,e_j)}{\partial s_j} \right) = \frac{\partial^2 A_j}{\partial y^2} \frac{\partial \gamma_j (s_j, e_j)}{\partial s_j} + \frac{\partial A_j}{\partial y} \frac{\partial^2 \gamma_j (s_j, e_j)}{\partial e_j \partial s_j} \]

We can further assume that the derivative
\[ (\text{III.27}) \]
are arbitrarily small. The rank of a matrix does not change if we add another arbitrarily small matrix to it. Hence to determine whether the matrix (\text{III.26}) has maximal rank we can assume (\text{III.27}) is zero. It is noted that if (\text{III.27}) is zero then the perturbation of \( \gamma_j(s_j,e_j) \) near \( \gamma_j(s_j,0) \) is very nearly a translation along the geodesics.

Another simplification is that by adding columns the maximal rank criterion becomes that
\[ d_{x,e} (F_1 - F_l, \ldots, F_{l-2} - F_l, F_{l-1} - F_l, d_{s_1}(F_1), \ldots, d_{s_l}(F_l)) \]
should have maximal rank. Filling in all the partial derivatives we get that
\[ (\text{III.28}) \]
should have rank \( l-1+1(n-1) \). In the matrix (\text{III.28}) there are \( n+nl \) rows and \( l-1+1(n-1) = nl - 1 \) columns. Hence there are more rows than columns.

The matrices \( d_y A_i, d_{s_j} \gamma_i \) all have rank \( n-1 \), moreover with \( d_y A_i \) they form a set of \( n \) independent vectors in \( T_{\gamma_i(s_j,0)} X \). Thus, the rows \( n+1 \) to \( n+(n-1)l \), i.e. those corresponding to \( d_{e_j}, j = 1, \ldots, l-1 \) are independent and so are the \( n-1 \) vectors
\[ \frac{\partial^2 A_2}{\partial y^2} \frac{\partial \gamma_i (s_l, e_l)}{\partial s_l} \]

Thus the lower \( nl \) rows of (\text{III.28}) make that the columns of (\text{III.28}) are independent. This is what we needed to prove.

Now we have only proven the statement of theorem III.30 for basepoints \( \gamma_j(s_j) \) with a point \( x \) on the conflict set. To make the statement global we need to cover \( X \times M_1 \times \cdots \times M_l \) with compact submanifolds, as is done in the proof of the Mather transversality theorem, proposition 3.3 in [Mat70b]. The countable intersection of dense subsets of \( \bigoplus_{i=1}^l \text{Emb}(M_i, X) \) will still be dense.

For the openness we can thus assume that the \( M_i \) are compact and that we need only consider that part of the conflict set that lies in a compactum \( X^o \subset X \). On these the perturbation of the matrix (\text{III.28}) can be made uniformly small over the compactum \( X^o \times (\times_{i=1}^l M_i) \). The proof of III.30 is complete.
III.4. $k$-jets of base manifolds determine $k$-jets of conflict sets

In this section we establish some results that were already imminent in the theorems of chapters one and two. We will apply the concepts introduced at the beginning of this chapter to the subject of chapter 1. In chapter 1 we concluded that in $\mathbb{R}^n$ with the euclidean metric away from caustics propagating waves retain their contact.

Let $M$ and $N$ be hypersurfaces in a manifold $X$. Suppose $p \in M \cap N$. Let $V$ be a linear subspace of $T_pM$. Recall the definition of $k$-contact from chapter 1 (Definition I.5). We modify it a little bit to read:

**Definition III.32.** $M$ and $N$ have $k$-contact, with $k > 0$ in the direction $V \subseteq (T_pN \cap T_pM)$ if there is a third manifold $L$ with $T_pL = V$ and such that $L$ has $(k)$-contact with both $M$ and $N$.

Instead of $V$ we consider a sequence of subspaces $\{V_i\}_{i=1,\ldots,m}$ each strictly contained in the other:

$$\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subset T_pM$$

with to each $V_i$ associated an integer

$$k_1 > k_2 > \cdots > k_m > 0$$

We call this a numbered flag $V = \{V_i, k_i\}_{i=1,\ldots,m}$. Two submanifolds $M$ and $N$ of $X$ have contact along $V$ if for each $1 \leq i \leq m$ $M$ and $N$ have $k_i$-contact in the direction $V_i$. Such a definition encompasses the notions above.

Contact is retained under diffeomorphisms. If we have a diffeomorphism $\phi$ from $X, p$ to $Y, q$ then $\phi(M)$ and $\phi(N)$ have $k$ contact in the direction $\phi_* V$ iff. $M$ and $N$ have $k$-contact in the direction $V$.

Let the conflict set be regular at a point $p$. At $p$ we have $l$ wavefronts originating from $M_i, q_i$ that have arrived there after time $t$. For simplicity we assume $t = 0$ and $q_i = p$ for $i = 1, \ldots, l$. The conflict set is a manifold $M_c$ and there is a projection $\pi_i: T_pM_c \to T_pM_i$. In this way a numbered flag $V$ of $T_pM_c$ projects to numbered flags $(\pi_i)_* V$ in each of the $T_pM_i$.

The lemmata in the proof of theorem I.2 can now be refined to read:

**Theorem III.33.** Let in addition to the above there be given

- $N_i$ with $T_pN_i = T_pM_i$ such that $M_i$ have at $p$ contact along the numbered flag $(\pi_i)_* V$ and
- $N_c$, germ at $p$ of the conflict set of the $N_i$.

Then $N_c$ and $M_c$ have contact along $V$.

**Proof.** It is enough to prove this for $k$ contact in a direction $V \subset T_pM_c$. Denote, as in chapter 1, $M^h_i$ for the big wave front. As before, there are three steps to take. Step 1, corresponding to lemma I.9. Near $p$ $M_i \times \mathbb{R}$ and $N_i \times \mathbb{R}$ have, as submanifolds of $X \times \mathbb{R}$ $k$-contact in the direction $W_i = \pi_i V \times T_0 \mathbb{R}$. For small $t$ the map $\Psi_i: t, x \mapsto \pi_X(\exp(\text{tvf}(H_i)))$, $t$ is a diffeomorphism. Thus their images $M^h_i$ and $N^h_i$ have $k$ contact in the directions $\Psi_i^* W_i$.

Step 2, corresponding to lemma I.10. Choose submersions $F_i$ for the $M^h_i$ at $p$. Then each of the $N^h_i$ individually has a $k$-contact along $\cap_{i=1}^l \Psi^*_i W_i$ with $M^h_i$, which we know to be a transversal intersection because we were assuming to be at a regular point of the conflict set. Now take for $M^h_c$ an immersion and for the $N^h_i$ submersions. Then it follows that $N^h_c$ has $k$ contact along $\cap_{i=1}^l \Psi^*_i W_i$ with $M^h_c$.

Step 3, corresponding to lemma I.11. Because the vectorspace $\cap_{i=1}^l \Psi^*_i W_i$ lies in general
position wrt. the projection to $T_pX \ N_c$ will have $k$-contact in the direction $\pi(\bigcap_{i=1}^d \Psi_i^* W_i) \subset T_pX$. □
IV Canonical relations for other geometrical constructions

IV.1. Introduction

In this chapter we will further exploit the concept of canonical relation and show how it can be used to create

- center sets,
- pedals and orthomtics, and
- billiards.

We will start by treating a few concepts that are in some sense dual to conflict sets. To understand the duality for the concept of center sets recall that in the construction of the conflict set by means of a canonical relation we looked at the diagonal in the base of the projection

$$(T^*\mathbb{R}^{n+1})^l \to (\mathbb{R}^{n+1})^l$$

When defining the center set below we will look at the diagonal in the fiber, i.e. we will look where the coordinates $\xi$ in each of the fibers $T^*\mathbb{R}^n \to \mathbb{R}^n$ are equal.

Kites, defined below, are in some sense dual to the conflict set because they describe tangents to the lifted conflict set.

However, it is noted that neither center sets nor kites are really dual to the conflict set: one cannot construct conflict sets directly from kites or center sets.

In the second section we will prove that the center set is generically Legendrian. We will prove anew the same statement for conflict sets, but in another way as promised in the previous chapter.

In the third section we recall the notion of orthomtic and pedal curve, as they are described in the book [BG92]. The orthomtic is a hypersurface defined using a point - mostly the origin - and another hypersurface conveniently called the mirror. The mirror is the conflict set of the orthomtic and the origin. The orthomtic turns out to be a reversed conflict set.

We also briefly touch upon the subject of billiards as described in [Tab95]. Our main objective is to show how a curvature formula used in the theory of billiards and in another form found in [BW59] is really the formula we found in chapter 1.

IV.2. “Dual” curves: kites, centers and normal chords

In this section we define a few more sets measuring what is in the middle. Here we meet more applications of the method where canonical relations represent a geometrical construction. All our constructions are in $\mathbb{R}^n$. They can also be carried out in other spaces than $\mathbb{R}^n$, for instance in symmetric spaces.

IV.2.1. Centers and centroids. In $\mathbb{R}^n$ all tangent planes can be identified with each other. Hence the equations

$$T^*_{(1,1)}\mathbb{R}^{2n} \setminus 0 = \{(x_1, x_2, \xi_1, \xi_2) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \mid \xi_1 = \xi_2 \neq 0\}$$

make sense on $T^*\mathbb{R}^{2n} \setminus 0$. The intersection

$$\left(T^*_{(1,1)}\mathbb{R}^{2n} \setminus 0\right) \cap (N^*M_1 \times N^*M_2)$$
describes those points $x_1$ on $M_1$ and $x_2$ on $M_2$ so that $M_1$ and $M_2$ have parallel tangent planes. Instead of considering the pair $x_1, x_2$ we can consider the midpoint of the line segment connecting the two.

**Definition IV.1.** The midpoints of the segments connecting two points, one on a manifold $M_1 \subset \mathbb{R}^n$ and another on $M_2 \subset \mathbb{R}^n$, and having parallel tangent planes form the center set.

**Theorem IV.2.** If

\[
\Gamma_{(1,1)}^n(\mathbb{R}^n \times \mathbb{R}^n) \cap N^*M_1 \times N^*M_2
\]

then the center set is Legendrian.

**Proof.** Let $S_1$ be $T^*\mathbb{R}^n$. Equip $S_1$ with coordinates $(y, \eta)$. Let $S_2$ be $T^*(\mathbb{R}^n \times \mathbb{R}^n)$. Equip $S_2$ with coordinates $(x_1, x_2, \xi_1, \xi_2)$.

Let $G_1 \subset S_1 \times S_2$ be

\[
G_1 = \{(y, \eta, x_1, \xi_1, x_2, \xi_2) \mid y = \frac{x_1 + x_2}{2}, \quad \eta = 2\xi_1, \quad \xi_1 = \xi_2 \neq 0\}
\]

A straightforward calculation shows that $G_1$ is a canonical relation. If we compose this canonical relation with $N^*M_1 \times N^*M_2 \subset S_2$ we will get the center set. Apply proposition III.9 to obtain the desired conclusion. \qed

**Remark IV.3.** In contrast to conflict sets one can calculate center sets in many cases. For instance, the graph of a function $f$ whose derivative is invertible and any other curve that is non-vertical curve $\gamma$ leads to a simple calculation of conflict sets. The tangents should be parallel, so we have

\[
\langle (1, f'), J(\gamma') \rangle = f'\gamma_x'(t) - \gamma_y'(t) = 0
\]

From this we can solve $x$ because $f'$ is invertible. Filling this in in

\[
\left( \frac{1}{2}(x + \gamma_x(t)), \frac{1}{2}(f(x) + \gamma_y(t)) \right)
\]

we get a curve. In figure IV.1 we see such a center set. The function $f$ is the parabola. Its derivative is invertible.

![Figure IV.1. Two center sets.](image)

As with conflict sets we can consider the center set of more than two submanifolds. This will be the centroid set.

In $\mathbb{R}^l$ consider the enlarged simplex

\[
\Delta_l : \quad \sum_{i=1}^{l} t_i = 1
\]
The $t_i$ are the components of a weight vector. Fix such a weight vector.

**Definition IV.4.** The weighted locus

$$y = \sum_{i=1}^{l} t_i x_i$$

of $l$-tuples of points $(x_1, \cdots, x_l)$ where the tangent planes to the $l$ surfaces $M_i$ are parallel, form the **centroid set**.

Criteria for when the centroid set is Legendrian are easily written down. Consider the canonical relation

$$(IV.4) \quad G_1 = \{ y = \sum_{i=1}^{l} t_i x_i \quad \eta t_i = \xi_i \} \subset T^* \mathbb{R}^n \times (T^* \mathbb{R}^n)^l$$

If we compose this with a product of the $\{N^* M_i\}_{i=1,\ldots,l}$ we get the centroid set with weights $t_i \neq 0$.

**Proposition IV.5.** The centroid set is Legendrian when

$$\times_{i=1}^{l} N^* M_i \pitchfork T^* (1, \cdots, 1) \mathbb{R}^{nl}$$

**Proof.** Immediate from (IV.4) and proposition III.9. \qed

In general the centroid set will have dimension $n - 1$.

**Figure IV.2.** A centroid set of three surfaces

**IV.2.2. Centroids in certain symmetric spaces.** Instead of $y = \sum_{i=1}^{l} t_i x_i$ we can consider a more general relation.

$$(IV.5) \quad y = Y(x_1, \cdots, x_l),$$

where $Y = Y(x_1, \cdots, x_l)$ is a function from $X_1 \times \cdots \times X_l$ to $X_0$ such that the matrices $\frac{\partial Y}{\partial x_i}$ are invertible. The relation $y = \sum_{i=1}^{l} t_i x_i$ is a special case of relations (IV.5).

From the relation (IV.5) we construct $G_1$.

$$(IV.6) \quad G_1 = \{(y, \mu, x_1, \xi_1, \cdots, x_l, \xi_l) \mid y = Y(x_1, \cdots, x_l) \quad \xi_i = \frac{\partial Y}{\partial x_i} \mu\}$$
The manifold $G_1$ is a canonical relation between $T^*\mathbb{R}^n$ and $(T^*\mathbb{R}^n)^l$. Because this is a purely local notion a canonical relation like $G_1$ can also be defined between the cotangent bundle $T^*X_0$ of an $n$-dimensional manifold $X_0$ and

$$T^*X_1 \times \cdots \times T^*X_l$$

where $X_1$ to $X_l$ are all $n$-dimensional manifolds. Clearly the condition under which the more general centroid set is Legendrian is that

$$N^*M_1 \times \cdots \times N^*M_l \pitchfork T^*_{dY}(X_1 \times \cdots \times X_l)$$

where we denoted

$$T^*_{dY}(X_1 \times \cdots \times X_l) = \{x_1, \xi_1, \ldots, x_l, \xi_l \mid \left(\frac{\partial Y}{\partial x_i}\right)^{-1}\xi_i = \left(\frac{\partial Y}{\partial x_j}\right)^{-1}\xi_j\}$$

The previous considerations thus enable us to generalize the centroid set to other manifolds than $\mathbb{R}^n$. Let $X$ be a Lie group with a bi-invariant metric. Recall from [Mil63] §21, that on a Lie group with a bi-invariant metric there exists for every point $x \in X$ an involutive isometry $\sigma(x, \cdot): X \rightarrow X$ which in group notation reads $y \rightarrow xy^{-1}x$.

Such isometries make $X$ into a symmetric space. Among the many examples of Lie groups with a bi-invariant metric are the spheres $S^n = SO(n+1)/SO(n)$.

We can now use the canonical relation (IV.6) with $l = 2$ and $X_0 = X_1 = X_2 = X$. Two choices for $Y$ are interesting for our purposes:

- $Y(x_1, x_2) = \sigma(x_1, x_2)$ i.e. $x_2$ is reflected on $Y(x_1, x_2)$, this is the relation we will meet further on when discussing the billiard transformation on the space of rays,
- $x_2 = \sigma(y, x_1)$, in $\mathbb{R}^n$ this relation is (IV.3).

### IV.2.3. Normal chords

Instead of the midpoints of the segment joining two points with parallel planes we can also consider the coinciding normals themselves.

For this construction we will make essential use of the Gauss map. Suppose we apply the Gauss map $\nu_G$ componentwise to

$$N^*M_1 \times \cdots \times N^*M_l$$

We then obtain

$$(IV.7) \quad \nu_G(N^*M_1) \times \cdots \times \nu_G(N^*M_l) \subset \times_{i=1}^l T^*S^{n-1}$$

We defined the center set using parallel tangent planes. The tangent planes at $p_1 \in M_1$ and $p_2 \in M_2$ coincide when we can find $\xi \in N^*_{p_1}M_1$ that is also in $N^*_{p_2}M_2$, that is when the normals at $p_1$ and $p_2$ coincide.

Coinciding normals can thus also be found using the “diagonal in the base” $T^*_\Delta(S^{n-1})^l$ of $T^*(S^{n-1})^l$.

**Definition IV.6.** The normals at points $x_i$ of the centroid form the **normal chord set**.

For the centroid set we have the result analogous to theorem III.25.

**Proposition IV.7.** The normal chord set is Legendrian if

$$(IV.8) \quad \nu_G(N^*M_1) \times \cdots \times \nu_G(N^*M_l) \pitchfork T^*_\Delta(S^{n-1})^l$$

Returning to the case $l = 2$ note that the normals are not the chords connecting two points on the center set. In fact we have three objects:

- the normals at the $x_i$,
- the normal to the center set, and
• the chords connecting the \( x_1 \) and \( x_2 \).

One might ask whether the first and the second are the same and what these have to do with the chords connecting \( x_1 \) and \( x_2 \). We will now only answer the first part of the question and leave the answer to the second part of our question to the next subsection.

Suppose \( x_1 \in M_1 \) and \( x_2 \in M_2 \), with normals \( \xi_1 \) and \( \xi_2 \) respectively.

If we first compute the normal chord set

\[
(x_1, \xi_1, x_2, \xi_2) \xrightarrow{\nu_G \times \nu_G} \left( \frac{\xi_1}{\|\xi_1\|}, \ldots, \frac{\xi_2}{\|\xi_2\|}, \ldots \right)
\]

and then take the diagonal we get the same value as when we first take the center set and then map to \( T^* S^{n-1} \). The same happens with \( l \) surfaces, If we have \( (x_i, \xi) \in N^* M_i \) and take the center set, we get

\[
\left( \frac{1}{l} \sum x_i, \xi \right)
\]

in \( T^* \mathbb{R}^n \). When applying the Gauss map we get

\[
\left( \frac{\xi}{\|\xi\|}, \frac{1}{l} \sum x_i - \frac{1}{l} \sum x_i, \xi \right)
\]

On the other hand if we first apply the product of Gauss maps \( \nu_G \times \cdots \times \nu_G \) to the product of Legendrian submanifolds \( N^* M_1 \times \cdots \times N^* M_l \) and afterwards take the centroid set we get

\[
\left( \frac{\xi}{\|\xi\|}, \sum x_i - \frac{\sum x_i}{\|\xi\|} \xi \right)
\]

which is up to a factor \( l \) the same thing.

In other words the diagram

\[
\begin{array}{ccc}
T^* \mathbb{R}^n \times \cdots \times T^* \mathbb{R}^n & \xrightarrow{\nabla T^*_{(1, \ldots, 1)} \mathbb{R}^{nl}} & T^* \mathbb{R}^n \\
\downarrow \nu_G \times \cdots \times \nu_G & & \downarrow \nu_G \\
T^* S^{n-1} \times \cdots \times T^* S^{n-1} & \xrightarrow{\nabla T_{\Delta S^{(n-1)}}} & T^* S^{n-1}
\end{array}
\]

commutes. Properly speaking the arrows do not really represent maps: in the diagram the upper row is the pull-back from proposition IV.5. The lower row is the pull-back from proposition IV.7.

Hence the first and the second items are the same. We summarize

**Proposition IV.8.** The normal to the center set is the normal singled out by the normal chord set. If either of the transversality conditions (IV.2) or (IV.8) holds then the normal chord set is Lagrangian and the center set is Legendrian.

**Remark IV.9.** For the normal chord set we can write down maximal rank criteria as we did for the conflict set in section III.2.14. A phase function for the image of the Gauss map is

\[
F_i : S^{n-1} \times M_i \to \mathbb{R}
\]

(IV.9)

\[
(v, s) \mapsto \langle v, \gamma(s) \rangle
\]

The image of the Gauss map is described by

(IV.10)

\[
\frac{\partial F_i}{\partial s} = 0
\]
To get a maximal rank criterion under which the normal chord set is Lagrangian we use as in III.2.16 a special phase function:

\[ F_1(v, s_1) + F_2(v, s_2) \]

And the maximal rank criterion that is equivalent to the transversality in (IV.8) is that the matrix

\[ d_{v,s_1,s_2}(d_{s_1,s_2} F) \]

has maximal rank where

\[ d_{s_1,s_2} F = 0 \]

If so, the normal chord set is Lagrangian. If one chooses local coordinates on \( S^{n-1} \), as is done in [BGM82], this is a nicely computable criterion.

**IV.2.4. Centers as wavefronts, chords as caustics.** We still need to make sense of the relationship between the normal chords and the lines from \( x_1 \) to \( x_2 \). These lines form an \( n-1 \) dimensional family. They remind us of the normals to a hypersurface. Normals to a hypersurface have the caustic as an envelope.

Consider the

\[
\mathbb{R} \times \left( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \cap T^*_{(1,1)}(\mathbb{R}^n \times \mathbb{R}^n) \right) \rightarrow \mathbb{R} \times \mathbb{R}^n \\
(t, x_1, \xi, x_2, \xi) \rightarrow (t, tx_1 + (1-t)x_2)
\]

(IV.11)

The points \( \ell(x_1, x_2) \) that are the image of

\[ t \rightarrow tx_1 + (1-t)x_2 \]

form a line. The envelope of the lines \( \ell(x_1, x_2) \) are the singular points of the projection that is the last arrow in

\[
\mathbb{R} \times \left( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \cap T^*_{(1,1)}(\mathbb{R}^n \times \mathbb{R}^n) \right) \rightarrow \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \\
(t, x_1, \xi, x_2, \xi) \rightarrow (t, tx_1 + (1-t)x_2) \rightarrow tx_1 + (1-t)x_2
\]

**Definition IV.10.** The image of \( \mathbb{R} \times N^*M_1 \times N^*M_2 \) in \( \mathbb{R} \times \mathbb{R}^n \) under (IV.11) is the big center set. The projection of its singular points to \( \mathbb{R}^n \) we will call the center caustic.

The center set is the intersection of the big center set with the plane \( t = \frac{1}{2} \). Trivially one has:

**Proposition IV.11.** If the intersection (IV.2) is transversal then the big center set is Legendrian in \( (J^1\mathbb{R}^n, dt - \sum_{i=1}^n \xi_i \, dx_i) \) Hence in that case the center caustic is Lagrangian.

If we fix \( t \) we get a curve like the center set, some sort of relative center set. Thus we have the following correspondences

<table>
<thead>
<tr>
<th>big center set</th>
<th>big wavefront</th>
</tr>
</thead>
<tbody>
<tr>
<td>center set</td>
<td>wavefront</td>
</tr>
<tr>
<td>center caustic</td>
<td>caustic</td>
</tr>
</tbody>
</table>

These concepts are illustrated in figure IV.3. At the right hand side of this figure we see a zoomed in version of the caustic at the center point. It can be calculated that the center caustic in the picture does not have any other singularity than ordinary cusps. The center caustic of \( M_1 \) and itself is known as the center symmetry set. This center symmetry set has been studied by many authors, two recent advances are [Jan96] and [GH99].

Generalizations of this construction to \( l > 2 \) hypersurfaces are straightforward. Let us briefly indicate how to proceed.
IV.2. “DUAL” CURVES: KITES, CENTERS AND NORMAL CHORDS

Take \( l \) hypersurfaces in \( \mathbb{R}^n \), the centroid set is \( n - 1 \)-dimensional. Its cuspidal edges are \( n - 2 \) dimensional. In \( \mathbb{R}^l \) consider once more the enlarged simplex

\[
\Delta_l : \sum_{i=1}^{l} t_i = 1
\]

and for every \( l \)-tuple \((x_1, \cdots, x_l)\) of points in \( \mathbb{R}^n \) map \( \Delta_l \) to \( \mathbb{R}^n \) by

\[
t_1, \cdots, t_l \mapsto t_1 x_1 + \cdots + t_l x_l
\]

In this way we have a map from

\[
\Delta_l \times (\mathbb{R}^n)^l \hookrightarrow \mathbb{R}^n
\]

Apply the map to the intersection in proposition IV.5. In this way for each point on the corresponding centroid you get an \( l - 1 \)-plane. So we have an \( n - l + 1 \)-parameter family of \( l - 1 \)-planes. Its envelope will be \( n - 1 \) dimensional. This envelope is the center caustic of \( l \) hypersurfaces in \( \mathbb{R}^n \). To generalize the notion of a big center set note that it will appear in product of \( \mathbb{R}^n \) and \( \mathbb{R}^l \). As each \( l \)-tuple \((t_1, \cdots, t_l)\) determines a point in \( \mathbb{P}^l \) by

\[
t_1, \cdots, t_l \mapsto [1; t_1; \cdots; t_l]
\]

it is slightly more natural to consider the big center set as a subset of \( \mathbb{P}T^*\mathbb{P}^l \times T^*\mathbb{R}^n \), with canonical 1 form \( \pi_1^* \alpha - \pi_2^*(\sum \xi_i d x_i) \).

**IV.2.5. Kites.** For a moment let us refocus attention to the simple case of two curves in \( \mathbb{R}^2 \) equipped with the euclidean metric. In figure IV.4 we see the image of two circles and their conflict set augmented with a number of kites. The kites consist of the normals from the basepoints to the conflict set as well as the tangent lines to the base points. The curve traced out by the intersection of the tangent lines is a straight line. In figure IV.5 we see the same construction, though now carried out with two circles contained in each other. There also a straight line is obtained.

**DEFINITION IV.12.** Let \( p \) be a regular point on the conflict set of two curves. The locus of the intersection points of the two tangent lines to the basepoints form the **kite curve**.
Remark IV.13. The kite curve can have very bad singularities: think of the kite curve of two lines. This will be a curve not at all: it is a point, namely the intersection of the two lines.

Example IV.14. The kite curve of any curve and a line will be contained in the line.

We will now describe a generalization of the kite curve to the case of $n$ hypersurfaces in $\mathbb{R}^n$. In that case with a regular point on the conflict set there are $n$ tangent planes intersecting in a single point. Thus there is also a kite curve.

In figure IV.6 another way of obtaining the kite curve in this case is pictured. Here we have constructed big wavefronts, and thus constructed a lifted conflict set $\pi_{n+1}L^h$ in $\mathbb{R}^{n+1}$. The developable surface of tangents to $\pi_{n+1}(L^h)$ intersects the plane $x_0 = 0$. The intersection coincides with the kite curve.

Proposition IV.15. The kite curve is a section of a developable surface.

It is possible to extend the kite curve over the singularities of $\pi_{n+1}L^h$ where there are at each point of $\pi_{n+1}L^h$ $n$ linearly independent normals. The extension is most conveniently done with the help of the big wavefronts $N^*M_i^h$ (or $N^*M_b^h$) though we have to take care to use the distance and not its square. Denote $\bar{y}$ a coordinate in $\mathbb{R}^{n+1}$. If $\bar{y}$ is on a tangent line to $\pi_{n+1}L^h$ we have the following relations

\begin{equation}
\langle \bar{y} - \bar{x}_i, \xi_i \rangle = 0 \quad i = 1, \ldots, l
\end{equation}
The section of the developable surface we are interested in is $y_0 = 0$. The relations (IV.12) become

\[(IV.13) \quad \langle y - x_i, \xi_i \rangle = x_0 \xi_{0,i} \quad i = 1, \ldots, l\]

This construction shows that with $n$ spheres in sufficiently general position the kite curve is a line, because the intersection of the big wavefronts will lie in a plane.

The definition of the kite curve as a section of a developable surface has several disadvantages. From chapter 2 we know that the curvature of the conflict of $n$ surfaces in $\mathbb{R}^n$ vanishes iff. all the curvatures at the basepoints are equal. If the curvature of the conflict set is zero then so is (see chapter 1) the curvature of the lifted conflict set. Thus in that case the lifted conflict set is a space curve in $\mathbb{R}^3$ with zero curvature. The kite curve is a section of its tangent developable.

Let us now see what happens to the kite curve if the curvature of the conflict set is zero, when $n = l = 2$. The lifted conflict set is a space curve in $\mathbb{R}^3$. A space curve in $\mathbb{R}^3$ looks like

$$\gamma(s) = (s, \frac{\kappa s^2}{2} + a_1 s^3 + \cdots, \frac{\kappa \tau s^3}{6} + b_1 s^4 + b_2 s^5 + \cdots)$$

According to [Shc84] the singularities of the dual start where $\tau = 0$. If $\kappa = 0$ then the singularities of the dual fall outside of the classification presented in [Shc84].

For the lifted conflict set $\tau = 0$ happens for instance when the derivative of the curvature on one base manifold equals the derivative of the curvature of the other base manifold. In
that case the kite curve is not singular, as the example of two circles shows.
The kite curve is singular when \( \kappa = 0 \), this happens when the curvatures at the basepoints are equal. But in that case the singularities of the dual immediatly no longer form part of the list of [Shc84].

**IV.3. Genericity of the transversality condition**

Once again we have \( M_i, i = 1, \cdots, l \) in an ambient manifold \( X \). Let there also be \( l \) quadratic Hamiltonians on \( T^*X \), satisfying the demands from III.1.9. The images \( \Psi_i(N^*M_i \times 0_{\mathbb{R}}) \) are the big wavefronts, see section III.1.11. The transversality condition under which the conflict set of \( M_1, \cdots, M_l \) is Legendrian is (III.16). The transversality is a generic property, according to theorem III.30. Here we will prove the genericity directly, i.e. we will not use the maximal rank criterion, we will prove the genericity of (III.16) directly. We will however assume that our Hamiltonians come from pseudo-Riemannian metrics.

**Theorem IV.16.** Under perturbations of \( l - 1 \) of the basepoints the transversality condition is satisfied for a residual subset of \( \bigoplus_{i=1}^l \text{Emb}(M_i, X) \).

**Proof.** The perturbations are as in the proof of III.30

\[
\gamma_i(s_i) = \gamma_i(s_i, 0) \mapsto \gamma_i(s_i, e_i), \quad i = 1, \cdots, l
\]

where

\[
\frac{\partial \gamma_i(0, 0)}{\partial e} = I_n \quad \text{and} \quad \frac{\partial^2 \gamma_i(0, 0)}{\partial e_i \partial s_i} = 0
\]

Thus we can look at the family

\[
x_1, \xi_1, \cdots, x_l, \xi_l, t_1, \tau_1, \cdots, t_l, \xi_l, \delta x_1, \cdots, \delta x_{l-1} \mapsto \\
\Psi^l_t(x_1 + \delta x_1, \xi_1, t_1), \cdots, \Psi^l_t(x_l + \delta x_l, \xi_l, t_l)
\]

This should be transversal to \( T^*_\Delta (X \times \mathbb{R})^l \). It will be enough to show that

\[
\bigotimes_{i=1}^l \pi_X \Psi^l_t
\]

is transversal to \( \Delta \subset X^l \). This in turn makes it clear that is enough to show that

\[
\pi_X \Psi^l_t(\gamma(s, e), \xi, t, \tau)
\]

is a submersion. Locally \( \pi_X \circ \Psi^l \) looks like

\[
\gamma(s, e) + t \frac{\partial H}{\partial \xi}
\]

in \( \mathbb{R}^n \). The derivative wrt. \( e \) is

(IV.14)

\[
I_n + t \frac{\partial H}{\partial x \partial \xi}
\]

We work locally and can choose normal coordinates, as in equation III.8 in which it will hold that at \( (x_0, \xi_0) \).

\[
\frac{\partial g^{ij,l}}{\partial x} = 0 \quad \text{or} \quad \frac{\partial H_l}{\partial x} = 0
\]

So the derivative in (IV.14) is simply the unit matrix \( I \). The rest of the proof is identical to the proof of III.30. \( \square \)

Now we return to the centre set and the centroid set. The criterion for these to be Legendre is in proposition IV.5.
Theorem IV.17. For a countable intersection of open and dense subsets of
\[ \bigoplus_{i=1}^{l} \text{Emb}(M_i, \mathbb{R}^n) \]
the center set is Legendrian (and the normal chord set therefore Lagrangian)

Proof. We need a covering \( \{U_\alpha\} \) of \( M_1 \times \cdots \times M_l \) and in each \( \{U_\alpha\} \) perturb the tangent space a little, as indicated in figure IV.7. It is enough to prove that, if \( \vec{n}_i \) is the map that assigns the normal to \( M_i \), that the map \( (\vec{n}_1, \cdots, \vec{n}_l) \) is transverse to the diagonal. We will first show that locally families exist that are indeed transverse to the diagonal. Denote by

Figure IV.7. The map \( \phi_{r,A,p,\gamma(s')} \circ \gamma(s) \)

\[ \phi_{r,A,p}, \ r \in \mathbb{R}, \ A \in SO(n, \mathbb{R}), \ p \in \mathbb{R}^n \]
a diffeomorphism, which is the identity on \( \mathbb{R}^n \) where we are outside the sphere of radius \( 2r \) round \( p \) and equal to \( q \to A(x - q) \), inside a circle of radius \( r \) round \( q \). Now compose an embedding \( \gamma: M \to \mathbb{R}^n \) with the map \( \phi_{r,A,p(\alpha)} \) and we will get a map that in some environment \( U'_\alpha \) of \( p(\alpha) \in M \) is submersive. Looking at a product \( \phi_{r,A,q(\alpha)} \circ \gamma_1, \phi_{r',A',p(\alpha)} \circ \gamma_2 \) we see that in a neighborhood \( U'_\alpha \) the transversality condition is satisfied. Indeed, at \( p(\alpha) \) the normal looks like \( A\vec{n} \).

One can pick a countable number of points \( p(\alpha) \) so that the \( U'_\alpha \) cover \( M_1 \times \cdots \times M_l \).

IV.4. “Reversed” sets: billiards and orthomtics

Several authors, notably Bruce & Giblin & Gibson [BG92] and [BGG83], Tabachnikov [Tab95] have studied geometrical constructions that can be generalized by means of canonical relations. In this section we want to show how close these constructions are to some form of “reversed conflict set”. Most of the material here is sketchy and serves mainly to illustrate that the conflict set is far from an isolated problem.

IV.4.6. Some constructions of curves in the plane. We will start with the simplest context: a curve \( \gamma \) in the plane \( \mathbb{R}^2 \). Fix a point in the plane and call it the origin \( O \). The curve \( \gamma \) will be the mirror for rays coming from \( O \). Denote \( \vec{n} \) a unit normal to \( \gamma \).

Definition IV.18 (Orthomtic). The orthomtic of \( \gamma \) and \( O \) is the curve defined by
\[ O + 2(\gamma(t) - O, \vec{n})\vec{n}. \]
Definition IV.19 (Pedal curve). The pedal curve of $\gamma$ and $O$ is the curve defined by

$$O + \langle \gamma(t) - O, \vec{n} \rangle \vec{n}.$$  

Conflict and center sets are related to orthomtics and pedal by the following proposition.

Proposition IV.20. The curve $\gamma$ is contained in the conflict set of the orthomtic of $\gamma$ and the origin $O$. The pedal curve is the center set of the origin and the orthomtic.

Proof. Clear from the definitions. $\square$

We can define wrt. to the mirror $\gamma$ and the origin $O$ a kite curve. We provide a definition of the kite curve with the drawing IV.8. A practical formula for drawing the kite curve starting from the pedal is

$$\text{kite}(\gamma) = \text{pedal}(\gamma) + \left( \frac{\langle J(\gamma - O), \gamma' \rangle}{\langle \gamma - O, \gamma' \rangle} \right)^2 (\text{pedal}(\gamma) - \gamma)$$

In the above formula $J$ is the usual complexification mapping.

Two other curves can also be defined, see figure IV.8 for a drawing of the billiard curve and the contrapedal curve wrt. to a parabola. Where in the definition of the pedal curve one lets down the segment $\gamma - O$ onto the normal $\vec{n}$ to $\gamma$, in the definition of the contrapedal we let down the same segment $\gamma - O$ to the tangent vector $\vec{t} = \gamma'/||\gamma'||$.

Definition IV.21 (Billiard curve). The billiard curve of $\gamma$ and $O$ is the curve defined by

$$O + 2\langle \gamma(t) - O, \vec{t} \rangle \vec{t}.$$  

Definition IV.22 (Contrapedal curve). The contrapedal curve of $\gamma$ and $O$ is the curve defined by

$$O + \langle \gamma(t) - O, \vec{t} \rangle \vec{t}.$$  

Figure IV.8. Pedal, Orthomtic, Kite, Contrapedal, Billiard

Let us now turn to a comparison between some existing results and results on the conflict set.
The origins of the curvature formula. The curvature formula (I.8) is in fact a very old formula. We can trace it back to a classic book in geometrical optics: the book [BW59]. There on p. 173 we find the formula we find in chapter 1, and more. Also, the formulas I.9 and I.10 are already written there. To establish the connection between conflict sets and ray systems in optics consider figure IV.9. On the left side rays from a circular arc hit a curved mirror. On the right side we see the same rays from a circular arc but now in the setting of conflict sets.

Figure IV.9. Left: incoming rays (dashed lines) reflected to outgoing rays. Right: the mirror is a conflict set.

Singular points of pedal and orthomtics. All these curves and the relations between them form a rich object of study. For instance it is well-known that the pedal curve has a singularity where the curvature of the mirror $\gamma$ is zero, hence where $\gamma$ has an inflection. Thus the orthomtic also has a singular point where $\gamma$ has an inflection, and vice versa. It can be derived also from the curvature formula that the orthomtic has a singularity when the mirror $\gamma$ has an inflection. We proceed as follows: suppose that the mirror $\gamma$ has an inflection $\kappa = 0$ at $p \in \gamma$. The curvature formula says that if this is the case, the wavefront coming from the orthomtic has the same curvature $\|p - O\|^{-1}$ as the wavefront coming from the origin $O$. This means that at distance $\|p - O\|$ the orthomtic meets its own caustic, which is what we needed to show.

The billiard is a symplectic transformation. In the drawing IV.8 we have introduced a billiard curve. The billiard curve defined here is not exactly the billiard studied in [Tab95]. In [Tab95] the author studies the transformation on the space of rays induced by the reflection. The billiard in that sense is a map

$$\text{Billiard: } T^* S^1 \to T^* S^1.$$  

It turns out that the graph of the billiard is a canonical relation between $T^* S^1$ and $T^* S^1$. Tabachnikov proves this directly using coordinates. We can prove the assertion in the general case where the mirror is embedded in $\mathbb{R}^n$ in at least two ways. Firstly, the billiard transformation on the space of rays is an instance of the construction with symmetric spaces indicated in section IV.5. Secondly, if we can prove that the orthomtic
is Legendrian we have also proven in view of theorem III.10 that the transformation on the space of rays preserves the symplectic form. In the next section we will derive a maximal rank criterion, (IV.19) under which the orthomtic is Legendrian.

- **Contrapedal.** The following are immediate, though they can also be obtained by a lengthy explicit calculation.

**Proposition IV.23.** The contrapedal curve is the center set of the origin and the billiard.

**Proposition IV.24.** The billiard curve and the contrapedal are singular where the evolute of the mirror contains the origin.

We will include another picture. Figure IV.10 beautifully illustrates the wavefront nature of the billiard: when the origin is the cuspidal point of the caustic the billiard and the contrapedal have a $4/3$ Lipschitz smoothness point. When the origin lies on a smooth part of the caustic the the contrapedal has a cusp.

The pictures here were made with a minor modification of the software available with the book [Gra93].

**IV.4.7. Higher dimensional analogues and generalizations.** With the exception of the kite curve the pedal, the contrapedal and the billiard and the orthomtic have straightforward generalizations to the $n$-dimensional case, where the mirror is a hypersurface in $\mathbb{R}^n$.

It is not hard to see that the corresponding statements remain true:

- the mirror is contained in the conflict set of the origin $O$ and the orthomtic,
- the pedal is the center set of the orthomtic and the origin,
- the contrapedal is the center set of the billiard surface and the origin,
- the pedal surface is singular where the mirror has zero Gaussian curvature
- the contra pedal surface is singular at the origin when the origin lies on a focal surface of the mirror,

Our main interest lies in the obvious generalization: replace the origin by some Legendrian submanifold $L_O$, cut $L_O$ with $\{H = 1\}$, follow the rays up to the mirror, and follow the reflected ray (billiard) or its opposite (orthomtic), during the same amount of time. The rays from the origin form a beam. If we replace the origin by some suitable $L_O$ we can get a parallel beam. This clearly has some optical significance.

With rays from the origin $O$ replaced by rays from $L_O$ do the statements above still hold? In particular, under which conditions is the transformation on the space of rays, induced by the
reflection symplectic, or in other words, when is the “reversed” conflict set, the orthomtic, Legendrian?
Let us first deduce a maximal rank criterion. A practical criterion is easily established. Under the condition that

\[ F(x, s) = \|x - \gamma(s)\|^2 - \|\gamma(s)\|^2 \]

defines a non-degenerate phase function, the orthomtic of the mirror \( \gamma \) and the origin \( O \) is Legendrian. Replace the origin with the rays that are normals from a submanifold \( \mu(t) \). For a parallel bundle \( \mu(t) \) is linear and for a point source \( \mu(t) \) is constant.

A phase function for the orthomtic is:

\[ F(x, s, t) = \frac{1}{2} \|x - \gamma(s)\|^2 - \frac{1}{2} \|\gamma(s) - \mu(t)\|^2 \]

For the orthomtic to be Legendrian we need that the matrix

\[
K(F) = \frac{1}{2} \langle \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{I} \rangle
\]

has maximal rank where \((x, s, t) \in \Sigma(F)\), i.e. when \( F = 0 \), \( d_s F = 0 \), \( d_t F = 0 \). The singularities of the orthomtic are inspected with the matrix

\[
\frac{d^2_{s,t} F}{d^2_{s,t} F}
\]

If \( \mu(t) \) is an embedded hypersurface, (IV.18) is a \((2n - 2) \times (2n - 2)\) matrix.

The matrix \( K(F) \) becomes

\[
K = \begin{pmatrix}
\frac{1}{2} \|x - \gamma(s)\|^2 & 0 & 0 \\
0 & d_s F & d_t F \\
0 & d_t F & d_t d_s F & d_t^2 F
\end{pmatrix}
\]

To evaluate it note that if we write

\[ F_1(x, s) = \frac{1}{2} \|x - \gamma(s)\|^2 \]

and \( F_2(x, t) = \frac{1}{2} \|x - \mu(t)\|^2 \)

that it follows

\[ F(x, s, t) = F_1(x, s) - F_2(\gamma(s), \mu(t)) \]

but also

\[
F(x, s, t) = F_1(x, s) - F_1(\mu(t), s)
\]

We will now calculate \( d_s^2 F \). To this end we write a Monge form for the mirror \( \gamma(s) \), i.e.

\[
\gamma(s) = (s, \frac{1}{2} \langle \mathbf{I} \gamma(s, s) \rangle)
\]

We also use that the vector from \( \mu(t) - \gamma(s) \) is mirrored through the tangent plane to the mirror to \( x - \gamma(s) \). Denote \( \phi \) the angle between \( x - \gamma(s) \) and the tangent plane to the mirror \( \gamma \). The derivative \( d_s^2 F \) then evaluates to

\[ 2 \sin(\phi) \|x - \gamma(s)\| \mathbf{I} \gamma. \]

The derivative \( d_s d_t F = \dot{\gamma}^T \dot{\mu} \) and \( d_t^2 F \) becomes the second fundamental form of the wavefront at \( \gamma(s) \) coming from the source surface \( \mu(t) \).

\[
K = \begin{pmatrix}
\dot{\gamma}^T \dot{\mu} & \frac{1}{2} \|x - \gamma(s)\| \mathbf{I} \gamma \\
\frac{1}{2} \|x - \gamma(s)\| \mathbf{I} \gamma & (\mathbf{I} \mu - \|\mu(t) - \gamma(s)\| \mathbf{I} \mu)^{-1} \mathbf{I} \mu
\end{pmatrix}
\]
Equation (IV.19) is important when dealing with the singularities of the orthomtic. As said a parallel bundle is one with $d^2 t \mu = 0$ whilst $d_t \mu$ also written as $\dot{\mu}$ is a constant $n$ by $n-1$ matrix. Hence the matrix $K$ for a parallel bundle becomes:

$$K = \begin{pmatrix} x - \gamma & 0 & 0 \\ d_s \gamma & 2 \sin(\phi) \| x - \gamma(s) \| \mathbf{II}_\gamma & \dot{\gamma}_T \dot{\mu} \\ 0 & \dot{\gamma}_T \dot{\mu} & 0 \end{pmatrix}$$
V Classification of singularities

V.1. Introduction

Throughout the previous chapters we have often encountered the view that the conflict set of \( l \) hypersurfaces is the projection of the intersection of their big wavefronts. In chapter one we used this view to prove \( C^j \)-smoothness at regular points. In chapter three we used it to formulate a criterion (III.16) under which the conflict set is Legendrian. In this chapter we will apply the same technique to classifying singularities of conflict sets.

We will first of all have to study the big wavefront somewhat more closely. Secondly we look at the intersection, and lastly at the projection.

Our object is to first show that the big wavefront can be stratified in some suitable way. The stratification will contain bad strata and good ones. The good strata will have low codimension, and the bad strata will have high codimension.

As before, we have an embedding of a hypersurface into an ambient space: \( \gamma: M \hookrightarrow X \). As a subset of \( X \times \mathbb{R} \) the big wavefront is given by

\[
F(x, s) = A(x, \gamma(s)) - x_0 = 0 \quad \frac{\partial F}{\partial s} = 0
\]

We will stick with the four assumptions on \( X \) and \( H \) of subsection III.1.9, so that \( F \) is a globally defined non-degenerate phase function.

Below we will define the notion of codimension for a germ. Doing so allows us to define for every \( \bar{x} = (x_0, x) \) and \( s \).

\[
\text{codim}(\bar{x}, s) = \text{codim}(F(x, \cdot))
\]

The codimension might well be infinite. For our purposes it will be enough to consider small codimensions \( \leq 6 \). Germs of finite codimension are finitely determined. So we can fix some \( N \) for which the germs of codimension \( N \) are \( V \)-equivalent iff. there \( N \)-jets are \( V \)-equivalent.

Also for the codimension \( \leq 6 \) orbits we know that these are “simple”, their orbits in the jet space are submanifolds and finitely many. Now suppose that the map

\[
(\bar{x}, s) \to j^N(x_0 - F(x, s)) \in J^N(M)
\]

is transverse to the structure of simple orbits \( \mathcal{A} \). The pullback of the structure \( \mathcal{A} \) exists in \( X \times \mathbb{R} \times M \) and inherits the nice structure.

This structure projects with injective differential to the big wavefront in \( X \times \mathbb{R} \). To ensure a transversal intersection of the projection of the pulled back orbits of \( \mathcal{A} \) we will also impose a multi-transversality condition.

\[
(\bar{x}, s^{(1)}, \ldots, s^{(p)}) \to (p)j^N(x_0 - F(x, s)) \in (p)J^N(M)
\]

A set of codimension \( \geq 7 \) on the big wavefront is not included in this structure.

Now suppose that we have \( l \) of these big wavefronts. It is necessary that the structures of orbits \( \leq 6 \) intersect transversely, and that the intersection does not include any bad strata. The maximal codimension of simple strata is 6. If we want only those in the intersection we have to impose \( n - l \leq 4 \). Thus the range of nice dimensions is \( n - l \leq 4 \). In these dimensions the conflict set only has combinations of the well-known ADE singularities.

From our reasoning it is also clear that outside this nice range there is going to be trouble. Indeed, suppose we extended our stratified structure to codimensions \( > 6 \). Here moduli arise and the constant codimension stratum is no longer a union of finitely many orbits in the jet-space.

In this chapter we first present some generalities on stability of Legendrian embeddings and
then we apply these to the big wavefronts and the conflict set. We take some time to explain these generalities, in order to arrive at a practical criterion for Legendrian stability stated in theorem V.15.

In the last section of this chapter we carry out some calculations that exhibit the singularities of conflict sets with \( n - l \leq 4 \) as certain singularities of wavefronts in \( \mathbb{R}^{n-l+2} \). We analyse the combinations of ADE singularities there and exhibit the singularities of the conflict set as non-versal unfoldings of singularities of higher codimension.

Thus if we start with stable big fronts in general position the singularities of the conflict set are not necessarily V-stable. This is due to the fact that we allow only perturbations of the base-manifolds and not of the conflict set itself. We always keep a separation between the variables \( s_i \).

Let us mention one more rather surprising consequence of our results, already pointed out in the introduction. The generic singularities of the conflict set of \( n \) hypersurfaces in \( X \), with \( \dim X = n \), are \( A_2^1 \) and \( A_2 \). Locally we always have the picture:

![Figure V.1. Generic local forms of conflict sets of \( n \) surfaces in \( X \).](image)

**V.2. Stability of Legendrian immersions.**

**V.2.1. V-equivalence.** Let \( S \) be some non-empty finite subset of \( \mathbb{R}^k \). We can speak of germs of \( \mathcal{C}^\infty \) functions at \( S \). Such germs form a ring \( \mathcal{C}^\infty(S) \). If \( S \) consists of a single point \( s_0 \) this ring is local. If \( S \) is a finite set we have

\[
\mathcal{C}^\infty(S) = \bigotimes_{s \in S} \mathcal{C}^\infty(s)
\]

**Definition V.1.** Two germs of maps \( f_1 \) and \( f_2 \) in \( \mathcal{C}^\infty((\mathbb{R}^k, S), \mathbb{R}^t) \) are called V-equivalent if there are germs of diffeomorphisms \( h \) and \( H \) that make the following diagram commutative:

\[
\begin{array}{ccc}
\mathbb{R}^k, S & \xrightarrow{\text{gr}(f_1)} & \mathbb{R}^k \times \mathbb{R}^t, S \times f_1(S) & \xrightarrow{\text{proj}} & \mathbb{R}^k, S \\
\downarrow h & & \downarrow H & & \downarrow h \\
\mathbb{R}^k, S & \xrightarrow{\text{gr}(f_2)} & \mathbb{R}^k \times \mathbb{R}^t, S \times f_2(S) & \xrightarrow{\text{proj}} & \mathbb{R}^k, S
\end{array}
\]

**Remark V.2.** Some authors do not use the term V-equivalence, but instead speak of contact-equivalence or \( \mathcal{K} \)-equivalence. The same notion was discussed in chapter 1 of this thesis.
We specialize further to where $t = 1$.
Attached to a germ $f \in C^\infty(s_0)$ is its tangent space $Tf$ to the orbit of $f$ under the action of V-equivalence. The tangent space $Tf$ is an ideal in the ring $C^\infty(s_0)$:

$$Tf = (f) + \left( \frac{\partial f}{\partial s_i} \right),$$

where “+” is a summation of ideals. The codimension of a germ is the codimension of the ideal $Tf$, that is it is the dimension of the quotient space $Q_f$

$$(V.3) \quad Q_f = \frac{C^\infty(s_0)}{Tf} \quad \text{codim } f = \dim Q_f$$

The codimension of a germ $f$ might well be infinite. For instance if $f(s_1, s_2) = s_1^2s_2$ is the germ at zero of a function on $\mathbb{R}^2$ then the ideal $Tf$ does not contain $s^k_2$ for all $k \geq 1$. So all of the monomials $s^k_2$ are required as a basis for the quotientring $Q_f$. However if the codimension of $f$ is finite then $Q_f \sim Q_g$ as $C^\infty(s_0)$ algebras is equivalent to $f \not\sim g$.

Our definition of "codimension" has the property that a quadratic form in the "$s$" variables has codimension one. This is a little unusual but natural in our case.

Denote $\mathcal{M}(s_0)$ the unique maximal ideal in $C^\infty(s_0)$. We can define the ring $J^r(s_0)$ of jets of order $r$ by setting:

$$J^r(s_0) = \frac{C^\infty(s_0)}{\mathcal{M}^{r+1}(s_0)}$$

The projection maps the ideal $Tf$ to $J^r(s_0)$. As an image we obtain $T^r f$. We set

$$d_r = \dim \frac{J^r(s)}{T^r f}$$

The sequence $d_r$ is non-decreasing with upper bound $\text{codim}(f)$. Consequently there is some minimal number $r_0$ for which $r > r_0 \Rightarrow d_r = d_r$. We’ll call it $\rho(f)$.

**Definition V.3.** A map germ $f$ is called finitely V-determined if there exists a $r_0 \in \mathbb{N}$ such that for every other germ $g$:

$$r \geq r_0, \quad j^r f = j^r g \Rightarrow f \not\sim g$$

We have - [Mat70a], theorem 3.7 - that $f$ is finitely determined iff. $\text{codim}(f) < \infty$.
Functions that do not have a critical point are locally V-equivalent to a linear form. This is codimension 0. If there are critical points we find quadratic forms in codimension 1. After that we look at ternary forms, and so on. Up to codimension 6 there are only finitely many orbits in $C^\infty(s)$, those are the ADE singularities. From codimension 7 there are infinitely many orbits with the same codimension.

Germs of codimension $< 7$ are determined by their 7-jet as a look at the ADE list shows. So a closed part of $J^6(s_0)$ can be Whitney stratified by the simple orbits.

**V.2.2. Unfoldings wrt. V-equivalence.**

**Definition V.4.** An unfolding of a germ $f \in C^\infty((\mathbb{R}^k, S), (\mathbb{R}^t, T))$ consists of a ( germ of a ) parameter manifold $\mathbb{R}^n, 0$ and a function $F: \mathbb{R}^k \times \mathbb{R}^n, S \times \{0\} \rightarrow \mathbb{R} \times \mathbb{R}^n, f(S) \times \{0\}$ such
that \( g|_{\mathbb{R}^k \times \{0\}} \equiv f \) and
\[
\mathbb{R}^k \times \mathbb{R}^n, S \times \{0\} \xrightarrow{F \times \text{id}} \mathbb{R}^t \times \mathbb{R}^n
\]
commutes.

Next, we will consider \( t = 1 \). Morphisms between two unfoldings \( F \) to \( F' \) are germs of maps that make the following diagram commutative:

\[
\begin{array}{c}
\mathbb{R}^{n'+k}, \{0\} \times S \xrightarrow{\text{id} \times \text{gr}(F')} \mathbb{R}^{n'+k+1} \\
\mathbb{R}^k, S \xrightarrow{h} \mathbb{R}^{s+1} \xrightarrow{H} \mathbb{R}^{n+k+1} \xrightarrow{\text{proj}} \mathbb{R}^{n'+k} \xrightarrow{\text{proj}} \mathbb{R}^{n'}, \{0\} \times S \xrightarrow{h'} \mathbb{R}^n, 0
\end{array}
\]

(V.4)

If \( h \) is the unfolding of a diffeomorphism on \( (\mathbb{R}^{n+k}, \{0\} \times S) \) then the two unfoldings are said to be isomorphic. If \( h' \) is a diffeomorphism then the two unfoldings are equivalent. If we have a map \( h': \mathbb{R}^{n'} \to \mathbb{R}^n \) then we can form the induced unfolding \( h^* F \):
\[
h^* F: \mathbb{R}^{n'} \times \mathbb{R}^k \to \mathbb{R} \quad h^* F = F \circ (h, \text{id})
\]
The unfolding is called trivial if it is isomorphic to the unfolding \((x, s) \to (x, f(s))\). If the codimension of the germ \( f \) is \(< \infty \) there is a universal object. A versal unfolding \( F \) has the property that every other unfolding \( F' \) of \( f \) is isomorphic to an induced unfolding of \( F \). If \( F \) has the minimal number of parameters among those unfoldings having this universal property then it is called miniversal.

The above diagram is unnecessarily complicated. If we write
\[
H: \mathbb{R}^{n'+k+t} \to \mathbb{R}^{n+k+t} \quad H = (H_n, H_k, H_t)
\]
\[
h: \mathbb{R}^{n'+s} \to \mathbb{R}^{n+k} \quad h = (h_n, h_k)
\]
we can derive relations between these maps.
\[
h_n(x') = h'(x') \\
H_k(x', s) = h_k(x', s) \\
H_n = h'(x')
\]
So that the diagram (V.4) reduces to a relation between \( F \) and \( F' \), nl.
\[
H(x', s, F'(x', s)) = (h'(x'), h_k(x', s), F(h'(x'), h_k(x', s)))
\]
In particular, this shows that \( F'(x', s) \) and \( F(h(x'), s) \) are V-equivalent and that we consequently have an identity (see [Mat70a])
\[
A(x', s)F(h'(x'), h_k(x', s)) = F'(x', s)
\]
where \( A: \mathbb{R}^{n'+k} \to \mathbb{R} \) is some smooth map.
V.2.3. Legendrian embeddings and $V$-equivalence. A Legendrian embedding is an embedding of an $n$-dimensional manifold in a contact manifold of dimension $2n + 1$, such the image is an integral manifold of the contact structure. As such we can speak of germs of Legendrian manifolds at some point in a fibered contact manifold. The singularities of their projections can be classified up to maps from the contact manifold to itself that preserve the fibering and the contact form.

For these germs we have the following,

**Lemma V.5.** If $\Psi: P^*X \to P^*X$ is a diffeomorphism that preserves the fibering and the contact form then $\Psi = g^*$ for some diffeomorphism $g: X \to X$.

Such germs of Legendrian manifolds are conveniently constructed with non-degenerate phase functions, as we saw in the previous chapter. Non-degenerate phase functions are special cases of unfoldings. If two non-degenerate phase functions are $V$-equivalent as unfoldings then they determine equivalent Legendrian manifolds. We want to show that the converse holds.

Let $F(x, s): \mathbb{R}^{n+k} \to \mathbb{R}$ be a non-degenerate phase function for a Legendrian manifold. If $\text{rk} \frac{\partial^2 F}{\partial s^2} = i$ then we can apply the parametric Morse lemma to write $F$ in the form

$$F'(x, s') + Q(s'') \# s'' = i \quad \text{rk} \frac{\partial^2 F'}{\partial s'^2} = 0$$

where $Q(s'')$ is a non-degenerate quadratic form. We will find that the corresponding Legendrian manifolds $L_F$ and $L_{F'}$ are the same. The new phase function has a minimum number of variables.

**Remark V.6.** The operation of adding an auxiliary $s$ variables as opposed to the reduction carried out above is called a “doubling” of the hypersurface $F = 0$. The terminology stems from the fact that in the complex domain \{ $u^2 = F(x, s)$ \} is a double cover of \{ $F = 0$ \} under the projection $x, s, u \mapsto x, s$.

Having minimized the number of $s$ variables by elimination of successive doublings of the hypersurface $F = 0$ we also have to touch upon the subject of suspension. Namely if $L \to T^*X \to X$ is the germ of a Legendrian immersion and $]-a, a[$ is some open interval then also

$$L \times ] - a, a [ \to T^*(X \times ] - a, a [) \to X \times ] - a, a [$$

is for any $a > 0$ the germ of a Legendrian immersion. As for the image in $X$, an example would be a cusp in $X$ becoming a cuspidal edge in $X \times ] - a, a [$. This occurs when $h'$ in (V.4) is a submersion, i.e. when some parameters are trivial in the unfolding. Submersions can be written locally in the form $(x, x') \mapsto x$. The restriction of $(x', s) \mapsto (x', d_x F)$ to \{ $x', s \mid F(x', s) = d_s F' = 0$ \} is then a suspension of the map $(x, s) \mapsto (x, d_x F)$ restricted to \{ $x, s \mid F(x, s) = d_s F(x, s) = 0$ \}.

If $F$ and $G$ are isomorphic as unfoldings then they are linked by a fibered equivalence.

(V.5) $$F(x, s) = A(x, s)G(\tilde{x}(x), \tilde{s}(x, s))$$

where $A$, $\tilde{s}$ and $x$ are to satisfy the usual requirements, namely $\tilde{s}(0, s) = s$, $\tilde{x}$ is a diffeomorphism and the smooth map $\tilde{s}(s, x_0)$ is for every $x_0$ a diffeomorphism. If also $F$ defines a germ of a Legendrian submanifold then $G$ also defines a germ of a Legendrian submanifold which is equivalent to $L_F$. 
\textbf{Theorem V.7.} Let $F \in \mathcal{C}^\infty(x_0, s_0)$ unfold some $f \in \mathcal{C}^\infty(s_0)$ and let $G$ be another unfolding of $f$ such that $F$, $G$ are non-degenerate phase functions and $L_F$ and $L_G$ are equivalent Legendrian manifolds, then $F$ and $G$ are isomorphic as $V$-unfoldings after possibly carrying out a number of doublings of the hypersurfaces $F = 0$, $G = 0$.

This is a purely local theorem. We can speak of $\mathbb{R}^k$ instead of $M$ and of $\mathbb{R}^n$ of $X$.

Two expositions of the proof of this theorem, in [Zak76] and [AGZV85] reduce the theorem to the corresponding one for Lagrangian submanifolds of $T^*X$: any two non-degenerate phase functions near $(x_0, s_0) \in \mathbb{R}^{n+k}$ defining the same Lagrangian manifold and having the same signature $\frac{\partial^2 F}{\partial s^2}$ are $R^+$-equivalent as unfoldings.

We will proceed as in the proof of the Lagrangian version of V.7. First of all, we may assume that

$$\frac{\partial^2 F}{\partial s^2} = 0$$

Secondly, we may also assume that the number of "$s$"-variables in $F$ is equal to the number of "$s$"-variables for $G$.

Thirdly, we may assume that $L_F = L_G$: they are equivalent via a diffeomorphism $\tilde{x}: \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, instead of $G$ we can consider $G(\tilde{x}, s)$.

Let us start by proving a lemma, mimicked from [Hörı71]. As before denote $\Sigma(F) = (F, d_s F)^{-1}(0)$.

\textbf{Lemma V.8.} There is diffeomorphism from $\mathbb{R}^{n+k}, (x_0, s_0)$ to itself that maps $\Sigma(F)$ to $\Sigma(G)$ and that preserves the fibering $\mathbb{R}^{n+k} \to \mathbb{R}^n$.

\textbf{Proof.} For this consider the map $\mathbb{R}^{n+k} \to \mathbb{R}^{n+1+k} \times \mathbb{P}^{n-1}$:

\begin{equation}
(V.6) \quad \mathbb{R}^{n+k} \ni (x, s) \to x, F, d_s F, [d_x F]
\end{equation}

We claim that when restricted to $\Sigma(F)$ this is a diffeomorphism onto its image $L_F \subset \mathbb{P}T^*\mathbb{R}^n$.

Call the map in (V.6) $j_F$. It holds that $(j_F)_* \delta s = 0$ implies $\delta s = 0$, because $F$ is a non-degenerate phase function.

Hence there exists a map

$$\Psi_F: (x, u, w, [\xi]) \to s \in \mathbb{R}^k$$

such that $(\Psi_F \circ j_F)(x, s) = s$. For the non-degenerate phase function $G$ we have similar maps $j_G$ and $\Psi_G$.

Mingling the two maps we claim that a diffeomorphism from $\mathbb{R}^{n+k}$ to $\mathbb{R}^{n+k}$ is

\begin{equation}
(V.7) \quad x, s \mapsto x, \Psi_G(j_F(x, s))
\end{equation}

That (V.7) is a diffeomorphism follows when we calculate

$$d_s(\Psi_G(j_F(x, s))) = d_u(\Psi_G \circ j_F) d_s F + d_w(\Psi_G \circ j_F) d_s^2 F + d_{[\xi]}(\Psi_G \circ j_F) d_s d_x F$$

which in $(x_0, s_0) \in \Sigma(F)$ reduces to

$$d_s(\Psi_G(j_F(x, s))) = d_{[\xi]}(\Psi_G \circ j_F) d_s d_x F$$

Thus (V.7) is a diffeomorphism. We have

$$(x_0, \Psi_G(j_F(x_0, s_0))) \in \Sigma(G) \iff (x_0, [d_F(x_0, s_0)]) \in L,$$

which is equivalent to $(x_0, s_0) \in \Sigma(F)$. \hfill \square
We are now in a position where we can assume near \((x_0, s_0)\) \(\Sigma(F) = \Sigma(G)\), and \(L_F = L_G\). Besides \(F\) and \(G\) also \(F(t, x, s) = tF(x, s) + (1 - t)G(x, s)\) is an unfolding. It should also be a non-degenerate phase function for \(L_F\). This need not be the case for if \(F(x_1, x_2, s_1, s_2) = x_1 + s_1^2 + s_2^2 + s_2^2\) and \(G(x_1, x_2, s_1, s_2) = x_1 + s_1^2 - s_2^2\). Then \(\Sigma(F)\) is \(\Sigma(G)\) and \(L_F, L_G\) are also identical but still \(F_t\) will not be a non-degenerate phase function for \(t = \frac{1}{2}\).

We will show that this is the only complication that can occur.

**Lemma V.9.** Assume that there are a minimum number of variables, i.e. that \(\frac{\partial^2 F}{\partial s^2} = 0\). Then \(\frac{\partial^2 G}{\partial s^2} = 0\) and \(F_t(x, s)\) is a non-degenerate phase function for every \(t\).

**Proof.** Because \(L_F = L_G\), possibly after multiplying \(G\) with some non-zero function, we have on \(\Sigma\) that \(d_x F = d_x G\). Thus on \(\Sigma\) the function \(F - G\) vanishes to second order. The fact that \(F\) is a non-degenerate phase function means that we can use \(F\) and \(d_s F\) as \(k + 1\) of the \(n + k\) coordinate functions in \(\mathbb{R}^{n+k}\). It follows that we can write \(F - G\) as

\[
G(x, s) - F(x, s) = \sum_{0 \leq i,j \leq k} \frac{1}{2} a_{ij}(x, s) \phi_i(x, s) \phi_j(x, s)
\]

where

\[
A = (a_{ij})_{1 \leq i,j \leq k}, \quad \phi_0(x, s) = F(x, s), \quad \phi_1(x, s) = \frac{\partial F}{\partial s_1}
\]

Differentiating twice one obtains on \(\Sigma\)

\[
d_s^2 G = d_s^2 F + A d_s^2 F d_s^2 F
\]

Thus \(d_s^2 F(x_0, s_0) = 0\) implies \(d_s^2 G(x_0, s_0) = 0\). In the same way we have \(d_x d_s F(x_0, s_0) = d_x d_s G(x_0, s_0)\). Thus for every fixed \(t\) the function \(F_t(x, s) = tF(x, s) + (1 - t)G(x, s)\) is a non-degenerate phase function.

We have assembled the ingredients for a proof of theorem (V.7).

**Proof of Theorem V.7.** We will use what normally is called the homotopy method. For an exposition of this see the “lemme de réduction” in [Mar76].

Consider the equation

\[
A_t F_t(x, s_t(x, s)) = F(x, s)
\]

Differentiation wrt. to \(t\) yields

\[
A_t (F - G) + \frac{\partial A(t, s, x)}{\partial t} F_t + \Xi(x, s, t) F_t = 0
\]

where \(\Xi(x, s, t)\) is a vector field

\[
\Xi(x, s, t) = \sum_{i=1}^{k} \frac{\partial s_i(t, s, x)}{\partial t} \frac{\partial}{\partial s_i}
\]

We have already seen in the above that \(F - G\) annuls to second order on \(\Sigma\), but \(\Sigma = \Sigma(F_t)\) as well. Thus we have an expression as in (V.8). This means that we can solve (V.9) and find the vector field \(\Xi\) and the function \(d_t A\).

Henceforth we will study the stability of the unfolding \(F\). It is the same as stability of the diagram \(L \rightarrow \mathbb{P}T^*X \rightarrow X\).

The theory of stability of unfoldings wrt. \(V\)-equivalence and stability of Legendrian immersions proceeds as the theory of stability of unfoldings wrt. to \(R^+\) equivalence and stability of
Lagrangian immersions. Proofs of these statements can be obtained by doing exactly what is described in [Dui74]. We will not do this completely but will concentrate on giving a criterion for global stability of a $V$-unfolding. It is our aim to show that in dimensions $\leq 6$, i.e. $\dim X \leq 6$, generic embeddings of compact manifolds give rise to wavefronts that are Legendre-stable.

DEFINITION V.10. If an unfolding $F$ of a function $f$ is such that there exists a neighborhood $U$ of $F$ in $C^\infty(X \times M, \mathbb{R})$ so that $G \in U$ implies $F$ $V$-equivalent as unfolding to $G$, then we say that $F$ is stable.

With $S \subset M$ we denote some finite subset of $M$, in other words a point $(s^{(1)}, \ldots, s^{(p)})$ of $M^{(p)}$.

PROPOSITION V.11. If the unfolding $F$ is stable then the graph of

$$X \times M^{(p)} \xrightarrow{\gamma^r} (p) J^r(M, \mathbb{R})$$

is transversal to every $X \times M^{(p)} \times O$ for every orbit $O$ of the action of $V$-equivalence on functions in $(p) J^r(M, \mathbb{R})$.

PROOF. This is proposition 2.1.2 in [Dui74]. Take an arbitrary orbit $O \subset (p) J^r(M)$. Unfoldings whose graphs are transversal to an orbit lie dense. Let $F$ be stable then a nearby unfolding $G$ is $V$-equivalent to $F$ and $G$ can be chosen to lie transverse to $O$.

We have

$$F(x, s^{(i)}) = A(x, s^{(i)})G(\tilde{x}(x), \tilde{s}(x, s^{(i)}))$$

near $x_0, S$. The mappings $\tilde{x}, \tilde{s}$ and $A$ define near $x_0, (p) j^r F(x_0, S)$ a diffeomorphism $j_{\tilde{x}, \tilde{s}, A}$ from $X \times (p) J^r(M)$ to itself that preserves the manifolds $X \times O$.

The diffeomorphism also maps the graph of $(p) j^r G$ to the graph of $(p) j^r F$. Hence the graph of $(p) j^r F$ lies transverse to the manifolds $X \times O$. \hfill $\Box$

The orbits have a tangent space and the graph of $(p) j^r F$ has a tangent space. That the orbits lie transverse to the graph of $(p) j^r F$ can near $x, S$ be expressed as an algebraic criterion: the tangent space to the orbit at $x_0, (p) j^r F(x_0, S)$ can be determined if we look at $f(s) = F(x_0, s)$. If we multiply this by something close to the identity, say $1 + \epsilon g(s)$, we get

(V.10) $$\frac{\partial}{\partial \epsilon} ((1 + \epsilon g(s))f(s))_{\epsilon=0} = g(s)f(s)$$

If we allow diffeomorphisms close to the identity we get

(V.11) $$\frac{\partial}{\partial \epsilon} ((f(s + \epsilon s))_{\epsilon=0} = \frac{s}{s} \frac{\partial f}{\partial s}$$

so that the tangent space to the orbit in $(p) J^r(S)$ is

$$C^\infty(x, S)(F) + M(S) \frac{\partial F}{\partial s} + M(S)^{r+1}$$

The tangent space to the graph of $(p) j^r F$

$$C^\infty(x, S) \frac{\partial F}{\partial s} + \mathbb{R} \frac{\partial F}{\partial x} + M(S)^{r+1}$$

Thus stability of an unfolding implies the algebraic criterion that at $x, S$:

(V.12) $$\forall r \quad C^\infty(S) = C^\infty(S) \left( F, \frac{\partial F}{\partial s} \right) + \mathbb{R} \left( \frac{\partial F}{\partial x} \right) + M(S)^{r+1}$$
The proof that the converse holds is a standard argument, which we will not repeat in detail. The first step is to call an unfolding \textit{inf-stable} if

\begin{equation}
C^\infty(M \times X) = C^\infty(M \times X) \left(F, \frac{\partial F}{\partial s}\right) + C^\infty(X) \left(\frac{\partial F}{\partial x}\right)
\end{equation}

Equation (V.13) implies (V.12). If (V.12) holds for sufficiently large $r$ and $x$, the inverse application can also be established. This needs an application of the Malgrange-Mather preparation theorem in the way we use it in the next paragraph to determine how large $r$ needs to be.

The second step is to recover stability from (V.13). This inverse statement is the one known as “infinitesimal stability implies stability”.

\textbf{V.2.4. Local stability of the unfoldings.} We indicate how the Malgrange-Mather theorem is usually used to eliminate tails.

Let $(x_0, s_0) \in \mathbb{R}^{n+k}$. Local stability of an unfolding $F: \mathbb{R}^{n+k} \to \mathbb{R}$ at $(x_0, s_0)$ is defined as follows.

\begin{equation}
C^\infty(s_0) = \sum_{i=1}^{k} \frac{\partial F}{\partial s_i}(x_0, \cdot) \cdot C^\infty(s) + C^\infty(s_0) \cdot F(x_0, \cdot) + \sum_{j=1}^{n} \mathbb{R} \cdot \frac{\partial F}{\partial x_j}(x_0, \cdot) + \mathcal{M}(s_0)^{r+1}
\end{equation}

holds as an identity between ideals for some $r \geq n$.

\textbf{THEOREM V.12.} The identity (V.14) is satisfied iff. the identity

\begin{equation}
C^\infty(x_0, s_0) = C^\infty(x_0, s_0) \cdot TF(x_0, \cdot) + \sum_{j=1}^{n} C^\infty(x_0) \cdot \frac{\partial F}{\partial x_j}
\end{equation}

holds.

\textbf{REMARK V.13.} Let $f: (X, p) \to (Y, q)$ be some $C^\infty$ map between two manifolds. This induces a map $f^*: C^\infty((Y, q), \mathbb{R}) \to C^\infty((X, p), \mathbb{R})$. Both these rings are local by the Hadamard lemma, see [Mat70a] lemma (1.4). The map $f^*$ makes every $C^\infty((X, p), \mathbb{R})$ module into an $C^\infty((Y, q), \mathbb{R})$ module. The Malgrange Mather preparation theorem answers the following question: “When is a module $A$ that is finitely generated over $C^\infty((X, p), \mathbb{R})$ finitely generated over $C^\infty((Y, q), \mathbb{R})$?”. This is the case if

\[
\frac{A}{f^*\mathcal{M}((Y, q))A}
\]

is a finite-dimensional vectorspace over $\mathbb{R}$. Again, see [Mat70a], theorem (1.10). An application which shows the multiple usages of this theorem can be found in [GG73], example (B) to theorem (3.6).

\textbf{PROOF OF THEOREM (V.12).} Let $B$ be the $\mathbb{R}$-“module”.

\[
\sum_{j=1}^{n} \mathbb{R} \cdot \frac{\partial F}{\partial x_j}
\]

and let $A$ be the module

$C^\infty(s_0) \cdot TF(x_0, \cdot)$.

Let $C = C^\infty(s_0)$. Saying that $A + B = C$ is saying that (V.15) holds. We consider $D = C/A$. This is a finitely generated $C^\infty(s_0)$ module. We want it to be finitely generated as an $\mathbb{R}$-“module”, where the generators should be the ones we have for $B$. We will use theorem
(3.10) from [GG73], which states that this is the case if and only if the projections of these generators in
\[ D' = \frac{D}{\mathcal{M}^{n+1}(s_0)D} \]
generate the module \( D' \). This is an equivalent statement to (V.14).

\[ \Box \]

V.2.5. Equisingularity manifolds. We set forth to translate (V.12) into a geometric criterion.

Let us pose for sufficiently large \( r \).
\[ \mathcal{E}^r(x_0, s_0) = \{(x, s) \mid j_s^r F(x, s) \in \mathcal{O}\} \subset X \times M \]
where \( \mathcal{O} \) is the orbit of \( j^r F \) under the action of \( V \)-equivalence. This orbit is a manifold near \( x_0, s_0 \). Also the graph of
\[ (V.16) \quad x, s \mapsto (F, \frac{\partial F}{\partial s}, \cdots, \frac{\partial^r F}{\partial s^r}) \in j^r(M) \]
is of course a smooth manifold.

If the graph of (V.16) intersects the orbit \( \mathcal{O} \) transversely the equisingularity manifolds are indeed submanifolds of dimension \( n + k - \text{codim} \mathcal{O} \). For sufficiently large \( r \) the intersection no longer depends on \( r \).

The tangent space to the orbit is
\[ (V.17) \quad j^r \left( C^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k M(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot) \right) \]
To determine \( \text{codim} \mathcal{O} \) in \( J^r(s_0) \) for sufficiently large \( r \) we need to calculate
\[ \text{codim} \mathcal{O} = \dim \mathbb{R} \left( \frac{C^\infty(s_0)}{M(s_0)^{r+1} + C^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k M(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot)} \right) \]
\[ = \dim \mathbb{R} \left( \frac{C^\infty(s_0)}{M(s_0)^{r+1} + C^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k C^\infty(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot)} \right) + \dim \mathbb{R} \left( \frac{M(s_0)^{r+1} + C^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k M(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot)}{M(s_0)^{r+1} + C^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k M(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot)} \right) \]
\[ = \text{codim}(f) + k \]

Proposition V.14. If the graph of (V.16) hits the orbit \( \mathcal{O} \) of \( f \) in \( J^r(M) \) transversally then
- the equisingularity manifolds are smooth and of codimension \( \text{codim}(f) + k \) in \( \mathbb{R}^{n+k} \),
- the equisingularity manifolds project immersively to \( \mathbb{R}^n \).

Proof. In view of the above, only the last statement remains to be proved.

We will show that if the vector \((0, \delta s)\) lies in the tangent space to \( \mathcal{E}(x_0, s_0) \) then it is zero.
The vector \((0, \delta s)\) lifts to \( J^r(k) \) by the graph of (V.16). So the lift of the vector is
\[ \sum_{j=1}^k j^r \left( \frac{\partial F}{\partial s_j}(x_0, \cdot) \right) \delta s_j \]
This vector should lie along the tangent space of the orbit \( O \). Taking into account the equation for the tangent space (V.17) we ask that:

\[
\sum_{j=1}^{k} \frac{\partial F}{\partial s_j}(x_0, \cdot) \delta s_j \in C^\infty(s_0) F(x_0, \cdot) + \sum_{j=1}^{k} \mathcal{M}(s_0) \frac{\partial F}{\partial s_j}(x_0, \cdot) + \mathcal{M}(s_0)^{r+1}
\]

Here \( r \) can be made large, say \( r > \text{codim}(f) \) and this will imply that the \( \delta s_j \) are all zero. We conclude that the projection of the \( \mathcal{E}(x_0, s_0) \) to \( X \) is immersive. \( \square \)

The equisingularity manifolds corresponding to codimension 1 are defined in \( \mathbb{R}^{n+k} \) by \( d_s F = 0 \) and \( F = 0 \). This is just \( \Sigma(F) \). Regular points on the wavefront are thus always of codimension 1. The wavefront is at regular points \( \mathcal{E}_X = \pi_X(\mathcal{E}(x_0, s_0)) \) - where \( \pi_X \) is the projection from \( X \times M \rightarrow X \). The codimension of \( \mathcal{E}_X \) in \( X \) is exactly the codimension of \( f \) if the orbit of \( f \) is hit transversally by the unfolding \( F \).

This situation is different to what happens in the Lagrangian case with the \( R^+ \)-equivalence. There the definition of codimension is such that the points with codimension one form the caustic. Most points of the Lagrangian manifold have codimension 0.

We come to the theorem which relates the equisingularity manifolds and the local stability to global stability. This theorem provides the practical criteria by which one decides whether stability holds.

**Theorem V.15.** Let \( F \in C^\infty(X \times M) \) be such that the map \( \Sigma(F) \rightarrow X \) is proper (and hence finite to one) \( F \) is a stable if

- \( F \) is locally stable at every \( (x, s) \in X \times M \)
- For a fiber \( (x, s^{(1)}, \ldots, s^{(p)}) \) of \( \Sigma(F) \rightarrow X \) the projections of equisingularity manifolds to \( X \) at each \( (x, s^{(i)}) \) intersect transversally at \( x \in X \).

Let us indicate the differences between this theorem and the theorem on Lagrangian stability that is stated in [Dui74], proposition 2.2.4.

These are the usual criteria for global stability. For instance in the result on generic mappings from the plane to the plane one asks that the curves along which folds occur intersect transversally.

The Lagrangian version of the theorem on global stability contains a third demand, namely that

- **Affine Independence.** The \( d_x F(x, s^{(i)}) \) for \( i = 1, \ldots, p \) are affinely independent, as linear operators on the tangent space to intersection of the equisingularity manifolds.

This demand though a little technical has a geometric interpretation. The vector \( d_x F \) may be tangent to the caustic. This will happen for instance on the regular part of a focal sheet from a surface in \( \mathbb{R}^n \). Hence two sheets of the caustic may come to lie as in figure V.2: the equisingularity manifolds (here the regular part of focal sheets) intersect transversally.

Intuitively we see that this is not a stable situation. From the proof of proposition 2.2.4 in [Dui74] we can also conclude that it is not globally stable.

However in the Legendrian case it is not necessary to impose a similar demand. By definition equisingularity manifolds in \( X \) cannot have codimension 0 in the Legendrian case. As vectors \( d_x F(x, s^{(i)}) \) it follows from the non-degeneracy condition on \( F \) that they are normals to the wavefronts and hence to the equisingularity manifolds. So if the equisingularity manifolds in the Legendrian case intersect transversally then as vectors the \( d_x F(x, \cdot) \) are linearly independent.
Proof of Theorem V.15. We have seen that stability is equal to (V.12) at all $x,S$. Thus we have to check that the conditions in the theorem imply (V.12) and vice-versa.

The tangent vectors $\delta x$ in $V_i = T \mathcal{E}_X(x_0, s^i_0)$ are those that
\[
\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x_0, \cdot)\delta x_i \in C^\infty(s_0^i)F + C(s_0^i)\frac{\partial F}{\partial s}(x_0, \cdot) + M(s_0^i)^{r+1}
\]
Thus
\[
(V.18) \quad \delta x \rightarrow \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x_0, \cdot)\delta x_i
\]
is a map from $\mathbb{R}^n$ to $C^\infty(s_0^i)$, and also to
\[
(V.19) \quad W_i = \frac{C^\infty(s_0^i)}{C^\infty(s_0^i)F + C(s_0^i)\frac{\partial F}{\partial s}(x_0, \cdot) + M(s_0^i)^{r+1}}
\]
These maps are surjective due to the local stability.
In this way the tangent space $V_i$ to each equisingularity manifold is the kernel of a map $\mathbb{R}^n \rightarrow W_i$. The $V_i$ intersect transversally iff.
\[
(V.20) \quad \mathbb{R}^n \ni \delta x \rightarrow W_1 \oplus \cdots \oplus W_p
\]
is surjective. Indeed if the map in (V.20) is surjective the dimension of its kernel is $n$ minus the sum of the dimensions of $W_i$. Thus the codimension of the kernel is the sum of the dimensions of the $W_i$, hence the sum of the codimensions of the $F(x_0, \cdot)$ at the $s_0^i$.
This is exactly the dimension that the intersection of the $V_i$ should have in order to have a transversal intersection of the equisingularity manifolds.
On the other hand, if (V.20) is surjective then the criterion (V.12) holds. \qed

V.3. Statement and proof of the main theorem
In this section we will assume some knowledge of stratifications. A standard reference is [GWdPL76].
Consider $J^r(s_0)$ the space of $r$-th order jets of functions at $s_0 \in M$. The action of $V$-equivalence divides $J^r(s_0)$ into orbits. Denote $W^r(s_0, m)$ the set of jets whose codimension is $> m$. For $N \leq \min(6, n)$ the complement $J^r(s_0) \setminus W(s_0, N)$ has only finitely many strata.
The set $W(s_0, N)$ is an algebraic set and this algebraic set has some Whitney stratification also that can be refined to fit together nicely with the stratification of the complement. Strata with codimension $\leq N$ correspond to finitely many orbits of the $V$-action. These are called the good strata, denote them $\mathcal{B}$. Together with the algebraic stratification $\cup \mathcal{B}$ of $W(s_0, N)$ that fits with $\mathcal{B}$ we have thus stratified $J^r(s_0)$.

**Proposition V.16.** For a generic embedding of a compact manifold the big wavefront $M^h_i = \pi_{n+1}(N^*M^h_i)$ has a Whitney stratified subset that consists of of equisingularity sub-manifolds of codimension at most $N = \min(n, 6)$. The strata correspond to singularity types of individual fronts. In particular, if $n \leq 6$ the strata miss only isolated points on the big wavefront.

**Proof.** Again write $F(x, s) = A(x, \gamma(s))$. The phase function for the big front is $G(\bar{x}, s) = x_0 - F(x, s)$. Consider

\[(V.21) \quad X \times M \rightarrow X \times J^r(M, \mathbb{R}), \quad (x, s) \mapsto x, j^r(G)\]

At each $(\bar{x}_0, s_0)$ the map is generically transverse to the stratification, because of the demands we put on our distance function in section III.1.9. As before with the equisingularity manifolds the strata project immersively to $X \times \mathbb{R}$. To get the strata to project in general position we would need that all strata that are above $x_0$ together intersect the diagonal stratification transversally. That is if

\[(V.22) \quad (\bar{x}, s^{(1)}, \cdots, s^{(p)}) \mapsto (j^r_s G(x, s^{(1)}), \cdots, j^r_s G(x, s^{(p)}))\]

to be transverse to the diagonal stratification $D(pJ^r(M))$. That this all works is again due to the fact that our distance functions have a nowhere zero first derivative.

Now, because $M$ is compact, there are only a finite number of points above one $\bar{x}_0$. Thus if we prove the multitransversality for $p = n + 1$ we have proven the multi transversality for all $p$. In that case the stratification of the part determined by the good jets can be refined. $\square$

**Remark V.17.** Note that our transversality requirements come in two steps corresponding to the maps $(V.21)$ and $(V.22)$.

**Remark V.18.** The distance functions $F(x, s) = A(x, \gamma(s))$ are for compact manifolds $M$ in $C_{pr}^\infty(M \times X)$. The function space $C_{pr}^\infty(M \times X)$ consists of those functions for which the projection of the surface

\[\Sigma(F) = \{F = 0, \frac{\partial F}{\partial s} = 0\} \subset M \times X\]

to $X$ is a proper map. It is in this space that we have the genericity results for Legendrian and Lagrangian mappings, cf. theorem V.15.

**V.3.6. Generic intersection of $l$ big fronts.** Let us now consider $l$ conic Lagrangian manifolds $N^*M_i$ in $T^*X$. These can be mapped to give $l$ big wavefronts manifolds $N^*M^h_i$ in $T^*(X \times \mathbb{R})$. Returning to conflict sets our object of interest is the intersection:

\[(V.23) \quad N^*M^h_i \times \cdots \times N^*M^h_i \cap T^*_\Delta(X \times \mathbb{R})^l\]

The partial stratification of each of the big wavefronts is dependent on some integer $N$. We can arrange this $N$ to be such that in the intersection $(V.23)$ there are only good strata.
Theorem V.19. Suppose $n - l \leq 4$. For a residual set of embeddings in $\bigoplus_{i=1}^l \Emb(M, X)$ the intersection V.23 and hence the conflict set only has combinations of simple singularities. If $\#A_1$ is the number of smooth big fronts in the intersection.

- singularities of conflict sets appear in generic $n - l$ parameter families of fronts in $\mathbb{R}^{n-\#A_1}$, if $l - \#A_1 = 2$ and
- in generic $\sigma(n - l)$ parameter families of fronts in $\mathbb{R}^{n-l+2}$, if $l - \#A_1 > 2$.

It holds $\sigma(2) = 2$, $\sigma(3) = 6$ and $\sigma(4) = 12$. The list of singularities of conflict sets is finite for $n - l \leq 4$. If $n - l > 4$ we expect moduli.

Proof. For the same reasons as in the proof of proposition V.16 the stratifications of the big fronts can be made to intersect transversally. This is the third transversality criterion we impose.

Suppose the big fronts meet at some point $(x_0, x)$. To find the highest codimension singularity we can encounter we assume that $l - 1$ of the $N^* M^i$ project to $X \times \mathbb{R}$ as a smooth hypersurface. Suppose the remaining one has at $(x_0, x)$ a codimension $\mu$ singularity. Then the stratum on which it lies will have codimension $\mu$. Adding codimensions we should have in the generic case that

$$(l - 1) + \mu \leq n + 1$$

Rewriting this, we obtain

$$\mu \leq n - l + 2$$

For $\mu \leq 6$ we have only simple stable singularities. Thus if $n - l + 2 \leq 6$, that is $n - l \leq 4$, is in the domain of the nice dimensions. On the other hand if $n - l > 4$ we will encounter strata of codimension higher then 5. It follows that $n - l \leq 4$ is the domain of nice dimensions.

The fourth and last criterion we require for the proof of our main theorem, has to do with projection of the intersection (V.23) to $X$. We want this projection to be a map with regular intersections.

We ask that the $p$-fold projection

$$\pi^{(p)} : \left( \mathbb{R}^{n+1} \right)^{(n)} \to (\mathbb{R}^n)^p$$

is transverse to the diagonal stratification when restricted to the intersection of the big fronts. This is also achieved with first order perturbations.

Once we know that the stratified big wavefronts intersect transversally we want to determine what sort of singularities occur in the finite list we have. We need to prove our claim that estimates the number of parameters needed to produce the fronts from the finite list in $\mathbb{R}^{n-l+2}$. The estimate will follow from a codimension and modality count, to be carried out in the next section. \(\square\)

V.4. Geometrical description of different cases

A “local model” for a singularity is a universal unfolding for it. Local models for all the simple singularities are well-known. A front with an $A_3$ singularity can for instance be made with $s_1^4 + x_1 s_1^2 + x_2 s_1 + x_3$. The fronts that we consider are big wavefronts. The singular points on them that we are considering have a tangent space to the stratum on which they lie of dimension at least one. This means, see the list of examples 6.4 in [Arn76], that the time function is a trivial parameter in the unfolding for the big wavefront. Locally the non-degenerate phase function for the big fronts that we are considering can be written:

$$x_0 = F(x, s)$$
where $F(x, s)$ is a versal unfolding from the ADE list.

However there is more than one big wavefront. What we know of the big wavefronts is that their equisingularity strata intersect transversally. Hence we can use these equisingularity strata to define coordinates.

For the description of these singularities the main distinction is the difference $n - l$. Indeed, if $(\mu_1, \mu_2, \cdots, \mu_l)$ is the list of codimensions then we seek $\mu_i$ with $1 \leq \mu_i$ and $\sum_{i=1}^l \mu_i \leq n + 1$. Those $\mu_i$ that are 1 correspond to smooth hypersurfaces. This is because the $A_1$ singularity is just a Morse function and the unfolding is

$$G_1: x_0 = s_1^2 + x_1$$

The equations $G_1 = 0$ and $d_x G_1 = 0$ imply $x_0 = x_1$. Because of the transversal intersection of the equisingularity manifolds $x_1$ can be discarded as a coordinate. Every $A_1$ singularity presents a reduction of $n$ and $l$ by 1.

If $n - l$ is fixed then for arbitrary $n$ a certain number of parts in the partition have to be 1. Let $k$ be the number that is $> 1$, thus at least 2. It follows that $2k + (l - k) \leq n + 1$ so that a maximum of $n - l + 1$ codimensions is $> 1$. The others are 1. In the following list of codimensions we have already eliminated the $A_1$ possibilities.

- **$n - l = 0$**
  - If $n = l$ then at most 1 of the $\mu_i$ is $> 1$. So the only case to consider is $l = 1$.
  - We can have only two cases: (1), (2).

- **$n - l = 1$**
  - At most 2 of the codimensions are $> 1$. So it is enough to consider $n = 3$, $l = 2$.
  - In addition to the above combinations we will have: (2, 2) and (3).

- **$n - l = 2$**
  - The relevant dimensions are: $n = 5$, $l = 3$. The new cases are: (4), (3, 2) and (2, 2, 2).

- **$n - l = 3$**
  - Dimensions: $n = 7$, $l = 4$. New cases: (5), (4, 2), (3, 3), (3, 2, 2), (2, 2, 2, 2)

- **$n - l = 4$**
  - Dimensions: $n = 9$, $l = 5$. New cases: (6), (5, 2), (4, 3), (4, 2, 2), (3, 3, 2), (3, 2, 2, 2) and (2, 2, 2, 2, 2).

For each of the strata there are only a limited number of singularities, from the ADE list. The conflict set has dimension $n - l + 1$. The codimension of a singularity on a generic front of dimension $n - l + 1$ is maximally $n - l + 2$. If we look at the above list we see that on the conflict set the codimension can add up to $2(n - l + 1)$.

**V.4.7.** $n - l = 0$. If $n - l = 0$ the singularities of the conflict set are the generic singularities of 2-dimensional fronts. Those are $A_1$, $A_2^2$ and $A_2$. Note the marked difference with the case of symmetry sets. Their list - see p. 168 of [JB85] - contains two more normal forms, namely $A_3$ and $A_3^3$.

The $A_3$ is an “endpoint”. One can imagine conflict sets where this happens. One could take $M_1 = M_2$, but this is surely no generic situation. The singularity $A_1^2$ happens when a symmetry set on the big wavefront of a curve $M_1$ in $\mathbb{R}^2$ gets cut by a smooth big wavefront. In the case of symmetry sets the big wavefronts all come from one curve and there are thus three branches meeting. On the conflict set $A_1^2$ is the sum of $A_1^2$ and $A_1$, thus there will only be two branches meeting.

**V.4.8.** $n - l = 1$. If $n - l = 1$ the codimension can add up to 4. The cases to consider are $A_2 A_2^2$, $A_2^3 A_2$, $A_2^3 A_2^2$. All other singularities are just those of generic 2-dimensional fronts. Pictures are partly in [JB85]

The $A_2 A_2$ singularity is a generic projection of two transversely intersecting cuspidal edges in $\mathbb{R}^4$. These cuspidal edges can intersect in two ways. This is indicated in figure V.3. One
way is that only a point remains, another way is where the adjacent $A_1$ strata intersect. To obtain a picture we will take two copies of our previous example

$$G_1: x_0 = s_1^3 + x_1 s_1 + x_2 \quad G_2: x_0 = s_2^3 + x_3 s_2 - x_2$$

We take tangent spaces to the strata at $x = 0$. For $F = s^3 + A s + B$ the variety determined by $(F, d_s F) = 0$ is $\frac{A^3}{27} + \frac{B^2}{4} = 0$. Thus the $A_1$ stratum of $G_1$ is determined by $B = x_2 - x_0 = 0$. The $A_2$-stratum is $A = B = 0$, thus $x_2 - x_0 = x_1 = 0$. For $G_2$ we have the $A_1$-stratum $x_2 + x_0 = 0$ and the $A_2$ stratum $x_3 = 0$. At zero these intersect transversally. We project the intersection along the time axis $x_0$ to $\mathbb{R}^3$. The surface we get is in figure V.4. This surface is also known as $D_4^+$ if we view it as a metamorphosis of a wavefront in $\mathbb{R}^3$. Recall that a metamorphosis is a one dimensional family of fronts, see [Arn90]. The name $D_4^+$ is chosen because the surface is also obtained with an unfolding

$$G_1 - G_2 = s_1^3 - s_2^3 + x_1 s_1 - x_3 s_2 + 2 x_2$$

The unfolding $G_1 - G_2$ is not a versal unfolding. If we want to unfold the $D_4^+$ germ $s_1^3 + s_2^3$ with a V-versal unfolding we need 4 parameters. In the unfolding $G_1 - G_2$ the term $s_1 s_2$ misses.

The picture $D_4^+$ is in [AGZV85], §22. In [JB85] it is on p. 174.

We proceed to discuss the differences between the list of singularities of symmetry sets in
[JB85] and our list. Again for symmetry sets the list is larger. There are the endpoints $D_4^\pm$. As mentioned in the introduction they do not occur on conflict sets. We do have $A_4^1$ because we have to consider 1 parameter metamorphoses. On the symmetry set $A_4^1$ appears as the intersection of $\binom{6}{2}$ planes. On the conflict set two planes are not present. Confirm also the picture 3 in the introduction.

On both the conflict set and the symmetry set we have $A_2^1 A_2$. The $A_2^1 A_2$ singularity is on the level of big fronts an intersection of a segment of a symmetry set with a cuspidal edge of a big front. On the conflict set it looks as the right hand side of V.3.

The picture the authors of [JB85] mention as $A_1 A_3$ we do not have because in our case only a suspension of $A_3$ can occur on one big wave front. The other big wavefront is a hyperplane that cuts the suspension of $A_3$ transversely. Such an intersection is a normal swallowtail, which Bruce et. al. mention as $A_4$.

V.4.9. $n - l = 2$. If $n - l = 2$ we need at least $n = 4$ and $l = 2$ to obtain an interesting new local model. Indeed the case (4) has $A_4^1$ and $D_4^\pm$ and suspensions of the cases that occur with $n - l = 1$. So the first really new case is (3, 2). On this stratum we have amongst others $A_3 A_2$. This is a metamorphosis of a 3-dimensional front. Some sections of this surface are in figure V.5. In one them we actually see a swallowtail meeting a cuspidal edge. For $A_3 A_2$

![Figure V.5. Sections of $A_3 A_2$](image-url)
we have the same illustration of our main theorem as with $A_2A_2$. The $A_3A_2$ singularity can happen when $n - l = 2$ thus $n = 4$ for the first time.

Suppose we make an $A_3$ with $x_0 = s_1^4 + x_1s_1^2 + x_2s_1 + x_3$. The tangent space to the $A_1$ stratum of the front is $x_0 = x_3$. For the $A_2$-stratum and $A_1^2$ add $x_2 = 0$ and for $A_3$ add $x_1 = 0$.

We need an $A_2$ that transversally intersects these. We can choose it to be $x_0 = s_2^3 + x_4s_2 - x_3$. The unfolding which we obtain is again not versal. It occurs in a 2-parameter family. We see this from the unfolding:

$$(V.25) \quad s_1^4 + x_1s_1^2 + x_2s_1 + 2x_3 - s_2^3 - x_4s_2$$

A two dimensional family in which this unfolding exists is made by augmenting $x_5s_1^2s_2$ and $x_6s_1s_2$. Adding $x_5s_1^2s_2$ and $x_6s_1s_2$ to $(V.25)$ we get an unfolding of an $E_6$-germ.

The remaining interesting case if $n - l = 2$ is $A_2A_2A_2$. Here $n = 5$ and the surface itself is three dimensional. To get a universal unfolding we have to find three $A_2$ unfoldings that lie in general position.

We could take $G_1: x_0 = s_1^3 + x_1s_1 + x_4$ and $G_2: x_0 = s_2^3 + x_2s_2 + x_5$ and $G_3: x_0 = s_3^3 + x_3s_3 - x_5$. We have to verify that at 0 the big wavefronts $G_i$ lie transverse to each other. This is done in the same way as above.

The intersection can be projected along the $x_0$ axis at first. In the above we have proven that in addition to the time axis there must exist some other direction in which we can project. To determine it we will use the fact that the conflict set in the nice dimensions is locally Whitney stratified. Hence at the singular point we consider we can try to compute the tangent space.

The conflict surface can be parameterized in the following way: $x_1 = -3s_1^2$, $x_2 = -3s_2^2$, $x_3 = -3s_3^2$ and $x_4 - x_0 = 2s_1^3$, $x_5 - x_0 = 2s_2^3$, $-x_5 - x_0 = 2s_3^3$. The projection along the $x_0$ surface results in $x_5 = s_2^3 - s_3^3$ and $x_4 = 2s_1^3 - s_2^3 - s_3^3$. In 0 the limit of tangent spaces to the $A_1A_1A_1$ stratum is thus $\delta x_1$, $\delta x_2$ and $\delta x_3$. Thus we could take any direction except those to project along.

Let us determine an unfolding. In $\mathbb{R}^n = \mathbb{R}^5$ the unfolding is still an unfolding of a multigerm, that is two copies of the $A_2A_2$. Two parameters will be missing in this unfolding. We can project still further down to $\mathbb{R}^4$. A generic section and projection is $\delta x_5$, in the $x_1, x_2, x_3, x_4$ space this unfolding is

$$F = 2G_1 - G_2 - G_3 = 2s_1^3 - s_2^3 - s_3^3 + 2x_1s_1 - x_2s_2 - x_3s_3 - 2x_4$$

The germ $2s_1^3 - s_2^3 - s_3^3$ is a $T_{333}$ germ. Thus we have some sort of answer to a question posed by T. C. Wall in [Wal77]. He asked for “some geometrical discussion of the higher order singularities”. In a way such a geometrical discussion is already present in [AGZV85]. Here we obtained an interpretation of $T_{333}$ as a falling together of three cuspidal edges.

All pictures here were obtained with the help of the software [GPS01]. Sections of the $A_2A_2A_2$ surface apparently are too singular to be depicted. In the next subsection we will also no longer be able to obtain the right pictures.

The $T_{333}$ germ we get is contained in the list of wavefronts listed in [AGZV85], §21.8. The case we have is mentioned as $P^0_8$. As a wavefront it occurs generically in $\mathbb{R}^7$. Here we encounter it in $\mathbb{R}^4$. We summarize as follows:

**Proposition V.20.** If $n - l = 2$ then the Legendrian singularities occur in at most 2 parameter families in $\mathbb{R}^4$. 
V.4.10. $n - l > 2$. After the previous longer treatments of examples we will now be brief. If the singularities on the individual wavefronts correspond to germs $f_i$ then the singularity of the conflict set is the germ $\sum f_i$. (In the $A_2A_2A_2$ case the $f_i$ were $s_1^3$, $s_2^3$ and $s_3^3$.)

Our definition of codimension was not the usual one. What most authors call the codimension we will call the multiplicity. It is

$$\dim_{\mathbb{R}} \frac{C^\infty(s_0)}{C^\infty(s_0) \left( \frac{\partial f}{\partial s} \right)}$$

All the germs we consider are quasi-homogeneous, hence we have:

**Lemma V.21.**

$$\text{codim} \left( \sum_{i=1}^{l} f_i \right) = \prod_{i=1}^{l} \text{codim}(f_i)$$

**Proof.** As the $f_i$ can be given a normal form where they are quasi-homogeneous. If their weights are $(\alpha_1, \ldots, \alpha_{\text{corank}(f_i)})$ then their multiplicity is

$$\prod_{i=1}^{\text{corank}(f_i)} \left( \frac{1}{\alpha_i} - 1 \right)$$

The germ $\sum f_i$ associated to the conflict set thus has as multiplicity the product of the multiplicities of $f_i$. But in the case of quasi-homogeneous germs the multiplicity equals the codimension (i.e. Milnor and Tjurina number coincide.) □

The general picture sketched above for the examples $A_2A_2$, $A_2A_3$ and $A_2A_2A_2$ is that the big wavefronts can be assumed to have $l - 1$ equations

$$x_0 = f_1(s_1) + x_1 + R_1(x, s_1)$$
$$\cdots$$
$$x_0 = f_{l-1}(s_{l-1}) + x_{l-1} + R_2(x, s_2)$$
$$x_0 = f_l(s_l) - x_1 + R_l(x, s_l)$$

(V.26)

Here the $R_i$ have no $x$-variables in common. The $s_i$-variables look like

$$s_i = (s_{i,1}, \ldots, s_{i,\text{corank}(f_i)})$$

In $R_i$ we will meet at least $\text{corank}(f_i)$ different $x$-variables.

The $R_i$ split up in terms linear and non-linear in $s_i$. We have for instance:

$$R_1 = x_1s_{1,1} + \cdots + x_{l+\text{corank}(f_i)-1}s_{1,\text{corank}(f_i)} + S_1(x, s_1)$$

The terms linear in $s_1$ assure that the rank condition is not violated. The term $S_1$ has none of the $x$ coordinates that appear in the linear terms. The total $s_1$ degree of the $S_1$ term is strictly higher than 1, i.e. $S_1(x, s_1)$ contains no terms linear in the $s_1$. For the other $R_i$ we have a similar normal form. No two of the $R_i$ have any $x$-variables in common.

A normal form as in (V.26) assures that all the equisingularity manifolds intersect transversely.

The non-versal unfoldings we obtain always contain for all $s$ variables the terms $x_is_i$. Also they contain the constant term. When studying the versal unfolding one distinguishes between the basis elements $e_i$ and the elements $J_i$. The $J_i$ are those monomials that do not
affect the multiplicity, their weighted degree is $\geq 1$. In our non-versal unfolding the monomials $J_i$ do not occur. The number of monomials $\#J$ is called the inner modality.

If we then carry out $n - l + 1$ section and projection steps - as described in section III.2.17 - we get a non-degenerate phase function with germ $\sum f_i$ at $x = 0$.

Let us compare the versal unfolding of the germs $\sum f_i$ to the non-versal unfolding we get. We will take the liberty of speaking of “the basis” for a versal unfolding even though there is no unique or canonical basis.

Following §13.2 in [AGZV85] the number $\#J$ of monomials $J$ is equal to the modality, for the quasi-homogeneous germs that we have. Hence an upper estimate for the number of parameters necessary for a wavefront to occur in a family is the codimension minus $\#J$.

Hence if we know that a conflict set is a front in $\mathbb{R}^{n-l+2}$, the number of parameters to obtain such a front is less or equal to

$$\text{codim}(\sum f_i) - \#J - (n - l + 2)$$

V.4.11. $n - l = 3$. With $l = 2$ and thus $n = 5$ we have as a first case the $(4, 2)$ stratum. This leads to at least three cases: $(D_4^\pm, A_2)$ and $(A_4, A_2)$. We have to study the $D_4^\pm$ versal unfolding. Unfoldings for the umbilics $D_4^\pm$ are:

$$D_4^+ : \quad 0 = s_1^3 + s_2^3 + A s_1 s_2 + B s_1 + C s_2 + D$$

and

$$D_4^- : \quad 0 = s_1^3 - 3 s_1 s_2^2 + A (s_1^2 + s_2^2) + B s_1 + C s_2 + D$$

The limit of the tangent planes at $0 \in \mathbb{R}^4$ to the wavefronts is $D = 0$. Big wavefronts we could choose in order to have transversal intersections of strata are

$$D_4^- : \quad x_0 = s_1^3 - 3 s_1 s_2^2 + x_1 (s_1^2 + s_2^2) + x_2 s_1 + x_3 s_2 + x_4$$

$$A_2 : \quad x_0 = s_3^3 + x_5 s_3 - x_4$$

The germ $s_1^3 - 3 s_1 s_2^2 - s_3^3$ has codimension 8. The corresponding conflict set is a hypersurface in $\mathbb{R}^5$.

The number of $J$-polynomials is in $(D_4^+ A_2)$-cases 1. We see that the $(D_4^+ A_2)$ arise in 2-parameter families of fronts in $\mathbb{R}^5$. Note that though $D_4^+ A_2$ results in the same germ as $A_2 A_2 A_2$ they have non-isomorphic unfoldings and thus according to theorem V.7 their wavefronts are not diffeomorphic.

On the $(4, 2)$-stratum we also have $(A_4 A_2)$. The germ we get is $s_1^3 - s_3^3$. This is still a simple singularity, namely $E_8$. Its codimension is 8. Here there are no $J$ polynomials. We need 3 parameters.

The following case is $(3, 3)$. It also occurs in $\mathbb{R}^5$. The germ becomes $s_1^3 + s_2^3$. Its codimension is 9. This is the singularity $X_9$ as the modality is 1 it will first occur generically as a 7-dimensional front in $\mathbb{R}^5$. Hence we need a 3 parameter family in $\mathbb{R}^5$.

In $\mathbb{R}^6$ we will meet $(3, 2, 2)$ for the first time. The germ is $s_1^3 + s_2^3 + s_3^3$. It has codimension 12. Its modality is 1. Hence this singularity happens in $12 - 1 - (3 + 2) = 6$ parameter families in $\mathbb{R}^5$.

In $\mathbb{R}^7$ we find a corank 4 singularity $(2, 2, 2, 2)$. It has codimension 16. Hence there are five elements of the basis of a versal unfolding whose weight is equal to or exceeds 1. The monomials $J_1$ to $J_5$ are

$s_1 s_2 s_3, s_2 s_3 s_4, s_1 s_3 s_4, s_1 s_2 s_4, s_1 s_2 s_3 s_4$
Hence the modality is 5 and this singularity happens in $16 - 5 - 5 = 6$ parameter families of fronts in $\mathbb{R}^5$.

**Proposition V.22.** If $n - l = 3$ then singularities of the conflict set happen in at most 6 parameter families of fronts in $\mathbb{R}^5$.

**V.4.12.** $n - l = 4$. The first new case is $(5, 2)$. Here we have a corank 3 germ for $D_5 A_2$. We also have corank 2 with $A_5 A_2$. In both cases the modality is 1 and the codimension 10. Hence these occur in 3 parameter families of fronts in $\mathbb{R}^6$. The germ we call $D_5 A_2$ is also known as $Q_{10}$ in the list of Arnold.

Then comes $n = 7$ with $(4, 2, 2)$. We can have three different triples of strata: $A_4 A_2 A_2$ and $D_{A_4}^{\pm} A_2 A_2$. The codimension here is 16. The germ $A_4 A_2 A_2$ has modality 2. The germ $D_{A_4}^{\pm} A_2 A_2$ has modality 5. Hence we expect $A_4 A_2 A_2$ in 8 parameter families of fronts in $\mathbb{R}^6$ and for $D_{A_4}^{\pm} A_2 A_2$ we need 5 parameters in $\mathbb{R}^6$.

A different corank 4 case comes with $n = 8$ and $(3, 2, 2, 2)$. This has codimension 24. It is of weighted degree 1 with weights $(\alpha_1, \cdots, \alpha_4) = (1/4, 1/3, 1/3, 1/3)$. The basis of the local algebra contains 6 monomials that are of weighted degree $\geq 1$. Hence we see that this singularity happens in 12 parameter families of fronts in $\mathbb{R}^6$.

Finally, there is the most singular one, which has codimension 32 with $n = 9$ and $l = 2 = 5$.

We calculate the stratum for which the multiplicity remains constant. This consist of 16 basis vectors in the local algebra. In this case $n - l = 4$ and $n - l + 2 = 6$ so that another 6 parameters are missing.

**Proposition V.23.** If $n - l = 4$ then the singularity of the conflict set appear in at most 12-parameter families in $\mathbb{R}^6$. 
## List of notations

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Samenvatting in het nederlands

Wat voorkennis


Het is gemakkelijk in te zien dat de symmetrieverzameling van een parabool een halve lijn is. De caustiek van een parabool is al een gecompliceerder verzameling. Een golffront dat van een parabool vertrekt en naar binnen beweegt is in eerste instantie nog glad. Echter na korte tijd al ontstaat er een knik in het golffront. Die knik gaat over in een zelfdoorsnijding en twee scherpe kanten. De ontwikkeling van het naar binnen bewegende golffront is dus: \[ \bigcup_{-} \bigcup_{-} \bigcup_{-} \bigcup_{-} \].

Een scherpe kant van een oppervlak of een kromme heet ook wel een singulier punt of een singulariteit. Zelfdoorsnijdingen heten ook wel multi-singulariteiten. De singulariteit op de caustiek en die op het golffront komende van een parabool heet een cusp, of spits. Al deze cuspen kunnen glad afgebeeld worden op de kromme \( y^3 = x^2 \).

Neem een willekeurig oppervlak en beschouw een golffront dat zich uitbreidt: hoe kunnen de singuliere punten eruit zien? Het antwoord op deze vraag is eenvoudig: hoe je maar wilt. Dat is als volgt in te zien.

Neem een willekeurig vreemd oppervlak met allerhande zelfdoorsnijdingen en scherpe kanten. Plaats op ieder punt van dat oppervlak een bol met straal 1. De gezamenlijke rand, of omhullende van al die bollen, is ook weer een oppervlak. Laat nu van de omhullende een golffront vertrekken. Dan is de golf na tijd 1 weer terug op het willekeurig gekozen oppervlak, met de willekeurige scherpe kanten.

Met wat meer moeite kan min of meer hetzelfde aangetoond worden voor de symmetrieverzameling en de focale verzameling. Ieder willekeurig gekozen oppervlak met ingewikkelde scherpe kanten en zelfdoorsnijdingen kan optreden als symmetrieverzameling of caustiek. Er lijkt geen enkele beperking te zijn. Het is des te verbazingwekkender dat, mits de dimensie van de ruimte waarin de golf zich voortplant kleiner dan 7 is, er in redelijk sterke zin slechts een stuk of tien soorten singulariteiten op golffronten bestaan.

Voor dit sterke resultaat over golffronten is het nodig een precies begrip te hebben van wat een golffront is. Ieder golffront beweegt zich voort. Op ieder punt van een golffront is er dus een goed gedefinieerde richting, die niet aan het golffront mag raken. Aan de andere kant is er de ruimte waarin het golffront zich voortplant, uitbreidt. Voeg nu aan ieder punt van deze ruimte alle richtingen toe waarin een golffront zich kan voortplanten. De nieuwe ruimte is de eenheidslengte coraakbundel. Een golffront kan gekarakteriseerd worden als de projectie van een bepaald type glad oppervlak in die eenheidslengte coraakbundel. Deze speciale gladde oppervlakken in de eenheidslengte coraakbundel heten Legendre variëteiten. Op de omslag van dit proefschrift staat een Legendre-variëteit voor de spits.
The geometry of conflict sets

Martijn van Manen

Uitnodiging
voor het bijwonen van de openbare verdediging van mijn proefschrift getiteld
“The geometry of conflict sets”
op maandag 8 september 2003
om 12.45 in de
Senaatszaal van het
Academiegebouw,
Domplein 29 te Utrecht

Na afloop is er een
receptie.

Martijn van Manen
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**Conflictverzamelingen**

Een conflictverzameling is de verzameling van punten die op gelijke afstand liggen van een aantal gegeven oppervlakken. Conflictverzamelingen in het vlak zijn het gemakkelijkst voor te stellen. De conflictverzameling van twee lijnen bestaat uit weer twee lijnen. De conflictverzameling van een cirkel en een lijn is een parabool, de conflictverzameling van twee cirkels is een aantal hyperbolen en/of ellipsen. Bekijk figuur I.1 voor wat voorbeelden.

Laat nu \( n \) de dimensie zijn van de ruimte \( X \) waarin gladde hyperoppervlakken \( M_1 \) tot \( M_l \) liggen. Neem tevens voor het gemak aan dat de ruimte waarin de oppervlakken liggen niet gekromd is. Noem de conflictverzameling van \( M_1 \) tot \( M_l \) in \( X M_n \).

In het algemeen bestaan conflictverzamelingen uit een aantal oppervlakken. Voor ieder punt \( q \) van de conflictverzameling ligt er een aantal punten op \( M_j \) die basispunten heten. De normaal vanuit een basispunt op \( M_j \) loopt door het punt \( q \) op de conflictverzameling. Voor ieder basispunt op ieder van de \( M_j \) is die afstand gelijk.

Het is ook mogelijk algemener afstandsfuncties toe te laten. Dan staat bijvoorbeeld de afstand tot \( q \) vanaf basispunten op \( M_j \) als \( \frac{\lambda_j}{\lambda_i} \) tot de afstand vanaf basispunten op \( M_i \). In formules uitgedrukt:

\[
\frac{\lambda_j}{\lambda_i} \text{afst}(p_i, q) = \text{afst}(p_j, q)
\]

waar \( p_i \) een basispunt op \( M_i \) is en \( p_j \) een basispunt op \( M_j \). Door deze verhoudingen te varieren ontstaat er een hele familie conflictverzamelingen.

Het voornaamste resultaat van dit proefschrift beschrijft de aard van een conflictverzameling van niet al te speciale oppervlakken. Het blijkt dat gegeven afstandsfuncties en “generieke” basis oppervlakken \( M_i \) conflictverzamelingen de bovengenoemde “Legendre”-eigenschap, karakteristiek voor golffronten, hebben en dat als het verschil \( n - l \) tussen de dimensie \( n \) van de omhullende ruimte \( X \) en het aantal oppervlakken \( l \) niet meer dan 4 is er op zulke generieke conflictverzameling op gladde equivalentie na slechts eindig veel verschillende singulariteiten bestaan. We bewijzen dus een analogon van de bekende classificatiestellingen voor golffronten, brandpuntverzamelingen en symmetrieverzamelingen.

In het vlak komt in bovenstaande familie van conflictverzamelingen slechts in geïsoleerde gevallen een niet Legendre punt voor.

Naast dit grotere resultaat worden er in dit proefschrift tal van kleinere en eenvoudiger dingen bewezen. In hoofdstuk 4 komen allerlei generalisaties en variaties op het begrip conflictverzameling aan bod. In de hoofdstukken 1 en 2 worden een aantal krommingsformules bewezen.

**Twee meetkundige constructies m.b.t. tot conflictverzamelingen**

Om na het tohu-bohu van de vorige paragraaf de niet-ingewijden toch nog een beetje een idee te geven van de zaken die in dit proefschrift aan de orde komen noem ik nu hier nog twee constructies, die beide iets zeggen over de krommingsformules van de eerste twee hoofdstukken.

Uit de geometrische optica is een formule bekend die beschrijft waar het brandpunt van een in een spiegel gereflecteerde stralenbundel komt te liggen, indien gegeven zijn de hoek van inval van de stralenbundel en de kromming van de spiegel.

De transformatie van een stralenbundel in zijn gereflecteerde is iets wat vaker bestudeerd wordt in de wiskunde. Meetkundig gezien is er weinig verschil tussen een lichtstraal die een spiegel raakt en een biljartbal die de rand van de biljarttafel raakt. Zo bestaat er een
uitgebreide hoeveelheid wiskundige theorie over katsende biljartballen en de baan die ze afleggen op een biljarttafel.

Om de baan van de gereflecteerde lichtstraal te bepalen construeert men het virtuele beeld van de lichtbron achter de spiegel. Dit virtuele beeld geniet in de wiskunde enige bekendheid als de “orthomtic”. In een omgekeerde wereld is de spiegel de conflictverzameling van de lichtbron en het virtuele beeld. De formule uit de geometrisch optica blijkt dus iets te zeggen over de kromming van conflictverzamelingen.

Een maat voor de kromming van een kromme is $\varepsilon$ gedeeld door de straal van de best rakende cirkel. Een cirkel met straal 2 heeft bijvoorbeeld een constante kromming van een half. We gaan nu uitgaande van de lichtbron en het virtuele beeld de best rakende cirkel aan de spiegel ofwel de conflictverzameling construeren, m.a.w. uitgaande van twee objecten gaan we de best rakende cirkel aan de conflictverzameling construeren.

Dat blijkt te kunnen met een klassieke constructie, die van de harmonische dubbelverhouding. Hoe dat werkt is te zien in de figuur op de achterkant van dit boekje. Het punt $O'$ ligt op de conflictverzameling van de twee gestippelde cirkels want er is een cirkel met middelpunt $O'$ die raakt aan beide gestippelde cirkels. De normaal aan de conflictverzameling is de lijn die de hoek tussen de twee normalen vanuit $A$ en $B$ naar $O'$ in tweeën deelt.

Laat nu de punten $A$ en $B$ neer op de normaal aan de conflictverzameling als in de figuur aangegeven. Dan ontstaan er op de normaal aan de conflictverzameling drie punten. Het vierde punt $C$ is nu het unieke punt zodanig dat de paren $(A', B')$ en $(O', C)$ de harmonische dubbelverhouding hebben, i.e. $A'C/A'O' = -B'C/B'O'$. De cirkel met middelpunt $C$ door $O'$ blijkt de best rakende cirkel aan de conflictverzameling te zijn, zoals gegeven door genoemde formule uit de geometrische optica. In het eerste hoofdstuk van dit proefschrift staan formules die het algemene n-dimensionale geval behandelen.

Het punt $C$ kan ook geconstrueerd worden met behulp van inversie. Neem de cirkel $c$ door $A'$ en $B'$ met middelpunt op de lijn $A'B'$. Inversie door een cirkel of door een bolschil is de afbeelding die alles binnenstebuiten keert: het middelpunt gaat naar oneindig, de cirkel of bolschil zelf blijft op zijn plaats en andere punten worden afgebeeld als in onderstaande figuur.

De definitie van inverse: het beeld onder inversie van $O'$ is $C$ en vice-versa.

De tweede constructie die aan bod komt gaat meer over het behandelde in hoofdstuk 2. In dat hoofdstuk staan formules die de best rakende bol vinden in het geval dat er drie basis oppervlakken in $\mathbb{R}^3$ zijn. In de figuur 1 zien we de eenvoudigste constellatie met drie basisoppervlakken. In dit geval blijkt het mogelijk om met inversie in te zien wat de conflictverzameling is. Laat de drie bollen uitdijen tot een punt waar twee van de drie bollen raken. In dat punt waar twee van de drie bollen raken plaatsen we een vierde bolschil waardoor
Figuur .1. Drie bollen: wat is de conflictverzameling?

we de drie bollen inverteren. Als we dat doen ontstaat figuur .2. De inversie afbeelding is

Figuur .2. Dezelfde drie bollen, nu geïnverteerd

een gladde één-op-één afbeelding buiten het centrum van de bolschil waardoor de inversie plaatsvindt. Dat betekent in het bijzonder dat de beelden van elkaar rakende objecten elkaar raken. Een vijfde bolschil die raakt aan de drie bolschillen van figuur .1 heeft als beeld dus een bolschil die raakt aan de twee vlakken en de bol in figuur .2. Maar het middelpunt van de vijfde bolschil is een punt van de conflictverzameling. Dus is het beeld onder inversie van de conflictverzameling een cirkel die zweeft tussen de twee vlakken en draait om de bolschil van figuur .2. De conflict verzameling zelf is dus ook een kromme die in een vlak ligt, voor een plaatje zie figuur II.2.
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Michiel, kijk! Je staat in een boekje, daar waar je vader had moeten staan.

Souki, entre les lignes t’as ajouté beaucoup.

Gedurende zijn studietijd was hij lid van verscheidene besturen en commissies. Hij was onder andere voorzitter van de studievereniging WEIS en lid van het faculteitsbestuur. In het studiejaar 1993-1994 volgde hij een aantal vakken maitrise en behaalde hij het diplôme d’études approfondies mathématiques pures aan de USTL te Lille, Frankrijk.