The geometry of conflict sets

De meetkunde van conflict verzamelingen
(met een samenvatting in het Nederlands)

Proefschrift

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Pourquoi donc, reprit le Sirien, citez-vous un certain Aristote en grec ? – C'est, répliqua le savant, qu'il faut bien citer ce qu'on ne comprend point du tout dans la langue qu'on entend le moins.

Voltaire, Micromegas

Schüler:	Kann Euch nicht eben ganz verstehen.
Mephistopheles:	Das wird nächstens schon besser gehen, Wenn Ihr lernt alles reduzieren Und gehörig klassifizieren.

Goethe, Faust

Preface

Sets in the middle

Conflict sets are sets where two wavefronts coming from different objects meet. To see a conflict set throw not one, but two stones in a pond. The interference pattern you get is a conflict set.
We will study how these are sets are curved, if they are smooth at all. When they are not smooth we will study their singularities. The main theme is that conflict sets are very much like wavefronts themselves.
The study of conflict sets was motivated by some very tangible geometric notions. The first of these is the notion of a Voronoi diagram. A Voronoi diagram is a division of the plane in different regions $\{V_i\}_{i=1,\cdots,l}$, such that for $x \in V_j$ the closest of a number of points $\{p_i\}_{i=1,\cdots,l}$ is p_j. Voronoi diagrams arise in many sciences, for instance the p_i can be roots of plants and the V_i the regions that each of the plants can take their water from.
The second of these is the idea of a skeletal set describing the form of an object in n-space. If we have a closed compact hypersurface M in $\mathbb{R}^n$, for instance the surface of a dog bone, the distance from some interior point to M can have non-unique absolute minima: the minimal distance is d_0 and there are at least two points on M for which it is attained. Normally, there is just one absolute minimum and thus the locus of points where there are two minima forms a codimension one subset in the interior of M. The set so obtained is commonly called, the medial axis or the central set. The two concepts are related as we can see from figure 1.
For the definition of the conflict set we will replace the points p_i in the definition of the

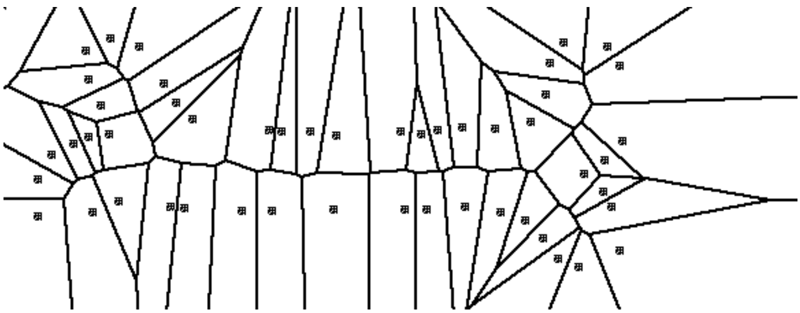

FIGURE .1. The medial axis of a dog bone approximated by a Voronoi diagram

Voronoi diagram by compact manifolds M_i and the critical values of the distance function are no longer required to be absolute minima. In this way we will obtain a set in the middle: the conflict set.

Generic differential geometry

Let M be an embedded manifold in an Riemannian manifold X. In differential geometry one studies the properties of such an embedding that are invariant under isometries of the ambient manifold X.

In this way a lot of results can be obtained. Even more can be said when one restricts attention to certain embeddings that we call "generic". As the name suggests most embeddings are generic. Genericity is defined using transversality conditions. For instance, a generic line in $\mathbb{R}^3$, intersects the XY-plane in a single point. A line contained in the XY-plane can be rotated slightly in space. The rotated line intersects the XY-plane in just one point.
Another example: most quadruples of points span a positive volume. Again, if the points do not span up a positive volume we can move one just a little bit and obtain four points that do span a positive volume.
What the transversality condition actually is varies from problem to problem but it is a method by which strong results can be obtained under mild conditions.
Here is one such result, concerning the envelope of the normal lines from a curve in the plane. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an embedding of the circle in the plane. We may think of an ellipse. At every point s_0 the curvature $\kappa(s_0)$ and the normal $\nu(s_0)$ are defined. The focal set or evolute, or caustic, is defined as the set of points in the plane traced out by $\gamma(s) + \kappa^{-1}(s)\nu(s)$. It is the envelope of the normals to the curve γ. Let us mention a well-known theorem.

THEOREM .1. *For a generic embedding γ the focal set F has the following property: every point $p \in F$ has a neighborhood in $p \in U \subset \mathbb{R}^2$ such that the intersection $U \cap F$ is diffeomorphic to one of the following algebraic varieties $x_2 = 0$, $x_1x_2 = 0$ or $x_2^3 - x_1^2 = 0$.*

In this theorem we have characterized all generic local forms of the focal set. Locally only a few singular situations can arise. In chapter four we obtain a similar characterization of conflict sets in low dimensions.

Cut locus, Maxwell strata, central set, medial axis, symmetry set

In generic differential geometry one often studies singular sets which are associated to a distance function from a submanifold and which measure some sort of symmetry.
The most well-known of these sets is probably the cut-locus associated to a Riemannian manifold X and a point p. In a neighborhood U of $0 \in T_pX$ the exponential map

$$\exp\colon T_pX \mapsto X$$

is a diffeomorphism: small vectors are mapped to geodesics that are globally minimizing. However, what is locally the shortest path need not be the shortest path globally.
For each $v \in T_pX$ with $\|v\| = 1$ let $t = t(v)$ be the largest number such that for all $0 \leq s_0 \leq t(v)$ the path

$$\{s \in [0, s_0] \mid \exp(p, sv)\}$$

is the shortest path from p to $\exp(p, s_0 v)$. Clearly $t(v)$ might be infinite. If $t(v)$ is finite the point $\exp(p, t(v)v)$ is called the cut point of p, v.
All cut points together form the cut-locus of X wrt. p. At points of the cut-locus the exponential map $\exp(p, \cdot)$ is no longer injective or its differential is no longer injective. If X is compact the cut-locus is a deformation retract of $X \setminus \{p\}$.
Another such set is related to an embedded manifold $M \subset X$, where X is supplied with a Riemannian metric. For $(p, v) \in NM$, where $\|v\|$ is small, that a neighborhood of p, v is mapped diffeomorphically to X. Where the exponential map is no longer a injective or immersive we have a locus called the central set or medial axis. It is what Thom called the "cut-locus d'une variété plongée", see [**Tho72**]. If M is a parabola and X is the plane it lies in we can see that the central set measures symmetry.
The central set of an ellipse in the plane consists of just one line segment. But the ellipse has two symmetry axes. So how do we incorporate the other axis?

The other axis is contained in the symmetry set. The symmetry set is the closure of those points $p \in X$ where the distance function $M \ni q \to d(p,q)$ has two non-degenerate critical points with the same critical value. When we consider the symmetry set of the ellipse in the plane we will get the desired two line segments.
There is a drawback here. The definitions of cut-locus and central set presuppose that we work in an ambient manifold X where the unit speed geodesics are defined for all times t. We ask that X is geodesically complete.
Examples of geodesically complete spaces are compact Riemannian manifolds, say an ellipsoid $\subset \mathbb{R}^3$, with three different axes. A geodesic on an ellipsoid can be non-periodic. In fact most geodesics on ellipsoids are non-periodic. Hence, symmetry sets of curves on the ellipsoid might become quite awkward, because not just the absolute minimum but every critical value is considered. Completeness of X is not enough for the definition of symmetry set set to give reasonable results.
To avoid unwanted behavior rather strict conditions have to be imposed on the ambient spaces that we consider. In this thesis these conditions are stated in section III.1.9. Basically X is assumed to be complete and all points in X are assumed to have an empty cut-locus.

It is not only in this respect that the symmetry set is different from the central set. Lo-

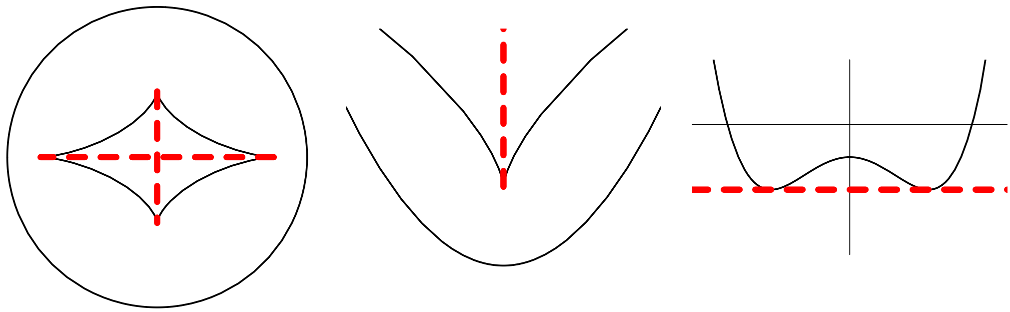

FIGURE .2. Two symmetry sets and the graph of a distance function having two critical points with the same critical value

cally the two can exhibit quite different behaviour. Consider a compact manifold without boundary in $\mathbb{R}^3$. Suppose the surface is the surface of a generic smooth dog bone. At one of the sides we will find a point where the central set looks like we can see at the left hand side of figure 3. There are $\binom{4}{2} = 6$ planes intersecting in 4 lines meeting at a point. Centered at that point of the central set there is a sphere completely contained inside the dog bone that touches the surface of the dog bone in four points. Such a singularity happens generically. It cannot be removed by deforming the dog bone. However the symmetry set looks quite different at such a point. On the symmetry set the six planes continue: there are still critical values but they are no longer absolute minima.
Both the symmetry set and the central set can have what we call endpoints. Endpoints are points where the axes of symmetry of the ellipse stop. Such endpoints are really singular points of a focal set. Though interesting, we will not consider these endpoints here. Instead we will measure symmetry between several manifolds. We take l manifolds M_i in our ambient manifold X and define the conflict set as the closure of those points $p \in X$ for which

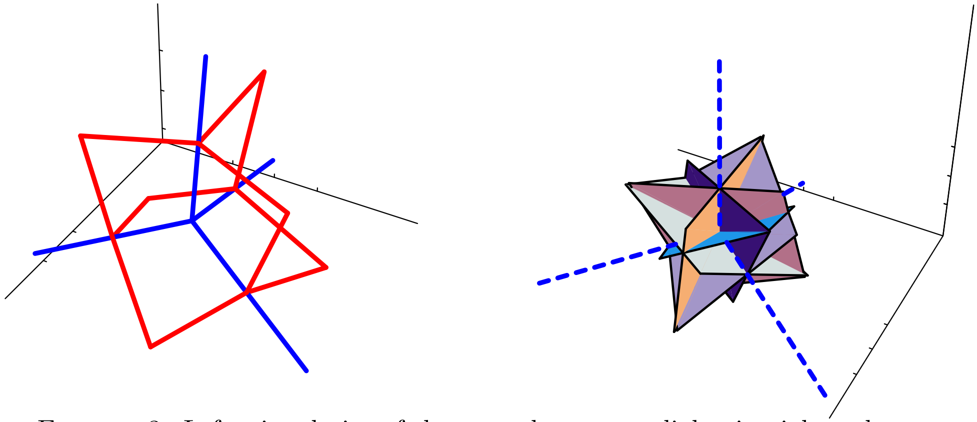

FIGURE .3. Left: singularity of the central set or medial axis, right: what the symmetry set looks like at such a point

there exist $q_i \in M_i$ such that the distance functions

$$d_{i,p} \colon M_i \to \mathbb{R} \qquad q_i \mapsto d_i(p, q_i)$$

all have a non-degenerate critical point and *at the same level*, that is, with the same critical value. Such points q form the conflict set. Pictures of the conflict set are in the figures I.1, II.2 and a singular one in figure III.5.
To develop some intuition for the differences between the conflict set and the symmetry set consider again the example of figure 3. The sphere that touches four patches of surface can come up in three different types of conflict sets. First of all there is the conflict set of four (compact) surfaces in $\mathbb{R}^3$. Such a conflict set generically contains only isolated points. Hence when $l = 4$ and $\dim X = 3$ figure 3 reduces to the intersection point of six planes.
In the setting of three surfaces in 3-space their conflict set is a line. Figure 3 will happen when two of the four points lie on one surface M_1 and the other two lie each on a different surface: M_2 and M_3. The symmetry set of M_1 and the conflict set of M_1, M_2 and M_3 intersect. Figure 3 reduces to two intersecting curves.
In the setting of two surfaces in 3-space we can get a sphere touching four pieces of surface when two symmetry sets meet on a conflict set. From the six planes in figure 3 four remain. Hence, there is a correspondence between singularities of central sets, symmetry sets and conflict sets, but there are also differences.

Symplectic and contact geometry

The focal set we introduced above is an instance of a much studied class of manifolds: Lagrangian submanifolds of a symplectic manifold. We will use Lagrangian manifolds to study the singularities of conflict sets.
Lagrangian submanifolds arise as one tries to solve first order partial differential equations. The standard example of such a PDE is one that describes equidistants to a submanifold. If $u \colon X \to \mathbb{R}$ is the function that describes the distance from a point x to a submanifold $M \subset X$ then at each point $x \in X$ we will have that the $\frac{\partial u}{\partial x}$ are unit length normal vectors. Thus we would need that

$$\|\frac{\partial u}{\partial x}\| = 1 \tag{.1}$$

Equation (1) describes a PDE, with initial condition $u_{|M} = 0$.
The solutions of the PDE are well-defined except at points of the caustic or the symmetry set. Hence the function that describes the distance to a submanifold is not really a function: we consider not the function but what would be the graph of $\frac{\partial u}{\partial x}$, this is a Lagrangian submanifold of T^*X, or the graph of $u, \frac{\partial u}{\partial x}$, this is a Legendrian submanifold in $J^1(X, \mathbb{R})$. If we project these graphs to X the singular points of the projection form the focal set.
Lagrangian and Legendrian manifolds are usually constructed in a down to earth way. Let $d\colon X \times X \to \mathbb{R}$ be a metric on X. If $\gamma\colon M \to X$ is an embedding of M then the distance from $s \in M$ to $x \in X$ is $F(x, s) = d(\gamma(s), x)$. The distance function has a critical point where

$$\frac{\partial F}{\partial s} = \frac{\partial d(\gamma(s), x)}{\partial s} = 0 \tag{.2}$$

Then where (2) holds the derivative of the distance "function" u above is $\frac{\partial F}{\partial x}$.
In general if we have a family of functions $F(x, s)$ such that the set $\frac{\partial F}{\partial s} = 0$ still behaves reasonably - i.e. it is a manifold - then the "graph" over X of $\frac{\partial F}{\partial x}$ is

$$\Lambda_F = \{x, \frac{\partial F}{\partial x} \mid \frac{\partial F}{\partial s} = 0\}.$$

This "graph" is called a Lagrangian manifold. In chapter three we will elaborate on these remarks. It turns out that for conflict sets such a family of functions exists as well and generically it behaves well.

Outline of the results

The contents and results of this thesis are as follows.

- In chapter one we will define the conflict set of l oriented hypersurfaces in $\mathbb{R}^n$. A remark concerning the smoothness of the conflict set is made. In chapters one and two we will study curvature and torsion properties of conflict sets, in terms of the properties of the base manifolds M_i. We will prove a formula for the curvature in the case where $l = 2$, and where $l = n = 3$. We will indicate how to proceed in the case $l = n > 3$. Chapter two concludes with a long computation that through an example proves the curvature formula for $l = n = 3$ in another way.
- The torsion of a conflict set is calculated in chapter two. The fact that in the case of spheres it is generically zero is the subject of a theorem in chapter 1.
- In chapter three we will prove that the singularities of conflict sets generically are of the same nature as the singularities of wave fronts. We also discuss what happens when the singularity of the conflict set is non-generic. The chapter starts with a review of the results from symplectic and contact geometry that we need. Part of the review is a slightly novel treatment of the Gauss map. We also use the language of symplectic geometry to rephrase some of the results of chapter one.
- In chapter four we discuss some notions similar to the conflict set. We introduce the center set, and show that it generically has the structure of a wavefront as well. We also discuss its relation to the center symmetry set introduced by Janeczko. In the last section we relate the notions of conflict set and center set to existing notions such as orthomtic, billiard transformation and pedal curve.
- In chapter five we classify singularities of conflict sets. In low dimensions local models for the conflict set are fabricated using non-versal deformations of sums of the well-known ADE-singularities. We determine explicitly in which dimensions there is a finite list

of singularities. The methods employed here are very similar to ones used to classify singularities of caustics and wavefronts. We thus try to put our results in that perspective, reviewing some of these things. It turns out that our analysis still gives rise to some surprises. For instance from the generic point of view the singularities of the conflict set of n surfaces in $\mathbb{R}^n$ are the same as those of 3 surfaces in $\mathbb{R}^3$.

The reader is notified that a shorter version of chapters one and two is in [**vM03**]. Chapters three, four and five are an extension and improvement of [**vM**]. Chapter 3 was the subject of talks of the author at conferences in the second half of 2000 in Cambridge and Liverpool.

I Curvature for two surfaces

I.1. Introduction

Let $M_1, \cdots, M_l$ be l manifolds in $\mathbb{R}^n$, all of smoothness $\mathcal{C}^j$ with $j \geq 1$ and of dimension $n-1$. Suppose the M_i are orientable. The orientations supply us with $n_1, \ldots, n_l$, unit sections of the conormal bundle $N^*(M_i)$ of the M_i in $\mathbb{R}^n$. We can speak of wavefronts that emanate from the l manifolds in distinct directions as "time" increases or decreases.
The **conflict set** M_c of $(M_1, n_1), (M_2, n_2), \ldots, (M_l, n_l)$ is defined as the set of points where these wavefronts meet. The **conflict set** of $M_1, M_2, \ldots, M_l$ is defined as the union of all the conflict sets associated to particular configurations of orientations. The **symmetry set** of (M_i, n_i) is defined as the set of points traced out by the self intersections of the wavefronts of M_i.
Given the second fundamental forms of M_1 and M_2, can we determine the second fundamental form $\mathbf{I\!I}_c$ of M_c?
In the next few theorems and their proofs we will assume that the distance function for each of the k manifolds has only regular points on the conflict set and that - again on the conflict set - the wavefronts travel in different directions, or if they travel in the same direction, they should do so at different speeds. This is to say that the sphere with center c on the conflict set and touching one of the M_i, touches at one point p only and c is not one of the centers of curvature of p. A more precise notion is provided in paragraph I.3. The regular part of the conflict set makes up what one might call the A_1^l stratum. It consists of those points that do not correspond to ones on the symmetry and/or focal set of the M_i.

THEOREM I.1. *Let M_i, $i = 1, \cdots, l \leq n$, be hypersurfaces in $\mathbb{R}^n$ of smoothness $\mathcal{C}^j$, $j \geq 1$. Let wavefronts at constant speeds λ_i emanate from M_i. The conflict set M_c is at least $\mathcal{C}^j$ at regular points.*

THEOREM I.2. *If M_1 and M_2 have second fundamental forms $\mathbf{I\!I}_1$ and $\mathbf{I\!I}_2$ then M_c has second fundamental form $(2\cos\phi)^{-1}\mathbf{P}^T(\mathbf{I\!I}_1^* - \mathbf{I\!I}_2^*)\mathbf{P}$, where 2ϕ is the angle that the tangent spaces encompass, $\mathbf{I\!I}^*$ denotes the second fundamental form at the point where the wavefronts meet, and $\mathbf{P}\colon T_p\mathbb{R}^n \to T_pM_c$ is the projection - restricted to the tangent space of the wavefronts of the M_i. The two manifolds are required to have smoothness $\mathcal{C}^2$.*

THEOREM I.3. *Let wavefronts emanate from M_i, $i = 1, \cdots, l$, spheres with different radii in $\mathbb{R}^n$, $l \leq n$. Let the convex hull of the centers of these scaled spheres be at least $l-1$ dimensional. The A_1^l stratum of M_c is a conic section in an $(n-l+2)$-dimensional affine subspace of $\mathbb{R}^n$.*

EXAMPLE I.4. Consider two circles α_1 and α_2 in the plane, with centers $(0, b_1)$ and $(0, -b_2)$ and radii r_1, r_2, such that $0 < b_1 < b_2$ and $b_1 + b_2 \neq \pm r_1 \pm r_2$. For each configuration of normals a separate conflict set arises. In concordance with theorem I.3 the conflict set is with all configurations a conic section. See the next chapter for a more meaningful example.

In the papers [**Sie99**] and [**SSG99**] similar results concerning curves in $\mathbb{R}^2$ and convex hypersurfaces in $\mathbb{R}^3$ are proved. Our results however are more general and our method of proof is completely different. Similar calculations are in [**Ber95**] and [**BW59**]. In section IV.4 we will explain their relevance and discuss some applications of the curvature formula.
The main lemmas we use to prove theorem I.2 are reexamined at the end of chapter three. The proofs here are lengthy but elementary and thus may provide some more insight.

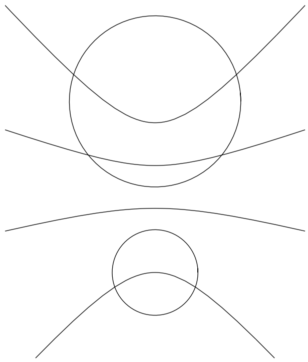

FIGURE I.1. The conflict set from example I.4

I.2. Preliminaries on contact between submanifolds

Let M and N be two manifolds in $\mathbb{R}^n$ both of dimension m. If $M \cap N$ is not empty we can study the contact between M and N at points p in $M \cap N$. Let σ_N be locally a submersion for N and ι_M be an immersion for M near p. The **contact map** for M and N at p is $\kappa_{N,M} = \sigma_N \circ \iota_M$. If $d^i\kappa_{N,M}$ is zero for $i < k$ then we say that M and N have **k-contact** at p.

The **κ-class** of $\kappa_{M,N}$ is the equivalence class of functions $f\colon \mathbb{R}^m \mapsto \mathbb{R}^{n-m}$ that are contact equivalent to $\kappa_{M,N}$. If K and L are also m-manifolds in $\mathbb{R}^n$ and they have such a function as contact map at q then there is a diffeomorphism ϕ carrying a neighborhood of p to a neighborhood of q and such that $\phi(M) = K$ and $\phi(N) = L$.

It is also possible to consider contacts between anequidimensional manifolds. Let M_1 have dimension m_1 and M_2 dimension m_2, and $m_1 < m_2$. Put

$$\kappa_{M_1,M_2} = \sigma_{M_1} \circ \iota_{M_2}\colon \mathbb{R}^{m_2} \mapsto \mathbb{R}^{n-m_1}$$

To compare two contacts one considers the κ-class of κ_{M_1,M_2}. A result of Montaldi says that κ_{M_1,M_2} is a trivial unfolding of κ_{M_2,M_1}, see [**Mon83**].

If M_1 and M_2, with $\dim M_1 \leq \dim M_2$, have at least 2-contact, then the tangent space of M_1 at p will be contained in $T_p(M_2)$. In this case we also employ the notion of $k+1$-contact in a certain direction V of $T_p(M_1)$. V will in general be a subspace of the tangent space.

DEFINITION I.5. M_1 and M_2 have $(k+1)$-contact, with $k > 0$ in the direction $V \subseteq (T_pM_1 \cap T_pM_2)$ if

(1) M_1 and M_2 have k-contact at the point p and,
(2) there is a third manifold L_1 with $T_pL_1 = V \neq (T_pM_1 \cup T_pM_2)$ and such that L_1 has $(k+1)$-contact with both M_1 and M_2.

Our main interest will be in two manifolds that have 3-contact at p in a certain direction. Here the second derivative of the contact map $d^2\kappa_{M_1,M_2}$ and its pushforward along ι_{M_1} is a bilinear form - possibly vector valued - on the tangent space of $\mathbb{R}^n$ at p with values in a euclidean space. We say that a vector $v \in T_p\mathbb{R}^n$ is **in the kernel of** $\iota^*\,\mathrm{d}^2\,\kappa_{M_1,M_2}$ if

$$v^{\mathrm{T}}\iota^* d^2\kappa_{M_1,M_2} v = 0.$$

EXAMPLE I.6. Let a hypersurface M in $\mathbb{R}^{n+1}$ be given by an immersion $\iota_M = (t, f(t))$, with $f\colon \mathbb{R}^n, 0 \mapsto \mathbb{R}, 0$. Let N be another hypersurface with immersion $\iota_N = (t, g(t))$ and $g\colon \mathbb{R}^n, 0 \mapsto \mathbb{R}, 0$. The hypersurfaces have k-contact iff. the first $k-1$ derivatives of f and g agree in 0.

LEMMA I.7. *Two manifolds M_1 and M_2 have 3-contact at the point p in the directions $W \subset (T_pM_1 \cap T_pM_2)$ iff. $v^{\mathrm{T}}\iota^* \mathrm{d}^2 f_i v = 0$, $\forall v \in W$ and for all components f_i of the contact map κ_{M_1,M_2}*

PROOF. "$\Longrightarrow$" Let L be the manifold that has 3-contact with both M_1 and M_2. Let $\dim L = l \le m_1 = \dim M_1$ and $m_1 \le m_2 = \dim M_2$. We can assume that $L = \mathbb{R}^l \times \{0\}$. Because, at $p = 0$ the tangent space of L is contained in the tangent space of M_i we can write the immersions of M_1 and M_2 as

$$\iota_{M_1} = (s_1, \cdots, s_{m_1}, f_{m_1+1}(s_1, \cdots, s_{m_1}), \cdots, f_n(s_1, \cdots, s_{m_1}))$$

and

$$\iota_{M_2} = (s_1, \cdots, s_{m_2}, g_{m_2+1}(s_1, \cdots, s_{m_2}), \cdots, g_n(s_1, \cdots, s_{m_2}))$$

Because

$$d^2\kappa_{L,M_2} = 0$$

and

$$d^2\kappa_{M_1,L} = 0$$

are zero by hypothesis it follows that

$$d_l^2\kappa_{M_1,M_2} = 0$$

in the point $p = 0$. Here d_l denotes the derivative with respect to $(x_1, \cdots, x_l)$. This is exactly what we needed to prove.

"$\Longleftarrow$" Choose an orthonormal basis for T_pM_1 so that W is spanned by $e_1, \ldots, e_k$, the first k basis vectors. L is then the manifold that is the image of

$$\iota_{M_1}(x_1, \ldots, x_k, 0, \ldots, 0).$$

□

From the proof it is clear that a similar statement holds for higher order contacts.

I.3. Proof of theorem I.1

If the speeds of the wavefronts are all equal we can think of spheres centered at the conflict set having at least 2-contact with each of the M_i. Note that there may be more than one sphere with center on the conflict set, i.e. the conflict set of oriented hypersurfaces may have self-intersections.

When the speeds of the wavefronts are different, that is λ_i, each of the spheres centered on the conflict set decomposes into l different spheres. The ratio of the radii of these spheres will be $\lambda_1 : \ldots : \lambda_l$.

The spheres in both cases will be called the **kissing spheres**, the terminology coming from the two-dimensional case. The points where the kissing spheres $(\alpha_j)_i$ of a point p on the conflict set M_c and M_i touches M_i will be called the **basepoints** p_j. At each basepoint we have a singularity type of the contact between the kissing sphere and the corresponding M_i. This is the same as the singularity type of the distance function $d(p, \cdot)$ on M_i. The following definition ensures that we are away from focal sets and also away from selfintersections.

DEFINITION I.8. A point p on the conflict set M_c is called regular if

- For each i the distance function $s \to d(p,s)$ $d\colon M_i \to \mathbb{R}$ is smooth and has a non-degenerate critical point at the basepoint $s = p_i$,
- The vectors
$$n_i^* = \left(\lambda_i, \frac{\partial d}{\partial x}_{|x=p}\right)$$
are linearly independent.

Even in the generic case - we won't specify what that means until chapter 3 - we expect other than regular points on the conflict set.
The definition of regularity assures that the conflict set is an immersed submanifold at p. It does not assure that the conflict set is an embedded submanifold.
A weaker version of the theorem I.1 is in fact a known result, see [**JB85**]. A short version of this argument goes as follows.
The distance functions induce a mapping of

$$\mathbb{R}^n \times \prod_{i=1}^{l} M_i \mapsto \prod_{i=1}^{l} J^1(M_i, \mathbb{R}) \tag{I.1}$$

The projection of the A_1^l stratum on to the first factor is an immersed $\mathcal{C}^{j-1}$ manifold. This stratum represents exactly those points q that are the center of a sphere having 2-contact with all manifolds M_i.
However, this proves only that the conflict set is $\mathcal{C}^{j-1}$.
We have conormal bundles $N^*(M_i)$. Denote by $*_i$ a map $N^*(M_i) \mapsto \mathbb{R}^{n+1}$ defined by

$$*_i\colon (p,\xi) \mapsto (p + \xi\vec{n}(p), \lambda_i \xi)$$

The images of these maps in $\mathbb{R}^{n+1}$ are n-manifolds, possibly singular. In the literature one sometimes uses the term **big wavefront** for the image of $N^*(M_i)$ by $*_i$. To check whether the image is a manifold around a point $*_i(p,\xi)$ we have to check two things. Firstly the map should be injective and secondly the derivative should have rank n. The first condition fails when there is more than one basepoint. Hence we have to be away from the symmetry set of M_i. The second condition fails when the first n coordinates represent the point on the focal surface corresponding to the basepoint or one of the basepoints, see [**Mil63**]. This is exactly the definition of a regular point.
Now the intersection of the k images in $\mathbb{R}^{n+1}$ is a transversal intersection at all but the above mentioned points, because $\vec{n}_i\lambda_i \neq \vec{n}_j\lambda_j$. Furthermore it is $\mathcal{C}^{j-1}$. The projection onto the first n coordinates has no critical points because the vector that is the direction of projection

$$(0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$$

never lies in the tangent space of the big wavefront. The projection is also the conflict set and thus we have established that the conflict set is $\mathcal{C}^{j-1}$. But in fact we shall see that the big wavefront is a $\mathcal{C}^j$ manifold. And thus the conflict set is $\mathcal{C}^j$.
A nice proof of the fact that the big wave front of a hypersurface M_1 at points where the distance function has regular values has the same smoothness as M_1 was supplied by J. J. Duistermaat.
Let p be a point on the original hypersurface M. Let q be a point on the wavefront W at distance d. If q is a regular point of the distance function on W with p being the basepoint of q then the conormal bundle N^*W has the same smoothness as N^*M around these points. They both have smoothness $\mathcal{C}^{j-1}$.
But the conormal bundle of a manifold W has smoothness $\mathcal{C}^{j-1}$ iff. the the manifold itself has

smoothness $\mathcal{C}^j$. Thus the wavefronts have equal smoothness at regular points of the distance function. As the big wavefront is a union of these wavefronts, smoothly parameterized, the big wave front is also $\mathcal{C}^j$.

I.4. Proof of theorem I.2

The proof of the theorem will depend on some lemmas. Basically we will construct for each basepoint a Meusnier sphere to the M_i it lies on. The Meusnier spheres N_i will have 3-contacts in appropriate directions V_i. As a consequence the intersections of the images of $N^*(N_i)$ and those of $N^*(M_i)$ in $\mathbb{R}^{n+1}$ will also have a 3-contact in some direction V_c. With this we can find the second fundamental form of the conflict set.

LEMMA I.9. *Let M and N have 3-contacts in the direction $V \subset T_pM$. Then $M^* = *_M(M)$ and $N^* = *_N(N)$ have 3 contact in the directions of the linear span of $(n,1), n \in N_pM$ and $\{(v,0) \mid v \in V\}$*

PROOF. It is enough to consider the Monge forms - that is : the second order Taylor approximations of the immersions and submersions of M and N. Thus choose coordinates such that

$$\iota_M(t_1,\dots,t_{n-1}) = (t_1,\dots,t_{n-1},(t_1,\dots,t_{n-1})^T\mathbf{A}(t_1,\dots,t_{n-1}))$$

where $\mathbf{A}$ is a symmetric matrix.
In the same fashion presume that

$$\sigma_N(x_1,\dots,x_n) = x_n - (x_1,\dots,x_{n-1})^T\mathbf{B}(x_1,\dots,x_{n-1})$$

The contact map between the manifolds is then

$$(x_1,\dots,x_{n-1})^T(\mathbf{A}-\mathbf{B})(x_1,\dots,x_{n-1})$$

For vectors in $v \in V$ we will have $v^T\,(\mathbf{A}-\mathbf{B})\,v = 0$. We shall write $(t_1,\dots,t_n) = (t',t_n) = (t',t'')$ and $(s_1,\dots,s_n) = (s',s_n) = (s',s'')$. The normal n_M to M in a neighbourhood of 0 is:

$$n_M = (-\sum_k \mathbf{A}_{k1}t_1,\dots,-\sum_k \mathbf{A}_{k,n-1}t_{n-1},1) = (-\mathbf{A}t',1)$$

Although this normal does not have unit length we can use it to write $*_M$ in coordinates: $*_M = (\iota_M + t_n n_M, t_n)$.
We make a transformation on the first n coordinates of $\mathbb{R}^{n+1}$ by

$$H : \begin{cases} s_i = (*_M)_i & i = 1,\dots,n-1 \\ s_n = t_n \end{cases}$$

This transformation is not singular in some neighborhood of 0 for $\frac{\partial H}{\partial t}(0) = \mathbf{I}$. Its inverse can be calculated when t is small:

$$t' = \left(\mathbf{I} + s_n\mathbf{A} + \frac{s_n^2}{2}\mathbf{A}^2 + \dots\right)s'$$

It is also convenient that the transformation H is the identity map on the tangent space of M, N, M^* and N^*.
We are interested in the Monge form of $*_M$ and $*_N$ in these new coordinates. The map $*_M$ has the following particularly simple form:

$$*_M = (s_1,\dots,s_{n-1},(s_1,\dots,s_{n-1})^T\mathbf{A}(s_1,\dots,s_{n-1}) + s_n, s_n),$$

We will compute a Monge form for N^*. The first $n-1$ components of $*_N$ are

$$(\mathbf{I} - t_n \mathbf{B})\, t' = s' + s_n\, (\mathbf{A} - \mathbf{B})\, s' + o(t^2)$$

The other components are simply ${s'}^T \mathbf{B} s' + s_n$ and s_n. The submersion for M^* is

$$\sigma_{M^*} = x_n - x_{n+1} - (x_1, \dots, x_{n-1})^T \mathbf{A} (x_1, \dots, x_{n-1})$$

and as a consequence

$$\kappa_{M*,N*} = \sigma_{M^*} \circ *_N = {s'}^T \mathbf{B} s' - {s'}^T \mathbf{A} s'$$

whereby the lemma is proved. □

At $p = 0$ we have 2 pairs of hypersurfaces: M_1, N_1 and M_2, N_2. Each element of a pair intersects transversally with elements of the other pair and the elements in one pair have identical tangent spaces. We have a third pair of manifolds: $M_c = M_1 \cap M_2$ and $N_c = N_1 \cap N_2$.

Lemma I.10. *If M_1 and N_1 have 3-contact in the directions V_a and M_2 and N_2 have 3-contact in the directions V_b then M_c and N_c have 3-contact in the directions $V_c = V_a \cap V_b$.*

Proof. As in the proof of the previous lemma we will only consider Monge forms. Of course we want to calculate the contact map. The submersion for M_c is quickly found:

$$\sigma_{M_c} = (\sigma_{M_1}, \sigma_{M_2})$$

To find an immersion for N_c first notice that the immersion for N_1 and N_2 can be written in the form:

$$\iota_{N_1} = (t_1, \dots, t_{n-2}, \mathbf{R}(\phi)(t_{n-1}, {t'}^T \mathbf{A} t')) \tag{I.2}$$

$$\iota_{N_2} = (t_1, \dots, t_{n-2}, \mathbf{R}(\psi)(t_{n-1}, {t'}^T \mathbf{B} t')) \tag{I.3}$$

Here we wrote $\mathbf{R}(\phi)$ for a rotation by an angle ϕ. The points of N_c are the points where

$$\iota_{N_1}(t') = \iota_{N_2}(s')$$

from which we conclude that

$$s_i = t_i, \quad i = 1, \dots, n-2$$

We have $s_{n-1} = s_{n-1}(s_1, \dots, s_{n-2})$ and $t_{n-1} = t_{n-1}(t_1, \dots, t_{n-2})$. With these relations we obtain two different immersions for ι_{M_c}, number one is derived from ι_{N_1} and $t_{n-1} = t_{n-1}(t_1, \dots, t_{n-2})$, number two from ι_{N_2} and $s_{n-1} = s_{n-1}(s_1, \dots, s_{n-2})$. We combine this with the above obtained to obtain a contact map.

$$\kappa_{M_c,N_c} = (\sigma_{M_1} \circ \iota_{N_1|t_{n-1}=t_{n-1}(t_1,\dots,t_{n-2})}, \sigma_{M_2} \circ \iota_{N_2|t_{n-1}=s_{n-1}(t_1,\dots,t_{n-2})})$$

Now we want to calculate the second derivative of this map. We will start with the first component.

$$\frac{\partial \kappa_{M_1,N_1|t_{n-1}=t_{n-1}(t_1,\dots,t_{n-2})}}{\partial(t_1, \dots, t_{n-2})} = \frac{\partial \kappa_{M_1,N_1}}{\partial(t_1, \dots, t_{n-2})} + \frac{\partial \kappa_{M_1,N_1}}{\partial t_{n-1}} \frac{\partial t_{n-1}}{\partial(t_1, \dots, t_{n-2})} \tag{I.4}$$

and

$$\frac{\partial^2 \kappa_{M_1,N_1|t_{n-1}=t_{n-1}(t_1,\dots,t_{n-2})}}{\partial(t_1, \dots, t_{n-2})^2} = \frac{\partial^2 \kappa_{M_1,N_1}}{\partial(t_1, \dots, t_{n-2})^2} + \frac{\partial \kappa_{M_1,N_1}}{\partial t_{n-1}} \frac{\partial^2 t_{n-1}}{\partial(t_1, \dots, t_{n-2})^2} + \frac{\partial^2 \kappa_{M_1,N_1}}{\partial t_{n-1}^2} \frac{\partial t_{n-1}}{\partial(t_1, \dots, t_{n-2})}$$

Because M_1 and N_1 have 3-contact in some direction we will surely have

$$\frac{\partial \kappa_{M_1,N_1}}{\partial t_{n-1}} = 0$$

and thus we are to evaluate the derivative

$$\frac{\partial t_{n-1}}{\partial(t_1,\ldots,t_{n-2})} \tag{I.5}$$

in $p=0$. As the surface $x_{n-1}=x_n=0$ is the tangent space of M_c in 0 we can see that the derivative (I.5) evaluates to 0 in $p=0$. If we write for a $n-1\times n-1$ matrix $\mathbf{C}$

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{(n-2,n-2)} & \mathbf{C}_{(n-1)} \\ \mathbf{C}_{(n-1)} & \mathbf{C}_{n-1,n-1} \end{pmatrix}$$

then the second derivative of the contact map of M_c and N_c in 0 equals

$$\left(d^2\kappa_{M_1,N_1\,(n-2,n-2)}, \mathrm{d}^2\,\kappa_{M_2,N_2\,(n-2,n-2)}\right)$$

This proves the lemma. □

LEMMA I.11. *Let M and N be manifolds $\mathbb{R}^{n+k}$ with a 3-contact in the direction $V\subset T_p\mathbb{R}^{n+k}$. Let* $\mathrm{pr}:\mathbb{R}^{n+k}\to\mathbb{R}^n$*, such that T_pM does not lie in the direction of the projection:* $\dim V = \dim \mathrm{pr}(V)$ *then* $\mathrm{pr}(M)$ *and* $\mathrm{pr}(N)$ *have 3 contact in the direction* $\mathrm{pr}(V)$ *at* $\mathrm{pr}(p)$.

PROOF. The proof consist of a simple verification with the contact map. □

After these lemmas we are ready to complete the proof I.2. Suppose we are at a point p in M_c.
There are two base points p_a and p_b on M_1 and M_2 respectively. Propagate the wave fronts until the three points p_a, p_b and p_c fall together, say p. Now take an arbitrary direction in the tangent space of M_c, say v_c. The direction v_c projects to directions v_1 and v_2 in T_pM_1 and T_pM_2 respectively.
We will proceed to construct the big wavefront of M_1 and M_2 : M_1^h and M_2^h. Because p is a regular point for the distance function at $(p,0)\in\mathbb{R}^{n+1}$ they intersect transversally in a manifold M_c^h. The manifold M_c^h projects down to M_c in $\mathbb{R}^n$.
There are Meusnier spheres to M_1 and M_2 in the directions v_1 and v_2 : N_1 and N_2. Their big wavefronts can also be constructed : N_1^h and N_2^h.
We apply lemma I.9 to obtain that N_1^h and M_1^h have 3-contacts in the direction

$$V_1 = \mathrm{sp}\left(\begin{pmatrix} v_1 \\ 0\end{pmatrix}, \begin{pmatrix} n_1 \\ 1\end{pmatrix}\right).$$

In a similar vein we define V_2.
We define N_c^h as the transversal intersection of N_1^h and N_2^h at $(p,0)$. By lemma I.10 M_c^h and N_c^h have 3-contact in the direction $V_c=V_1\cap V_2$. The directions in V_c can be projected down to $\mathbb{R}^n$ by a projection pr.
We claim that

$$\mathrm{pr}(V_1\cap V_2) = v_c. \tag{I.6}$$

To prove equation I.6 we first try to solve the equation

$$\lambda_1\begin{pmatrix} v_1 \\ 0\end{pmatrix} + \lambda_2\begin{pmatrix} n_1 \\ 1\end{pmatrix} = \lambda_3\begin{pmatrix} v_2 \\ 0\end{pmatrix} + \lambda_4\begin{pmatrix} n_2 \\ 1\end{pmatrix}$$

If $n_1 = n_2$ then also $v_1 = v_2$. In this case $\lambda_2 = \lambda_4 = 0$ and equation I.6 will hold. Thus we will assume $\lambda_2 = \lambda_4 \neq 0$ and we can write (with different λ_i)

$$\lambda_1 \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} n_2 \\ 1 \end{pmatrix} \tag{I.7}$$

Because $n_1 \neq n_2$ we can write $n_c = n_1 - n_2$. Now v_1 and v_2 are projections of v_c onto the tangent planes of T_pM_c. The tangent planes of M_1 and M_2 make an equal angle with M_c. Therefore we can solve I.7 if we put $\lambda_1 = \lambda_2$. The line that is thus in V_c is of the form

$$v_c^h = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} n_2 \\ 1 \end{pmatrix}$$

Equation I.6 is proved.
We also know that $\mathrm{pr}(v_c^h)$ makes an equal angle with v_1 and v_2 so that we will have

$$v_c^h = \begin{pmatrix} v_1 + v_2 \\ \mu \end{pmatrix}$$

Projecting N_h^c down to N_c we see that this projection is regular around p and that, by lemma I.11, the manifold N_c will have 3-contact with M_c at p in the direction v_c. Thus the curvature of M_c in the direction v_c is that of N_c. Indeed, intuitively, this is the direction in which a minimal change in chord length is achieved.
We now see that it is enough to calculate the curvature of the spheres N_1 and N_2 in the directions v_1 and v_2 and then to compute the curvature of the conflict set of the spheres in the direction v_c. In particular, for two wavefronts starting from a plane curve, it will be enough to calculate the curvature of the conflict set of two curvature circles. Example I.4 is characteristic. The conflict line is parameterized by $(\sqrt{b_1 b_2}\sinh t, 2^{-1}(b_1 - b_2)(1 - \cosh t))$. Its curvature is

$$\frac{1}{2}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\cos\phi \tag{I.8}$$

where T_1 and T_2 are the distances from the conflict set to the curvature centers, and ϕ is half on the angle that n_1 and n_2 thus their inverses are the curvatures of the circles at the point where the wavefronts meet.

Remark I.12. For a further discussion of (I.8) and its applications see section IV.4.

Remark I.13. Sometimes it is more convenient to choose n_1 and n_2 so that $n_c = n_1 + n_2$. In this case the minus sign in equation I.8 becomes a plus sign. Thus, in equation I.8 the distances T_1 and T_2 are measured with respect to the direction of the normals n_1 and n_2. If one chooses the normals to the wavefronts as in the setup then the formula holds but when one chooses the distances differently the mentioned sign change occurs. In the sequel we will stick to the original setup where the sign change does not occur.

Example I.14. In figure I.2 one sees two circles with their conflict set. The conflict set is an ellipse. The normals at the left side are the normals that give this ellipse as conflict set. But, chosen thus T_1 and T_2 will have different signs.
The normals as chosen in the picture on the right do not correspond to the conflict set as shown. However if we use these normals to compute the curvature of the conflict set as shown in the picture we need to change the minus sign in equation I.8 into a plus sign.

With two spheres in $\mathbb{R}^n$ the conflict set is a hyperboloid or ellipsoid. v_1, v_2, v_c and n_c all lie in one 2-dimensional plane. Thus the above formula also applies to each direction in T_pM_c. When $v_c \in T_pM_1 \cap T_pM_2$ the angle γ will be 0. In that case we can simply add the

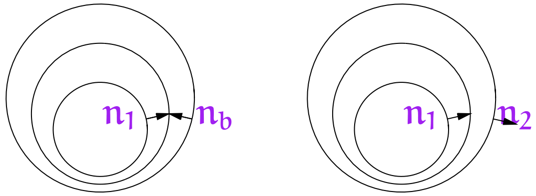

FIGURE I.2. Two ways of choosing the normals

curvatures of the circles we obtain. The spheres have radius T_1 and T_2. The corresponding circles have radius $T_1 \cos\phi$ and $T_2 \cos\phi$ where ϕ is the angle between n_1 and n_c. In the other case $v_c \notin T_pM_1 \cap T_pM_2$ and we choose v_c to lie orthogonally to this intersection. Then v_1, v_2 and v_c lie in the plane spanned by n_1, n_2 and n_c. The angle v_c and v_1 make in this case is also ϕ. Thus we have formulas for the curvature, in the two cases an illustration of which is in figure I.3.

$$\kappa_c = \frac{1}{2}\left(\frac{\kappa_1}{\cos\phi} - \frac{\kappa_2}{\cos\phi}\right) \tag{I.9}$$

$$\kappa_c = \frac{1}{2}(\kappa_1 - \kappa_2)\cos\phi \tag{I.10}$$

The final step in the proof consists of linking these two formulas to the elegant matrix repre-

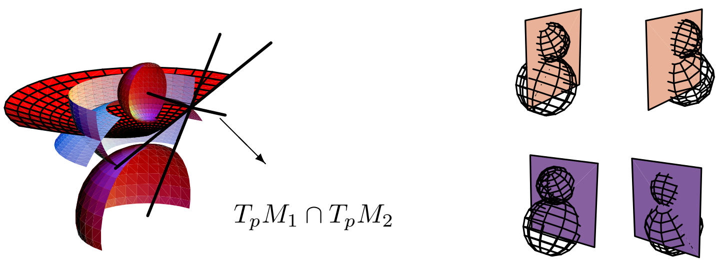

FIGURE I.3. Two orthogonal directions in T_pM_c account for two different curvature formulas

sentation. Choose an orthonormal basis $\{e_1, \ldots, e_{n-2}\}$ for $T_pM_1 \cap T_pM_2$ and choose vectors e_a, e_a and e_c that complete to orthonormal bases for T_pM_1, T_pM_2 and T_pM_c respectively. e_a, e_b and e_c correspond with each other.
The projection of $T_p\mathbb{R}^n$ onto T_pM_c restricted to T_pM_1 is an isomorphism. It is the identity on $\mathrm{sp}\{e_1, \ldots, e_{n-1}$ and e_a is mapped to $e_c \cos\phi$. Call the matrix representation of this $\mathbf{P}$.
The function that assigns a curvature to each direction of the tangent space of a $\mathcal{C}^2$ hypersurface is in fact a symmetric bilinear form on the tangent space. It is the inverse of the first fundamental form multiplied by the second fundamental form. Thus the representation for this function is a matrix. One such matrix for T_pM_c that gives the correct curvatures in

the directions $e_1, \dots, e_{n-2}$ and e_c would be

$$\frac{1}{2\cos\phi}\mathbf{P}^T(\mathbf{A}-\mathbf{B})\mathbf{P}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the matrices of the curvature functions on the tangent spaces T_pM_1 and T_pM_2. To prove that this is in fact the curvature function for T_pM_c all we need to do is write the equations for the curvature in other directions than $e_1, \dots, e_{n-2}$ and e_c. We will omit this calculation.

We now want to pass on from the curvature functions to second fundamental forms. For M_1 and M_2 we can take immersions as the ones in equations I.2, replacing ψ by $-\phi$. We now see that their first fundamental forms in intrinsic coordinates at 0 are simply identity matrices. The first fundamental form of the conflict set is the identity matrix. We have proven that at 0 the second fundamental form of the conflict set $\mathbb{I}_c$ is:

$$\mathbb{I}_c = \frac{1}{2\cos\phi}\mathbf{P}^T\left(\mathbb{I}_1 - \mathbb{I}_2\right)\mathbf{P}$$

I.5. Proof of theorem I.3

The special case of l spheres in $\mathbb{R}^n$ the conflict set has an appealing property: each of its components is a conic section.

So let us have l spheres in $\mathbb{R}^n$, each having center p_i, $i = 1, \dots l$. Denote their radii by r_i. Each component is given by $l-1$ equations:

$$\|x - p_i\| = \|x - p_1\| + d_i, \qquad i = 2, \dots, l \tag{I.11}$$

The equations (I.11) describe conic sections.

Each d_i will be written

$$d_{1,i} = d_i = a_i - a_1$$

where $a_i = \pm r_i$. Thus we have in total a maximum of 2^l components of the complete conflict set. We will consider a fixed component, that is we will fix a_i for the rest of the proof. Other equations are

$$\|x - p_j\| = \|x - p_i\| + d_{i,j}$$

with $d_{i,j} = d_{1,j} - d_{1,i}$.

Now we square both sides of these equations.

$$\langle x, x\rangle - 2\langle p_j, x\rangle + \langle p_j, p_j\rangle = \langle x, x\rangle - 2\langle p_i, x\rangle + \langle p_i, p_i\rangle + 2d_{i,j}\|x - p_i\| + d_{i,j}^2 \qquad i, j = 1, \dots, l$$

and we obtain

$$2d_{i,j}\|x - p_i\| = 2\langle x, p_i - p_j\rangle + \|p_j\|^2 - \|p_i\|^2 - d_{i,j}^2, \qquad i, j = 1, \dots, l$$

When $d_{i,j} = \pm\|p_i - p_j\|$ the solution to the equation I.11 is a line. So the component of the conflict set we consider lies in a line and the theorem is trivially true. We can safely exclude this case and all the others where

$$d_{i,j} = \pm\|p_i - p_j\|$$

If they exist choose mutually disjoint subsets $J_1, \cdots, J_k$ of $\{1, \cdots, n\}$ so that

- $a_{i_1} = a_{i_2}$ for all $i_1, i_2 \in J_i$
- the cardinalities $j_i = | J_i |$ are ordered $j_1 \geq j_2 \geq \cdots \geq j_k$.
- $j_i \geq 2$

The complement of the union of these subsets will be denoted by J_0:

$$\cup_{i=0}^{k} J_i = \{1, \cdots, l\}$$

The number of elements j_0 in J_0 is possibly 0. If all a_i are different then $j_0 = n$, and $k = 0$. It is now convenient to put

$$\begin{aligned} P(i,j) &= 2d_{i,j}\|x - p_i\| \\ &= 2\langle x, p_i - p_j\rangle + \|p_j\|^2 - \|p_i\|^2 - d_{i,j}^2 \end{aligned}$$

We finally relabel the points such that

$$J_0 = \{1, \cdots, j_0\}, J_1 = \{j_0 + 1, \cdots, j_0 + j_1\}$$

and so on. Let us first assume that $k \geq 1$ and $j_0 \geq 3$.
From each of the J_m we have $j_m - 1$ linear equations

$$P(j_0 + \cdots + j_m, i) = 0 \qquad i \in J_m \backslash \{j_0 + \cdots + j_m\} \tag{I.12}$$

We also have $j_0 - 2$ linear equations

$$\frac{P(1,2)}{d_{1,2}} = \frac{P(1,i)}{d_{1,i}} \qquad i = 3, \ldots, j_0 \tag{I.13}$$

There is a third set of m linear equations that read

$$\frac{P(1,2)}{d_{1,2}} = \frac{P(1,i)}{d_{1,i}} \qquad i = j_0 + j_1, j_0 + j_1 + j_2, \cdots, j_0 + j_1 + \cdots + j_m \tag{I.14}$$

In total we have $l - 2$ linear equations. These equations all have the form

$$\langle v_i, x\rangle = w_i \tag{I.15}$$

From I.12, I.13 and I.14 there are three sets of v_i. The first set is

$$p_{j_0+j_1+\cdots+j_m} - p_i \qquad i \in J_m \backslash \{j_0 + \cdots + j_m\} \tag{I.16}$$

The second set is

$$\frac{p_1 - p_2}{d_{1,2}} - \frac{p_1 - p_i}{d_{1,i}} \qquad i = 3, \ldots, j_0 \tag{I.17}$$

The third set is

$$\frac{p_1 - p_2}{d_{1,2}} - \frac{p_1 - p_i}{d_{1,i}} \qquad i = j_0 + j_1, j_0 + j_1 + j_2, \cdots, j_0 + j_1 + \cdots + j_m \tag{I.18}$$

Each of the equations determines an affine hyperplane in $\mathbb{R}^n$ The hypersurfaces will intersect transversally if the vectors v_i are linearly independent. The $l - 2$ vectors are linearly independent iff. there are 2 more vectors such that these l will be independent. Two vectors that will do are p_1 and p_2 because by hypothesis the p_i span up a $l-1$ dimensional simplex. If $k = 0$ then all the a_i are different and the vectors are given by equations as in I.13 and I.17. If $k > 0$ and $j_0 = 2$ the v_i are given by equations I.12, I.14 and I.16, I.18. If $k > 0$ and $j_0 = 1$ j_1 will equal 2. The case $j_0 = 1$ is similar. When $l = 2$ the theorem is trivial.
Saying that the conflict set is a conic section is equivalent to saying that the conflict set is the zero-set of some quadratic equations. The quadratic equations are given by the $P(i,j)$. This completes the proof of theorem I.3.

Remark I.15. Theorem I.3 should also hold with other conditions. There are examples of 3 spheres in $\mathbb{R}^n$, with their centers lying on a line, such that a component of the conflict set lies in a hyperplane and that it is a conic section in this plane. Caution is needed though because there are also examples of four spheres in $\mathbb{R}^n$ - i. e. two concentric spheres mirrored through some hyperplane - where the theorem is not true.

Remark I.16. If $l = n = 3$ we have a special case, that was studied a lot classically. The 1 parameter family of spheres with center on the conflict set that touch the three given spheres has a special surface , called a Dupin cyclide, as an envelope. By an appropriate inversion the conflict set - which is on the symmetry set of the cyclide - is mapped onto a circle. Under this inversion the cyclide is mapped to a torus. Thus the conflict set itself is a conic section in a hyperplane in $\mathbb{R}^3$, see [**Cox52**]. For cyclides one might consult several works of T.E. Cecil.

II Three and more surfaces

II.1. Introduction

In this chapter we will elaborate on the results of the previous chapter. We will obtain a formula for the curvature and the torsion of the conflict set of three surfaces. The relevant formulas are in theorems II.6 and II.10. The last section of this chapter - which can be safely skipped - contains an illustration of the formulas for curvature and torsion and of theorem I.3.

II.2. The tangent space to the conflict set

We would like to give a description of the tangent space of the conflict set of (M_1, n_1), (M_2, n_2), ..., (M_l, n_l) at regular points, where the M_i are hypersurfaces in $\mathbb{R}^n$.
So at regular points we have - as in the previous chapter - a set of basepoints p_i on each of the M_i. The tangent spaces T_pM_i define l affine hyperplanes W_i in $\mathbb{R}^n$ that intersect transversally. The conflict set of the W_i has the same tangent space as the conflict set of the M_i. So we might as well assume that the M_i are affine hyperplanes, with a point p in their intersection.
Here the tangent space will be $n - l + 1$ dimensional. We will split it into two components. First of all a part of it will correspond to the intersection of the M_i. Let $V_c = \bigcap_{i=l}^{l} T_pM_i$, then $V_c \subset T_pM_c$. This part of the tangent space is readily calculated and so we can study the projection

$$\pi\colon T_p\mathbb{R}^n \mapsto T_p\mathbb{R}^n/V_c.$$

The $\pi(n_i)$ form a basis for $T_p\mathbb{R}^n/V_c$ and we also have that $\pi(n_i) \perp \pi(T_pM_i)$. Denote M_c^π the conflict set of the $\pi(M_i)$. This will be just a line. And this line corresponds to the second part of the tangent space. We thus have

$$V_c \oplus \text{“line”} = T_pM_c$$

Proposition II.1. *$\pi(T_pM_c) = T_pM_c^\pi$ and $V_c \oplus T_pM_c^\pi = T_pM_c$*

Proof. Consider a vector x that is in T_pM_c. It has a lift to $\mathbb{R}^{n+1}$ where it has to be in the intersection of the tangent spaces to the big fronts. Thus the vectors x that are in T_pM_c all have

$$\langle x, n_i \rangle = \langle x, n_j \rangle$$

From here it follows that

$$\langle \pi(x), \pi(n_i) \rangle = \langle \pi(x), \pi(n_j) \rangle$$

On the other hand if $\pi(x) \in T_pM_c^\pi$ then either $x \in V_c$ or we may assume that $x \perp V_c$, in which case the above two equations are equivalent. □

For this second part of the tangent space it is thus enough to look at the case where $l = n$ and $V_c = \{0\}$. In this case we have a formula. In order to write this formula we introduce the following notations. If i, j, k are integers then we write

$$q(i, j, k) = 1 + ((i + j - 1) \bmod k)$$

This function is a “circulator” over a finite index set $\{1, \cdots, k\}$. Also, if we have $n - 1$ vectors in $\mathbb{R}^n$ we define a cross product, that generalizes the cross product of 2 vectors in

$\mathbb{R}^3$. The cross-product of $n-1$ vectors v_i in $\mathbb{R}^n$ is defined by the requirement that for each vector $w \in \mathbb{R}^n$ we have

$$\langle \times_{i=1}^{n-1} v_i, w\rangle = \det(v_1, \cdots, v_{n-1}, w)$$

PROPOSITION II.2. *If $l = n$ then the vector*

$$t_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} n_{q(i,j,l)}$$

defines the tangent space to the conflict set.

PROOF. All we need to verify is that $\langle t_c, n_i\rangle = \langle t_c, n_j\rangle$. □

We summarize our discussion in a theorem:

THEOREM II.3. *The tangent space to the conflict set of (M_1, n_1), $(M_2, n_2), \dots, (M_l, n_l)$, all hypersurfaces in $\mathbb{R}^n$ is spanned by*

$$V_c \oplus \left(\sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} \pi(n_{q(i,j,l)}) \right)$$

provided that we are at regular points of the conflict set.

II.3. A curvature formula

We next want to calculate the curvature of the conflict set of (M_1, n_1), $(M_2, n_2), \dots, (M_l, n_l)$ at regular points in terms of the curvature at the basepoints. We calculate the curvature for the case $l = n = 3$. The result is in theorem II.6.

The curvature formula will be obtained by differentiation. As before we write:

$$t_c = n_1 \times n_2 + n_2 \times n_3 + n_3 \times n_1 \tag{II.1}$$

We will also write

$$e = \frac{t_c}{\|t_c\|}$$

We will denote t_i the unit vector of the projection of t_c to T_pM_i. For the curvature of the conflict set we need to calculate, according to the Frenet-Serret equation, the derivative $\nabla_e e$:

$$\nabla_e e = \nabla_e \frac{t_c}{\|t_c\|} = \frac{\|t_c\|^2 \nabla_e t_c - \langle t_c, \nabla_e t_c\ \rangle t_c}{\|t_c\|^3}$$

By ∇_v we mean the directional derivative

$$\nabla_v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$$

It is clear that

$$\langle \nabla_e e, t_c\rangle = 0$$

We can choose coordinates so that e is $(0,0,1)$. In particular this means that $\nabla_e e$ has a zero component in the last coordinate, the "z" coordinate. The vector t_c has only non-zero components in all but this last coordinate. Now denote $(\centerdot)_i$ the i-th component of a vector. We have just concluded that:

$$(\nabla_e e)_3 = 0$$

and that thus:

$$\kappa_c^2 = \frac{(\nabla_e t_c)_1^2 + (\nabla_e t_c)_2^2}{\|t_c\|^2}$$

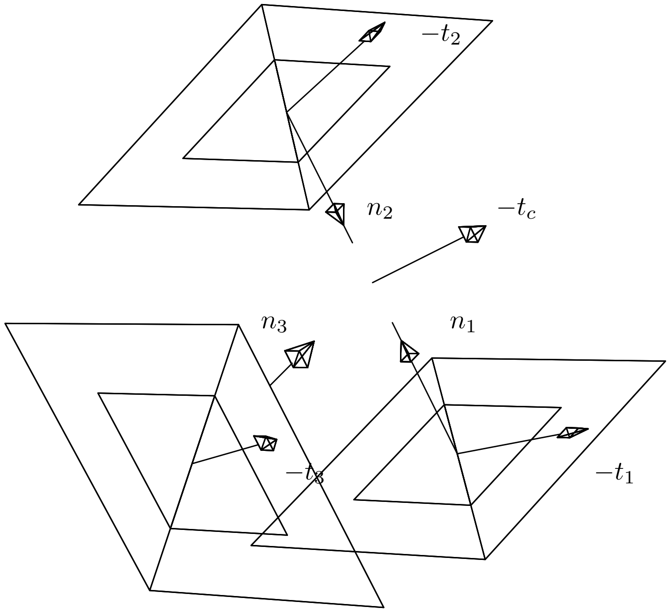

FIGURE II.1. The geometry of the three tangent planes

REMARK II.4. We will calculate the square of the curvature. In this way we avoid "sign" problems. There seems to be no natural choice of frame for the conflict set. For instance when we interchange n_1 and n_2 in (II.1) t_c changes sign. For the first normal to the conflict set things get even more complicated.

We compute $\nabla_e t_c$ through a further choice of coordinates namely:

$$e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad n_i = \begin{pmatrix} \sin\alpha \sin\beta_i \\ \sin\alpha\cos\beta_i \\ \cos\alpha \end{pmatrix} \quad t_i = \begin{pmatrix} -\cos\alpha\sin\beta_i \\ -\cos\alpha\cos\beta_i \\ \sin\alpha \end{pmatrix} \tag{II.2}$$

Furthermore

$$\nabla_e n_i = \kappa_i t_i \sin\alpha \tag{II.3}$$

where κ_i is the normal curvature of the wavefront of M_i at the point of the conflict set that we are considering.

REMARK II.5. Suppose a curve γ is contained in a hypersurface $M \subset \mathbb{R}^3$. This curve has a so-called normal curvature κ_n and a so-called geodesic curvature κ_g. The geodesic curvature depends only on the interior geometry of M and the normal curvature depends only on the exterior geometry of M. The curvature of γ as a space curve is κ. Between κ and the aforementioned curvatures we have the following relation:

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

Thus equation (II.3) merely says that the derivative only depends on the exterior geometry.

With these choices

$$\begin{aligned} \|t_c\|^2 &= \sin^4\alpha\,(\sin(\beta_1-\beta_2)+\sin(\beta_2-\beta_3)+\sin(\beta_3-\beta_1))^2 \\ &= 16\sin^4\alpha\sin^2(\frac{\beta_1-\beta_2}{2})\sin^2(\frac{\beta_2-\beta_3}{2})\sin^2(\frac{\beta_3-\beta_1}{2}) \end{aligned} \tag{II.4}$$

and

$$(\nabla_e t_c)_1 = \sin^3\alpha\,(\kappa_1(\cos\beta_3-\cos\beta_2)+\kappa_2(\cos\beta_1-\cos\beta_3)+\kappa_3(\cos\beta_2-\cos\beta_1))$$

Similarly

$$(\nabla_e t_c)_2 = \sin^3\alpha\,(\kappa_1(\sin\beta_3-\sin\beta_2)+\kappa_2(\sin\beta_1-\sin\beta_3)+\kappa_3(\sin\beta_2-\sin\beta_1))$$

This is because

$$\begin{aligned} \nabla_e t_c &= (\nabla_e n_1)\times(n_2-n_3)+(\nabla_e n_2)\times(n_3-n_1)+(\nabla_e n_3)\times(n_1-n_2) \\ &= \kappa_1\sin\alpha t_1\times(n_2-n_3)+\kappa_2\sin\alpha t_2\times(n_3-n_1)+\kappa_3\sin\alpha t_3\times(n_1-n_2) \end{aligned}$$

Now we have

$$\begin{aligned} (\nabla_e t_c)_1^2+(\nabla_e t_c)_2^2 = \sin^6\alpha\,\big(&(\kappa_1-\kappa_2)^2+(\kappa_2-\kappa_3)^2+(\kappa_3-\kappa_1)^2 \\ &+2(\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2)(\kappa_2-\kappa_3)(\kappa_3-\kappa_1) \\ &+2(\cos\beta_2\cos\beta_3+\sin\beta_2\sin\beta_3)(\kappa_3-\kappa_1)(\kappa_1-\kappa_2) \\ &+2(\cos\beta_3\cos\beta_1+\sin\beta_3\sin\beta_1)(\kappa_1-\kappa_2)(\kappa_2-\kappa_3)\big) \\ = \sin^6\alpha\,\big(&(\kappa_1-\kappa_2)^2+(\kappa_2-\kappa_3)^2+(\kappa_3-\kappa_1)^2 \\ &+2\cos(\beta_1-\beta_2)(\kappa_2-\kappa_3)(\kappa_3-\kappa_1) \\ &+2\cos(\beta_2-\beta_3)(\kappa_3-\kappa_1)(\kappa_1-\kappa_2) \\ &+2\cos(\beta_3-\beta_1)(\kappa_1-\kappa_2)(\kappa_2-\kappa_3)\big) \end{aligned}$$

This is finally equal to

$$= -4\sin^6\alpha\sum_{i=1}^{3}\sin^2\left(\frac{\beta_{q(i,1,3)}-\beta_{q(i,2,3)}}{2}\right)(\kappa_{q(i,2,3)}-\kappa_i)(\kappa_i-\kappa_{q(i,1,3)}) \tag{II.5}$$

We have proven the following theorem.

THEOREM II.6. *The squared curvature of the conflict set of* 3 *hypersurfaces is at regular points given by the formula:*

$$\kappa_c^2 = -\frac{\sin^2\alpha}{4}\sum_{i=1}^{3}\frac{(\kappa_i-\kappa_{q(i,1,3)})(\kappa_{q(i,1,3)}-\kappa_{q(i,2,3)})}{\sin^2(\frac{\beta_i-\beta_{q(i,1,3)}}{2})\sin^2(\frac{\beta_{q(i,1,3)}-\beta_{q(i,2,3)}}{2})} \tag{II.6}$$

We can also find a more coordinate free form of formula (II.6). We reformulate theorem II.6:

$$\kappa_c^2 = \sin^6(\alpha)\sum_{i=1}^{3}\frac{(\kappa_i-\kappa_{q(i,1,3)})(\kappa_{q(i,1,3)}-\kappa_{q(i,2,3)})}{(1-\langle n_i, n_{q(i,1,3)}\rangle)(\langle n_{q(i,1,3)}, n_{q(i,2,3)}\rangle-1)} \tag{II.7}$$

We used that:

$$1-\langle n_i, n_j\rangle = 2\sin^2(\alpha)\sin^2\left(\frac{\beta_i-\beta_j}{2}\right),$$

an identity that follows from (II.2). We also have the following:

PROPOSITION II.7. *It holds* $\kappa_c = 0$ *iff.* $\kappa_i = \kappa_j$ *for* $1 \le i, j \le 3$.

PROOF. In view of a theorem on contact, see theorem III.33 or [**vM03**], it will be enough to prove the theorem in the case where the base manifolds are spheres. The theorem then follows from the long calculation below or from the proof of theorem I.3. □

II.4. Higher derivatives: Torsion

Theorem I.3 says that the conflict set with the three spheres has zero torsion. We are interested in computing in general the torsion of the conflict set when $l = n = 3$. To compute the torsion we need the Frenet-Serret equations, see [**Spi**], vol. 2. The Frenet-Serret equations can be applied once we know the first and second normal to the conflict set.

We contend that the first normal to the conflict set is

$$n_c = \kappa_1 e \times (n_2 - n_3) + \kappa_2 e \times (n_3 - n_1) + \kappa_3 e \times (n_1 - n_2) \tag{II.8}$$

This is proven by using the polar coordinates and comparing each term in the cyclic sum to the corresponding terms in

$$\mathrm{pr}_e(\nabla_e \frac{t_c}{\|t_c\|})$$

We can also use a lemma:

LEMMA II.8. *If* $\langle u, v\rangle = 0$ *then*

$$\mathrm{pr}_u(w \times v) = \frac{\langle w, u\rangle}{\|u\|^2} u \times v \text{ and } u \times (u \times v) = -\|u\|^2 v$$

PROOF. Set $u = (0, 0, 1)$. □

We can apply this lemma to verify the assertion of equation (II.8). Let $v = n_2 - n_3$, $u = e_l$ and $w = t_1$. Then

$$\mathrm{pr}_e(\kappa_1 t_1 \times (n_2 - n_3)) = \kappa_1 \sin\alpha \; e \times (n_2 - n_3)$$

which in view of (II.1) proves (II.8).

To find the third vector in the Frenet frame - up to sign - we take the outer product:

$$b_c = n_c \times e$$

Using $e \times (e \times (n_i - n_{q(i,1,3)})) = n_{q(i,1,3)} - n_i$ we can write:

$$b_c = \kappa_1(n_2 - n_3) + \kappa_2(n_3 - n_1) + \kappa_3(n_1 - n_2)$$

This further entails that $\|b_c\| = \|n_c\|$. Recalling the Frenet-Serret equations (see [**Spi**]) the derivative we are interested in is

$$\nabla_e \frac{b_c}{\|b_c\|} = -\tau_c \frac{n_c}{\|n_c\|}$$

We can now compute this derivative and the torsion

$$\nabla_e \frac{b_c}{\|b_c\|} = \frac{\|b_c\|^2 \nabla_e b_c - \langle b_c, \nabla_e b_c\rangle b_c}{\|b_c\|^3} = -\tau_c \frac{n_c}{\|n_c\|}$$

This leads to

$$\tau_c = -\frac{\langle \nabla_e b_c, n_c\rangle}{\|b_c\|^2}$$

As a first step we will derive some expression for $\|b_c\|^2$.

$$\begin{aligned}
\|b_c\|^2 &= \sum_{i=1}^{3}(\kappa_i - \kappa_{q(i,1,3)})^2 + 2\cos(\beta_i - \beta_{q(i,1,3)})(\kappa_{q(i,1,3)} - \kappa_{q(i,2,3)})(\kappa_{q(i,2,3)} - \kappa_i) \\
&= -4\sin^2\alpha \sum_{i=1}^{3} \sin^2\left(\frac{\beta_1 - \beta_{q(i,1,3)}}{2}\right)(\kappa_{q(i,1,3)} - \kappa_{q(i,2,3)})(\kappa_{q(i,2,3)} - \kappa_i) \\
&= \frac{\kappa_c^2\|t_c\|^2}{\sin^4\alpha}
\end{aligned}$$

The derivative of interest is thus

$$\begin{aligned}
\nabla_e b_c = \sin\alpha(&\kappa_1(\kappa_2 t_2 - \kappa_3 t_3) + \kappa_2(\kappa_3 t_3 - \kappa_1 t_1) + \kappa_3(\kappa_1 t_1 - \kappa_2 t_2) \\
&+ \nabla_e\kappa_1(n_2 - n_3) + \nabla_e\kappa_2(n_3 - n_1) + \nabla_e\kappa_3(n_1 - n_2))
\end{aligned}$$

Now in this derivative we encounter terms of the form $\nabla_e\kappa_i$. Such terms embody two variations. One is the change of curvature on the wavefront. The other one is the distancing between the point on the conflict set and the corresponding basepoint. Heuristically we feel that the last influence should have no effect on the conflict set. The curvature of the conflict set should only depend on the relative position of the base manifolds.
Note that by only taking into account the distance variation we have:

$$\nabla_e\kappa_i = \nabla_e\frac{1}{T_i} \approx -\frac{\nabla_e T_i}{T_i^2} = -\kappa_i^2\nabla_e T_i = -\kappa_i^2\cos\alpha$$

Here we have used the notation T_i for the distance between the point under consideration on the conflict set and the center of the Meusnier sphere of the surface M_i at the basepoint on M_i. Because we are at regular points this distance is never zero. Taking both variations into account we thus have:

$$\nabla_e\kappa_i = -\kappa_i^2\cos\alpha + \sin\alpha\nabla_{t_i}\kappa_i'$$

Here κ_i' is the normal curvature of the traced out curve on the wavefront. There seems to be no obvious expression for its derivative.
REMARK II.9. The meaning of the term $D_{t_i}\kappa_i'$ is as follows. It is the derivative of the normal curvature along the basecurve on the equidistant at p. The relation between normal curvature and normal curvature on an equidistant is clear. At distance d the normal curvature is

$$\frac{\kappa}{1 + d\kappa}$$

Where their derivatives are concerned this is much less the case. Let us start by clarifying how the 3-jet of a hypersurface determines the derivative of the normal curvature.
In case of an immersion that up to a quadratic form looks like

$$(s,t) \mapsto (s, t, \frac{1}{6}(As^3 + 3Bs^2t + 3Cst^2 + Dt^3))$$

and a curve with tangent $(\cos\mu, \sin\mu)$ the derivative is

$$A\cos^3\mu + 3B\cos^2\mu\sin\mu + 3C\cos\mu\sin^2\mu + D\sin^3\mu.$$

This means that - as was to be expected - if two surfaces have 2 contact and 4 contact in the direction $\vec{v}$ curves in that direction on the surfaces have equal derivative of the normal curvature.
The relation between the derivative of the normal curvature at the base manifold and at its

equidistant is most conveniently found by just using curves in $\mathbb{R}^2$. If we want a curve with prescribed derivative of the curvature α at 0 we can take

$$\gamma\colon t \mapsto (t, \frac{\kappa}{2}t^2 + \frac{\alpha}{6}t^3)$$

At distance d its derivative of the normal curvature is

$$\frac{\alpha}{(1+d\kappa)^3}$$

We write the derivative of b_c as a sum of three components:

$$\begin{aligned}\nabla_e b_c &= \sin\alpha(\kappa_1(\kappa_2 t_2 - \kappa_3 t_3) + \kappa_2(\kappa_3 t_3 - \kappa_1 t_1) + \kappa_3(\kappa_1 t_1 - \kappa_2 t_2)) \\ &\quad - \cos\alpha(\kappa_1^2(n_2 - n_3) + \kappa_2^2(n_3 - n_1) + \kappa_3^2(n_1 - n_2)) \\ &\quad + \sin\alpha(\nabla_{t_1}\kappa_1'(n_2 - n_3) + \nabla_{t_2}\kappa_2'(n_3 - n_1) + \nabla_{t_3}\kappa_3'(n_1 - n_2)) \\ &\quad I_1 + I_2 + II\end{aligned}$$

With some perseverance we obtain $\langle I_1 + I_2, n_c\rangle = 0$. We thus have

$$\begin{aligned}\langle \nabla_e b_c, n_c\rangle = \langle II, n_c\rangle = \sin\alpha\|t_c\|(&\kappa_1(\nabla_{t_2}\kappa_2' - \nabla_{t_3}\kappa_3') + \\ &\kappa_2(\nabla_{t_3}\kappa_3' - \nabla_{t_1}\kappa_1') + \kappa_3(\nabla_{t_1}\kappa_1' - \nabla_{t_2}\kappa_2'))\end{aligned}$$

Consequently

THEOREM II.10. *With notations as above the torsion of the conflict set is given by:*

$$\tau_c = \pm\frac{\sin^5\alpha}{\|t_c\|\kappa_c^2}(\kappa_1(\nabla_{t_2}\kappa_2' - \nabla_{t_3}\kappa_3') + \kappa_2(\nabla_{t_3}\kappa_3' - \nabla_{t_1}\kappa_1') + \kappa_3(\nabla_{t_1}\kappa_1' - \nabla_{t_2}\kappa_2'))$$

REMARK II.11. Compare this formula to a classical formula for the torsion of a space curve with nonzero curvature κ and a unit parameterization $c(t)$ (see [**Spi**])

$$\tau = \frac{1}{\kappa^2}\langle\frac{\mathrm{d}\,c}{\mathrm{d}\,t} \times \frac{\mathrm{d}^2\,c}{\mathrm{d}\,t^2}, \frac{\mathrm{d}^3\,c}{\mathrm{d}\,t^3}\rangle$$

We see that the term $\frac{1}{\kappa_c^2}$ is natural in this respect.

II.5. Higher dimensional analogues

The same differentiation techniques can be used to determine the curvature of l surfaces in $\mathbb{R}^l$. The trouble is that it is less clear what angles and geometrical data other than just the normals we should take in order to get useful formulae.

The tangent to the conflict set is

$$t_c = \sum_{i=1}^{l}(-1)^{(i+1)(l+1)} \times_{j=1}^{l-1} n_{q(i,j,l)}$$

If we differentiate t_c wrt. arclength we obtain

$$\nabla_e t_c = \sum_{i=1}^{l}(-1)^{(i+1)(l+1)}(\nabla_e n_i) \times_{j=1}^{l-2} (n_{q(i,j,l)} - n_{q(i,j+1,l)})$$

Here we have written again:

$$e = \frac{t_c}{\|t_c\|},$$

and we have used the formula

$$\nabla_e n_i = \kappa_i t_i \sin\alpha$$

We know that the first normal in the Frenet frame (up to sign and length) to the conflict curve is

$$n_c = \sum_{i=1}^{l} (-1)^{(i+1)(l+1)} \kappa_i e \times_{j=1}^{l-2} (n_{q(i,j,l)} - n_{q(i+1,j,l)})$$

This means that we might also calculate the curvature in the following way:

$$\kappa_c = \frac{\langle \nabla_e t_c, n_c \rangle}{\|n_c\| \|t_c\|}$$

It is also remarked that proposition II.7 generalizes to higher $l = n$.

II.6. An example: three disjoint spheres

Conflict sets are very difficult to calculate, both algebraically and numerically. Here we will present an example with three surfaces in $\mathbb{R}^3$ in order to have a non-trivial example of the situation encountered in theorem II.6, in which we can actually compute explicitly a parameterization of the conflict set. Our example is so typical that the calculations we do to compute the curvature almost provide a new proof of (II.6).
The example we are talking about is the example of three disjoint spheres in $\mathbb{R}^3$ with outward pointing normals. Their conflict set can be explicitly calculated if we make one further assumption: we will demand that the convex hull of their centers is a triangle with acute angles only.

LEMMA II.12. *Let p_1, p_2, p_3 be three points in $\mathbb{R}^3$, such that their convex hull is a triangle with acute angles. These points are in general position wrt. to linear subspaces of $\mathbb{R}^3$. Coordinates can be so chosen that mutual distances are preserved and $p_1 = (b_1, 0, 0)$, $p_2 = (0, b_2, 0)$, $p_3 = (0, 0, b_3)$.*

PROOF. Through each pair of points there passes a sphere, so that the two points are poles of the sphere. The three spheres have one point of intersection because the three points are in general position. This point of intersection is chosen to be the origin O. The lines through O and p_i will be the three coordinate axes. □

If the centers of the spheres do not span up a triangle with acute angles such a coordinate representation is not possible. In that case one could revert to the following more general representation. If we have n points in $\mathbb{R}^n$ that lie in general position this is a $\frac{n(n-1)}{2}$ dimensional space. Coordinates for it can be written

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & \cdots & 1 \\ b_{21} & b_{22} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Proof is by induction. For $n = 2$ the statement is clear.

II.6.1. A parameterization for the conflict set. We will continue with the diagonal representation of 3 points. We choose the three points as centers of mutually disjoint spheres with radii r_i. The mutual disjointness can be achieved with a condition on the radii:

$$r_i + r_j < \sqrt{b_i^2 + b_j^2}$$

We know that the conflict set of these spheres with outward normals lies in a plane. For the sake of brevity we will write this conflict set as $M_{p,r}$. The equations for the lift of $M_{p,r}$ to $\mathbb{R}^4$ are

$$x_0 = \|x - p_i\| - r_i \qquad 1 \leq i \leq 3$$

They can be rewritten in algebraic form:

$$(x_0 + r_i)^2 = \|x - p_i\|^2$$

It seems that these equations will add another algebraic component to the conflict set. They do but this component will be just another conflict set, namely the one with the orientations reversed.
We will frequently use $\bar{x} = (x_0, x)$ for a point in $\mathbb{R}^{n+1}$ whose projection to $\mathbb{R}^n$ is x. In this section $n = 3$. It is also useful to introduce a quadratic form of sign $(1, 3)$ on $\mathbb{R}^4$ namely

$$\|x_0, x\|_{-1} = x_0^2 - \|x\|^2$$

and the corresponding form

$$\langle (x_0, x), (y_0, y) \rangle_{-1} = x_0 y_0 - \langle x, y \rangle$$

The unique plane through the points p_1, p_2, p_3 is given by

$$\sum_{i=1}^{3} \frac{x_i}{b_i} = 1 \tag{II.9}$$

LEMMA II.13. *If q lies on $M_{p,r}$ then its reflection through the above plane also lies on the conflict set.*

PROOF. The reflection through the plane (II.9) is an isometry that maps spheres centered at p_i to themselves. The reflection thus leaves the conflict set $M_{p,r}$ also invariant. □

The intersection of this plane with the conflict set is a point α. The point lifts to a point $\bar{\alpha} = (\alpha_0, \alpha)$ in $\mathbb{R}^4$. Here α_0 is the "time" at which this component of starts appearing. So we have for $\bar{\alpha}$ that

$$\|(\alpha_0 + r_i, \alpha - p_i)\|_{-1} = 0 \; i = 1, \cdots, 3 \text{ and } \sum_{i=1}^{n} \frac{\alpha_i}{b_i} = 1$$

From the equations we can find two solutions for $\bar{\alpha}$. The solutions take the form

$$\alpha_i = b_i \frac{f_i \pm g_i \sqrt{D}}{2N_i} \; i = 1, \cdots, 3$$

Here D is a discriminant.

$$D = \left(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2\right)\left(b_1^2 + b_2^2 - (r_1 - r_2)^2\right)$$
$$\times \left(b_1^2 + b_3^2 - (r_1 - r_3)^2\right)\left(b_2^2 + b_3^2 - (r_2 - r_3)^2\right)$$

The equation $D = 0$ corresponds exactly to the degeneracy condition in the previous chapter, in the last section. If $D = 0$ then we will have for instance that

$$b_1^2 + b_2^2 = \|p_1 - p_2\|^2 = (r_1 - r_2)^2$$

so that

$$\pm \|p_1 - p_2\| = r_1 - r_2$$

In this case the conflict set is reduced to a line, but by our previous demand this is impossible. The condition corresponds exactly to the two cones in 4-space having 2-contact. For $i = 1, \cdots, 3$ the nominator N_i is

$$\left(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2\right) \left(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2 - b_3^2 (r_1 - r_2)^2 - b_2^2 (r_1 - r_3)^2 - b_1^2 (r_2 - r_3)^2\right)$$

The most obvious geometrical significance of the term

$$F = b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2$$

is that the surface of the triangle spanned by the p_i is

$$\frac{1}{2}\sqrt{b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2}$$

This can be proven using Archimedes formula.
The other term is

$$E = b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2 - b_3^2 (r_1 - r_2)^2 - b_2^2 (r_1 - r_3)^2 - b_1^2 (r_2 - r_3)^2.$$

In fact this term introduces no new geometry, it is in the ideal generated by

$$(b_1^2 + b_2^2 - (r_1 - r_2)^2, b_2^2 + b_3^2 - (r_2 - r_3)^2, b_3^2 + b_1^2 - (r_3 - r_1)^2)$$

This can be checked by computing a standard basis. Or - a little less obvious - one can remark that if we write $d_{i,j} = b_i^2 + b_j^2 - (r_i - r_j)^2$ that

$$E = -\frac{d_{1,2}^2 + d_{2,3}^2 + d_{3,1}^2}{4} + \frac{d_{1,2}d_{2,3} + d_{2,3}d_{3,1} + d_{3,1}d_{1,2}}{2}$$

The f_i are

$$\begin{aligned}
f_1 &= b_1(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2)(b_2^2 + b_3^3 - (r_2 - r_3)^2)(b_1^2 + (r_1 - r_2)(r_3 - r_1)) \\
f_2 &= b_2(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2)(b_3^2 + b_1^3 - (r_3 - r_1)^2)(b_1^2 + (r_2 - r_3)(r_1 - r_2)) \\
f_3 &= b_3(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2)(b_1^2 + b_2^3 - (r_1 - r_2)^2)(b_3^2 + (r_2 - r_3)(r_3 - r_1))
\end{aligned}$$

The g_i are

$$\begin{aligned}
g_1 &= r_1(b_2^2 + b_3^2) - r_2 b_3^2 - r_3 b_2^2 \\
g_2 &= r_2(b_3^2 + b_1^2) - r_3 b_1^2 - r_1 b_3^2 \\
g_3 &= r_3(b_1^2 + b_2^2) - r_1 b_2^2 - r_2 b_1^2
\end{aligned}$$

We remark that all f_i and g_i are in the ideal generated by $(d_{1,2}, d_{2,3}, d_{3,1}$ and that also $b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2$ is in this ideal. In fact f_i can be expressed in terms of the generators of the ideal. We have

$$\begin{aligned}
f_1 &= \frac{d_{1,2}d_{2,3} + d_{2,3}d_{3,1} - d_{3,1}^2}{2} \\
f_2 &= \frac{d_{2,3}d_{3,1} + d_{3,1}d_{1,2} - d_{1,2}^2}{2} \\
f_3 &= \frac{d_{3,1}d_{1,2} + d_{1,2}d_{2,3} - d_{2,3}^2}{2}
\end{aligned}$$

The g_i can be expressed as multiples of the differences $r_i - r_{q(i,1,3)}$ and the $d_{i,j}$.

$$\alpha_0 = \frac{\sqrt{F d_{1,2} d_{2,3} d_{3,1}} + \sum_{i=1}^{3} (r_i + r_{q(i,1,3)})(b_{q(i,2,3)}^2 (r_i - r_{q(i,1,3)})^2 - b_i^2 b_{q(i,1,3)}^2)}{2E}$$

If again, $d_{i,j} \neq 0$ then each of the two components calculated can be parameterized with a parameterization

$$\bar{\sigma}(t) = (\sigma_0, \sigma) = (\alpha_0, \alpha) + (\beta_0, \beta)(\cosh(t) - 1) + (\gamma_0, \gamma)\sinh(t)$$

The γ vector will be orthogonal to the β vector, and the β vector will lie in the plane defined by the points p_1, p_2, p_3. We thus have the relations:

$$\begin{aligned} \langle \beta, \gamma \rangle &= 0 \\ \sum_{i=1}^{n} \frac{\beta_i}{b_i} &= 0 \end{aligned} \tag{II.10}$$

Furthermore as at $t = 0$ the spheres will be disjoint we will have to assume that $\gamma_0 = 0$. Once we have found the solutions for $\bar{\alpha}$ we can use these to find the parameterization. Namely, writing u for $\cosh(t) - 1$ and v for $\sinh(t)$,

$$\|(\alpha_0 + \beta_0 u + \gamma_0 v + r_i, \alpha + \beta u + \gamma v - p_i)\|_{-1} - \|(\alpha_0 + r_i, \alpha - p_i)\|_{-1} = 0$$

and we get using the $\langle \cdot, \cdot \rangle_{-1}$ inner product.

$$\langle (2\alpha_0 + 2r_i + \beta_0 u + \gamma_0 v, \alpha - p_i + \beta u + \gamma v), (\beta_0 u + \gamma_0 v, \beta u + \gamma v) \rangle_{-1} = 0 \tag{II.11}$$

Now instead of v we can also use $-v$ in this equation. Adding the two we get

$$2\langle (\alpha_0 + r_i, \alpha - p_i) + (\beta_0 u, \beta u), (\beta_0 u, \beta u) \rangle_{-1} + \|(\gamma_0 v, \gamma v)\|_{-1}$$

What we now do is to insert several values of u, v in order to obtain a simpler system of equations, one that separates the questions of finding β and γ. We can use the values:

$$u = 1,\ v = \sqrt{3} \qquad u = 2,\ v = \sqrt{8}$$

The

$$2u\langle (\alpha_0 + r_i, \alpha - p_i), (\beta_0, \beta) \rangle_{-1} + u^2 \|\bar{\beta}\|_{-1} + v^2 \|\bar{\gamma}\|_{-1} = 0 \tag{II.12}$$

This is only a rank 2 system of equations, because $v^2 = u^2 + 2u$. We thus conclude a set of equations:

$$\langle (\alpha_0 + r_i, \alpha - p_i), (\beta_0, \beta) \rangle_{-1} - \|(\beta_0, \beta)\|_{-1} = 0 \quad i = 1, \cdots, 3 \tag{II.13}$$

We now take the identity (II.10) and the first of these sets to find the $\bar{\beta}$ vector. From the same source we obtain the

$$\|(\beta_0, \beta)\|_{-1} + \|(\gamma_0, \gamma)\|_{-1} = 0$$

This does not determine γ but we have that γ is a multiple of $(b_1^{-1}, b_2^{-1}, b_3^{-1})$ and that from (II.11) we can also conclude - now by subtracting the equation for v from the one for $-v$:

$$\langle (\alpha_0 + r_i + \beta_0 u, \alpha - p_i + \beta u), (\gamma_0, \gamma) \rangle_{-1} = 0$$

This gives rise to

$$\langle (\gamma_0, \gamma), (\beta_0, \beta) \rangle_{-1} = 0$$

So that $\gamma_0 = 0$ - which was to be expected.

One might remark that a solution for $\bar{\beta}$ is constructed from $\bar{\alpha}$, of which there are two. Also the equations for $\bar{\beta}$ are quadratic so that you would expect four solutions for $\bar{\beta}$. In fact this is not a problem. For each solution $\bar{\alpha}$ we have only one non-zero $\bar{\beta}$ and the solutions for the

two $\bar{\alpha}$ differ only by a sign change. This agrees with the geometry.
Our results are that

$$\|\beta\|^2 = \frac{d_{1,2}d_{2,3}d_{3,1}(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2 - E)}{4E} \tag{II.14}$$

and that

$$\|\bar{\beta}\|_{-1} = \frac{d_{1,2}d_{2,3}d_{3,1}}{4E} \tag{II.15}$$

Furthermore

$$\begin{aligned}
\beta_0 &= \frac{\sqrt{Fd_{1,2}d_{2,3}d_{3,1}}}{2E} \\
\beta_1 &= -\frac{b_1(r_1(b_2^2+b_3^2) - b_3^2r_2 - b_2^2r_3)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}} = -\frac{b_1g_1\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{F}} \\
\beta_2 &= -\frac{b_2(r_2(b_3^2+b_1^2) - b_3^2r_1 - b_1^2r_3)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}} \\
\beta_3 &= -\frac{b_3(r_3(b_1^2+b_2^2) - b_1^2r_2 - b_2^2r_1)\sqrt{d_{1,2}d_{2,3}d_{3,1}}}{2E\sqrt{(b_1^2b_2^2 + b_2^2b_3^2 + b_3^2b_1^2)}}
\end{aligned}$$

Note that the directions of the β and γ vectors can also be found from the proof of theorem I.3 . It seems useful to check whether the two results obtained from different reasonings do indeed coincide, as they should.
According to the proof of theorem I.3 the normal vector to the plane in which the conflict set lies in is - if defined:

$$\frac{p_1 - p_2}{r_2 - r_1} - \frac{p_1 - p_3}{r_3 - r_1} \tag{II.16}$$

All cyclic permutations of (II.16) are also allowed, but these are just collinear. Taking the outer product of the vector (II.16) and the vector $(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3})$ we obtain the direction of the β vector.
In figure II.2 the two components of the conflict set of three spheres are depicted.

II.6.2. The curvature. We can compute the curvature of the conflict set of the three spheres by using the parameterization. Once this has been done we compare the outcome with the outcome of (II.6). These should be the same.
The curvature of this curve will only depend on the β and the γ coefficients. In fact it will only depend on their euclidean norms. If a curve in $\mathbb{R}^2$ has parameterization $(B(\cosh(t) - 1), C\sinh(t))$ its curvature is

$$\kappa(t) = \frac{BC}{\left(B^2\sinh^2(t) + C^2\cosh^2(t)\right)^{\frac{3}{2}}}$$

Here we will have that

$$B = \|\beta\| \qquad C = \|\gamma\|$$

because the conflict set is in fact a plane curve.
As before it is more convenient to consider the square of the curvature.

$$\kappa_c^2(t) = \frac{\|\beta\|^2\|\gamma\|^2}{\left(\|\beta\|^2\sinh^2(t) + \|\gamma\|^2\cosh^2(t)\right)^3} = \frac{\|\beta\|^2\|\gamma\|^2}{\|\sigma'(t)\|^6}$$

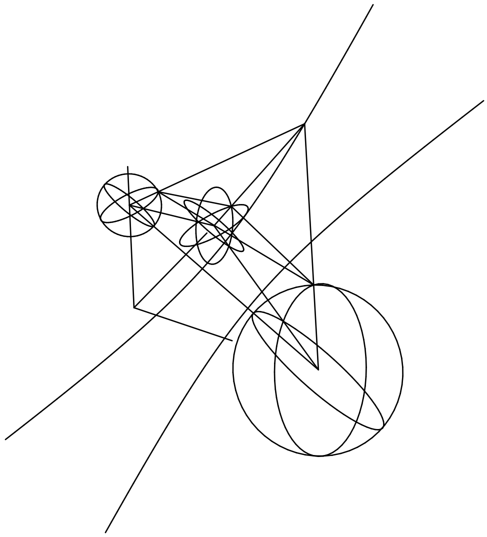

FIGURE II.2. The conflict set of three spheres

So that for the curvature we only need to determine the euclidean lengths of the β and γ vectors in $\mathbb{R}^3$. However this expression can still be considerably simplified. In the above we have seen that

$$\|(\beta_0, \beta)\|_{-1} + \|(\gamma_0, \gamma)\|_{-1} = 0 \tag{II.17}$$

and that we have for geometrical reasons:

$$\gamma_0 = 0$$

(This is because time will only increase or decrease on one component.)
We have obtained the formula for the curvature in terms of the b_i and the r_i but for the formula to be useful we would like to obtain this formula with other parameters. The first of these are the T_i, the distances from the point on the conflict set to the three centers of curvatures.

$$T_i^2 = \|\alpha + u\beta + v\gamma - p_i\|^2 = (\alpha_0 + r_i + u\beta_0)^2$$

Now we note that we also have $T_i - T_j = r_i - r_j$.
We strive for a representation in terms of these distances and some angles. Denote t_c the tangent to the conflict set and for further simplicity the parametrization of the conflict set in $\mathbb{R}^3$ will be written:

$$\sigma(t) = \alpha + \beta u + \gamma v$$

We also denote n_i the normal from the "center of curvature" to the conflict set, thus:

$$n_i = \frac{\sigma(t) - p_i}{T_i}$$

It is clear that

$$\sigma'(t) = \beta v + \gamma u + \gamma \tag{II.18}$$

Both vectors t_c and $\sigma'(t)$ lie along the conflict set.
We now calculate the inner product $\langle \sigma'(t), n_i \rangle$

LEMMA II.14. $\langle \sigma'(t), n_i \rangle = \|\sigma'(t)\| \cos\alpha = v\beta_0$

PROOF.

$$\begin{aligned}\langle \sigma'(t), n_i \rangle &= \|\sigma'(t)\| \cos\alpha \\ &= \langle \frac{\sigma - p_i}{T_i}, \beta v + \gamma u + \gamma \rangle \\ &= \frac{1}{T_i} \langle \alpha + \beta u + \gamma v - p_i, \beta v + \gamma u + \gamma \rangle \\ &= \frac{1}{T_i} \left(\langle \alpha - p_i, \beta v \rangle + uv\|\beta\|^2 + uv\|\gamma\|^2 + v\|\gamma\|^2 \right)\end{aligned}$$

Now we use the identity (II.17) which is in fact:

$$\beta_0^2 = \|\beta\|^2 + \|\gamma\|^2$$

Consequently:

$$\begin{aligned}&= \frac{1}{T_i} \left(\langle \alpha - p_i, \beta v \rangle + uv\beta_0^2 + v\|\gamma\|^2 \right) \\ &= \frac{1}{T_i} \left((\alpha_0 + r_i)\beta_0 v + uv\beta_0^2 \right) \\ &= \frac{1}{T_i} (\alpha_0 + u\beta_0 + r_i) v\beta_0 = v\beta_0\end{aligned}$$

In this last calculation we used that

$$\langle \alpha - p_i, v\beta \rangle + v\|\gamma\|^2 = \langle \alpha_0 + r_i, v\beta_0 \rangle$$

which is a straight consequence of (II.13) and (II.17). □

We try to find the sine of the angle α. We note that

$$\|\sigma'(t)\|^2 = \|\beta\|^2 v^2 + \|\gamma\|^2 (u+1)^2 = \beta_0^2 v^2 + \|\gamma\|^2$$

We combine the above with the lemma to find

$$\|\sigma'(t)\|^2 \sin^2\alpha = \|\gamma\|^2$$

We can use this to further simplify the expression for κ_c.

$$\kappa_c^2 = \sin^2\alpha \frac{\|\beta\|^2}{\|\sigma'(t)\|^4} \tag{II.19}$$

We now want to check whether formula (II.19) agrees with the previous calculations. First of all we check our description of the tangent spaces.
The lines from the p_i to σ meet the spheres around the p_i in points q_i.

$$q_i = \frac{r_i}{T_i}(\sigma - p_i) + p_i = \sigma - (\alpha_0 + \beta_0 u) n_i$$

They span up a simplex with the point σ.

LEMMA II.15. *The median starting from the vertex σ is the tangent line to the conflict set.*

PROOF. From an analysis of the tangent space to the conflict set in $\mathbb{R}^n$ of k hypersurfaces we know that $\sigma'(t)$ lies along $n_1 \times n_2 + n_2 \times n_3 + n_3 \times n_1$. This last vector should be orthogonal to $q_i - q_j = (\alpha_0 + \beta_0 u)(n_j - n_i)$. Indeed, the outer product is zero. □

This median is in itself again a conflict set of the tangent planes to the spheres at the points q_i. The center of the circumradius of the points (q_1, q_2, q_3) is the intersection of this median with the plane through (q_1, q_2, q_3). We would like to calculate this radius. The conflict set projects on each sphere. The projection is in each case a circle. For a fixed σ the tangent line is

$$\sigma'(t) - \langle \sigma'(t), n_i \rangle n_i$$

One can easily prove (using a cosine formula) that the distance from q_i to q_j is

$$\sqrt{\frac{d^2 d_{i,j}}{T_i T_j}}$$

where $d = \alpha_0 + \beta_0 u$.
Somewhat more work is involved in comparing (II.19) and (II.6). We recall that for four vectors in $\mathbb{R}^3$ we have the following identity

$$\langle a \times b, c \times d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle$$

With this we can calculate the cosine of the angles $\beta_i - \beta_j$ that are the angles between the planes spanned by $\sigma'(t), n_i$ and $\sigma'(t), n_j$.

$$\langle \sigma'(t) \times n_i, \sigma'(t) \times n_j \rangle = \|\sigma'(t)\|^2 \langle \frac{\sigma - p_i}{T_i}, \frac{\sigma - p_j}{T_j} \rangle - \|\sigma'(t)\|^2 \cos^2 \alpha$$

On the other hand using (II.2) we have:

$$\langle \sigma'(t) \times n_i, \sigma'(t) \times n_j \rangle = \|\sigma'(t)\|^2 \sin^2 \alpha \cos(\beta_i - \beta_j)$$

We combine the two to obtain

$$\cos(\beta_i - \beta_j) = \frac{\langle n_i, n_j \rangle - \cos^2 \alpha}{\sin^2 \alpha}$$

And $1 - \cos(\beta_{q(i,1,3)} - \beta_{q(i,2,3)})$ is

$$\frac{1 - \langle n_{q(i,1,3)}, n_{q(i,2,3)} \rangle}{\sin^2 \alpha} \tag{II.20}$$

We also write

$$\begin{aligned}
\sin^2(\frac{\beta_1 - \beta_2}{2}) &= \frac{1 - \cos(\beta_1 - \beta_2)}{2} \\
&= \frac{1 - \langle n_1, n_2 \rangle}{2 \sin^2 \alpha} \\
&= \frac{T_1 T_2 - \langle \sigma - p_1, \sigma - p_2 \rangle}{2 T_1 T_2 \sin^2 \alpha} \\
&= \frac{2 T_1 T_2 - \|\sigma - p_1\|^2 - \|\sigma - p_2\|^2 + b_1^2 + b_2^2}{4 T_1 T_2 \sin^2 \alpha} \\
&= \frac{d_{1,2}}{4 T_1 T_2 \sin^2 \alpha}
\end{aligned}$$

Now we switch back to the formulas in paragraph II.3

$$\|t_c\|^2 = \frac{1}{4 \sin^2 \alpha} \frac{d_{1,2} d_{2,3} d_{3,1}}{T_1^2 T_2^2 T_3^2}$$

And we also recalculate

$$(\nabla_e \vec{t_c})_1^2 + (\nabla_e \vec{t_c})_2^2 = \sin^4 \alpha \frac{b_1^2(r_2 - r_3)^2 + b_2^2(r_3 - r_1)^2 + b_3^2(r_1 - r_2)^2}{T_1^2 T_2^2 T_3^2}$$

Using the previous identities for $\|\beta\|^2$ and $\|\gamma\|^2$ in equations (II.14) and (II.15) it follows that the curvature formula (II.19) agrees with the result obtained in (II.6).

III The conflict set as a wavefront

III.1. Some symplectic and contact geometry

The equidistants and focal sets we have encountered in the previous chapter can be studied from a slightly more abstract point of view, that of symplectic geometry. It turns out that this is a very fruitful approach. In this chapter we start to apply this theory to the geometry of conflict sets. A standard introduction to symplectic geometry is [**Arn86**]. See also [**Dui96**].

III.1.1. Lagrangian manifolds and symplectic manifolds. A symplectic manifold is a manifold with a closed non-degenerate 2-form ω on it. Cotangent bundles T^*X are symplectic manifolds. On them there is defined a canonical 1-form σ such that $\omega = \mathrm{d}\,\sigma$. If $X = \mathbb{R}^n$ the canonical 1-form is $\sigma = \sum \xi_i \,\mathrm{d}\,x_i$ and $\omega = \sum \mathrm{d}\,\xi_i \wedge \mathrm{d}\,x_i$.

Definition III.1. A submanifold L of T^*X is called *Lagrangian* if the restriction of ω to TL is 0 and $\dim L = n$. If in addition the restriction of $\sigma = \sum \xi_i \,\mathrm{d}\,x_i$ to TL is zero then L is a conic Lagrangian manifold.

From a conic Lagrangian manifold we can always remove the zero section of $T^*X \to X$, the resulting manifold will still be a conic Lagrangian manifold.
For local considerations we can put $X = \mathbb{R}^n$. There are several ways to construct Lagrangian submanifolds of $T^*\mathbb{R}^n$.

- *Generating functions:* Divide the index set $\{1, \cdots, n\}$ into two disjoint parts I and J. Let $S\colon \mathbb{R}^n \to \mathbb{R}$ be a real-valued function: $S = S(x_I, \xi_J)$. Then $\{(x, \xi) \in T^*\mathbb{R}^n \mid x_J = \partial S/\partial \xi_J \quad \xi_I = -\partial S/\partial x_I\}$ is lagrange.
- *Phase functions:* Let $F(x, s)\colon \mathbb{R}^{n+k} \to \mathbb{R}$ be such that a $\mathrm{d}_s F$ has a regular value at 0. Then $(\mathrm{d}_s F)^{-1}(0)$ is an n-dimensional submanifold of $\mathbb{R}^{n+k}$ whose image under the mapping $(x, \mathrm{d}_x F)$ is an immersed Lagrangian submanifold of $T^*\mathbb{R}^n$.

All Lagrangian submanifolds of $T^*\mathbb{R}^n$ can be constructed in both ways, at least locally. The two representations are connected by the following. If L is represented around $(x_0, \xi_0) \in T^*\mathbb{R}^n$ by $S(x_I, \xi_J)$ then there is a phase function $F(x, s) = S(x_I, s) + \langle s, x_J\rangle$ that also represents L.

III.1.2. Legendrian manifolds and contact manifolds. A contact manifold is a manifold M with a contact structure. The contact structure is defined using a 1-form σ, appropriately called the contact form.
The contact form σ is to satisfy two demands. The derivative $\mathrm{d}\,\sigma$ is not degenerated on the hypersurfaces in the tangent space of M that result from $\sigma = 0$. From this condition it already follows that the manifold has to be odd-dimensional. Indeed, non-degenerate 2-forms only exist on even-dimensional spaces. So the tangent space to the contact-manifold has odd dimension, say $2n + 1$.
The second condition is that the 1-form is maximally non-integrable. In earthly terms this is equivalent to saying that $\sigma \wedge (\mathrm{d}\,\sigma)^n$ is a volume form.
It will also be equivalent to an integral manifold of σ having maximal dimension $n - 1$.
The contact structure on M is the hyperplane subbundle $\ker\sigma$ of TM.

Definition III.2. Submanifolds of contact manifolds that are integral manifolds of maximal dimension of $\ker\sigma$ are called *Legendrian submanifolds.*

Let the contact manifold be fibered such that each fiber is a Legendrian submanifold of the contact manifold. We will only consider such *fibered contact manifolds.*

The most important example of such a contact manifold is the following. Let N be any manifold. At a point q of N consider the set of all hyperplanes through the origin in the tangent space T_qN. This is a projective space $\mathbb{P}T^*_qN$. To construct a field of hyperplanes on the manifold $\mathbb{P}T^*N$ consider a point $(x_1, ..., x_n, \xi_1; ...; \xi_n)$ on it. This point defines a hyperplane in the tangent space of the base manifold N. This plane lifts to an $n-1$ dimensional plane in the $2n-1$ dimensional tangent space to the total space $\mathbb{P}T^*N$ in $(x_1, ..., x_n, \xi_1; ...; \xi_n)$. Join this plane with the vertical directions and one obtains a field of hyperplanes in $T(\mathbb{P}T^*N)$. This field of hyperplanes defines a contact structure.
It is verified that each fiber of the fibration $\mathbb{P}T^*N \mapsto N$ is an integral manifold of the contact structure. Indeed, all tangent directions to fibers are vertical directions. They always lie in the hyperplanes of the contact structure.
We have thus made $\mathbb{P}T^*N$ into a fibered contact manifold. Two other examples of contact manifolds are important.

- $J^1(N)$: Every germ is defined by its function value z and derivative y in a point x. The 1-form $\sigma = \mathrm{d}\, z - y\, \mathrm{d}\, x$ defines a 1-form on $J^1(N)$. This 1-form defines a contact structure because $\sigma \wedge (\mathrm{d}\, \sigma)^n$ defines a volume form.
- ST^*N: If we coorient each contact element in $\mathbb{P}T^*N$ we obtain a double covering of that space. This is ST^*N.

We have the following generalization of the Darboux lemma

LEMMA III.3. *Every contact manifold is locally contactomorphic to $J^1(\mathbb{R}^n)$, for some n.*

As an example we take $\mathbb{P}T^*\mathbb{R}^n$. Around a point in $\mathbb{P}T^*\mathbb{R}^n$ coordinates can be written

$$(q_1, \cdots, q_n, p_1; \cdots, p_{n-1}; 1).$$

Thus there is a local contactomorphism to $J^1(\mathbb{R}^{n-1})$. In these coordinates the contact form will be $\mathrm{d}\, q_n - \sum_{i=1}^{n-1} p_i\, \mathrm{d}\, q_i$.
As with Lagrangian manifolds Legendrian manifolds can also be constructed using both generating functions and phase functions. Before getting to this point we have to explain both symplectization and contactization.

III.1.3. Symplectization. Symplectization is a canonical construction that associates to a contact manifold V a symplectic manifold and to a Legendrian submanifold L a Lagrangian submanifold Λ.
The contact form that determines the field of hyperplanes in the tangent space is determined up to a multiple. Instead of $\mathrm{d}\, z - p\, \mathrm{d}\, q$ we might as well write $\lambda\, \mathrm{d}\, z - \lambda p\, \mathrm{d}\, q$. Here λ is some independent variable $\neq 0$.
The form $(\mathrm{d}(\lambda p\, \mathrm{d}\, q))^{n+1}$ is indeed a volume form so that $\mathbb{R} \times V$ is a symplectic manifold. If L is a Legendrian submanifold of V then $\Lambda = \mathbb{R} \times L$ is lagrange in $\mathbb{R} \times V$. Infact, Λ will not just be lagrange, Λ will be conic lagrange.

EXAMPLE III.4. Let L be a Legendrian submanifold of $\mathbb{P}T^*\mathbb{R}^n$. The symplectization of $\mathbb{P}T^*\mathbb{R}^n$ will be $T^*\mathbb{R}^n$. The $n-1$ dimensional Legendrian manifold is send to a conic Lagrangian manifold.

III.1.4. Contactization. Contactization is a canonical way of constructing a contact manifold out of a symplectic one. First we notice that on the symplectic manifold there is defined the canonical 1-form σ. We take the direct product of $\mathbb{R}$ with the contact manifold. This is an additional coordinate z. Lagrangian submanifolds Λ carry over to Legendrian

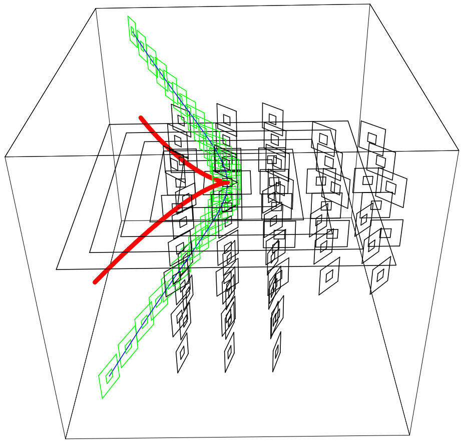

FIGURE III.1. Field of hyperplanes that makes a cusp smooth.

ones by an integration

$$z = \int \sigma = \int \xi \, \mathrm{d}\, x$$

This is how we go from $T^*\mathbb{R}^n$ to $\mathrm{J}^1(\mathbb{R}^n, \mathbb{R})$.

III.1.5. Constructions with phase functions. In section III.1.1 we have seen how to use phase functions to construct Lagrangian submanifolds of $T^*\mathbb{R}^n$. There are essentially two ways of constructing Legendrian submanifolds from phase functions.
The first construction is connected with contactization. Take a Lagrangian Λ submanifold of $T^*\mathbb{R}^n$. Let Λ be given by a phase function $F(x,s)$. If $\mathrm{d}_s\, F$ has a regular value at 0 then $(d_s F)^{-1}(0) \ni (x,s) \mapsto (x, \mathrm{d}_x\, F(x,s))$ has a Lagrangian manifold in $T^*\mathbb{R}^n$ as an image Λ_F. The contactization process now gives a Legendrian submanifold in the contactization of $T^*\mathbb{R}^n$, that is $J^1(\mathbb{R}^n)$. This submanifold is given by the image of:

$$(\mathrm{d}_s\, F)^{-1}(0) \ni (x,s) \mapsto (x, F(x,s), \mathrm{d}_x\, F(x,s))$$

In this way we have constructed a Legendrian submanifold starting from what is a phase function in the Lagrangian sense.
The second construction is connected with symplectization.

DEFINITION III.5. If we have a phase function such that $F, \mathrm{d}_s\, F(x,s)$ has a regular value in 0 then it is called non-degenerate.

REMARK III.6. Except in isolated cases (e. g. in subsection IV.2.3) this is what we will mean by a non-degenerate phase function.
The image $L_F \subset \mathbb{P}T^*\mathbb{R}^n$ of

$$(F(x,s), \mathrm{d}_s\, F)^{-1}(0) \ni (x,s) \mapsto (x, \mathrm{d}_x\, F(x,s)) \tag{III.1}$$

is $n-1$ dimensional and isotropic. We map

$$\mathbb{R} - \{0\} \times L_F \to T^*\mathbb{R}^n \tag{III.2}$$

or

$$\mathbb{R}_{>0} \times L_F \to T^*\mathbb{R}^n$$

by

$$(\tau, q, p) \mapsto (q, \tau p)$$

the image of which is a conic Lagrangian manifold and can be interpreted in the first case as a submanifold of $\mathbb{P}T^*\mathbb{R}^n$ and in the second case as a submanifold of $ST^*\mathbb{R}^n$.
There is another way of looking at this. We can regard $F(x,s)$ as a family of (oriented) hypersurfaces in $T\mathbb{R}^{n+k}$: for each $(x_0,s_0) \in \mathbb{R}^{n+k}$ we have the hyperplane $\mathrm{d}_{x,s} F(x_0,s_0)(\delta x, \delta s) = 0$ in the tangent space $T_{(x_0,s_0)}\mathbb{R}^{n+k}$. The ones that project as hyperplanes in $T\mathbb{R}^n$ are those that have $\frac{\partial F}{\partial s}(x_0,s_0) = 0$. From the requirement that $(F, \mathrm{d}_s F)$ has a regular value at 0 it follows that we can always write locally

$$F(x,s) = x_n - \tilde{F}(x_1, \cdots, x_{n-1}, s)$$

so that indeed a Legendrian submanifold is constructed, either in $\mathbb{P}T^*\mathbb{R}^n$ or in $ST^*\mathbb{R}^n$.

III.1.6. Note on terminology. Now that the notions “conic Lagrange”, “Legendre” and “Lagrange” have been clearly established we will for the sake of readability start using phrases such as: “... hence C is conic lagrange in X ... ” , when we mean to say that there exists a conic Lagrangian manifold $L \subset T^*X$ such that the projection of L to X is C: $\pi_X L = C$ In this sort of phrases it does not matter very much whether C is conic Lagrange or Legendre, hence we will use both terminologies. What is meant will always be clear from the context.

III.1.7. Canonical relations. From one conic Lagrangian manifold one constructs a new one by means of “a section and a projection”. This idea is made precise by the notion of a canonical relation. For canonical relations we refer to the book [**Hör85**] , pages 289 *et seq.*, in particular theorem 21.2.14.

Definition III.7. A canonical relation between two symplectic manifolds (S_1, ω_1) and (S_2, ω_2) is a Lagrangian submanifold of $(S_1 \times S_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$.

By π_1 and π_2 we mean the projections $\pi_1 \colon S_1 \times S_2 \to S_1$ and $\pi_2 \colon S_1 \times S_2 \to S_2$.
Here are a few ways to construct canonical relations:

- A Lagrangian submanifold N of $(S_1 \times S_2, \omega_1 + \omega_2)$ gives rise to a canonical relation between S_1 and S_2 by the accent mapping $'$:

$$(N)' = \{x_1, \xi_1, x_2, \xi_2 \in S_1 \times S_2 \ \mid (x_1, \xi_1, x_2, -\xi_2) \in N\}$$

- If S_2 is a point a canonical relation between S_1 and S_2 is any Lagrangian manifold of S_1.
- If $f \colon S_1 \to S_2$ is a symplectomorphism, then the graph of f is a canonical relation between S_1 and S_2.

The composition of two maps can be done via their graph. If we have $f \colon S_1 \to S_2$ and $g \colon S_2 \to S_3$ symplectomorphisms then the graph of their composition is

$$\pi_{S_1 \times S_3}(\mathrm{gr}(f) \times \mathrm{gr}(g) \cap S_1 \times \Delta(S_2 \times S_2) \times S_3) \subset S_1 \times S_3$$

This is also how canonical relations are composed.

Theorem III.8. *Let $S_i, i = 1 \cdots 3$ be three symplectic manifolds. Let G_1 be a canonical relation between S_1 and S_2 and G_2 one between S_2 and S_3. If $G_1 \times G_2$ intersects $S_1 \times \Delta(S_2) \times S_3$ transversally then the image G_3 under the projection $S_1 \times S_2 \times S_2 \times S_3 \mapsto S_1 \times S_3$ is a canonical relation between S_1 and S_3. We call this the composition $G_1 \circ G_2$ of G_1 and G_2.*

In our applications we will usually apply this theorem there where S_3 is a point. So we use the next proposition, that rephrases the demand in the theorem of Hörmander.

PROPOSITION III.9. *Let G_1 be a canonical relation between S_1 and S_2 whose projection to S_2 is an immersion and let G_2 be a canonical relation between S_2 and a point. Then the composition $G_1 \circ G_2$ is a canonical relation , and thus a Lagrangian manifold in S_1, if $\pi_2(G_1) \pitchfork G_2$.*

PROOF. We need that

$$G_1 \times G_2 \pitchfork S_1 \times \Delta(S_2) \tag{III.3}$$

This intersection is contained in the graph of the projection $\pi_2 \colon G_1 \to G_2$. We have that (III.3) holds iff.

$$\mathrm{gr}(\pi_2) \pitchfork G_1 \times G_2$$

this in turn is true iff.

$$\pi_2(G_1) \pitchfork G_2$$

□

III.1.8. The Gauss map. The space of oriented lines in $\mathbb{R}^n$ can be realized as the symplectic manifold T^*S^{n-1}. Namely if ℓ is a directed line in $\mathbb{R}^n$ then this line has a direction $v \in S^{n-1}$.
The direction v determines a hyperplane through the origin:

$$H_\ell = \{x \in \mathbb{R}^n \mid \langle v, x\rangle = 0\} \subset \mathbb{R}^n$$

The hyperplane H_ℓ can be identified with the tangent plane $T_v S^{n-1}$, to be identified with $T_v^* S^{n-1}$ through the Legendre mapping. The intersection point of H_ℓ and ℓ determines thus a point in $T_v^* S^{n-1}$. The normal to an oriented $(M, \vec{n})$ hypersurface in $\mathbb{R}^n$ is a directed line.

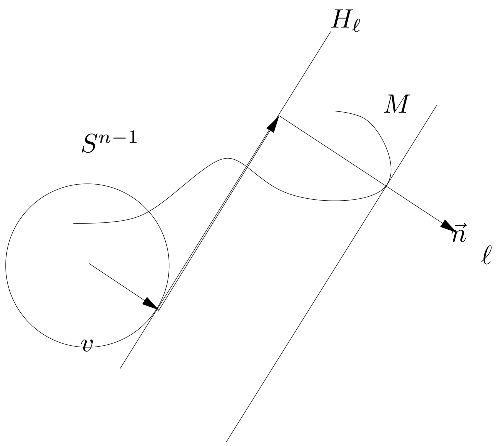

FIGURE III.2. Construction of space of directed lines

We thus have a map $N^*M \mapsto T^*S^{n-1}$. As a map $\vec{n} \colon M \to S^{n-1}$ it is known as the Gauss map. The image of this map is Lagrangian in T^*S^{n-1}, see [**Arn90**].

THEOREM III.10. *There is a canonical relation G between T^*S^{n-1} and $T^*\mathbb{R}^n \setminus \{0\}$ such that its composition with any conic Lagrangian manifold in $T^*\mathbb{R}^n \setminus \{0\}$ yields a Lagrangian submanifold of T^*S^{n-1}. In particular, the composition of G with the conormal bundle N^*M of a submanifold $M \subset \mathbb{R}^n$ yields a Lagrangian submanifold of T^*S^{n-1} that coincides with the image of the Gauss map.*

PROOF. We write (v, μ) for coordinates on T^*S^{n-1}. They are really coordinates on $T^*\mathbb{R}^n$ but we will always have $\langle v, \mu\rangle = 0$ and $\|v\| = 1$ so that they can be used as coordinates on T^*S^{n-1}. The canonical symplectic form is $\mathrm{d}\, v \wedge \mathrm{d}\, \mu$.
Consider the following subset G of $T^*S^{n-1} \times (T^*\mathbb{R}^n \setminus 0)$

$$\{(v, \mu, x, \xi) \mid v = \frac{\xi}{\|\xi\|}, \quad \mu = x - \frac{\langle x, \xi\rangle \xi}{\|\xi\|^2}, \quad \|\xi\| = C\}$$

(Here $C > 0$ is some constant.)
The subset G mimicks exactly the geometric construction that associates to a point on a conormal N^*M a directed line in T^*S^{n-1}. Our proof will consist of two steps:

- that G is a canonical relation from $T^*\mathbb{R}^n \setminus 0$ to T^*S^{n-1}
- that conic Lagrangian manifolds are exactly those that can be pulled back to T^*S^{n-1} by G.

Step 1. Remark that instead of $\omega_2 = \sum_{i=1}^n \mathrm{d}\, x_i \wedge \mathrm{d}\, \xi_i$ we can take any multiple $\lambda\omega_2$ of ω_2 as a symplectic form, because $\omega_2 = 0$ on a tangent space iff. a nonzero multiple of it is zero. Both forms give the same structure in the tangent space.
Accordingly if we prove that $\lambda\omega_2 = \mathrm{d}\, v \wedge \mathrm{d}\, \mu$ on every T_pG we have shown that G is a canonical relation, between $T^*S^{n-1}, \mathrm{d}\, v \wedge \mathrm{d}\, \mu$ and $T^*\mathbb{R}^n \setminus \{0\}, \omega_2$.
We are to prove that for two tangent vectors $(\delta v, \delta\mu, \delta x, \delta\xi)$ and $(\delta v', \delta\mu', \delta x', \delta\xi')$ at $p = (v, \mu, x, \xi)$ we have

$$\lambda \sum_{i=1}^n \delta x_i \delta\xi_i' - \delta x_i' \delta\xi_i = \sum_{i=1}^n \delta v_i \delta\mu_i' - \delta v_i' \delta\mu_i$$

We first get rid of the v coordinates.

$$v = \frac{\xi}{\|\xi\|} \quad \Rightarrow \quad \delta v_i = \frac{\delta\xi_i}{C}$$

So that we are left with:

$$\sum_{i=1}^n \left(\lambda\delta x_i + \frac{\delta\mu_i}{C}\right)\delta\xi_i' = \sum_{i=1}^n \left(\lambda\delta x_i' + \frac{\delta\mu_i'}{C}\right)\delta\xi_i \tag{III.4}$$

The next candidates for removal are the μ coordinates:

$$\begin{aligned} \delta\mu_i &= \delta x_i - \frac{\langle x, \xi\rangle}{C^2}\delta\xi_i - \frac{\xi_i}{C^2}\left(\sum_{j=1}^n \xi_j \delta x_j + x_j \delta\xi_j\right) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned} \tag{III.5}$$

The terms coming from III inserted in each side of (III.4) annul because they are multiples of $\sum \xi_i \delta\xi_i$. The terms from II are identical at both sides of (III.4). Finally choose $\lambda = \frac{-1}{C}$.
Step 2: G is a canonical relation. Its projection to $T^*\mathbb{R}^n$ is $\|\xi\| = C$. The projection is an immersion. We can apply proposition III.9: if $L \subset T^*\mathbb{R}^n$ intersects $\|\xi\| = C$ transversely we have that via G we can define in T^*S^{n-1} an "image of the Gauss map". But $L \pitchfork \{\|\xi\| = C\}$ amounts to saying that L is conic lagrange. □

REMARK III.11. Though probably known to experts we have not found this Gauss canonical relation for arbitrary conic Lagrangian manifolds in the literature.

III.1.9. Wavefront from a submanifold. In the sequel we will encounter a distance function on a $\mathcal{C}^\infty$ manifold X. If $X = \mathbb{R}^n$ and we use the standard euclidean metric it is clear what is meant. In other cases where we define a distance function by means of a Hamiltonian we need to impose rather strong conditions on X and the Hamiltonian H.

DEFINITION III.12. A *homogeneous Hamiltonian* is a function on the slit cotangent bundle $T^*X \setminus 0$ that is homogeneous of degree 2 in the coordinates of the fiber of $T^*X \setminus \{0\} \mapsto X$. In other words, it is the continuous assignment to each fiber of a function homogeneous of degree 2.

EXAMPLE III.13. • Any finsler metric on X gives rise to a homogeneous Hamiltonian.
• If $g_{ij}(x)$ is the matrix of a Riemannian metric on X a homogeneous Hamiltonian is

$$H\colon (x,\xi) \mapsto \sum_{i,j} g^{ij}\xi_i\xi_j$$

Throughout this chapter and the next we will assume

- **1** that all our Hamiltonians are $\mathcal{C}^\infty$. In physics this will not always be the case. For instance, on the border of two different media there is refraction. Here the Hamiltonian will not be $\mathcal{C}^\infty$.
- **2** that the quadratic form

$$\frac{\partial^2 H}{\partial \xi^2}$$

  is non-degenerated, for all x, ξ.

To contrast our approach with others we explicitly remark that our quadratic Hamiltonians include pseudo-Riemannian metrics.

DEFINITION III.14. For any function $K\colon T^*X \to \mathbb{R}$ the Hamiltonian vectorfield $\mathrm{vf}(K)$ is defined by

$$\omega(\mathrm{vf}(K),\cdot) = -\,\mathrm{d}\,K \tag{III.6}$$

The Hamiltonian vectorfield $\mathrm{vf}(K)$ has an induced exponential mapping

$$\exp_K\colon T^*X \times \mathbb{R} \to T^*X$$

The map $\exp_K(x,\xi,t)$ is called the *Hamiltonian flow*.
Take any conic Lagrangian submanifold L of $T^*X \setminus 0$. Because H is homogeneous in the fiber coordinates the intersection $L \cap H^{-1}(1)$ is transversal. For a fixed $t = t_0$ we can flow out $L \cap H^{-1}(1)$ with the Hamiltonian vectorfield of H. The image is

$$\exp_H(L \cap H^{-1}(1), t_0)$$

is isotropic and of dimension $n-1$.

REMARK III.15. Note that it does not matter much whether we flow out an isotropic manifold by the Hamiltonian vectorfield of H or by $\mathrm{vf}(H^p)$. From equation (III.6) we can deduce that the one vector $\mathrm{vf}(H^p)$ is a multiple of the other $\mathrm{vf}(H)$ when restricted to $H = 1$.

If we also "multiply the fibers" of $\exp_H(L \cap H^{-1}(1), t_0)$

$$\mathbb{R}_{>0} \times T^*X \to T^*X \qquad (\lambda, x, \xi) \to (x, \lambda\xi)$$

we obtain a map

$$L \mapsto L_t = \exp_H^{\mathbb{R}}(L, t)$$

that sends conic Lagrangian submanifolds to conic Lagrangian submanifolds. In particular, we may apply $\exp_H^{\mathbb{R}}$ to the fiber $T^*_{x_0}X$ over x_0 of $T^*X \setminus 0$. Denote

$$S(x_0, t) = \exp_H^{\mathbb{R}}(T^*_{x_0}X, t)$$

With the assumptions we make $S(x_0, t)$ is a $\mathcal{C}^\infty$ manifold. Note that the manifold $S(x_0, t)$ is not necessarily symmetric or connected. The manifolds $S(x_0, t)$ are smooth. Their projection to X is not always smooth.

We will now make an additional assumptions about the pair X, H.

- **3** The integral curves of $\mathrm{vf} H$ are defined for all $t \in \mathbb{R}$.

In case H is a Finsler metric the integral curves of $\mathrm{vf}(H)$ are geodesics. Thus, the assumption amounts to saying that X is complete, see [**BCS00**]. This assertion is called the Hopf-Rinow theorem.

Consider the integral

$$A = A(x_0, x_1) = \int_\Gamma \xi \,\mathrm{d}\, x \tag{III.7}$$

where Γ is a curve in $T^*X \cap \{H = 1\}$ that makes this action stationary and whose projection to the X space goes from x_0 to x_1. The solution curves to this problem are locally uniquely defined due to the two assumptions above. Thus the integral A depends only on x_0 and x_1 - and on the energy-level $H = 1$ chosen.

The manifolds $S(x_0, t)$ are level sets of the function $x \mapsto \pi_X(A(x_0, x))$. This makes $A(x_0, x_1)$ the perfect candidate for a distance or time function, as will be shown in the next section. The function $A(x_0, x_1)$ is $\mathcal{C}^\infty$ on $X \times X \setminus \Delta$. We will also call it the **work function**.

It is the analogue of $\|x_0 - x_1\|$ in the euclidean case.

Our last and final assumption is:

- **4** For any pair x_0 and x_1 there is at most one curve Γ as above.

Now we have assembled all we needed for our analysis. The $S(x_0, t)$ are defined for all x_0 and t by the third assumption and their projection to X is smooth by the fourth assumption. Sadly there is a catch here: these demands are very strong. When $X = \mathbb{R}^n$ and we use the standard Riemannian metric they are trivially satisfied. There is one other well-known case where these assumptions are satisfied. We cite the Cartan-Hadamard theorem.

Theorem III.16. *If X is a simply connected complete Riemannian manifold with all sectional curvatures negative or 0 then any two points can be joined with a unique geodesic. The exponential map $\exp\colon T_pM \to X$ is a diffeomorphism for every $p \in X$.*

The Cartan-Hadamard theorem also holds in the finsler setting. It singles out an exceptional situation because already when $X = S^n$ our assumptions are no longer satisfied because closed geodesics abound there. This is clearly not what we want.

We are thus led to contemplating whether we can do a little more. For instance we could impose our conditions in some open subset of $X \times \mathbb{R}$. So if $X = S^n$ then we could leave out say the north-pole $\{0\}$ and consider for each $x \in S^n \setminus \{0\}$ only those times for which the north pole is not attained. However $S^n \setminus \{0\}$ can be spread out and in that case it will look just like $\mathbb{R}^n$ and such additional time and space restrictions will certainly not ease notation.

Instead we note that if we have on $\mathbb{R}^n$ a quadratic Hamiltonian independent of x then all our demands are satisfied. Such quadratic Hamiltonians are often called translation invariant. In the case where $H = \sum g^{ij}(x)\xi_i\xi_j$ it is possible to choose local coordinates such that

$$\frac{\partial g^{ij}}{\partial x_k}(0) = 0 \tag{III.8}$$

Thus in some sense H does not depend on x here. For a proof of this assertion, see [**Hör85**], part III, appendix C. We conclude that nearly all quadratic Hamiltonians are near to translation invariant ones.

We will now turn our attention to wavefronts not just from a point, but from a submanifold M of X with its accompanying Hamiltonian H the wavefront of M at time t.

Definition III.17. For $\gamma \in \mathrm{Emb}(M, X)$ define

$$\mathrm{WF}(t, M) = \bigcup_{s \in M} S(\gamma(s), t) \subset M \times T^*X$$

The singular values of the projection of $\mathrm{WF}(t, M)$ to T^*X are called the wavefront $\Sigma(t, M)$ of M at time t.

The wavefront $\Sigma(t, M) \subset T^*X$ is thus defined as an envelope, and not as a flow-out. The following lemma relates the flow-out and the envelope.

Lemma III.18 (Huygens principle).

$$\Sigma(t, M) = \exp_H^{\mathbb{R}}(N^*M)$$

If $M \subset X$ is cooriented and a hypersurface and H is positive definite the manifold $N^*M \cap \{H = 1\}$ has two components, each corresponding to one sign of the orientation. We could so speak of $\Sigma^+(t, M)$ and $\Sigma^-(t, M)$, assuming t is positive.

Example III.19. Take $X = \mathbb{R}^2$ and $H = \xi_1^2 - \xi_2^2$ then the "distance" from $p_0 = (x_0, y_0)$ to $p_1 = (x_1, y_1)$ is $\sqrt{|(x_0 - x_1)^2 - (y_0 - y_1)^2|}$, if we choose energy levels ± 1. Indeed, some points cannot even be connected, for instance the origin and $(1, 1)$. However, our four assumptions are satisfied. In figures III.3 and III.4 we see the envelopes for different values

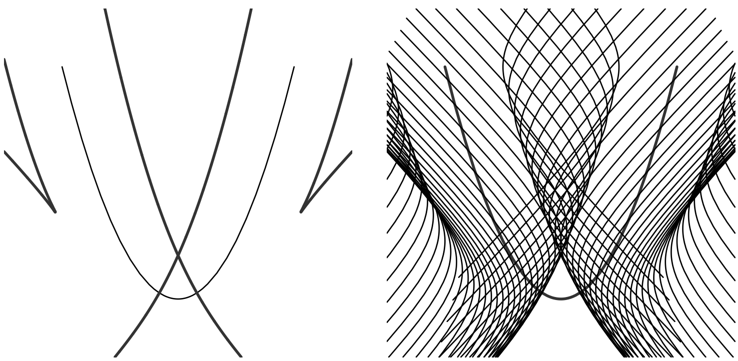

Figure III.3. Fronts for $\xi_1^2 - \xi_2^2 = -1$

of the energy level. Below are all the wavefronts from different points. The envelopes are above.

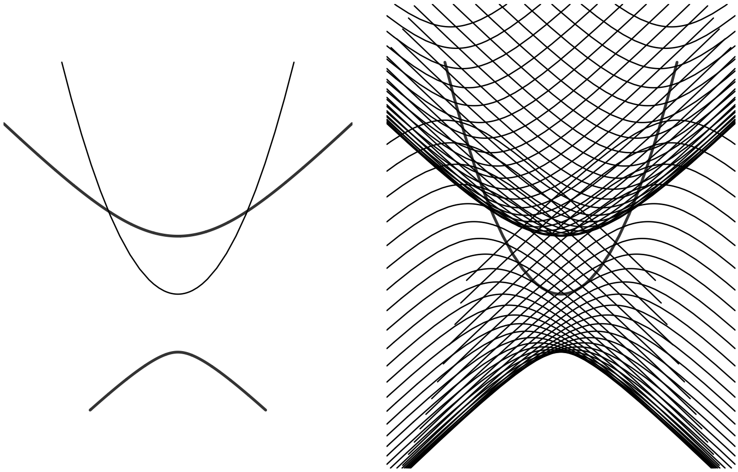

FIGURE III.4. Fronts for $\xi_1^2 - \xi_2^2 = 1$

Several articles among which [**IPS00**] deal with pseudo-Riemannian metrics to investigate questions in generic differential geometry. With our definition of distance we do not need to distinguish between the case of pseudo-Riemannian geometry and the case of Riemannian geometry.
We repeat that the combination of the work function A with an embedding γ allows us to produce a globally defined non-degenerate phase function $F(x, s) = A(x, \gamma(s))$ for $\Sigma(t, M)$. This will be essential in the sequel.
In the literature one finds the assertion that the existence of a global non-degenerate phase function implies some conditions on the cohomology of the ambient manifold, see for instance [**Zak84**], p. 2733. In this way the conclusion of the Cartan-Hadamard theorem is not so surprising.

III.1.10. Quadratic Hamiltonians and the Legendre transform. We have already seen that the integral curves of the Hamiltonian vectorfield $\mathrm{vf}(H)$ correspond to geodesics. We would like to explain in some detail the relation between the Hamiltonian viewpoint and the more traditional geometric viewpoint. For the geometry we refer to [**BCS00**]. The main change that we make is that we do not require that our "metrics" are positive definite.
The Hamiltonian dynamics take place in the cotangent space and the differential geometry in the tangent space. There is a pairing between the two:

$$T_x^* X \times T_x X \to \mathbb{R} \qquad (\xi, v) \mapsto \xi(v)$$

For a fixed $v \in T_x X$ we can look for critical values of the function

$$\xi(v) - H(x, \xi)$$

These critical values are attained there where

$$v = \frac{\partial H}{\partial \xi} \tag{III.9}$$

Because

$$\det(\frac{\partial^2 H}{\partial \xi^2}) \neq 0$$

we know that (III.9) defines a local diffeomorphism from T^*X to TX. It is the **Legendre transform**. When v is the Legendre transform of ξ we write $v = \xi^\flat$.
The critical value is uniquely determined by the Legendre transform and we can introduce the Lagrangian $\mathcal{L}$.

$$\mathcal{L}(x,v) = \operatorname*{extr}_{\xi \in T_x^*X} (\xi(v) - H(x,\xi))$$

If $v = \xi^\flat$ then with (III.9) we have:

$$\mathcal{L}(x,v) = H(x,\xi)$$

If we have a finsler metric $\mathcal{F}$ on TX we can put

$$\mathcal{L}(x,v) = \frac{1}{2}\mathcal{F}^2(x,v)$$

Example III.20. The case with Riemannian metrics permits more explicit comparisons between H and $\mathcal{L}$. As usual denote g_{ij} a Riemannian metric. We have $H = \frac{1}{2}\langle g^{ij}\xi, \xi\rangle$. And the Lagrangian is $\mathcal{L}(x,v) = \frac{1}{2}\langle g_{ij}v_i, v_j\rangle$. The Legendre transform is

$$v = g^{ij}\xi$$

The inverse of the Legendre transform can be used to obtain from the canonical 1-form on $T(T^*X)$ a form on $T(TX)$

$$\xi \,\mathrm{d}\, x \quad \Rightarrow \xi^\flat \,\mathrm{d}\, x$$

When restricted to the surface $\mathcal{L}(x,v) = 1$ this form is sometimes known as the Hilbert form. We denote it $\alpha^\flat$. Contrary to what is done in [**BCS00**] the form we introduce here does not have the property that it is invariant under rescaling of the v coordinate, we do not care because we fix the manifold over which we work.

$$\alpha^\flat(x, \lambda v) = \lambda \alpha^\flat$$

The Hilbert form can be used just as the integral $\int \xi \,\mathrm{d}\, x$ to obtain extrema of path length. With Riemannian metrics we carry out the following calculation.

$$\int_\Gamma \xi \,\mathrm{d}\, x = \int_{\Gamma'} g_{ij} v_i \,\mathrm{d}\, x$$

Here Γ is the curve from (III.7) and Γ' is the image of Γ under the Legendre transform. Denote the length of a curve $\gamma\colon [a,b] \to X$ as

$$L_a^b = \int_a^b \sqrt{g_{ij}\frac{\mathrm{d}\,\gamma_i}{\mathrm{d}\,t}\frac{\mathrm{d}\,\gamma_j}{\mathrm{d}\,t}}\,\mathrm{d}\,t$$

whilst the energy is

$$E_a^b = \int_a^b g_{ij}\frac{\mathrm{d}\,\gamma_i}{\mathrm{d}\,t}\frac{\mathrm{d}\,\gamma_j}{\mathrm{d}\,t}\,\mathrm{d}\,t$$

It follows that the extremum of the integral of $\xi \,\mathrm{d}\, x$ is both equal to the length and the energy of the curve, because $\pi_X(\Gamma)$ is unit parameterized.

III.1.11. The big wavefront. There is one special kind of wave front that we still need to investigate. This is the big wavefront, sometimes also called the graph of the time function.
To define it take a submanifold $M \subset X$, where X satisfies all four assumptions stated in section III.1.9. The union of all wavefronts $\Sigma(t, M)$ as a subset of $\mathbb{R} \times T^*X$ is Legendrian submanifold of

$$\mathrm{J}^1(X, \mathbb{R}) = \mathbb{R} \times T^*X$$

with contact form $\mathrm{d}\, x_0 - \sum \xi \,\mathrm{d}\, x$. We can view $\mathrm{J}^1(X, \mathbb{R})$ as a coordinate patch of $\mathbb{P}T^*(X \times \mathbb{R})$. We can apply the symplectization and obtain from the union of the $\Sigma(t, M)$ a conic Lagrangian manifold in $T^*(X \times \mathbb{R})$. This is the big wavefront N^*M^h.
Let us show another way to define the big wavefront. Denote $0_{\mathbb{R}}$ the zero section of $T^*\mathbb{R}$. We observe that $0_{\mathbb{R}} \times N^*M \subset T^*(X \times \mathbb{R}) \setminus 0$ is a conic Lagrangian manifold in $T^*(X \times \mathbb{R})$. Again, in particular $0_{\mathbb{R}} \times T^*_{x_0}X$ is a conic Lagrangian manifold.

Lemma III.21. *Let $H\colon T^*X \to \mathbb{R}$ be a Hamiltonian, positively homogeneous of degree 1. Let $\phi^t\colon T^*X \times \mathbb{R} \to T^*X$ be the corresponding flow. The map $\Psi\colon T^*(X \times \mathbb{R}) \to T^*(X \times \mathbb{R})$ given by*

$$(x, \xi, t, \tau) \xrightarrow{\Psi} (\phi^t(x, \xi), t, \tau - H(x, \xi)) \tag{III.10}$$

preserves the symplectic form.

Proof. The map Ψ is symplectic iff.

$$\omega(\mathrm{d}\,\cdot, \mathrm{d}\,\cdot') = \omega(\cdot, \cdot')$$

We write $p = (x, \xi)$, and thus

$$\mathrm{d}_{p,t,\tau}\, \phi^t(\delta p, \delta t, \delta\tau) = \mathrm{d}_p\, \phi^t \delta p + \mathrm{vf}(H)\delta t$$

Writing this out we thus need to prove that

$$\begin{aligned}
&\omega((\mathrm{d}_p\, \phi^t \delta p + \mathrm{vf}(H)\delta t, \delta t, \delta\tau - \nabla H\delta p), (\mathrm{d}_p\, \phi^t \delta p' + \mathrm{vf}(H)\delta t', \delta t', \delta\tau' - \nabla H\delta p'))\\
&= \omega(\mathrm{d}_p\, \phi^t \delta p + \mathrm{vf}(H)\delta t, \mathrm{d}_p\, \phi^t \delta p' + \mathrm{vf}(H)\delta t')\\
&+ \omega((\delta t, \delta\tau - \nabla H\delta p)(\delta t', \delta\tau' - \nabla H\delta p'))\\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

equals $\omega((\delta p, \delta t, \delta\tau), (\delta p', \delta t', \delta\tau'))$. We calculate part I.

$$\begin{aligned}
\mathrm{I} &= \omega(\mathrm{d}_p\, \phi^t \delta t, \mathrm{d}'_p\, \phi^t \delta t') + \omega(\mathrm{vf} H\delta t, \mathrm{d}'_p\, \phi^t \delta t')\\
&+ \omega(\mathrm{d}_p\, \phi^t \delta t, \mathrm{vf}(H)\delta t',) + \delta t\delta t' \omega(\mathrm{vf}(H), \mathrm{vf}(H))\\
&= \omega(\delta p, \delta p') + \omega(\mathrm{vf}(H)\delta t, \mathrm{d}_p\, \phi^t \delta p') - \omega(\mathrm{vf}(H)\delta t', \mathrm{d}_p\, \phi^t \delta p)\\
&= \omega(\delta p, \delta p')
\end{aligned}$$

Next we calculate part II. Because

$$\mathrm{II} = \omega((\delta t, \delta\tau)(\delta t', \delta\tau'))$$

the proof is complete. ☐

Remark III.22. In our case we have a quadratic Hamiltonian but on the intersection

$$0_{\mathbb{R}} \times T^*_{x_0}X \cap H = 1$$

we can take $\sqrt{H}$ and apply Ψ from the lemma to get an isotropic manifold of dimension n in $T^*(X \times \mathbb{R})$. Its projection to $X \times \mathbb{R}$ is already the big wavefront. To get a conic Lagrangian manifold we multiply the fibers. As before this gives us the big wavefront.

When a fixed orientation $\vec{n}$ is given and H takes on only positive values - as in the case of a Riemannian metric - the intersection $\{H = 1\} \cap N^*M$ contains two components. There is the side corresponding to n in $\{H = 1\} \cap N^*M$ and a second side. We may decide to flow out only one of these and in such a way obtain a big wavefront we'll call N^*M^b. The projection of this to $X \times \mathbb{R}$ is what was studied in chapter one.

However, for the local considerations that follow this distinction is not relevant.

III.2. The conflict set via a canonical relation

In this section we apply the notion of a canonical relation to conflict sets in order to prove that the conflict set has a Legendre singularity if a certain transversality condition is satisfied.

III.2.12. Definition of conflict set. As in the previous section we have the following setup: Let M_i, $i = 1, 2$ be two smooth manifolds of dimension $n - 1$, equipped with an orientation n_i. Suppose that by $\gamma_i \colon M_i \mapsto X$, $i = 1, 2$, the manifolds are smoothly embedded in a smooth manifold X of dimension n. Also let $H_i \colon X \mapsto \mathbb{R}$, $i = 1, 2$ be two homogeneous Hamiltonians.

Definition III.23. The conflict set of two submanifolds $M_1, M_2 \subset X$ relative to two quadratic Hamiltonians $H_1, H_2 \colon T^*X \to \mathbb{R}$ is the set

$$C = \{x \in X \mid \exists t \in \mathbb{R} \;\; x \in \pi_X(\Sigma(t, M_1)) \;\; x \in \pi_X(\Sigma(t, M_2))\}$$

The set C is the projection of the intersection of big wavefronts, in the way we have encountered it in chapter 1.

Just as with the wavefronts there is a phase function defined on the whole of X for the conflict set C of two submanifolds of X. It is given by

$$F(x, s_1, s_2) \colon X \times M_1 \times M_2 \to \mathbb{R}$$
$$F(x, s_1, s_2) = A_1(\gamma_1(s_1), x) - A_2(\gamma_2(s_2), x)$$

Here $A(\gamma_i(s_i), x)$ is the work function for the Hamiltonian H_i.

III.2.13. A transversality condition. In the theorem below we will employ the notation:

$$T^*_\Delta(X^l) \setminus 0 = \{(x_1, \cdots, x_l, \xi_1, \cdots, \xi_l) \in T^*(X^l) \setminus 0 \;\mid\; x_i = x_j \;\, i \neq j\} \tag{III.11}$$

Theorem III.24. *If the flow out of $(N^*M_1 \times N^*M_2)' \cap ((H_1 - H_2)^{-1}(0)$ by $H_1 - H_2$ intersects $T^*_\Delta(X \times X)$ transversally then the conflict set is a conical Lagrangian submanifold of T^*X.*

Proof. In the setting of theorem (III.8) we choose $S_1 = T^*\Delta$, $S_2 = T^*(X \times X)$ and S_3 a point. The canonical relation in $S_1 \times S_2$ we choose is

$$G_1 = \{(x_1, \xi_1, x_2, x_3, \xi_2, \xi_3 \mid x_1 = x_2 = x_3 \quad \xi_1 = \xi_2 + \xi_3\}$$

We will now choose an appropriate Legendrian submanifold in S_2 so that we can apply proposition III.9.

The manifolds N^*M_1 and N^*M_2 are two conical Lagrangian submanifolds in T^*X. Apply the accent mapping $'$ to the product $N^*M_1 \times N^*M_2$. The non-degeneracy of the Hamiltonians implies that $\{H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0\}$ is a hypersurface in $T^*(X \times X)$. If we flow out the intersection

$$(N^*M_1 \times N^*M_2) \cap \{H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0\}$$

by the Hamiltonian vectorfield of $H_1 - H_2$ we stay inside the hypersurface

$$\{H_1(x_1, \xi_1) = H_2(x_2, \xi_2) \neq 0\}.$$

Call the flow-out N. The conclusion of the theorem follows if we take the composition $G_1 \circ N$. $\square$

Thus we have a very general criterion under which the conflict set resulting from two homogeneous Hamiltonians is Legendrian.

III.2.14. Examples, I. Stated as in theorem III.24 the criterion is not very computable. To obtain a computable criterion we first have to restrict our attention to a computable situation. One of the few situations in which we can calculate wavefronts is in $\mathbb{R}^n$. To (locally) find equations for a wavefront we take as a phase function F the squared distance from x to $\gamma(s)$. A phase function for the conflict set is

$$\begin{aligned} F(x, s_1, s_2) &= \|x - \gamma_1(s_1)\|^2 - \|x - \gamma_2(s_2)\|^2 \\ &= F_1(x, s_1) - F_2(x, s_2) \end{aligned}$$

so that the equations for a conflict set are

$$F(x, s_1, s_2) = F_2(x, s_2) - F_1(x, s_1) = 0 \quad \frac{\partial F_1}{\partial s_1} = 0 \quad \frac{\partial F_2}{\partial s_2} = 0$$

Now the demand that F is non-degenerate phase function wrt. both s_1 and s_2, so that the matrix

$$\text{(III.12)} \qquad \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F_1}{\partial s_1} & -\frac{\partial F_2}{\partial s_2} \\ \frac{\partial^2 F_1}{\partial s_1 \partial x} & \frac{\partial^2 F_1}{\partial s_1^2} & 0 \\ -\frac{\partial^2 F_2}{\partial s_2 \partial x} & 0 & -\frac{\partial^2 F_2}{\partial s_2^2} \end{pmatrix}$$

has maximal rank.
Because we can write

$$F(x, s_1, s_2) = 2\langle x, \gamma_2(s_2) - \gamma_1(s_1)\rangle + \|\gamma_1(s_1)\|^2 - \|\gamma_2(s_2)\|^2$$

the matrix (III.12) can be written at points of the conflict set:

$$\mathbf{K} = \begin{pmatrix} \gamma_2(s_2) - \gamma_1(s_1) & 0 & 0 \\ -\dot{\gamma_1}(s_1) & \langle \dot{\gamma_1}(s_1), \dot{\gamma_1}(s_1)\rangle - \langle x - \gamma_1(s_1), \ddot{\gamma_1}\rangle & 0 \\ -\dot{\gamma_2}(s_2) & 0 & \langle \dot{\gamma_2}(s_2), \dot{\gamma_2}(s_2)\rangle - 2\langle x - \gamma_2(s_2), \ddot{\gamma_2}\rangle \end{pmatrix}$$

The rank $\operatorname{rk} \mathbf{K}$ of $\mathbf{K}$ is bounded from below by

$$1 + \operatorname{rk}\left(\frac{\partial^2 F_1}{\partial s_1^2}\right) + \operatorname{rk}\left(\frac{\partial^2 F_2}{\partial s_2^2}\right)$$

A few remarks are in order:

- The rank $\operatorname{rk} \mathbf{K}$ has to be at least $1 + \dim M_1 + \dim M_2$ for the conflict set to be Legendrian at x.

- The first n columns of the matrix are $\mathbf{K}$ linearly independent: if $\gamma_1(s_1) \neq \gamma_2(s_2)$ the vector $\gamma_2(s_2) - \gamma_1(s_1)$ cannot lie in the tangent plane of M_1 at $\gamma_1(s_1)$. Suppose this were the case. Then

$$\langle x - \gamma_1(s_1), \gamma_2(s_2) - \gamma_1(s_1)\rangle = 0 \tag{III.13}$$

for some x on the conflict set. Because x lies on the conflict set we would also have

$$\langle x - \gamma_1(s_1), x - \gamma_1(s_1)\rangle = \langle x - \gamma_2(s_2), x - \gamma_2(s_2)\rangle \tag{III.14}$$

Adding (III.13) and (III.14) we get that

$$\langle x - \gamma_1(s_1), x - \gamma_2(s_2)\rangle = \langle x - \gamma_2(s_2), x - \gamma_2(s_2)\rangle$$

and thus, $\gamma_1(s_1) = \gamma_2(s_2)$ which is impossible. On the other hand $\gamma_1(s_1) = \gamma_2(s_2)$ can be avoided because the conflict set does not change when we move both manifolds M_1 and M_2 by the same distance. We can use this argument only locally but then we are only looking locally.
- The square matrices

$$\mathrm{d}_{s_i^2} F_i = \langle \dot\gamma_i(s_i), \dot\gamma_i(s_i)\rangle - \langle x - \gamma_i(s_i), \ddot\gamma_i\rangle$$

have maximal rank if the wavefront of M_i at x is smooth. Note that the wavefront might well be $\mathcal{C}^1$ at points where $d^2_{s_i} F_i$ does not have maximal rank. In fact the equation $\det(\mathrm{d}^2_{s_i} F_i(x, s_i)) = 0$ defines the focal surfaces of M_i. The relationship between the singularities of the map $F_i(x, s_i)$ and the geometry of M_i is investigated in [**Por01**].

In chapter V we will apply the singularity theory of Lagrangian and Legendrian mappings to the problem of conflict sets. There it is always good to keep the matrix $\mathbf{K}$ in mind.

III.2.15. Generalization to three surfaces and more. The previous construction worked to obtain the conflict set of two submanifolds M_1, M_2 of an ambient manifold X. It also generalizes to the case where we have $l \leq n$ submanifolds of X.
We will use the big wavefronts from section III.1.11. Let

$$P = \prod_{i=1}^{l} N^* M_i^h \subset (T^*(X \times \mathbb{R}))^l \tag{III.15}$$

THEOREM III.25. *The conflict set of the M_i is Legendrian when*

$$P \pitchfork T^*_\Delta\left((X \times \mathbb{R})^l\right)$$

PROOF. We consider a canonical relation in $T^*X \times T^*\mathbb{R} \times (T^*X \times T^*\mathbb{R})^l$.

$$G_1 = \{(\bar y, \bar\eta, \bar x_1, \bar\xi_1, \cdots, \bar x_l, \bar\xi_l) \mid \bar y = \bar x_i \ i = 1, \cdots, l \ \bar\eta = \sum_{i=1}^{l} \bar\xi_i\}$$

To check that G_1 is a conic canonical relation we need to show that it is a conic Lagrangian manifold wrt. to the form $\bar\eta d\bar y - \sum_{i=1}^{l} \bar\xi_i \bar x_i$. This is the case.
The manifold P can be pulled back to $T^*X \times T^*\mathbb{R}$ if the conditions in theorem III.8 are fulfilled. I.e. we are to have a transversal intersection

$$G_1 \times P \pitchfork T^*X \times T^*\mathbb{R} \times \Delta\left((T^*X \times T^*\mathbb{R})^l\right)$$

This intersection is transversal iff.

$$P \pitchfork T^*_\Delta\left((X \times \mathbb{R})^l\right) \tag{III.16}$$

Let L^h be the conical Lagrangian manifold that is the pull back of P by G_1.
We proceed to pull L^h back to T^*X. The canonical relation we use for this is

$$G_2 = \{(x, \xi, \bar{y}, \bar{\eta}) \mid x = \pi_X \bar{y} \ \pi_X^* \xi = \eta \ \eta_0 = 0\}$$

Again we have that upon applying proposition III.9

$$G_2 \times L^h \pitchfork T^*X \times \Delta\left(T^*(X \times \mathbb{R})\right) \leftrightarrow W \pitchfork L^h \tag{III.17}$$

where

$$W = \{(\bar{y}, \bar{\eta}) \mid \eta_0 = 0\}$$

The proof will be complete with the following lemma. □

Lemma III.26. $P \pitchfork T^*_\Delta\left((X \times \mathbb{R})^l\right) \Rightarrow W \pitchfork L^h$

Proof. Suppose that we did not have $W \pitchfork L^h$. Because W is a hypersurface that would mean that at some point p in L^h the tangent space TL^h would be contained in TW. So it would hold

$$\langle (0, 0, 0, \delta\eta_0), \vec{w} \rangle = 0, \quad \forall \vec{w} \in TL^h \tag{III.18}$$

Let J denote the usual complexification mapping. In local coordinates we can write:

$$\omega(v, w) = \langle v, J(w) \rangle$$

As J maps δy_0 to $\delta \eta_0$ the equation (III.18) becomes

$$\omega((0, 0, \delta y_0, 0), w) = 0, \quad \forall w \in T_pL^h$$

with ω being the canonical symplectic structure. But L^h is Lagrangian, so we would have that this vector $(0, 0, \delta y_0, 0) \in TL^h$. But that is clearly impossible. □

It is verified that in the case $k = 2$ the above construct is the same as the one with the accent mapping.
We have that $\bar{x}_{i,0} = \bar{x}_{j,0}$. The construction with W implies that we have a fixed energy level, so that $\bar{x}_{i,0}$ is the "time traveled", i.e. $\bar{x}_{i,0} = A_i(x, \gamma_i(s_i))$. When $k = 2$ there is thus just one equation:

$$A_1(\gamma_1(s_1), x) - A_2(\gamma_2(s_2), x) = 0$$

Here we have the same phase function as the one we got with the accent mapping. As a consequence the constructions are identical.

III.2.16. Examples, II. Let $F(x, s)$ be a non-degenerate phase function for a Legendrian submanifold of $\mathbb{P}T^*X$. Suppose we want to construct the corresponding conical Lagrangian manifold with a phase function. We can pick $\lambda F(x, s)$. Indeed it holds - provided that $\lambda \neq 0$ - that

$$F = 0, \ d_s F = 0 \Leftrightarrow d_{\lambda, s}(\lambda F) = 0.$$

With this in mind we can start composing any number $l \leq n$ of Legendrian submanifolds of T^*X. Let $\{F_i(x, s_i)\}_{i=1}^l$ be their phase functions, we can consider the following

$$F = \lambda_1(F_1 - F_2) + \lambda_2(F_2 - F_3) + \cdots + \lambda_{l-1}(F_{l-1} - F_l) \tag{III.19}$$

as a phase function. Here $(\lambda_1, \cdots, \lambda_{l-1})$ lies in a conical subset of $\mathbb{R}^{l-1} \setminus \{0\}$. Properly speaking we are not composing Legendrian submanifolds, but phase functions. These phase functions are determined - as we have seen in the above - by a Hamiltonian, a submanifold

of the ambient manifold X and the fixed energy level $H = 1$ chosen.
For a phase function we could also have used the less symmetric

$$F = (F_1 - F_2) + \lambda_1(F_2 - F_3) + \cdots + \lambda_{l-2}(F_{l-1} - F_l)$$

It is instructive to calculate the matrix of derivatives

$$\mathrm{d}_{x,\lambda_1,\cdots,\lambda_{l-1},s_1,\cdots,s_l}\left(\mathrm{d}_{\lambda_1,\cdots,\lambda_{l-1},s_1,\cdots,s_l} F\right) \tag{III.20}$$

there where

$$\mathrm{d}_{\lambda_1,\cdots,\lambda_{l-1},s_l,\cdots,s_l} F = 0$$

or equivalently

$$F_i = F_{i+1},\ \frac{\partial F_i}{\partial s_i} = 0,\ i = 1,\cdots,l \tag{III.21}$$

Thus the matrix (III.20) evaluated at the points of the conflict set defined by (III.21) looks like

$$\mathbf{K} = \begin{pmatrix} \gamma_2 - \gamma_1 & & & & \\ \vdots & \mathbf{0} & & \mathbf{0} & \\ \gamma_l - \gamma_{l-1} & & & & \\ -\lambda_1\dot{\gamma_1} & & \lambda_1\,\mathrm{d}^2_{s_1} F_1 & 0 & \cdots & 0 \\ -(\lambda_2-\lambda_1)\dot{\gamma_2} & \mathbf{0} & 0 & (\lambda_2-\lambda_1)\,\mathrm{d}^2_{s_2} F_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \lambda_{l-1}\dot{\gamma_l} & & 0 & 0 & \cdots & -\lambda_{l-1}\,\mathrm{d}^2_{s_l} F_l \end{pmatrix} \tag{III.22}$$

If the matrix has maximal rank, that is $l-1+\sum \dim(M_i)$, then the conflict set is Legendrian. We can see that the λ_i do not matter a great deal, we can simply eliminate them from the matrix (III.22).
Let us analyze this matrix again in the way such matrices are analyzed in [**Por01**].

- The first column block contains the derivatives wrt. the x variable. These are n independent vectors, so that if the lower right block of the matrix III.22 has rank $l-1-n+l(n-1)$ then the conflict set is Legendrian. In particular it is Legendrian when it is smooth.
- We can see that when there is one umbilic - i. e. $\mathrm{d}^2_{s_1} F = 0$ - and the other wavefronts are smooth that we then have a Legendrian singularity at the conflict set. This is in concordance with the remarks above about non-Legendrian singularities. The matrix $\mathbf{K}$ looks like

$$\begin{pmatrix} \gamma_2 - \gamma_1 & 0 & 0 \\ -\dot{\gamma_1} & 0 & 0 \\ \dot{\gamma_2} & 0 & -\,\mathrm{d}^2_{s_2} F_2 \end{pmatrix}$$

  We might as well forget the middle column. Then it is clear from the lemma that the rank of this $\mathbf{K}$ is $n + \mathrm{rk}\, d^2_{s_2} F_2$, when $\gamma_2(s_2)$ and $\gamma_1(s_1)$ do not fall together - something which can be avoided. Because the second wavefront was regular at the conflict set we can conclude here that the conflict set is Legendrian.
- On the other hand if $\mathrm{rk}\, d^2_{s_2} F_2 < n-1$ the conflict set will have a non-Legendrian singularity.

- Another example is where $l = n = 3$. This is the simplest case with three hypersurfaces. Here the matrix $\mathbf{K}$ looks like:
$$\begin{pmatrix} \gamma_2 - \gamma_1 & 0 & 0 & 0 & 0 \\ \gamma_3 - \gamma_2 & 0 & 0 & 0 & 0 \\ -\dot{\gamma}_1 & 0 & \mathrm{d}^2_{s_1} F_1 & 0 & 0 \\ -\dot{\gamma}_2 & 0 & 0 & \mathrm{d}^2_{s_2} F_2 & 0 \\ \dot{\gamma}_3 & 0 & 0 & 0 & \mathrm{d}^2_{s_3} F_3 \end{pmatrix}$$
Disregarding the column with zeroes this is a 8×9 matrix. Let us first consider the first three columns. They are independent. Now if two of the wavefronts are smooth at the conflict set we can have the third one singular and obtain a singular space curve.
- The relation between caustics and wavefronts is that singular points of the wavefronts lie on the caustic. For the conflict set and the caustics of the base manifolds we have something similar. If a point on the conflict set is singular then not all the $\mathrm{d}^2_{s_i} F_i$ can have maximal rank. This means that singular points of the conflict set lie on the caustic of one the manifolds. The singular points of the conflict sets lie on a codimension $n - l$ set on each caustic.

Example III.27 (Birth and death of a component). Conflict sets are hard to calculate, both numerically and algebraically. Thus we have to restrict our attention to a very simple situation. Already this simple situation leads to some interesting conclusions.
Take
$$\gamma_1(s) = \left(s, \frac{1}{2}s^2\right) \text{ and } \gamma_2(t) = A(\phi)\,(\gamma_1(t) - b)$$
thus γ_2 is just an affine transform of γ_1. We know that non-Legendrian points occur when the rank condition fails. The rank condition fails when the conflict set, the caustic of γ_1 and the caustic of γ_2 meet. These are three curves and for them to meet we see from the matrix that we need a one parameter family. Different families can be fabricated: we could choose variations in γ_1 or γ_2 , or we could vary H_1 and H_2, that is we vary the metrics associated to γ_1 and γ_2. We will take
$$F_z = (1 + z)\|x - \gamma_1(s)\| - (1 - z)\|x - \gamma_2(t)\|$$
The quickest way to calculate the conflict set in this case is by repeated use of resultants. For $z = 0$ we get figure III.5. In the middle of figure III.5 we see a separate component. We will show how it comes to existence through the non-Legendrian point, and why it has exactly four cusps. When $z = 8/10$ the picture looks very different, see figure III.6 Here the component is not born yet. For the region inside the two caustics another four cusps need to be born. But this is about to happen, just below and just above the cuspidal point of the caustic of γ_1 there are two curve segments of the conflict set. When they have shifted through this cuspidal points the four cusps will be there. Just as with wavefronts cusps are born in pairs. This is what has happened in figure III.7. One now sees the cusps on the right side moving towards the intersection of the caustic of γ_1 and γ_2. They will meet there at the same time. At this point the separate component is born. Over there the two Legendrian manifolds corresponding to the wavefronts of γ_1 and γ_2 become tangent to each other. It can be traced that it dies indeed at the other point where the caustics of γ_1 and γ_2 intersect.
The transition described here is not entirely new in the literature. As a purely local phenomenon it is described in [**BG86**]. The A_2^2 case on p. 195 is the same as we have here. We will return to the A_2^2 case in the next chapter, see figure V.4.

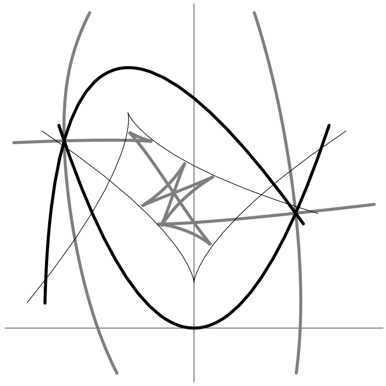

FIGURE III.5. $z = 0$: The curves γ_1 and γ_2 (thick black line), together with their evolutes (thin black line) and the conflict set (thick grey line) .

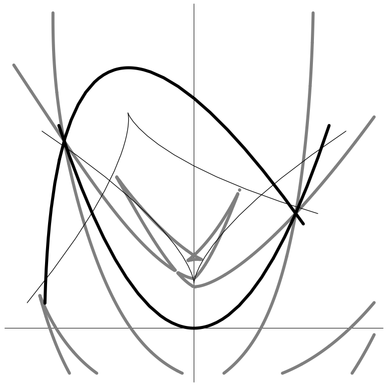

FIGURE III.6. $z = 8/10$: The curves γ_1 and γ_2, together with their evolutes and the conflict set.

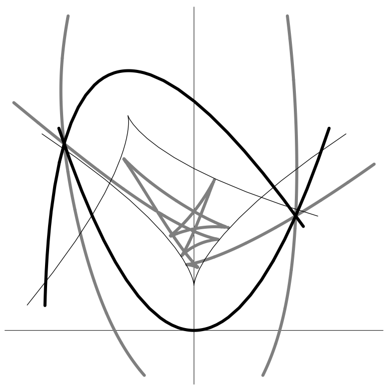

FIGURE III.7. $z = 5/10$: The curves γ_1 and γ_2, together with their evolutes and the conflict set.

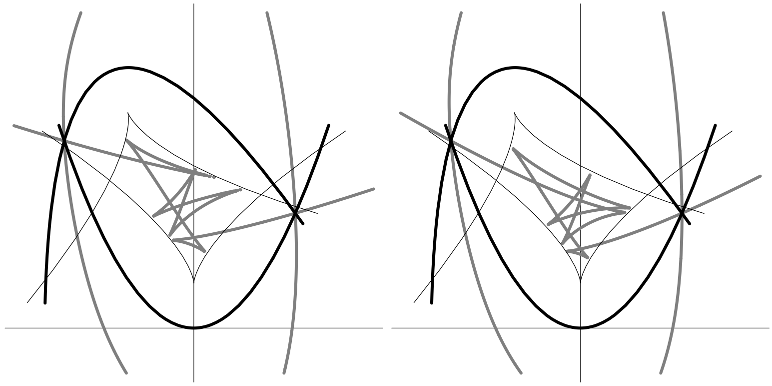

FIGURE III.8. Left $z = 2/10$ and right side $z = 35/100$

REMARK III.28. In the calculations carried out in the previous example resultants were much faster then elimination of the s-variables by Groebner bases. This has been our experience with a lot of examples. With two surfaces in $\mathbb{R}^3$ we have not seen the end of Groebner basis elimination or resultant methods even with fairly fast computers. The fastest of these systems [**GPS01**] would run for days without ending.

III.2.17. Further section and projection. All considerations in this subsection will be purely local, hence take $X = \mathbb{R}^n$ for simplicity. Consider a Legendrian submanifold L of $T^*\mathbb{R}^n \setminus 0$. At each $(x_0, \xi_0) \in L$ we can consider the dimension of $T_{x_0,\xi_0}L \cap T_{x_0}\pi^{-1}(x_0)$. This **fiber dimension** is an upper semi continuous function on L. Because the fiber dimension is integer valued there is some open neighborhood U of (x_0, ξ_0) such that on a dense subset of $U \cap L$ the fiber dimension takes on a single value. This number is the type of the Legendrian submanifold at (x_0, ξ_0). If the type is 1 the Legendrian submanifold we have is of general type.
The Legendrian manifolds we encounter if $l > 2$ are not of general type. Their fiber dimension is at regular points $l - 1$. We wish to show how we can apply the section and projection procedure to realize the conflict set as a Legendrian manifold of general type, or more generally how to realize a Legendrian manifold of higher type, as a Legendrian manifold of general type.
Thus let L be a Legendrian manifold of type $l - 1 > 1$ in $T^*\mathbb{R}^n$. For a vector v in $T_0\mathbb{R}^n$ we consider in $T^*\mathbb{R}^n$ the hypersurface

$$W_v = \{(x, \xi) \in T^*\mathbb{R}^n \quad | \quad \xi(v) = 0\}$$

By Sard's theorem for almost all v we have

$$W_v \pitchfork L \tag{III.23}$$

We have seen in the above (equation (III.17)) that (III.23) implies that we can project immersively L along the direction v. The projection of L along v to $T^*\mathbb{R}^{n-1}$ gives rise to a Legendrian manifold in $\pi_v(L)$, it will be of type $l - 2$. This process can be repeated until we have a Legendrian manifold of general type. We summarize.

THEOREM III.29. *A conic Lagrangian manifold of type $l - 1 > 1$ near $(0, \xi)$ in $T^*\mathbb{R}^n$ can be sectioned and then projected immersively to a Legendrian manifold of general type in $T^*\mathbb{R}^{n-l+2}$.*

III.3. Surjectivity of the jet mapping

In this section we will prove that the condition which assures that the conflict set has a Legendre singularity is generically satisfied under perturbations of the base manifolds. There are two ways of going about proving this. In this section we prove the genericity of the maximal rank criterion (III.22) for phase functions In the next chapter we prove the genericity of the criterion (III.16).

III.3.18. Maximal rank criterion. Recall from [**Hir94**] that a residual set in a topological space is a countable intersection of open and dense subsets. The space of $\mathcal{C}^\infty$ mappings from M_i to X equipped with the Whitney topology is a Baire space, meaning that countable intersections of open and dense sets are still dense. Hence residual sets lie dense.
For our purposes it is convenient to introduce the space $\mathrm{Emb}(M, X)$. It is the space of closed embeddings of the hypersurface M in X. Again from [**Hir94**] we know that the space of closed embeddings is open in the space $\mathcal{C}^\infty(M, X)$.

THEOREM III.30. *Let M_i be l closed compact hypersurfaces X. For a residual subset of embeddings in $\oplus_{i=1}^{l} \mathrm{Emb}(M_i, X)$ with the Whitney topology the conflict set only has Legendrian singularities.*

In the above we have made four assumptions on our Hamiltonian such that for each of the $A_i(x, y)$

$$\frac{\partial A_i}{\partial y} \neq 0 \tag{III.24}$$

Moreover, because the Hamiltonians are non-degenerate we will also have

$$\det\left(\frac{\partial^2 A_i}{\partial y^2}\right) \neq 0$$

Let each of the γ_i depend on an additional parameter $e_i \in \mathbb{R}^n$. So every embedding is replaced by a family of embeddings: $\gamma_i = \gamma_i(s_i, e_i)$.
In the euclidean case we take translations of the γ_i, there γ_i has the form $\gamma_i(s_i, e) = \gamma_i(s_i, 0) + e$. In the general case we assume that near any point coordinates can be chosen such that for s_i near $s_{i,0}$ and small e such that we can write in these coordinates:

$$\gamma_i(s_i, e) = \gamma(s_i) + \frac{\partial \gamma_i}{\partial e} e + O(\|e\|^2)$$

and

$$\frac{\partial \gamma_i}{\partial e}(s_{i,0}, 0) = \mathbf{I}_n$$

Let us cite lemma 3.2. in [**Mat70b**].

LEMMA III.31 (Lemma V.3.2.). *Let U be a submanifold of a manifold W. Let $\mathfrak{F}$ be a topological space and $j\colon \mathfrak{F} \to \mathcal{C}^\infty(V, W)$ a mapping, where V is a manifold. Suppose that for each $f \in \mathfrak{F}$ there exists a continuous mapping $\phi\colon (E, e_0) \to (\mathfrak{F}, f)$, where E is a manifold and $e_0 \in E$, such that the mapping $\Phi\colon E \times V \to W$ (defined by $\Phi(e, v) = j\phi(e)(v)$) is $\mathcal{C}^\infty$ and transversal to U. Then*

$$\{f \in \mathfrak{F} \mid j(f) \text{ is transversal to } U\}$$

is dense in $\mathfrak{F}$.

We want to apply the lemma to our present situation, where we need to prove theorem III.30.
Take

$$\mathfrak{F} = \oplus_{i=1}^{l} \operatorname{Emb}(M_i, X)$$

We need to prove that the matrix $\mathbf{K}$ from (III.22) generically has maximal rank. Hence the graph of

$$x, s_1, \cdots, s_l \to F_1 - F_2, \cdots, F_{l-1} - F_l, \frac{\partial F_1}{\partial s_1}, \cdots, \frac{\partial F_l}{\partial s_l} \tag{III.25}$$

is to intersect its zero-level transversally. We have to fill in that U is the zero-level. So U is

$$\{x, s_1, \cdots, s_l, 0, 0, \cdots, 0\}$$

in

$$X \times M_1 \times \cdots \times M_l \times \mathbb{R}^{l-1} \times \left(\mathbb{R}^{n-1}\right)^l$$

The simplest way to get a map $\Phi\colon E \times V \to W$ transverse to $U \subset W$ is to make Φ submersive. Let

$$V = M_1 \times \cdots \times M_l \times X$$

and let E be $(\mathbb{R}^n)^l$
The map Φ will be the graph of (III.25), where the F_i are the distance functions

$$F_i = A_i(x, \gamma(s_i, e_i))$$

If the map Φ is submersive then for a dense subset of $\oplus_{i=1}^{l} \operatorname{Emb}(M_i, X)$ the maximal rank criterion is fulfilled and the conflict set is generically Legendre.
To show that the map Φ is submersive we need to show that the matrix of derivatives

$$\mathrm{d}_{x,e,s}\left(F_1 - F_2, \cdots, F_{l-1} - F_l, \mathrm{d}_{s_1} F_1, \cdots \mathrm{d}_{s_l} F_l\right)$$

has rank $l - 1 + l(n-1)$. (We have written $s = (s_1, \cdots, s_l)$ and $e = (e_1, \cdots, e_l)$)
It will be enough to show that

$$\mathrm{d}_{x,e}\left(F_1 - F_2, \cdots, F_{l-1} - F_l, \mathrm{d}_{s_1}(F_1), \cdots, \mathrm{d}_{s_l}(F_l)\right) \tag{III.26}$$

has the required rank $l-1+l(n-1)$. Thus we are to calculate a number of partial derivatives

$$\mathrm{d}_{x,e_i}(F_j - F_{j+1}) \text{ and } \mathrm{d}_{x,e_i}(\mathrm{d}_{s_j}(F_j))$$

because of these partial derivatives the matrix in (III.26) is composed. We find:

$$\mathrm{d}_x(F_j - F_{j+1}) = \frac{\partial A_j}{\partial x} - \frac{\partial A_{j+1}}{\partial x}$$

Here

$$\frac{\partial A_j}{\partial x}$$

is the normal to the front from M_j.

$$\mathrm{d}_{e_i}(F_j - F_{j+1}) = \frac{\partial A_j}{\partial y}\frac{\partial \gamma_j(s_j, e_j)}{\partial e_i} - \frac{\partial A_{j+1}}{\partial y}\frac{\partial \gamma_{j+1}(s_{j+1}, e_{j+1})}{\partial e_i}$$

Above we chose our families so that in local coordinates

$$\frac{\partial \gamma_i(s_i, e_i)}{\partial e_i}$$

is the identity matrix. Hence

$$\mathrm{d}_{e_j}(F_j - F_{j+1}) = \frac{\partial A_j}{\partial y} = \mathrm{d}_y A_j \qquad \mathrm{d}_{e_{j+1}}(F_j - F_{j+1}) = -\frac{\partial A_{j+1}}{\partial y} = -\mathrm{d}_y A_{j+1}$$

The next partial derivatives to calculate are

$$\mathrm{d}_x(\mathrm{d}_{s_j}(F_j)) = \mathrm{d}_x(\frac{\partial A_j}{\partial y}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j}) = \frac{\partial^2 A_j}{\partial x \partial y}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j}$$

and

$$\mathrm{d}_{e_j}(\mathrm{d}_{s_j}(F_j)) = \mathrm{d}_{e_j}(\frac{\partial A_j}{\partial y}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j}) = \frac{\partial^2 A_j}{\partial y^2}\frac{\partial \gamma_j(s_j,e_j)}{\partial e_j}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j} + \frac{\partial A_j}{\partial y}\frac{\partial^2 \gamma_j(s_j,e_j)}{\partial e_j \partial s_j}$$

We can further assume that the derivative

$$\frac{\partial^2 \gamma_j(s_j,e_j)}{\partial e_j \partial s_j} \tag{III.27}$$

are arbitrarily small. The rank of a matrix does not change if we add another arbitrarily small matrix to it. Hence to determine whether the matrix (III.26) has maximal rank we can assume (III.27) is zero. It is noted that if (III.27) is zero then the perturbation of $\gamma_j(s_j,e_j)$ near $\gamma_j(s_j,0)$ is very nearly a translation along the geodesics.
Another simplification is that by adding columns the maximal rank criterion becomes that

$$\mathrm{d}_{x,e}\left(F_1 - F_l, \cdots, F_{l-2} - F_l, F_{l-1} - F_l, \mathrm{d}_{s_1}(F_1), \cdots, \mathrm{d}_{s_l}(F_l)\right)$$

should have maximal rank. Filling in all the partial derivatives we get that
(III.28)

$$\begin{pmatrix} \mathrm{d}_x A_1 - \mathrm{d}_x A_l & \cdots\cdots\cdots & & \mathrm{d}_x A_{l-1} - \mathrm{d}_x A_l & \frac{\partial^2 A_j}{\partial x \partial y}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j} & \cdots & \frac{\partial^2 A_j}{\partial x \partial y}\frac{\partial \gamma_j(s_j,e_j)}{\partial s_j} \\ \mathrm{d}_y A_1 & 0 & \cdots & 0 & \frac{\partial^2 A_1}{\partial y^2}\frac{\partial \gamma_1(s_1,e_1)}{\partial s_1} & \cdots & 0 \\ 0 & \mathrm{d}_y A_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\mathrm{d}_y A_l & -\mathrm{d}_y A_l & \cdots & -\mathrm{d}_y A_l & 0 & \cdots & \frac{\partial^2 A_2}{\partial y^2}\frac{\partial \gamma_l(s_l,e_l)}{\partial s_l} \end{pmatrix}$$

should have rank $l-1+l(n-1)$. In the matrix (III.28) there are $n+nl$ rows and $l-1+l(n-1) = nl-1$ columns. Hence there are more rows than columns.
The matrices $\mathrm{d}_y^2 A_i \,\mathrm{d}_{s_i} \gamma_i$ all have rank $n-1$, moreover with $\mathrm{d}_y A_i$ they form a set of n independent vectors in $T_{\gamma(s_j,0)}X$. Thus, the rows $n+1$ to $n+(n-1)l$, i.e. those corresponding to d_{e_j} , $j = 1, \cdots, l-1$ are independent and so are the $n-1$ vectors

$$\frac{\partial^2 A_2}{\partial y^2}\frac{\partial \gamma_l(s_l,e_l)}{\partial s_l}$$

Thus the lower nl rows of (III.28) make that the columns of (III.28) are independent. This is what we needed to prove.
Now we have only proven the statement of theorem III.30 for basepoints $\gamma_j(s_j)$ with a point x on the conflict set. To make the statement global we need to cover $X \times M_1 \times \cdots \times M_l$ with compact submanifolds, as is done in the proof of the Mather transversality theorem, proposition 3.3 in [**Mat70b**]. The countable intersection of dense subsets of $\oplus_{i=1}^{l} \mathrm{Emb}(M_i, X)$ will still be dense.
For the openness we can thus assume that the M_i are compact and that we need only consider that part of the conflict set that lies in a compactum $X^o \subset X$. On these the perturbation of the matrix (III.28) can be made uniformly small over the compactum $X^o \times (\times_{i=1}^{l} M_i)$. The proof of III.30 is complete.

III.4. k-jets of base manifolds determine k-jets of conflict sets

In this section we establish some results that were already imminent in the theorems of chapters one and two. We will apply the concepts introduced at the beginning of this chapter to the subject of chapter 1. In chapter 1 we concluded that in $\mathbb{R}^n$ with the euclidean metric away from caustics propagating waves retain their contact.

Let M and N be hypersurfaces in a manifold X. Suppose $p \in M \cap N$. Let V be a linear subspace of T_pM. Recall the definition of k-contact from chapter 1 (Definition I.5). We modify it a little bit to read:

Definition III.32. M and N have k-contact, with $k > 0$ in the direction $V \subseteq (T_pN \cap T_pM)$ if there is a third manifold L with $T_pL = V$ and such that L has (k)-contact with both M and N.

Instead of V we consider a sequence of subspaces $\{V_i\}_{i=1,\cdots,m}$ each strictly contained in the other:

$$\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subset T_pM$$

with to each V_i associated an integer

$$k_1 > k_2 > \cdots > k_m > 0$$

We call this a numbered flag $\mathcal{V} = \{V_i, k_i\}_{i=1,\cdots,m}$. Two submanifolds M and N of X have contact along $\mathcal{V}$ if for each $1 \leq i \leq m$ M and N have k_i-contact in the direction V_i. Such a definition encompasses the notions above.

Contact is retained under diffeomorphisms. If we have a diffeomorphism ϕ from X, p to Y, q then $\phi(M)$ and $\phi(N)$ have k contact in the direction $\phi_* V$ iff. M and N have k-contact in the direction V.

Let the conflict set be regular at a point p. At p we have l wavefronts originating from M_i, q_i that have arrived there after time t. For simplicity we assume $t = 0$ and $q_i = p$ for $i = 1, \cdots, l$. The conflict set is a manifold M_c and there is a projection $\pi_i \colon T_pM_c \to T_pM_i$. In this way a numbered flag $\mathcal{V}$ of T_pM_c projects to numbered flags $(\pi_i)_*\mathcal{V}$ in each of the T_pM_i.

The lemmata in the proof of theorem I.2 can now be refined to read:

Theorem III.33. *Let in addition to the above there be given*

- N_i *with* $T_pN_i = T_pM_i$ *such that* M_i *have at* p *contact along the numbered flag* $(\pi_i)_*\mathcal{V}$ *and*
- N_c*, germ at* p *of the conflict set of the* N_i*.*

Then N_c *and* M_c *have contact along* $\mathcal{V}$*.*

Proof. It is enough to prove this for k contact in a direction $V \subset T_pM_c$. Denote, as in chapter 1, M_i^h for the big wave front. As before, there are three steps to take. Step 1, corresponding to lemma I.9. Near p $M_i \times \mathbb{R}$ and $N_i \times \mathbb{R}$ have, as submanifolds of $X \times \mathbb{R}$ k-contact in the direction $W_i = \pi_i V \times T_0\mathbb{R}$. For small t the map $\Psi_i \colon t, x \mapsto \pi_X(\exp(t\mathrm{vf}(H_i))), t$ is a diffeomorphism. Thus their images M_i^h and N_i^h have k contact in the directions $\Psi_i^* W_i$.

Step 2, corresponding to lemma I.10. Choose submersions F_i for the M_i^h at p. Then each of the N_i^h individually has a k-contact along $\cap_{i=1}^l \Psi_i^* W_i$ with M_c^h, which we know to be a transversal intersection because we were assuming to be at a regular point of the conflict set. Now take for M_c^h an immersion and for the N_i^h submersions. Then it follows that N_c^h has k contact along $\cap_{i=1}^l \Psi_i^* W_i$ with M_c^h.

Step 3, corresponding to lemma I.11. Because the vectorspace $\cap_{i=1}^l \Psi_i^* W_i$ lies in general position wrt. the projection to $T_pX\ N_c$ will have k-contact in the direction $\pi(\cap_{i=1}^{l} \Psi_i^* W_i) \subset T_pX$. $\square$

IV Canonical relations for other geometrical constructions

IV.1. Introduction

In this chapter we will further exploit the concept of canonical relation and show how it can be used to create

- center sets,
- pedals and orthomtics, and
- billiards.

We will start by treating a few concepts that are in some sense dual to conflict sets. To understand the duality for the concept of center sets recall that in the construction of the conflict set by means of a canonical relation we looked at the diagonal in the base of the projection

$$\left(T^*\mathbb{R}^{n+1}\right)^l \to \left(\mathbb{R}^{n+1}\right)^l$$

When defining the center set below we will look at the diagonal in the fiber, i.e. we will look where the coordinates ξ in each of the fibers $T^*\mathbb{R}^n \to \mathbb{R}^n$ are equal.
Kites, defined below, are in some sense dual to the conflict set because they describe tangents to the lifted conflict set.
However, it is noted that neither center sets nor kites are really dual to the conflict set: one cannot construct conflict sets directly from kites or center sets.
In the second section we will prove that the center set is generically Legendrian. We will prove anew the same statement for conflict sets, but in another way as promised in the previous chapter.
In the third section we recall the notion of orthomtic and pedal curve, as they are described in the book [**BG92**]. The orthomtic is a hypersurface defined using a point - mostly the origin - and another hypersurface conveniently called the mirror. The mirror is the conflict set of the orthomtic and the origin. The orthomtic turns out to be a reversed conflict set.
We also briefly touch upon the subject of billiards as described in [**Tab95**]. Our main objective is to show how a curvature formula used in the theory of billiards and in another form found in [**BW59**] is really the formula we found in chapter 1.

IV.2. "Dual" curves: kites, centers and normal chords

In this section we define a few more sets measuring what is in the middle. Here we meet more applications of the method where canonical relations represent a geometrical construction. All our constructions are in $\mathbb{R}^n$. They can also be carried out in other spaces than $\mathbb{R}^n$, for instance in symmetric spaces.

IV.2.1. Centers and centroids. In $\mathbb{R}^n$ all tangent planes can be identified with each other. Hence the equations

$$T^*_{(1,1)}\mathbb{R}^{2n} \setminus 0 = \{(x_1, \xi_1, x_2, \xi_2) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \quad | \quad \xi_1 = \xi_2 \neq 0\} \tag{IV.1}$$

make sense on $T^*\mathbb{R}^{2n} \setminus 0$. The intersection

$$\left(T^*_{(1,1)}\mathbb{R}^{2n} \setminus 0\right) \cap \left(N^*M_1 \times N^*M_2\right)$$

describes those points x_1 on M_1 and x_2 on M_2 so that M_1 and M_2 have parallel tangent planes. Instead of considering the pair x_1, x_2 we can consider the midpoint of the line segment connecting the two.

Definition IV.1. The midpoints of the segments connecting two points, one on a manifold $M_1 \subset \mathbb{R}^n$ and another on $M_2 \subset \mathbb{R}^n$, and having parallel tangent planes form the **center set**.

Theorem IV.2. *If*

$$T^*_{(1,1)}(\mathbb{R}^n \times \mathbb{R}^n) \pitchfork N^*M_1 \times N^*M_2 \tag{IV.2}$$

then the center set is Legendrian.

Proof. Let S_1 be $T^*\mathbb{R}^n$. Equip S_1 with coordinates (y, η). Let S_2 be $T^*(\mathbb{R}^n \times \mathbb{R}^n)$. Equip S_2 with coordinates (x_1, x_2, ξ_1, ξ_2).
Let $G_1 \subset S_1 \times S_2$ be

$$G_1 = \{(y, \eta, x_1, \xi_1, x_2, \xi_2) \quad | \, y = \frac{x_1 + x_2}{2} \quad \eta = 2\xi_1 \quad \xi_1 = \xi_2 \neq 0\} \tag{IV.3}$$

A straightforward calculation shows that G_1 is a canonical relation. If we compose this canonical relation with $N^*M_1 \times N^*M_2 \subset S_2$ we will get the center set. Apply proposition III.9 to obtain the desired conclusion. □

Remark IV.3. In contrast to conflict sets one can calculate center sets in many cases. For instance, the graph of a function f whose derivative is invertible and any other curve non-vertical curve γ leads to a simple calculation of conflict sets. The tangents should be parallel, so we have

$$\langle (1, f'), J(\gamma') \rangle = f'\gamma'_x(t) - \gamma'_y(t) = 0$$

From this we can solve x because f' is invertible. Filling this in in

$$\left(\frac{1}{2}(x + \gamma_x(t)), \frac{1}{2}(f(x) + \gamma_y(t)) \right)$$

we get a curve. In figure IV.2.1 we see such a center set. The function f is the parabola. Its derivative is invertible.

Figure IV.1. Two center sets.

As with conflict sets we can consider the centre set of more than two submanifolds. This will be the centroid set.
In $\mathbb{R}^l$ consider the enlarged simplex

$$\Delta_l \colon \qquad \sum_{i=1}^{l} t_i = 1$$

The t_i are the components of a weight vector. Fix such a weight vector.

DEFINITION IV.4. The weighted locus

$$y = \sum_{i=1}^{l} t_i x_i$$

of l-tuples of points $(x_1, \cdots, x_l)$ where the tangent planes to the l surfaces M_i are parallel, form the **centroid set**.

Criteria for when the centroid set is Legendrian are easily written down. Consider the canonical relation

$$G_1 = \{y = \sum_{i=1}^{l} t_i x_i \qquad \eta t_i = \xi_i\} \subset T^*\mathbb{R}^n \times (T^*\mathbb{R}^n)^l \tag{IV.4}$$

If we compose this with a product of the $\{N^*M_i\}_{i=1,\cdots,l}$ we get the centroid set with weights $t_i \neq 0$.

PROPOSITION IV.5. *The centroid set is Legendrian when*

$$\times_{i=1}^{l} N^*M_i \pitchfork T^*_{(1,\cdots,1)}\mathbb{R}^{nl}$$

PROOF. Immediate from (IV.4) and proposition III.9. $\square$

In general the centroid set will have dimension $n-1$.

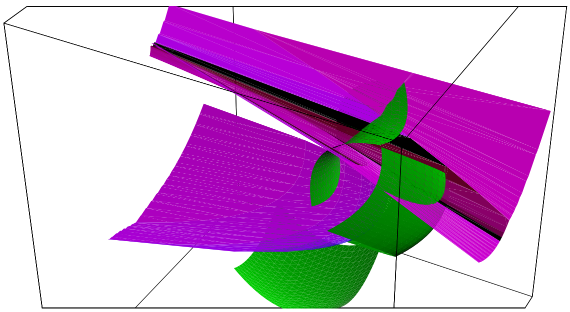

FIGURE IV.2. A centroid set of three surfaces

IV.2.2. Centroids in certain symmetric spaces. Instead of $y = \sum_{i=}^{l} t_i x_i$ we can consider a more general relation.

$$y = Y(x_1, \cdots, x_l). \tag{IV.5}$$

where $Y = Y(x_1, \cdots, x_l)$ is a function from $X_1 \times \cdots \times X_l$ to X_0 such that the matrices $\frac{\partial Y}{\partial x_i}$ are invertible. The relation $y = \sum_{i=}^{l} t_i x_i$ is a special case of relations (IV.5).
From the relation (IV.5) we construct G_1.

$$G_1 = \{(y, \mu, x_1, \xi_1, \cdots x_l, \xi_l) \mid y = Y(x_1, \cdots, x_l) \;\; \xi_i = \frac{\partial Y}{\partial x_i}\mu\} \tag{IV.6}$$

The manifold G_1 is a canonical relation between $T^*\mathbb{R}^n$ and $(T^*\mathbb{R}^n)^l$. Because this is a purely local notion a canonical relation like G_1 can also be defined between the cotangent bundle T^*X_0 of an n-dimensional manifold X_0 and

$$T^*X_1 \times \cdots \times T^*X_l$$

where X_1 to X_l are all n-dimensional manifolds. Clearly the condition under which the more general centroid set is Legendrian is that

$$N^*M_1 \times \cdots \times N^*M_l \pitchfork T^*_{\mathrm{d}\,Y}(X_1 \times \cdots \times X_l)$$

where we denoted

$$T^*_{\mathrm{d}\,Y}(X_1 \times \cdots \times X_l) = \{x_1, \xi_1, \cdots, x_l, \xi_l \mid \left(\frac{\partial Y}{\partial x_i}\right)^{-1} \xi_i = \left(\frac{\partial Y}{\partial x_j}\right)^{-1} \xi_j\}$$

The previous considerations thus enable us to generalize the centroid set to other manifolds than $\mathbb{R}^n$. Let X be a Lie group with a bi-invariant metric. Recall from [**Mil63**] §21, that on a Lie group with a bi-invariant metric there exists for every point $x \in X$ an involutive isometry $\sigma(x, \cdot)\colon X \to X$ which in group notation reads $y \to xy^{-1}x$.
Such isometries make X into a symmetric space. Among the many examples of Lie groups with a bi-invariant metric are the spheres $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.
We can now use the canonical relation (IV.6) with $l = 2$ and $X_0 = X_1 = X_2 = X$. Two choices for Y are interesting for our purposes:

- $Y(x_1, x_2) = \sigma(x_1, x_2)$ i.e. x_2 is reflected on $Y(x_1, x_2)$, this is the relation we will meet further on when discussing the billiard transformation on the space of rays,
- $x_2 = \sigma(y, x_1)$, in $\mathbb{R}^n$ this relation is (IV.3).

IV.2.3. Normal chords. Instead of the midpoints of the segment joining two points with parallel planes we can also consider the coinciding normals themselves.
For this construction we will make essential use of the Gauss map. Suppose we apply the Gauss map ν_G componentwise to

$$N^*M_1 \times \cdots \times N^*M_l$$

We then obtain

$$\nu_G(N^*M_1) \times \cdots \times \cdots \nu_G(N^*M_l) \subset \times_{i=1}^{l} T^*S^{n-1} \tag{IV.7}$$

We defined the center set using parallel tangent planes. The tangent planes at $p_1 \in M_1$ and $p_2 \in M_2$ coincide when we can find $\xi \in N^*_{p(1)}M_1$ that is also in $N^*_{p(2)}M_2$, that is when the normals at p_1 and p_2 coincide.
Coinciding normals can thus also be found using the "diagonal in the base" $T^*_\Delta(S^{n-1})^l$ of $T^*(S^{n-1})^l$.

Definition IV.6. The normals at points x_i of the centroid form the **normal chord set**.

For the centroid set we have the result analogous to theorem III.25.

Proposition IV.7. *The normal chord set is Legendrian if*

$$\nu_G(N^*M_1) \times \cdots \times \nu_G(N^*M_l) \pitchfork T^*_\Delta(S^{n-1})^l \tag{IV.8}$$

Returning to the case $l = 2$ note that the normals are not the chords connecting two points on the center set. In fact we have three objects:

- the normals at the x_i,
- the normal to the center set, and

• the chords connecting the x_1 and x_2.

One might ask whether the first and the second are the same and what these have to do with the chords connecting x_1 and x_2. We will now only answer the first part of the question and leave the answer to the second part of our question to the next subsection.
Suppose $x_1 \in M_1$ and $x_2 \in M_2$, with normals ξ_1 and ξ_2 respectively.
If we first compute the normal chord set

$$(x_1, \xi_1, x_2, \xi_2) \overset{\nu_G \times \nu_G}{\to} (\frac{\xi_1}{\|\xi_1\|}, \cdots, \frac{\xi_2}{\|\xi_2\|}, \cdots)$$

and then take the diagonal we get the same value as when we first take the center set and then map to T^*S^{n-1}. The same happens with l surfaces, If we have $(x_i, \xi) \in N^*M_i$ and take the center set, we get

$$\left(\frac{1}{l}\sum x_i, \xi\right)$$

in $T^*\mathbb{R}^n$. When applying the Gauss map we get

$$\left(\frac{\xi}{\|\xi\|}, \frac{1}{l}\sum x_i - \frac{\langle \frac{1}{l}\sum x_i, \xi\rangle}{\|\xi\|}\xi\right)$$

On the other hand if we first apply the product of Gauss maps $\nu_G \times \cdots \times \nu_G$ to the product of Legendrian submanifolds $N^*M_1 \times \cdots \times N^*M_l$ and afterwards take the centroid set we get

$$\left(\frac{\xi}{\|\xi\|}, \sum x_i - \frac{\langle \sum x_i, \xi\rangle}{\|\xi\|}\xi\right)$$

which is up to a factor l the same thing.
In other words the diagram

$$\begin{tikzcd}
T^*\mathbb{R}^n \times \cdots \times T^*\mathbb{R}^n \arrow[r, "\pitchfork T^*_{(1,\cdots,1)}\mathbb{R}^{nl}"] \arrow[d, "\nu_G \times \cdots \times \nu_G"] & T^*\mathbb{R}^n \arrow[d, "\nu_G"] \\
T^*S^{n-1} \times \cdots \times T^*S^{n-1} \arrow[r, "\pitchfork T_\Delta S^{(n-1)l}"] & T^*S^{n-1}
\end{tikzcd}$$

commutes. Properly speaking the arrows do not really represent maps: in the diagram the upper row is the pull-back from proposition IV.5. The lower row is the pull-back from proposition IV.7.
Hence the first and the second items are the same. We summarize

PROPOSITION IV.8. *The normal to the center set is the normal singled out by the normal chord set. If either of the transversality conditions (IV.2) or (IV.8) holds then the normal chord set is Lagrangian and the center set is Legendrian.*

REMARK IV.9. For the normal chord set we can write down maximal rank criteria as we did for the conflict set in section III.2.14. A phase function for the image of the Gauss map is

$$\begin{aligned} F_i \colon S^{n-1} \times M_i &\to \mathbb{R} \\ (v, s) &\mapsto \langle v, \gamma(s)\rangle \end{aligned} \tag{IV.9}$$

The image of the Gauss map is described by

$$\frac{\partial F_i}{\partial s} = 0 \tag{IV.10}$$

To get a maximal rank criterion under which the normal chord set is Lagrangian we use as in III.2.16 a special phase function:

$$F_1(v, s_1) + F_2(v, s_2)$$

And the maximal rank criterion that is equivalent to the transversality in (IV.8) is that the matrix

$$\mathrm{d}_{v,s_1,s_2}\left(\mathrm{d}_{s_1,s_2} F\right)$$

has maximal rank where

$$\mathrm{d}_{s_1,s_2} F = 0$$

If so, the normal chord set is Lagrangian. If one chooses local coordinates on S^{n-1}, as is done in [**BGM82**], this is a nicely computable criterion.

IV.2.4. Centers as wavefronts, chords as caustics. We still need to make sense of the relationship between the normal chords and the lines from x_1 to x_2. These lines form an $n-1$ dimensional family. They remind us of the normals to a hypersurface. Normals to a hypersurface have the caustic as an envelope.
Consider the

$$\begin{aligned} \mathbb{R} \times \Big(T^*\mathbb{R}^n \times T^*\mathbb{R}^n \cap T^*_{(1,1)}(\mathbb{R}^n \times \mathbb{R}^n)\Big) &\to \mathbb{R} \times \mathbb{R}^n \\ (t, x_1, \xi, x_2, \xi) &\to (t, tx_1 + (1-t)x_2) \end{aligned} \tag{IV.11}$$

The points $\ell(x_1, x_2)$ that are the image of

$$t \to tx_1 + (1-t)x_2$$

form a line. The envelope of the lines $\ell(x_1, x_2)$ are the singular points of the projection that is the last arrow in

$$\mathbb{R} \times \Big(T^*\mathbb{R}^n \times T^*\mathbb{R}^n \cap T^*_{(1,1)}(\mathbb{R}^n \times \mathbb{R}^n)\Big) \to \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(t, x_1, \xi, x_2, \xi) \to (t, tx_1 + (1-t)x_2) \to tx_1 + (1-t)x_2$$

Definition IV.10. The image of $\mathbb{R} \times N^*M_1 \times N^*M_2$ in $\mathbb{R} \times \mathbb{R}^n$ under (IV.11) is the **big center set**. The projection of its singular points to $\mathbb{R}^n$ we will call the **center caustic**.

The center set is the intersection of the big center set with the plane $t = \frac{1}{2}$. Trivially one has:

Proposition IV.11. *If the intersection (IV.2) is transversal then the big center set is Legendrian in* $(J^1\mathbb{R}^n, dt - \sum_{i=1}^n \xi_i \,\mathrm{d}\, x_i)$ *Hence in that case the center caustic is Lagrangian.*

If we fix t we get a curve like the center set, some sort of relative center set. Thus we have the following correspondences

big center set	big wavefront
center set	wavefront
center caustic	caustic

These concepts are illustrated in figure IV.3. At the right hand side of this figure we see a zoomed in version of the caustic at the center point. It can be calculated that the center caustic in the picture does not have any other singularity than ordinary cusps. The center caustic of M_1 and itself is known as the center symmetry set. This center symmetry set has been studied by many authors, two recent advances are [**Jan96**] and [**GH99**].
Generalizations of this construction to $l > 2$ hypersurfaces are straightforward. Let us briefly indicate how to proceed.

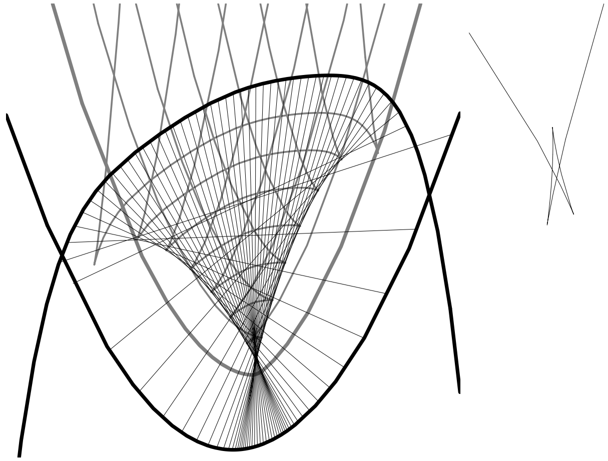

FIGURE IV.3. Left hand side: the projection of the big center set, right hand side:zoom in to the singular point of the caustic

Take l hypersurfaces in $\mathbb{R}^n$, the centroid set is $n-1$-dimensional. Its cuspidal edges are $n-2$ dimensional. In $\mathbb{R}^l$ consider once more the enlarged simplex

$$\Delta_l : \qquad \sum_{i=1}^{l} t_i = 1$$

and for every l-tuple $(x_1, \cdots, x_l)$ of points in $\mathbb{R}^n$ map Δ_l to $\mathbb{R}^n$ by

$$t_1, \cdots, t_l \mapsto t_1 x_1 + \cdots + t_l x_l$$

In this way we have a map from

$$\Delta_l \times (\mathbb{R}^n)^l \mapsto \mathbb{R}^n$$

Apply the map to the intersection in proposition IV.5. In this way for each point on the corresponding centroid you get an $l-1$-plane. So we have an $n-l+1$-parameter family of $l-1$-planes. Its envelope will be $n-1$ dimensional. This envelope is the center caustic of l hypersurfaces in $\mathbb{R}^n$. To generalize the notion of a big center set note that it will appear in product of $\mathbb{R}^n$ and $\mathbb{R}^l$. As each l-tuple $(t_1, \cdots, t_l)$ determines a point in $\mathbb{P}^l$ by

$$t_1, \cdots, t_l \mapsto [1; t_1; \cdots; t_l]$$

it is slightly more natural to consider the big center set as a subset of $\mathbb{P}T^*\mathbb{P}^l \times T^*\mathbb{R}^n$, with canonical 1 form $\pi_1^*\alpha - \pi_2^*(\sum \xi_i \,\mathrm{d}\, x_i)$.

IV.2.5. Kites. For a moment let us refocus attention to the simple case of two curves in $\mathbb{R}^2$ equipped with the euclidean metric. In figure IV.4 we see the image of two circles and their conflict set augmented with a number of kites. The kites consist of the normals from the basepoints to the conflict set as well as the tangent lines to the base points. The curve traced out by the intersection of the tangent lines is a straight line. In figure IV.5 we see the same construction, though now carried out with two circles contained in each other. There also a straight line is obtained.

DEFINITION IV.12. Let p be a regular point on the conflict set of two curves. The locus of the intersection points of the two tangent lines to the basepoints form the **kite curve**.

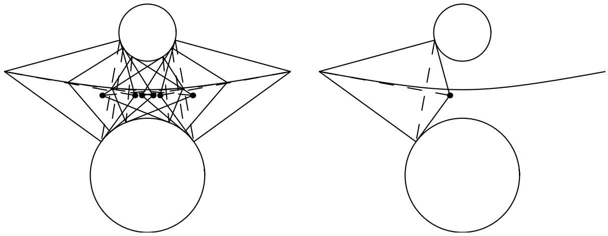

FIGURE IV.4. Oriented conflict sets and some kites

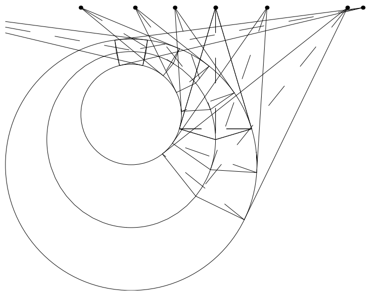

FIGURE IV.5. Kite curve of two circles contained in eachother

REMARK IV.13. The kite curve can have very bad singularities: think of the kite curve of two lines. This will be a curve not at all: it is a point, namely the intersection of the two lines.

EXAMPLE IV.14. The kite curve of any curve and a line will be contained in the line.

We will now describe a generalization of the kite curve to the case of n hypersurfaces in $\mathbb{R}^n$. In that case with a regular point on the conflict set there are n tangent planes intersecting in a single point. Thus there is also a kite curve.
In figure IV.6 another way of obtaining the kite curve in this case is pictured. Here we have constructed big wavefronts, and thus constructed a lifted conflict set $\pi_{n+1}L^h$ in $\mathbb{R}^{n+1}$. The developable surface of tangents to $\pi_{n+1}(L^h)$ intersects the plane $x_0 = 0$. The intersection coincides with the kite curve.

PROPOSITION IV.15. *The kite curve is a section of a developable surface.*

It is possible to extend the kite curve over the singularities of $\pi_{n+1}L^h$ where there are at each point of $\pi_{n+1}L^h$ n linearly independent normals. The extension is most conveniently done with the help of the big wavefronts $N^*M_i^h$ (or $N^*M_i^b$) though we have to take care to use the distance and not its square. Denote $\bar{y}$ a coordinate in $\mathbb{R}^{n+1}$. If $\bar{y}$ is on a tangent line to $\pi_{n+1}L^h$ we have the following relations

$$\langle \bar{y} - \bar{x}_i, \bar{\xi}_i \rangle = 0 \qquad i = 1, \cdots, l \tag{IV.12}$$

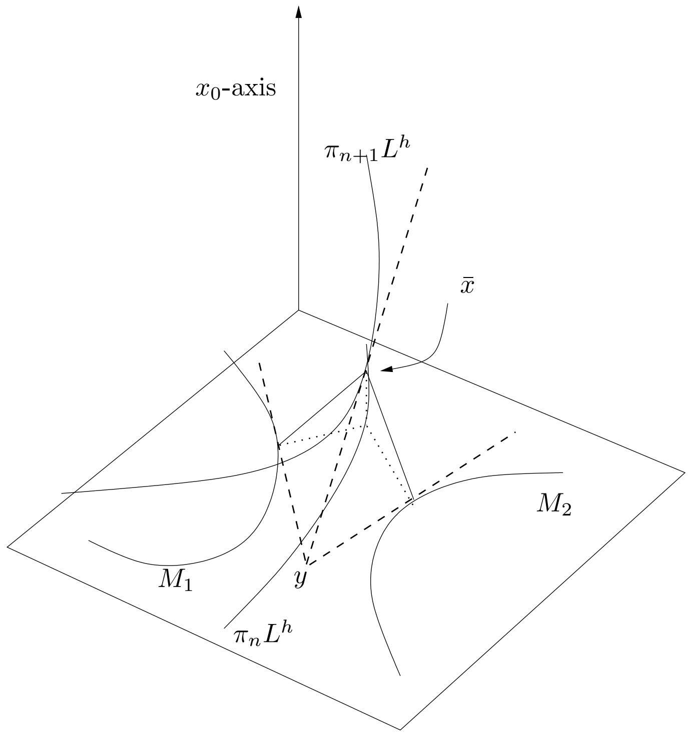

FIGURE IV.6. The construction of the kite curve.

The section of the developable surface we are interested in is $y_0 = 0$. The relations (IV.12) become

$$\langle y - x_i, \xi_i \rangle = x_0 \xi_{0,i} \qquad i = 1, \cdots, l \tag{IV.13}$$

This construction shows that with n spheres in sufficiently general position the kite curve is a line, because the intersection of the big wavefronts will lie in a plane.
The definition of the kite curve as a section of a developable surface has several disadvantages. From chapter 2 we know that the curvature of the conflict of n surfaces in $\mathbb{R}^n$ vanishes iff. all the curvatures at the basepoints are equal. If the curvature of the conflict set is zero then so is (see chapter 1) the curvature of the lifted conflict set. Thus in that case the lifted conflict set is a space curve in $\mathbb{R}^3$ with zero curvature. The kite curve is a section of its tangent developable.
Let us now see what happens to the kite curve if the curvature of the conflict set is zero, when $n = l = 2$. The lifted conflict set is a space curve in $\mathbb{R}^3$. A space curve in $\mathbb{R}^3$ looks like

$$\gamma(s) = (s, \frac{\kappa s^2}{2} + a_1 s^3 + \cdots, \frac{\kappa\tau s^3}{6} + b_1 s^4 + b_2 s^5 + \cdots)$$

According to [**Shc84**] the singularities of the dual start where $\tau = 0$. If $\kappa = 0$ then the singularities of the dual fall outside of the classification presented in [**Shc84**].
For the lifted conflict set $\tau = 0$ happens for instance when the derivative of the curvature on one base manifold equals the derivative of the curvature of the other base manifold. In

that case the kite curve is not singular, as the example of two circles shows.
The kite curve is singular when $\kappa = 0$, this happens when the curvatures at the basepoints are equal. But in that case the singularities of the dual immediatly no longer form part of the list of [**Shc84**].

IV.3. Genericity of the transversality condition

Once again we have $M_i,\ i = 1, \cdots, l$ in an ambient manifold X. Let there also be l quadratic Hamiltonians on T^*X, satisfying the demands from III.1.9. The images $\Psi_i(N^*M_i \times 0_{\mathbb{R}})$ are the big wavefronts, see section III.1.11. The transversality condition under which the conflict set of $M_1, \cdots, M_l$ is Legendrian is (III.16). The transversality is a generic property, according to theorem III.30. Here we will prove the genericity directly, i. e. we will not use the maximal rank criterion, we will prove the genericity of (III.16) directly. We will however assume that our Hamiltonians come from pseudo-Riemannian metrics.

THEOREM IV.16. *Under perturbations of $l-1$ of the basepoints the transversality condition is satisfied for a residual subset of $\bigoplus_{i=l}^{l} \mathrm{Emb}(M_i, X)$.*

PROOF. The perturbations are as in the proof of III.30

$$\gamma_i(s_i) = \gamma_i(s_i, 0) \mapsto \gamma_i(s_i, e_i), \qquad i = 1, \cdots, l$$

where

$$\frac{\partial \gamma_i(0,0)}{\partial e} = \mathbf{I}_n \text{ and } \frac{\partial^2 \gamma_i(0,0)}{\partial e_i \partial s_i} = 0$$

Thus we can look at the family

$$\begin{gathered} x_1, \xi_1, \cdots, x_l, \xi_l, t_1, \tau_1, \cdots, t_l, \xi_l, \delta x_1, \cdots, \delta x_{l-1} \mapsto \\ \Psi_1^t(x_1 + \delta x_1, \xi_1, t_1, \tau_1), \cdots, \Psi_l^t(x_l + \delta x_l, \xi_l, t_l, \tau_l) \end{gathered}$$

This should be transversal to $T^*_\Delta(X \times \mathbb{R})^l$. It will be enough to show that

$$\times_{i=1}^{l} \pi_X \Psi_i^t$$

is transversal to $\Delta \subset X^l$. This in turn makes it clear that is enough to show that

$$\pi_X \Psi^t(\gamma(s, e), \xi, t, \tau)$$

is a submersion. Locally $\pi_X \circ \Psi^t$ looks like

$$\gamma(s, e) + t\frac{\partial H}{\partial \xi}$$

in $\mathbb{R}^n$. The derivative wrt. e is

$$\mathbf{I}_n + t\frac{\partial H}{\partial x \partial \xi} \tag{IV.14}$$

We work locally and can choose normal coordinates, as in equation III.8 in which it will hold that at (x_0, ξ_0).

$$\frac{\partial g^{ij,l}}{\partial x} = 0 \text{ or } \frac{\partial H_l}{\partial x} = 0$$

So the derivative in (IV.14) is simply the unit matrix $\mathbf{I}$. The rest of the proof is identical to the proof of III.30. $\square$

Now we return to the centre set and the centroid set. The criterion for these to be Legendre is in proposition IV.5.

THEOREM IV.17. *For a countable intersection of open and dense subsets of*

$$\bigoplus_{i=1}^{l} \mathrm{Emb}(M_i, \mathbb{R}^n)$$

the center set is Legendrian (and the normal chord set therefore Lagrangian)

PROOF. We need a covering $\{U_\alpha\}$ of $M_1 \times \cdots \times M_l$ and in each $\{U_\alpha\}$ perturb the tangent space a little, as indicated in figure IV.7. It is enough to prove that, if $\vec{n}_i$ is the map that assigns the normal to M_i, that the map $(\vec{n}_1, \cdots, \vec{n}_l)$ is transverse to the diagonal. We will first show that locally families exist that are indeed transverse to the diagonal. Denote by

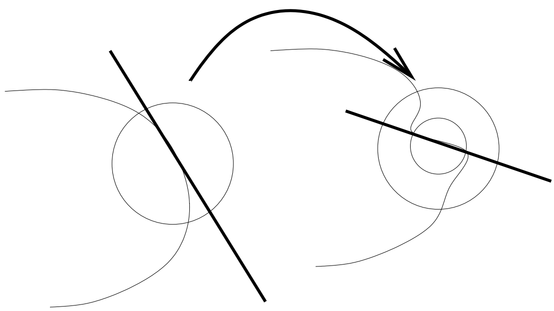

FIGURE IV.7. The map $\phi_{r,A,p=\gamma(s')} \circ \gamma(s)$

$$\phi_{r,A,p}, \quad r \in \mathbb{R}, \quad A \in SO(n, \mathbb{R}), \quad p \in \mathbb{R}^n$$

a diffeomorphism, which is the identity on $\mathbb{R}^n$ where we are outside the sphere of radius $2r$ round p and equal to $q \to A(x - q)$, inside a circle of radius r round q. Now compose an embedding $\gamma \colon M \to \mathbb{R}^n$ with the map $\phi_{r,A,p(\alpha)}$ and we will get a map that in some environment U'_α of of $p(\alpha) \in M$ is submersive. Looking at a product $\phi_{r,A,q(\alpha)} \circ \gamma_1, \phi_{r',A',p(\alpha)} \circ \gamma_2$ we see that in a neighborhood U_α the transversality condition is satisfied. Indeed, at $p(\alpha)$ the normal looks like $A\vec{n}$.
One can pick a countable number of points $p(\alpha)$ so that the U_α cover $M_1 \times \cdots \times M_l$. □

IV.4. "Reversed" sets: billiards and orthomtics

Several authors, notably Bruce & Giblin & Gibson [**BG92**] and [**BGG83**], Tabachnikov [**Tab95**] have studied geometrical constructions that can be generalized by means of canonical relations. In this section we want to show how close these constructions are to some form of "reversed conflict set". Most of the material here is sketchy and serves mainly to illustrate that the conflict set is far from an isolated problem.

IV.4.6. Some constructions of curves in the plane. We will start with the simplest context: a curve γ in the plane $\mathbb{R}^2$. Fix a point in the plane and call it the origin O. The curve γ will be the mirror for rays coming from O. Denote $\vec{n}$ a unit normal to γ.

DEFINITION IV.18 (Orthomtic). The orthomtic of γ and O is the curve defined by

$$O + 2\langle \gamma(t) - O, \vec{n} \rangle \vec{n}.$$

Definition IV.19 (Pedal curve). The pedal curve of γ and O is the curve defined by

$$O + \langle \gamma(t) - O, \vec{n} \rangle \vec{n}.$$

Conflict and center sets are related to orthomtics and pedal by the following proposition.

Proposition IV.20. *The curve γ is contained in the conflict set of the orthomtic of γ and the origin O. The pedal curve is the center set of the origin and the orthomtic.*

Proof. Clear from the definitions. □

We can define wrt. to the mirror γ and the origin O a kite curve. We provide a definition of the kite curve with the drawing IV.8. A practical formula for drawing the kite curve starting from the pedal is

$$\mathrm{kite}(\gamma) = \mathrm{pedal}(\gamma) + \left(\frac{\langle J(\gamma - O), \gamma' \rangle}{\langle \gamma - O, \gamma' \rangle} \right)^2 (\mathrm{pedal}(\gamma) - \gamma)$$

In the above formula J is the usual complexification mapping.
Two other curves can also be defined, see figure IV.8 for a drawing of the billiard curve and the contrapedal curve wrt. to a parabola. Where in the definition of the pedal curve one lets down the segment $\gamma - O$ onto the normal $\vec{n}$ to γ, in the definition of the contrapedal we let down the same segment $\gamma - O$ to the tangent vector $\vec{t} = \gamma' / \|\gamma'\|$.

Definition IV.21 (Billiard curve). The billiard curve of γ and O is the curve defined by

$$O + 2\langle \gamma(t) - O, \vec{t} \rangle \vec{t}.$$

Definition IV.22 (Contrapedal curve). The contrapedal curve of γ and O is the curve defined by

$$O + \langle \gamma(t) - O, \vec{t} \rangle \vec{t}.$$

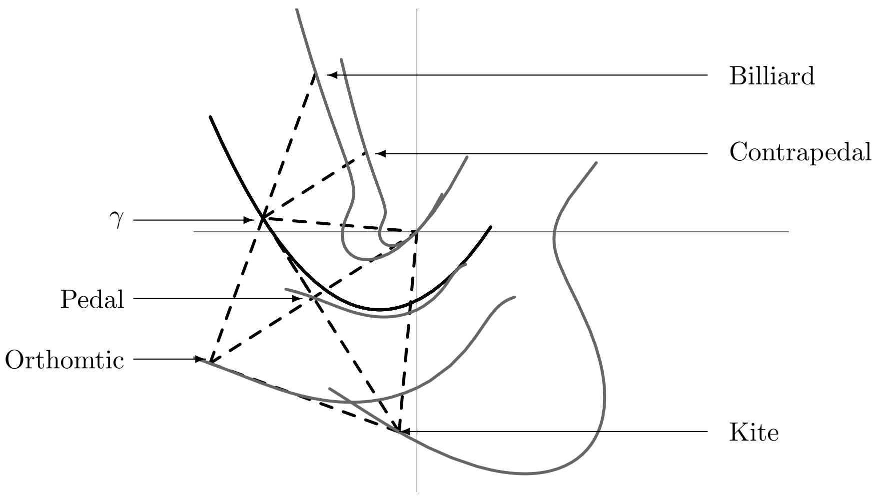

Figure IV.8. Pedal, Orthomtic, Kite, Contrapedal, Billiard

Let us now turn to a comparison between some existing results and results on the conflict set.

- *The origins of the curvature formula.* The curvature formula (I.8) is in fact a very old formula. We can trace it back to a classic book in geometrical optics: the book [**BW59**]. There on p. 173 we find the formula we find in chapter 1, and more. Also, the formulas $I.9$ and $I.10$ are already written there. To establish the connection between conflict sets and ray systems in optics consider figure IV.9. On the left side rays from a circular arc hit a curved mirror. On the right side we see the same rays from a circular arc but now in the setting of conflict sets.

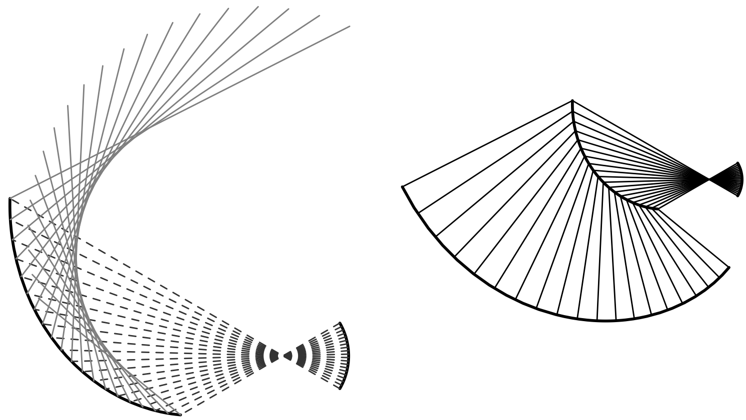

FIGURE IV.9. Left: incoming rays (dashed lines) reflected to outgoing rays. Right: the mirror is a conflict set.

- *Singular points of pedal and orthomtics.* All these curves and the relations between them form a rich object of study. For instance it is well-known that the pedal curve has a singularity where the curvature of the mirror γ is zero, hence where γ has an inflection. Thus the orthomtic also has a singular point where γ has an inflection, and vice versa.
  It can be derived also from the curvature formula that the orthomtic has a singularity when the mirror γ has an inflection. We proceed as follows: suppose that the mirror γ has an inflection $\kappa = 0$ at $p \in \gamma$. The curvature formula says that that if this is the case, the wavefront coming from the orthomtic has the same curvature $\|p - O\|^{-1}$ as the wavefront coming from the origin O. This means that at distance $\|p - O\|$ the orthomtic meets its own caustic, which is what we needed to show.
- *The billiard is a symplectic transformation.* In the drawing IV.8 we have introduced a billiard curve. The billiard curve defined here is not exactly the billiard studied in [**Tab95**]. In [**Tab95**] the author studies the transformation on the space of rays induced by the reflection. The billiard in that sense is a map

$$\text{Billiard}\colon T^*S^1 \to T^*S^1. \tag{IV.15}$$

  It turns out that the graph of the billiard is a canonical relation between T^*S^1 and T^*S^1. Tabachnikov proves this directly using coordinates. We can prove the assertion in the general case where the mirror is embedded in $\mathbb{R}^n$ in at least two ways. Firstly, the billiard transformation on the space of rays is an instance of the construction with symmetric spaces indicated in section IV.5. Secondly, if we can prove that the orthomtic is Legendrian we have also proven in view of theorem III.10 that the transformation on the space of rays preserves the symplectic form. In the next section we will derive a maximal rank criterion, (IV.19) under which the orthomtic is Legendrian.

- *Contrapedal.* The following are immediate, though they can also be obtained by a lenghty explicit calculation.

PROPOSITION IV.23. *The contrapedal curve is the center set of the origin and the billiard.*

PROPOSITION IV.24. *The billiard curve and the contrapedal are singular where the evolute of the mirror contains the origin.*

We will include another picture. Figure IV.10 beautifully illustrates the wavefront nature of the billiard: when the origin is the cuspidal point of the caustic the billiard and the contrapedal have a 4/3 Lipschitz smoothness point. When the origin lies on a smooth

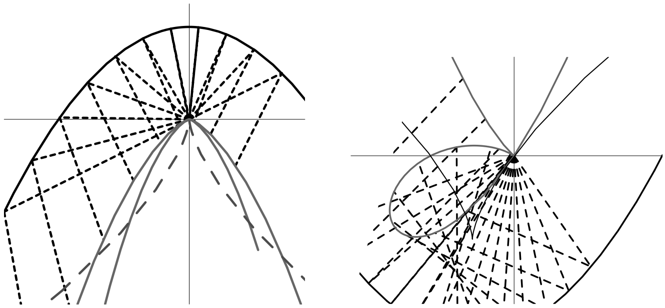

FIGURE IV.10. Billiard and contrapedal (both grey) , rays (finely dashed) , caustic (roughly dashed), and the mirror (solid, black)

part of the caustic the the contrapedal has a cusp.

The pictures here were made with a minor modification of the software available with the book [**Gra93**].

IV.4.7. Higher dimensional analogues and generalizations. With the exception of the kite curve the pedal, the contrapedal and the billiard and the orthomtic have straightforward generalizations to the n-dimensional case, where the mirror is a hypersurface in $\mathbb{R}^n$. It is not hard to see that the corresponding statements remain true:

- the mirror is contained in the conflict set of the origin O and the orthomtic,
- the pedal is the center set of the orthomtic and the origin,
- the contrapedal is the center set of the billiard surface and the origin,
- the pedal surface is singular where the mirror has zero Gaussian curvature
- the contra pedal surface is singular at the origin when the origin lies on a focal surface of the mirror,

Our main interest lies in the obvious generalization: replace the origin by some Legendrian submanifold L_O, cut L_O with $\{H = 1\}$, follow the rays up to the mirror, and follow the reflected ray (billiard) or its opposite (orthomtic), during the same amount of time.

The rays from the origin form a beam. If we replace the origin by some suitable L_O we can get a parallel beam. This clearly has some optical significance.

With rays from the origin O replaced by rays from L_O do the statements above still hold? In particular, under which conditions is the transformation on the space of rays, induced by the

reflection symplectic, or in other words, when is the "reversed" conflict set, the orthomtic, Legendrian?
Let us first deduce a maximal rank criterion. A practical criterion is easily established. Under the condition that

$$F(x,s) = \|x-\gamma(s)\|^2 - \|\gamma(s)\|^2$$

defines a non-degenerate phase function, the orthomtic of the mirror γ and the origin O is Legendrian. Replace the origin with the rays that are normals from a submanifold $\mu(t)$. For a parallel bundle $\mu(t)$ is linear and for a point source $\mu(t)$ is constant.
A phase function for the orthomtic is:

$$F(x,s,t) = \frac{1}{2}\|x-\gamma(s)\|^2 - \frac{1}{2}\|\gamma(s)-\mu(t)\|^2$$

For the orthomtic to be Legendrian we need that the matrix

$$K(F) = \mathrm{d}_{x,s,t}(F, \mathrm{d}_{s,t} F) \tag{IV.16}$$

has maximal rank where $(x,s,t) \in \Sigma(F)$, i.e. when $F = 0$, $d_s F = 0$, $d_t F = 0$. The singularities of the orthomtic are inspected with the matrix

$$\mathrm{d}^2_{s,t} F. \tag{IV.17}$$

If $\mu(t)$ is an embedded hypersurface, (IV.17) is a $(2n-2)\times(2n-2)$ matrix.
The matrix $K(F)$ becomes

$$\mathbf{K} = \begin{pmatrix} \mathrm{d}_x F & 0 & 0 \\ \mathrm{d}_x \mathrm{d}_s F & \mathrm{d}_s^2 F & \mathrm{d}_s \mathrm{d}_t F \\ \mathrm{d}_x \mathrm{d}_t F & \mathrm{d}_t \mathrm{d}_s F & \mathrm{d}_t^2 F \end{pmatrix}$$

To evaluate it note that if we write

$$F_1(x,s) = \frac{1}{2}\|x-\gamma(s)\|^2 \text{ and } F_2(x,t) = \frac{1}{2}\|x-\mu(t)\|^2$$

that it follows

$$F(x,s,t) = F_1(x,s) - F_2(\gamma(s),\mu(t))$$

but also

$$F(x,s,t) = F_1(x,s) - F_1(\mu(t),s) \tag{IV.18}$$

We will now calculate $\mathrm{d}_s^2(F)$. To this end we write a Monge form for the mirror $\gamma(s)$, i.e.

$$\gamma(s) = (s, \frac{1}{2}\langle \mathbf{II}_\gamma s, s\rangle)$$

We also use that the vector from $\mu(t)-\gamma(s)$ is mirrored through the tangent plane to the mirror to $x-\gamma(s)$. Denote ϕ the angle between $x-\gamma(s)$ and the tangent plane to the mirror γ. The derivative $\mathrm{d}_s^2 F$ then evaluates to

$$2\sin(\phi)\|x-\gamma(s)\|\mathbf{II}_\gamma.$$

The derivative $\mathrm{d}_s \mathrm{d}_t F = \dot\gamma^T\dot\mu$ and $\mathrm{d}_t^2 F$ becomes the second fundamental form of the wavefront at $\gamma(s)$ coming from the source surface $\mu(t)$.

$$\mathbf{K} = \begin{pmatrix} x-\gamma & 0 & 0 \\ \mathrm{d}_s \gamma & 2\sin(\phi)\|x-\gamma(s)\|\mathbf{II}_\gamma & \dot\gamma^T\dot\mu \\ 0 & \dot\gamma^T\dot\mu & (\mathbf{I}_\mu - \|\mu(t)-\gamma(s)\|\mathbf{II}_\mu)^{-1}\,\mathbf{II}_\mu \end{pmatrix} \tag{IV.19}$$

Equation (IV.19) is important when dealing with the singularities of the orthomtic. As said a parallel bundle is one with $\mathrm{d}_t^2\,\mu = 0$ whilst $\mathrm{d}_t\,\mu$ also written as $\dot{\mu}$ is a constant n by $n-1$ matrix. Hence the matrix $\mathbf{K}$ for a parallel bundle becomes:

$$\mathbf{K} = \begin{pmatrix} x-\gamma & 0 & 0 \\ \mathrm{d}_s\,\gamma & 2\sin(\phi)\|x-\gamma(s)\|\mathbf{I\!I}_\gamma & \dot{\gamma}^T\dot{\mu} \\ 0 & \dot{\gamma}^T\dot{\mu} & 0 \end{pmatrix} \tag{IV.20}$$

V Classification of singularities

V.1. Introduction

Throughout the previous chapters we have often encountered the view that the conflict set of l hypersurfaces is the projection of the intersection of their big wavefronts. In chapter one we used this view to prove $\mathcal{C}^j$-smoothness at regular points. In chapter three we used it to formulate a criterion (III.16) under which the conflict set is Legendrian. In this chapter we will apply the same technique to classifying singularities of conflict sets.
We will first of all have to study the big wavefront somewhat more closely. Secondly we look at the intersection, and lastly at the projection.
Our object is to first show that the big wavefront can be stratified in some suitable way. The stratification will contain bad strata and good ones. The good strata will have low codimension, and the bad strata will have high codimension.
As before, we have an embedding of a hypersurface into an ambient space: $\gamma\colon M \mapsto X$. As a subset of $X \times \mathbb{R}$ the big wavefront is given by

$$\text{(V.1)} \qquad F(x,s) = A(x,\gamma(s)) - x_0 = 0 \qquad \frac{\partial F}{\partial s} = 0$$

We will stick with the four assumptions on X and H of subsection III.1.9, so that F is a globally defined non-degenerate phase function.
Below we will define the notion of codimension for a germ. Doing so allows us to define for every $\bar{x} = (x_0, x)$ and s.

$$\operatorname{codim}(\bar{x}, s) = \operatorname{codim}(F(x,\cdot))$$

The codimension might well be infinite. For our purposes it will be enough to consider small codimensions ≤ 6. Germs of finite codimension are finitely determined. So we can fix some N for which the germs of codimension N are V-equivalent iff. there N-jets are V-equivalent. Also for the codimension ≤ 6 orbits we know that these are “simple”, their orbits in the jet space are submanifolds and finitely many. Now suppose that the map

$$(\bar{x}, s) \rightarrow \mathrm{j}^N(x_0 - F(x,s)) \in J^N(M)$$

is transverse to the structure of simple orbits $\mathcal{A}$. The pullback of the structure $\mathcal{A}$ exists in $X \times \mathbb{R} \times M$ and inherits the nice structure.
This structure projects with injective differential to the big wavefront in $X \times \mathbb{R}$. To ensure a transversal intersection of the projection of the pulled back orbits of $\mathcal{A}$. we will also impose a multi-transversality condition.

$$(\bar{x}, s^{(1)}, \cdots, s^{(p)}) \rightarrow {}_{(p)}\mathrm{j}^N(x_0 - F(x,s)) \in {}_{(p)}\,\mathrm{J}^N(M)$$

A set of codimension ≥ 7 on the big wavefront is not included in this structure.
Now suppose that we have l of these big wavefronts. It is necessary that the structures of orbits ≤ 6 intersect transversely, and that the intersection does not include any bad strata. The maximal codimension of simple strata is 6. If we want only those in the intersection we have to impose $n - l \leq 4$. Thus **the range of nice dimensions is** $n - l \leq 4$. In these dimensions the conflict set only has combinations of the well-known ADE singularities. From our reasoning it is also clear that outside this nice range there is going to be trouble. Indeed, suppose we extended our stratified structure to codimensions > 6. Here moduli arise and the constant codimension stratum is no longer a union of finitely many orbits in the jet-space.
In this chapter we first present some generalities on stability of Legendrian embeddings and

then we apply these to the big wavefronts and the conflict set. We take some time to explain these generalities, in order to arrive at a practical criterion for Legendrian stability stated in theorem V.15.
In the last section of this chapter we carry out some calculations that exhibit the singularities of conflict sets with $n - l \leq 4$ as certain singularities of wavefronts in $\mathbb{R}^{n-l+2}$. We analyse the combinations of ADE singularities there and exhibit the singularities of the conflict set as non-versal unfoldings of singularities of higher codimension.
Thus if we start with stable big fronts in general position the singularities of the conflict set are not necessarily V-stable. This is due to the fact that we allow only perturbations of the base-manifolds and not of the conflict set itself. We always keep a separation between the variables s_i.
Let us mention one more rather surprising consequence of our results, already pointed out in the introduction. The generic singularities of the conflict set of n hypersurfaces in X, with $\dim X = n$, are A_1^2 and A_2. Locally we always have the picture:

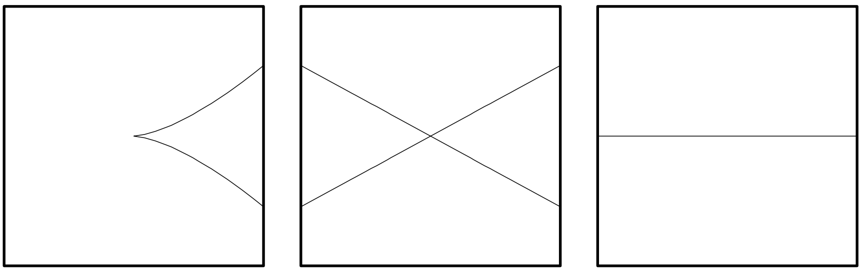

FIGURE V.1. Generic local forms of conflict sets of n surfaces in X.

V.2. Stability of Legendrian immersions.

V.2.1. V-equivalence. Let S be some non-empty finite subset of $\mathbb{R}^k$. We can speak of germs of $\mathcal{C}^\infty$ functions at S. Such germs form a ring $\mathcal{C}^\infty(S)$. If S consists of a single point s_0 this ring is local. If S is a finite set we have

$$\mathcal{C}^\infty(S) = \otimes_{s \in S} \mathcal{C}^\infty(s)$$

DEFINITION V.1. Two germs of maps f_1 and f_2 in $\mathcal{C}^\infty((\mathbb{R}^k, S), \mathbb{R}^t)$ are called V-equivalent if there are germs of diffeomorphisms h and H that make the following diagram commutative:

$$\begin{array}{ccccc}
\mathbb{R}^k, S & \xrightarrow{\mathrm{gr}(f_1)} & \mathbb{R}^k \times \mathbb{R}^t, S \times f_1(S) & \xrightarrow{\mathrm{proj}} & \mathbb{R}^k, S \\
\downarrow h & & \downarrow H & & \downarrow h \\
\mathbb{R}^k, S & \xrightarrow{\mathrm{gr}(f_2)} & \mathbb{R}^k \times \mathbb{R}^t, S \times f_2(S) & \xrightarrow{\mathrm{proj}} & \mathbb{R}^k, S
\end{array}$$

REMARK V.2. Some authors do not use the term V-equivalence , but instead speak of contact-equivalence or $\mathcal{K}$-equivalence. The same notion was discussed in chapter 1 of this thesis.

We specialize further to where $t = 1$.
Attached to a germ $f \in \mathcal{C}^\infty(s_0)$ is its tangent space $\mathrm{T}f$ to the orbit of f under the action of V-equivalence. The tangent space $\mathrm{T}f$ is an ideal in the ring $\mathcal{C}^\infty(s_0)$:

$$\mathrm{T}f = (f) + \left(\frac{\partial f}{\partial s_i}\right), \tag{V.2}$$

where "+" is a summation of ideals. The codimension of a germ is the codimension of the ideal $\mathrm{T}f$, that is it is the dimension of the quotient space Q_f

$$Q_f = \frac{\mathcal{C}^\infty(s_0)}{\mathrm{T}f} \qquad \operatorname{codim} f = \dim Q_f \tag{V.3}$$

The codimension of a germ f might well be infinite. For instance if $f(s_1, s_2) = s_1^2 s_2$ is the germ at zero of a function on $\mathbb{R}^2$ then the ideal $\mathrm{T}f$ does not contain s_2^k for all $k \geq 1$. So all of the monomials s_2^k are required as a basis for the quotientring Q_f. However if the codimension of f is finite then $Q_f \sim Q_g$ as $\mathcal{C}^\infty(s_0)$ algebras is equivalent to $f \overset{V}{\sim} g$.
Our definition of "codimension" has the property that a quadratic form in the "s" variables has codimension one. This is a little unusual but natural in our case.
Denote $\mathcal{M}(s_0)$ the unique maximal ideal in $\mathcal{C}^\infty(s_0)$. We can define the ring $J^r(s_0)$ of jets of order r by setting:

$$J^r(s_0) = \frac{\mathcal{C}^\infty(s_0)}{\mathcal{M}^{r+1}(s_0)}$$

The projection maps the ideal $\mathrm{T}f$ to $J^r(s_0)$. As an image we obtain $\mathrm{T}^r f$. We set

$$d_r = \dim \frac{J^r(s)}{\mathrm{T}^r f}$$

The sequence d_r is non-decreasing with upper bound $\operatorname{codim}(f)$. Consequently there is some minimal number r_0 for which $r > r_0 \Rightarrow d_{r_0} = d_r$. We'll call it $\rho(f)$.

Definition V.3. A map germ f is called finitely V-determined if there exists a $r_0 \in \mathbb{N}$ such that for every other germ g:

$$r \geq r_0, \quad j^r f = j^r g \Rightarrow \quad f \overset{V}{\sim} g$$

We have - [**Mat70a**], theorem 3.7 - that f is finitely determined iff. $\operatorname{codim}(f) < \infty$.
Functions that do not have a critical point are locally V-equivalent to a linear form. This is codimension 0. If there are critical points we find quadratic forms in codimension 1. After that we look at ternary forms, and so on. Up to codimension 6 there are only finitely many orbits in $\mathcal{C}^\infty(s)$, those are the ADE singularities. From codimension 7 there are infinitely many orbits with the same codimension.
Germs of codimension < 7 are determined by their 7-jet as a look at the ADE list shows. So a closed part of $J^6(s_0)$ can be Whitney stratified by the simple orbits.

V.2.2. Unfoldings wrt. V-equivalence.

Definition V.4. An unfolding of a germ $f \in \mathcal{C}^\infty((\mathbb{R}^k, S), (\mathbb{R}^t, T))$ consists of a (germ of a) parameter manifold $\mathbb{R}^n, 0$ and a function $F\colon \mathbb{R}^k \times \mathbb{R}^n, S \times \{0\} \to \mathbb{R} \times \mathbb{R}^n, f(S) \times \{0\}$ such that $g_{|\mathbb{R}^k\times\{0\}} \equiv f$ and

$$\begin{tikzcd}
\mathbb{R}^k \times \mathbb{R}^n, S \times \{0\} \arrow[rr, "F\times\mathrm{id}"] \arrow[dr, "\mathrm{proj}"'] & & \mathbb{R}^t \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}"] \\
& \mathbb{R}^n &
\end{tikzcd}$$

commutes.

Next, we will consider $t = 1$. Morphisms between two unfoldings F to F' are germs of maps that make the following diagram commutative:

(V.4)
$$\begin{tikzcd}
& \mathbb{R}^{n'+k}, \{0\}\times S \arrow[rr, "\mathrm{id}\times\mathrm{gr}(F')"] \arrow[dd, "h"] & & \mathbb{R}^{n'+k+1} \arrow[rr, "\mathrm{proj}"] \arrow[dd, "H"] & & \mathbb{R}^{n'+k}, \{0\}\times S \arrow[r, "\mathrm{proj}"] \arrow[dd, "h"] & \mathbb{R}^{n'}, 0 \arrow[dd, "h'"] \\
\mathbb{R}^k, S \arrow[ur] \arrow[dr] & & \mathbb{R}^{s+1} \arrow[ur] \arrow[dr] & & \mathbb{R}^k, S \arrow[ur] \arrow[dr] & & \\
& \mathbb{R}^{n+k}, 0\times\{S\} \arrow[rr, "\mathrm{id}\times\mathrm{gr}(F)"] & & \mathbb{R}^{n+k+1} \arrow[rr, "\mathrm{proj}"] & & \mathbb{R}^{n+k}, \{0\}\times S \arrow[r, "\mathrm{proj}"] & \mathbb{R}^n, 0
\end{tikzcd}$$

If h is the unfolding of a diffeomorphism on $(\mathbb{R}^{n+k}, \{0\} \times S)$ then the two unfoldings are said to be isomorphic. If h' is a diffeomorphism then the two unfoldings are equivalent. If we have a map $h'\colon \mathbb{R}^{n\prime} \to \mathbb{R}^n$ then we can form the induced unfolding h^*F:

$$h^*F\colon \mathbb{R}^{n\prime} \times \mathbb{R}^k \to \mathbb{R} \quad h^*F = F \circ (h, \mathrm{id})$$

The unfolding is called trivial if it is isomorphic to the unfolding $(x, s) \to (x, f(s))$. If the codimension of the germ f is $< \infty$ there is a universal object. A versal unfolding F has the property that every other unfolding F' of f is isomorphic to an induced unfolding of F. If F has the minimal number of parameters among those unfoldings having this universal property then it is called miniversal.

The above diagram is unnecessarily complicated. If we write

$$H\colon \mathbb{R}^{n'+k+t} \to \mathbb{R}^{n+k+t} \qquad H = (H_n, H_k, H_t)$$
$$h\colon \mathbb{R}^{n'+s} \to \mathbb{R}^{n+k} \qquad h = (h_n, h_k)$$

we can derive relations betweens these maps.

$$h_n(x') = h'(x') \qquad H_k(x', s) = h_k(x', s) \qquad H_n = h'(x')$$

So that the diagram (V.4) reduces to a relation between F and F', nl.

$$H(x', s, F'(x', s)) = (h'(x'), h_k(x', s), F(h'(x'), h_k(x', s)))$$

In particular, this shows that $F'(x', s)$ and $F(h(x'), s)$ are V-equivalent and that we consequently have an identity (see [**Mat70a**])

$$A(x', s)F(h'(x'), h_k(x', s)) = F'(x', s)$$

where $A\colon \mathbb{R}^{n'+k} \to \mathbb{R}$ is some smooth map.

V.2.3. Legendrian embeddings and V-equivalence. A Legendrian embedding is an embedding of an n-dimensional manifold in a contact manifold of dimension $2n+1$, such the image is an integral manifold of the contact structure. As such we can speak of germs of Legendrian manifolds at some point in a fibered contact manifold. The singularities of their projections can be classified up to maps from the contact manifold to itself that preserve the fibering and the contact form.

For these germs we have the following,

LEMMA V.5. *If $\Psi\colon \mathbb{P}T^*X \to \mathbb{P}T^*X$ is a diffeomorphism that preserves the fibering and the contact form then $\Psi = g^*$ for some diffeomorphism $g\colon X \to X$.*

Such germs of Legendrian manifolds are conveniently constructed with non-degenerate phase functions, as we saw in the previous chapter. Non-degenerate phase functions are special cases of unfoldings. If two non-degenerate phase functions are V-equivalent as unfoldings then they determine equivalent Legendrian manifolds. We want to show that the converse holds.

Let $F(x,s)\colon \mathbb{R}^{n+k} \to \mathbb{R}$ be a non-degenerate phase function for a Legendrian manifold. If

$$\operatorname{rk} \frac{\partial^2 F}{\partial s^2} = i$$

then we can apply the parametric Morse lemma to write F in the form

$$F'(x,s') + Q(s'') \qquad \#s'' = i \qquad \operatorname{rk} \frac{\partial^2 F'}{\partial s'^2} = 0$$

where $Q(s'')$ is a non-degenerate quadratic form. We will find that the corresponding Legendrian manifolds L_F and $L_{F'}$ are the same. The new phase function has a minimum number of variables.

REMARK V.6. The operation of adding an auxiliary s variables as opposed to the reduction carried out above is called a "doubling" of the hypersurface $F = 0$. The terminology stems from the fact that in the complex domain $\{u^2 = F(x,s)\}$ is a double cover of $\{F = 0\}$ under the projection $x, s, u \mapsto x, s$.

Having minimized the number of s variables by elimination of successive doublings of the hypersurface $F = 0$ we also have to touch upon the subject of suspension. Namely if

$$L \to T^*X \to X$$

is the germ of a Legendrian immersion and $]-a,a[$ is some open interval then also

$$L \times]-a,a[\to T^*(X \times]-a,a[) \to X \times]-a,a[$$

is for any $a > 0$ the germ of a Legendrian immersion. As for the image in X, an example would be a cusp in X becoming a cuspidal edge in $X \times]-a,a[$.

This occurs when h' in (V.4) is a submersion, i.e. when some parameters are trivial in the unfolding. Submersions can be written locally in the form $(x,x') \mapsto x$. The restriction of $(x',s) \mapsto (x', d_{x'}F)$ to $\{x',s \mid F(x',s) = \mathrm{d}_s F' = 0\}$ is then a suspension of the map $(x,s) \mapsto (x, \mathrm{d}_x F)$ restricted to $\{x,s \mid F(x,s) = \mathrm{d}_s F(x,s) = 0\}$.

If F and G are isomorphic as unfoldings then they are linked by a fibered equivalence.

$$F(x,s) = A(x,s)G(\tilde{x}(x), \tilde{s}(x,s)) \tag{V.5}$$

where A, $\tilde{s}$ and x are to satisfy the usual requirements, namely $\tilde{s}(0,s) = s$, $\tilde{x}$ is a diffeomorphism and the smooth map $\tilde{s}(s,x_0)$ is for every x_0 a diffeomorphism. If also F defines a germ of a Legendrian submanifold then G also defines a germ of a Legendrian submanifold which is equivalent to L_F.

THEOREM V.7. *Let $F \in \mathcal{C}^\infty(x_0, s_0)$ unfold some $f \in \mathcal{C}^\infty(s_0)$ and let G be another unfolding of f such that F, G are non-degenerate phase functions and L_F and L_G are equivalent Legendrian manifolds, then F and G are isomorphic as V-unfoldings after possibly carrying out a number of doublings of the hypersurfaces $F = 0$, $G = 0$.*

This is a purely local theorem. We can speak of $\mathbb{R}^k$ instead of M and of $\mathbb{R}^n$ of X.
Two expositions of the proof of this theorem, in [**Zak76**] and [**AGZV85**] reduce the theorem to the corresponding one for Lagrangian submanifolds of T^*X: any two non-degenerate phase functions near $(x_0, s_0) \in \mathbb{R}^{n+k}$ defining the same Lagrangian manifold and having the same signature $\frac{\partial^2 F}{\partial s^2}$ are $R+$-equivalent as unfoldings.
We will proceed as in the proof of the Lagrangian version of V.7. First of all, we may assume that

$$\frac{\partial^2 F}{\partial s^2} = 0$$

Secondly, we may also assume that the number of "s"-variables in F is equal to the number of "s"-variables for G.
Thirdly, we may assume that $L_F = L_G$: they are equivalent via a diffeomorphism $\tilde{x} \colon \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, instead of G we can consider $G(\tilde{x}, s)$.
Let us start by proving a lemma, mimicked from [**Hör71**]. As before denote $\Sigma(F) = (F, \mathrm{d}_s F)^{-1}(0)$.

LEMMA V.8. *There is diffeomorphism from $\mathbb{R}^{n+k}, (x_0, s_0)$ to itself that maps $\Sigma(F)$ to $\Sigma(G)$ and that preserves the fibering $\mathbb{R}^{n+k} \to \mathbb{R}^n$.*

PROOF. For this consider the map $\mathbb{R}^{n+k} \to \mathbb{R}^{n+1+k} \times \mathbb{P}^{n-1}$:

$$\mathbb{R}^{n+k} \ni (x, s) \to x, F, \mathrm{d}_s F, [\mathrm{d}_x F] \tag{V.6}$$

We claim that when restricted to $\Sigma(F)$ this is a diffeomorphism onto its image $L_F \subset \mathbb{P}T^*\mathbb{R}^n$. Call the map in (V.6) j_F. It holds that $(j_F)_*\delta s = 0$ implies $\delta s = 0$, because F is a non-degenerate phase function.
Hence there exists a map

$$\Psi_F \colon (x, u, w, [\xi]) \to s \in \mathbb{R}^k$$

such that $(\Psi_F \circ j_F)(x, s) = s$. For the non-degenerate phase function G we have similar maps j_G and Ψ_G.
Mingling the two maps we claim that a diffeomorphism from $\mathbb{R}^{n+k}$ to $\mathbb{R}^{n+k}$ is

$$x, s \mapsto x, \Psi_G(j_F(x, s)) \tag{V.7}$$

That (V.7) is a diffeomorphism follows when we calculate

$$\mathrm{d}_s(\Psi_G(j_F(x, s))) = \mathrm{d}_u(\Psi_G \circ j_F)\, \mathrm{d}_s F + \mathrm{d}_w(\Psi_G \circ j_F)\, \mathrm{d}_s^2 F + \mathrm{d}_{[\xi]}(\Psi_G \circ j_F)\, \mathrm{d}_s \mathrm{d}_x F$$

which in $(x_0, s_0) \in \Sigma(F)$ reduces to

$$\mathrm{d}_s(\Psi_G(j_F(x, s))) = \mathrm{d}_{[\xi]}(\Psi_G \circ j_F)\, \mathrm{d}_s \mathrm{d}_x F$$

Thus (V.7) is a diffeomorphism. We have

$$(x_0, \Psi_G(j_F(x_0, s_0))) \in \Sigma(G) \text{ iff. } (x_0, [\mathrm{d}_F(x_0, s_0)]) \in L,$$

which is equivalent to $(x_0, s_0) \in \Sigma(F)$. $\square$

We are now in a position where we can assume near (x_0, s_0) $\Sigma(F) = \Sigma(G)$, and $L_F = L_G$. Besides F and G also $F_t(x, s) = tF(x, s) + (1-t)G(x, s)$. is an unfolding. It should also be a non-degenerate phase function for L_F. This need not be the case for if $F(x_1, x_2, s_1, s_2) = x_1 + s_1^2 + s_2^2 + s_2^5$ and $G(x_1, x_2, s_1, s_2) = x_1 + s_1^2 - s_2^2$. Then $\Sigma(F)$ is $\Sigma(G)$ and L_F, L_G are also identical but still F_t will not be a non-degenerate phase function for $t = \frac{1}{2}$.
We will show that this is the only complication that can occur.

LEMMA V.9. *Assume that there are a minimum number of variables, i. e. that* $\frac{\partial^2 F}{\partial s^2} = 0$. *Then* $\frac{\partial^2 G}{\partial s^2} = 0$ *and* $F_t(x, s)$ *is a non-degenerate phase function for every* t.

PROOF. Because $L_F = L_G$, possibly after multiplying G with some non-zero function, we have on Σ that $\mathrm{d}_x F = \mathrm{d}_x G$. Thus on Σ the function $F - G$ vanishes to second order. The fact that F is a non-degenerate phase function means that we can use F and $\mathrm{d}_s F$ as $k+1$ of the $n+k$ coordinate functions in $\mathbb{R}^{n+k}$. It follows that we can write $F - G$ as

$$G(x, s) - F(x, s) = \sum_{0 \le i, j \le k} \frac{1}{2} a_{ij}(x, s)\phi_i(x, s)\phi_j(x, s) \tag{V.8}$$

where

$$A = (a_{ij})_{1 \le i, j \le k} \qquad \phi_0(x, s) = F(x, s) \qquad \phi_i(x, s) = \frac{\partial F}{\partial s_i}$$

Differentiating twice one obtains on Σ

$$\mathrm{d}_s^2 G = \mathrm{d}_s^2 F + A\, \mathrm{d}_s^2 F\, \mathrm{d}_s^2 F$$

Thus $\mathrm{d}_s^2 F(x_0, s_0) = 0$ implies $\mathrm{d}_s^2 G(x_0, s_0) = 0$. In the same way we have $\mathrm{d}_x \mathrm{d}_s F(x_0, s_0) = \mathrm{d}_x \mathrm{d}_s G(x_0, s_0)$. Thus for every fixed t the function $F_t(x, s) = tF(x, s) + (1-t)G(x, s)$ is a non-degenerate phase function. □

We have assembled the ingredients for a proof of theorem (V.7).

PROOF OF THEOREM V.7 . We will use what normally is called the homotopy method. For an exposition of this see the “lemme de réduction” in [**Mar76**].
Consider the equation

$$A_t F_t(x, s_t(s, x)) = F(x, s)$$

Differentiation wrt. to t yields

$$A_t(F - G) + \frac{\partial A(t, s, x)}{\partial t} F_t + \Xi(x, s, t) F_t = 0 \tag{V.9}$$

where $\Xi(x, s, t)$ is a vectorfield

$$\Xi(x, s, t) = \sum_{i=1}^{k} \frac{\partial s_i(t, s, x)}{\partial t} \frac{\partial}{\partial s_i}$$

We have already seen in the above that $F - G$ annuls to second order on Σ, but $\Sigma = \Sigma(F_t)$ as well. Thus we have an expression as in (V.8). This means that we can solve (V.9) and find the vector field Ξ and the function $\mathrm{d}_t A$.
Hence F and G are contact equivalent as unfoldings. □

Henceforth we will study the stability of the unfolding F. It is the same as stability of the diagram $L \to \mathbb{P}T^*X \to X$.
The theory of stability of unfoldings wrt. V-equivalence and stability of Legendrian immersions proceeds as the theory of stability of unfoldings wrt. to $R+$ equivalence and stability of

Lagrangian immersions. Proofs of these statements can be obtained by doing exactly what is described in [**Dui74**]. We will not do this completely but will concentrate on giving a criterion for global stability of a V-unfolding. It is our aim to show that in dimensions ≤ 6, i. e. $\dim X \leq 6$, generic embeddings of compact manifolds give rise to wavefronts that are Legendre-stable.

Definition V.10. If an unfolding F of a function f is such that there exists a neighborhood U of F in $\mathcal{C}^\infty(X \times M, \mathbb{R})$ so that $G \in U$ implies F V-equivalent as unfolding to G, then we say that F is stable.

With $S \subset M$ we denote some finite subset of M, in other words a point $(s^{(1)}, \cdots, s^{(p)})$ of $M^{(p)}$.

Proposition V.11. *If the unfolding F is stable then the graph of*

$$X \times M^{(p)} \stackrel{{}_p\mathrm{j}^r F}{\longrightarrow} {}_{(p)}\,\mathrm{J}^r(M, \mathbb{R})$$

is transversal to every $X \times M^{(p)} \times \mathcal{O}$ for every orbit $\mathcal{O}$ of the action of V-equivalence on functions in ${}_{(p)}\,\mathrm{J}^r(M, \mathbb{R})$

Proof. This is proposition 2.1.2 in [**Dui74**]. Take an arbitrary orbit $\mathcal{O} \subset {}_{(p)}\,\mathrm{J}^r(M)$. Unfoldings whose graphs are transversal to an orbit lie dense. Let F be stable then a nearby unfolding G is V-equivalent to F and G can be chosen to lie transverse to $\mathcal{O}$.
We have

$$F(x, s^{(i)}) = A(x, s^{(i)}) G(\tilde{x}(x), \tilde{s}(x, s^{(i)}))$$

near x_0, S. The mappings $\tilde{x}$, $\tilde{s}$ and A define near $x_0, {}_{(p)}\,\mathrm{j}^r F(x_0, S)$ a diffeomorphism $j_{\tilde{x},\tilde{s},A}$ from $X \times {}_{(p)}\,\mathrm{J}^r(M)$ to itself that preserves the manifolds $X \times \mathcal{O}$.
The diffeomorphism also maps the graph of ${}_{(p)}\,\mathrm{j}^r G$ to the graph of ${}_{(p)}\,\mathrm{j}^r F$. Hence the graph of ${}_{(p)}\,\mathrm{j}^r F$ lies transverse to the manifolds $X \times \mathcal{O}$. $\square$

The orbits have a tangent space and the graph of ${}_{(p)}\,\mathrm{j}^r F$ has a tangent space. That the orbits lie transverse to the graph of ${}_{(p)}\,\mathrm{j}^r F$ can near x, S be expressed as an algebraic criterion: the tangent space to the orbit at $x_0, {}_p\,\mathrm{j}^{(r)} F(x_0, S)$ can be determined if we look at $f(s) = F(x_0, s)$. If we multiply this by something close to the identity, say $1 + \epsilon g(s)$, we get

$$\frac{\partial}{\partial \epsilon}\left((1 + \epsilon g(s)) f(s)\right)_{|\epsilon=0} = g(s) f(s) \tag{V.10}$$

If we allow diffeomorphisms close to the identity we get

$$\frac{\partial}{\partial \epsilon}\left(f(s + \epsilon s)\right)_{|\epsilon=0} = s \frac{\partial f}{\partial s} \tag{V.11}$$

so that the tangent space to the orbit in ${}_{(p)}\,\mathrm{J}^r(S)$ is

$$\mathcal{C}^\infty(x, S)(F) + \mathcal{M}(S) \frac{\partial F}{\partial s} + \mathcal{M}(S)^{r+1}$$

The tangent space to the graph of ${}_{(p)}\,\mathrm{j}^r F$

$$\mathcal{C}^\infty(x, S) \frac{\partial F}{\partial s} + \mathbb{R} \frac{\partial F}{\partial x} + \mathcal{M}(S)^{r+1}$$

Thus stability of an unfolding implies the algebraic criterion that at x, S:

$$\forall r \qquad \mathcal{C}^\infty(S) = \mathcal{C}^\infty(S)\left(F, \frac{\partial F}{\partial s}\right) + \mathbb{R}\left(\frac{\partial F}{\partial x}\right) + \mathcal{M}(\mathcal{S})^{r+1} \tag{V.12}$$

The proof that the converse holds is a standard argument, which we will not repeat in detail. The first step is to call an unfolding **inf-stable** if

$$\mathcal{C}^\infty(M \times X) = \mathcal{C}^\infty(M \times X)\left(F, \frac{\partial F}{\partial s}\right) + \mathcal{C}^\infty(X)\left(\frac{\partial F}{\partial x}\right) \tag{V.13}$$

Equation (V.13) implies (V.12). If (V.12) holds for sufficiently large r and x, S the inverse application can also be established. This needs an application of the Malgrange-Mather preparation theorem in the way we use it in the next paragraph to determine how large r needs to be.
The second step is to recover stability from (V.13) . This inverse statement is the one known as "infinitesimal stability implies stability".

V.2.4. Local stability of the unfoldings. We indicate how the Malgrange-Mather theorem is usually used to eliminate tails.
Let $(x_0, s_0) \in \mathbb{R}^{n+k}$. Local stability of an unfolding $F\colon \mathbb{R}^{n+k} \to \mathbb{R}$ at (x_0, s_0) is defined as follows.

$$\mathcal{C}^\infty(s_0) = \sum_{i=1}^{k} \frac{\partial F}{\partial s_i}(x_0, \cdot) \cdot \mathcal{C}^\infty(s) + \mathcal{C}^\infty(s_0) \cdot F(x_0, \cdot) + \sum_{j=1}^{n} \mathbb{R} \cdot \frac{\partial F}{\partial x_j}(x_0, \cdot) + \mathcal{M}(s_0)^{r+1} \tag{V.14}$$

holds as an identity between ideals for some $r \geq n$.

THEOREM V.12. *The identity (V.14) is satisfied iff. the identity*

$$\mathcal{C}^\infty(x_0, s_0) = \mathcal{C}^\infty(x_0, s_0) \cdot \mathrm{T}F(x_0, \cdot) + \sum_{j=1}^{n} \mathcal{C}^\infty(x_0) \cdot \frac{\partial F}{\partial x_j} \tag{V.15}$$

holds.

REMARK V.13. Let $f\colon (X, p) \to (Y, q)$ be some $\mathcal{C}^\infty$ map between two manifolds. This induces a map $f^*\colon \mathcal{C}^\infty((Y, q), \mathbb{R}) \to \mathcal{C}^\infty((X, p), \mathbb{R})$. Both these rings are local by the Hadamard lemma, see [**Mat70a**] lemma (1.4). The map f^* makes every $\mathcal{C}^\infty((X, p), \mathbb{R})$ module into an $\mathcal{C}^\infty((Y, q), \mathbb{R})$ module. The Malgrange Mather preparation theorem answers the following question: "When is a module A that is finitely generated over $\mathcal{C}^\infty((X, p), \mathbb{R})$ finitely generated over $\mathcal{C}^\infty((Y, q), \mathbb{R})$?". This is the case if

$$\frac{A}{f^*\mathcal{M}((Y, q))A}$$

is a finite-dimensional vectorspace over $\mathbb{R}$. Again, see [**Mat70a**], theorem (1.10). An application which shows the multiple usages of this theorem can be found in [**GG73**], example (B) to theorem (3.6).

PROOF OF THEOREM (V.12). Let B be the $\mathbb{R}$-"module".

$$\sum_{j=1}^{n} \mathbb{R} \cdot \frac{\partial F}{\partial x_j}$$

and let A be the module

$$\mathcal{C}^\infty(s_0) \cdot \mathrm{T}F(x_0, \cdot).$$

Let $C = \mathcal{C}^\infty(s_0)$. Saying that $A + B = C$ is saying that (V.15) holds. We consider $D = C/A$. This is a finitely generated $\mathcal{C}^\infty(s_0)$ module. We want it to be finitely generated as an $\mathbb{R}$-"module", where the generators should be the ones we have for B. We will use theorem

(3.10) from [**GG73**] , which states that this is the case iff. the projections of these generators in

$$D' = \frac{D}{\mathcal{M}^{n+1}(s_0)D}$$

generate the module D'. This is an equivalent statement to (V.14). □

V.2.5. Equisingularity manifolds. We set forth to translate (V.12) into a geometric criterion.
Let us pose for sufficiently large r.

$$\mathcal{E}^r(x_0, s_0) = \{(x, s) \mid j^r_s F(x, s) \in \mathcal{O}\} \subset X \times M$$

where $\mathcal{O}$ is the orbit of $j^r F$ under the action of V-equivalence. This orbit is a manifold near x_0, s_0. Also the graph of

$$x, s \mapsto \left(F, \frac{\partial F}{\partial s}, \cdots, \frac{\partial^r F}{\partial s^r}\right) \in \mathrm{j}^r(M) \tag{V.16}$$

is of course a smooth manifold.
If the graph of (V.16) intersects the orbit $\mathcal{O}$ transversely the equisingularity manifolds are indeed submanifolds of dimension $n + k - \mathrm{codim}\mathcal{O}$. For sufficiently large r the intersection no longer depends on r.
The tangent space to the orbit is

$$j^r\left(\mathcal{C}^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^{k} \mathcal{M}(s_0)\frac{\partial F}{\partial s_j}(x_0, \cdot)\right) \tag{V.17}$$

To determine $\mathrm{codim}\,\mathcal{O}$ in $\mathrm{J}^r(s_0)$ for sufficiently large r we need to calculate

$$\begin{aligned}
\mathrm{codim}\,\mathcal{O} &= \dim_{\mathbb{R}} \frac{\mathcal{C}^\infty(s_0)}{\mathcal{M}(s_0)^{r+1} + \mathcal{C}^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k \mathcal{M}(s_0)\frac{\partial F}{\partial s_j}(x_0, \cdot)} \\
&= \dim_{\mathbb{R}} \frac{\mathcal{C}^\infty(s_0)}{\mathcal{M}(s_0)^{r+1} + \mathcal{C}^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k \mathcal{C}^\infty(s_0)\frac{\partial F}{\partial s_j}(x_0, \cdot)} \\
&+ \dim_{\mathbb{R}} \frac{\mathcal{M}(s_0)^{r+1} + \mathcal{C}^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k \mathcal{C}^\infty(s_0)\frac{\partial F}{\partial s_j}(x_0, \cdot)}{\mathcal{M}(s_0)^{r+1} + \mathcal{C}^\infty(s_0)F(x_0, \cdot) + \sum_{j=1}^k \mathcal{M}(s_0)\frac{\partial F}{\partial s_j}(x_0, \cdot)} \\
&= \mathrm{codim}(f) + k
\end{aligned}$$

Proposition V.14. *If the graph of (V.16) hits the orbit $\mathcal{O}$ of f in $\mathrm{J}^r(M)$ transversally then*
- *the equisingularity manifolds are smooth and of codimension* $\mathrm{codim}(f) + k$ *in* $\mathbb{R}^{n+k}$,
- *the equisingularity manifolds project immersively to* $\mathbb{R}^n$.

Proof. In view of the above, only the last statement remains to be proved.
We will show that if the vector $(0, \delta s)$ lies in the tangent space to $\mathcal{E}(x_0, s_0)$ then it is zero.
The vector $(0, \delta s)$ lifts to $J^r(k)$ by the graph of (V.16). So the lift of the vector is

$$\sum_{j=1}^{k} \mathrm{j}^r\left(\frac{\partial F}{\partial s_j}(x_0, \cdot)\right)\delta s_j$$

This vector should lie along the tangent space of the orbit $\mathcal{O}$. Taking into account the equation for the tangent space (V.17) we ask that:

$$\sum_{j=1}^{k} \frac{\partial F}{\partial s_j}(x_0,\cdot)\delta s_j \in \mathcal{C}^\infty(s_0)F(x_0,\cdot) + \sum_{j=1}^{k} \mathcal{M}(s_0)\frac{\partial F}{\partial s_j}(x_0,\cdot) + \mathcal{M}(s_0)^{r+1}$$

Here r can be made large, say $r > \operatorname{codim}(f)$ and this will imply that the δs_j are all zero. We conclude that the projection of the $\mathcal{E}(x_0, s_0)$ to X is immersive. □

The equisingularity manifolds corresponding to codimension 1 are defined in $\mathbb{R}^{n+k}$ by $\mathrm{d}_s F = 0$ and $F = 0$. This is just $\Sigma(F)$. Regular points on the wavefront are thus always of codimension 1. The wavefront is at regular points $\mathcal{E}_X = \pi_X(\mathcal{E}(x_0, s_0))$ - where π_X is the projection from $X \times M \to X$. The codimension of $\mathcal{E}_X$ in X is exactly the codimension of f if the orbit of f is hit transversally by the unfolding F.
This situation is different to what happens in the Lagrangian case with the R^+-equivalence. There the definition of codimension is such that the points with codimension one form the caustic. Most points of the Lagrangian manifold have codimension 0.
We come to the theorem which relates the equisingularity manifolds and the local stability to global stability. This theorem provides the practical criteria by which one decides whether stability holds.

Theorem V.15. *Let $F \in \mathcal{C}^\infty(X \times M)$ be such that the map $\Sigma(F) \to X$ is proper (and hence finite to one) F is a stable iff.*

- *F is locally stable at every $(x, s) \in X \times M$*
- *For a fiber $(x, s^{(1)}, \cdots, s^{(p)})$ of $\Sigma(F) \to X$ the projections of equisingularity manifolds to X at each $(x, s^{(i)})$ intersect transversally at $x \in X$.*

Let us indicate the differences between this theorem and the theorem on Lagrangian stability that is stated in [**Dui74**], proposition 2.2.4.
These are the usual criteria for global stability. For instance in the result on generic mappings from the plane to the plane one asks that the curves along which folds occur intersect transversally.
The Lagrangian version of the theorem on global stability contains a third demand, namely that

- Affine independence. The $\mathrm{d}_x F(x, s^{(i)})$ for $i = 1, \cdots, p$ are affinely independent, as linear operators on the tangent space to intersection of the equisingularity manifolds.

This demand though a little technical has a geometric interpretation. The vector $d_x F$ may be tangent to the caustic. This will happen for instance on the regular part of a focal sheet from a surface in $\mathbb{R}^n$. Hence two sheets of the caustic may come to lie as in figure V.2: the equisingularity manifolds (here the regular part of focal sheets) intersect transversally.
Intuitively we see that this is not a stable situation. From the proof of proposition 2.2.4 in [**Dui74**] we can also conclude that it is not globally stable.
However in the Legendrian case it is not necessary to impose a similar demand. By definition equisingularity manifolds in X cannot have codimension 0 in the Legendrian case. As vectors $\mathrm{d}_x F(x, s^{(i)})$ it follows from the non-degeneracy condition on F that they are normals to the wavefronts and hence to the equisingularity manifolds. So if the equisingularity manifolds in the Legendrian case intersect transversally then as vectors the $\mathrm{d}_x F(x, \cdot)$ are linearly independent.

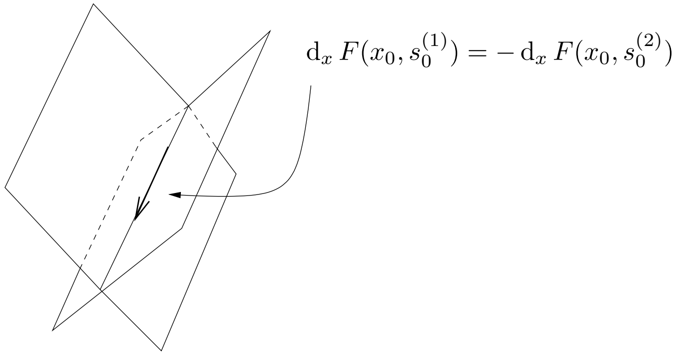

FIGURE V.2. Two sheets of caustic

PROOF OF THEOREM V.15. We have seen that stability is equal to (V.12) at all x, S. Thus we have to check that the conditions in the theorem imply (V.12) and vice-versa.
The tangent vectors δx in $V_i = T\mathcal{E}_X(x_0, s_0^{(i)})$ are those that

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x_0, \cdot)\delta x_i \in \mathcal{C}^\infty(s_0^{(i)})F + \mathcal{C}(s_0^{(i)})\frac{\partial F}{\partial s}(x_0, \cdot) + \mathcal{M}(s_0^{(i)})^{r+1}$$

Thus

$$\delta x \to \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x_0, \cdot)\delta x_i \tag{V.18}$$

is a map from $\mathbb{R}^n$ to $\mathcal{C}^\infty(s_0^{(i)})$, and also to

$$W_i = \frac{\mathcal{C}^\infty(s_0^{(i)})}{\mathcal{C}^\infty(s_0^{(i)})F + \mathcal{C}(s_0^{(i)})\frac{\partial F}{\partial s}(x_0, \cdot) + \mathcal{M}(s_0^{(i)})^{r+1}} \tag{V.19}$$

These maps are surjective due to the local stability.
In this way the tangent space V_i to each equisingularity manifold is the kernel of a map $\mathbb{R}^n \to W_i$. The V_i intersect transversally iff.

$$\mathbb{R}^n \ni \delta x \to W_1 \oplus \cdots \oplus W_p \tag{V.20}$$

is surjective. Indeed if the map in (V.20) is surjective the dimension of its kernel is n minus the sum of the dimensions of W_i. Thus the codimension of the kernel is the sum of the dimensions of the W_i , hence the sum of the codimensions of the $F(x_0, \cdot)$ at the $s_0^{(i)}$. This is exactly the dimension that the intersection of the V_i should have in order to have a transversal intersection of the equisingularity manifolds.
On the other hand, if (V.20) is surjective then the criterion (V.12) holds. □

V.3. Statement and proof of the main theorem

In this section we will assume some knowledge of stratifications. A standard reference is [**GWdPL76**].
Consider $\mathrm{J}^r(s_0)$ the space of r-th order jets of functions at $s_0 \in M$. The action of V-equivalence divides $\mathrm{J}^r(s_0)$ into orbits. Denote $W^r(s_0, m)$ the set of jets whose codimension is $> m$. For $N \leq \min(6, n)$ the complement $\mathrm{J}^r(s_0) \setminus W(s_0, N)$ has only finitely many strata.

The set $W(s_0, N)$ is an algebraic set and this algebraic set has some Whitney stratification also that can be refined to fit together nicely with the stratification of the complement. Strata with codimension $\leq N$ correspond to finitely many orbits of the V-action. These are called the good strata, denote them $\mathcal{B}$. Together with the algebraic stratification $\complement\mathcal{B}$ of $W(s_0, N)$ that fits with $\mathcal{B}$ we have thus stratified $\mathrm{J}^r(s_0)$.

PROPOSITION V.16. *For a generic embedding of a compact manifold the big wavefront $M_i^h = \pi_{n+1}(N^*M_i^h)$ has a Whitney stratified subset that consists of of equisingularity submanifolds of codimension at most $N = \min(n, 6)$. The strata correspond to singularity types of individual fronts. In particular, if $n \leq 6$ the strata miss only isolated points on the big wavefront.*

PROOF. Again write $F(x, s) = A(x, \gamma(s))$. The phase function for the big front is $G(\bar{x}, s) = x_0 - F(x, s)$. Consider

$$X \times M \mapsto X \times \mathrm{J}^r(M, \mathbb{R}), \qquad (x, s) \mapsto x, \mathrm{j}^r(G) \tag{V.21}$$

At each $(\bar{x}_0, s_0)$ the map is generically transverse to the stratification, because of the demands we put on our distance function in section III.1.9.
As before with the equisingularity manifolds the strata project immersively to $X \times \mathbb{R}$. To get the strata to project in general position we would need that all strata that are above x_0 together intersect the diagonal stratification transversally. That is if

$$(\bar{x}_0, s_0^{(i)}) \in \pi_{n+1}(L^h), \qquad i = 1, \cdots, p$$

then we want

$$(\bar{x}, s^{(1)}, \cdots, s^{(p)}) \mapsto (\mathrm{j}^r_s\, G(x, s^{(1)}), \cdots, \mathrm{j}^r_s\, G(x, s^{(p)})) \tag{V.22}$$

to be transverse to the diagonal stratification $\mathcal{D}({}_pJ^n(M))$. That this all works is again due to the fact that our distance functions have a nowhere zero first derivative.
Now, because M is compact, there are only a finite number of points above one $\bar{x}_0$. Thus if we prove the multitransversality for $p = n+1$ we have proven the multi transversality for all p. In that case the stratification of the part determined by the good jets can be refined. $\square$

REMARK V.17. Note that our transversality requirements come in two steps corresponding to the maps (V.21) and (V.22).

REMARK V.18. The distance functions $F(x, s) = A(x, \gamma(s))$ are for compact manifolds M in $\mathcal{C}^\infty_{\mathrm{pr}}(M \times X)$. The function space $\mathcal{C}^\infty_{\mathrm{pr}}(M \times X)$ consists of those functions for which the projection of the surface

$$\Sigma(F) = \{F = 0, \frac{\partial F}{\partial s} = 0\} \subset M \times X$$

to X is a proper map. It is in this space that we have the genericity results for Legendrian and Lagrangian mappings, cf. theorem V.15.

V.3.6. Generic intersection of l big fronts. Let us now consider l conic Lagrangian manifolds N^*M_i in T^*X. These can be mapped to give l big wavefronts manifolds $N^*M_i^h$ in $T^*(X \times \mathbb{R})$. Returning to conflict sets our object of interest is the intersection:

$$N^*M_i^h \times \cdots \times N^*M_i^h \pitchfork T^*_\Delta(X \times \mathbb{R})^l \tag{V.23}$$

The partial stratification of each of the big wavefronts is dependent on some integer N. We can arrange this N to be such that in the intersection (V.23) there are only good strata.

THEOREM V.19. *Suppose $n - l \leq 4$. For a residual set of embeddings in $\bigoplus_{i=1}^{l} \mathrm{Emb}(M, X)$ the intersection V.23 and hence the conflict set only has combinations of simple singularities. If $\#A_1$ is the number of smooth big fronts in the intersection.*

- *singularities of conflict sets appear in generic $n-l$ parameter families of fronts in $\mathbb{R}^{n-\#A_1}$, if $l - \#A_1 = 2$ and*
- *in generic $\sigma(n-l)$- parameter families of fronts in $\mathbb{R}^{n-l+2}$, if $l - \#A_1 > 2$.*

It holds $\sigma(2) = 2$, $\sigma(3) = 6$ and $\sigma(4) = 12$. The list of singularities of conflict sets is finite for $n - l \leq 4$. If $n - l > 4$ we expect moduli.

PROOF. For the same reasons as in the proof of proposition V.16 the stratifications of the bigfronts can be made to intersect transversally. This is the third transversality criterion we impose.
Suppose the big fronts meet at some point (x_0, x). To find the highest codimension singularity we can encounter we assume that $l-1$ of the $N^*M_i^h$ project to $X \times \mathbb{R}$ as a smooth hypersurface. Suppose the remaining one has at (x_0, x) a codimension μ singularity. Then the stratum on which it lies will have codimension μ. Adding codimensions we should have in the generic case that

$$(l-1) + \mu \leq n + 1$$

Rewriting this, we obtain

$$\mu \leq n - l + 2$$

For $\mu \leq 6$ we have only simple stable singularities. Thus if $n - l + 2 \leq 6$, that is $n - l \leq 4$, is in the domain of the nice dimensions. On the other hand if $n - l > 4$ we will encounter strata of codimension higher then 5. It follows that $n - l \leq 4$ is the domain of nice dimensions.
The fourth and last criterion we require for the proof of our main theorem, has to do with projection of the intersection (V.23) to X. We want this projection to be a map with regular intersections.
We ask that the p-fold projection

$$\pi^{(p)} \colon \left(\mathbb{R}^{n+1}\right)^{(p)} \to (\mathbb{R}^n)^p$$

is transverse to the diagonal stratification when restricted to the intersection of the big fronts. This is also achieved with first order perturbations.
Once we know that the stratified big wavefronts intersect transversally we want to determine what sort of singularities occur in the finite list we have. We need to prove our claim that estimates the number of parameters needed to produce the fronts from the finite list in $\mathbb{R}^{n-l+2}$. The estimate will follow from a codimension and modality count, to be carried out in the next section. □

V.4. Geometrical description of different cases

A "local model" for a singularity is a universal unfolding for it. Local models for all the simple singularities are well-known. A front with an A_3 singularity can for instance be made with $s_1^4 + x_1 s_1^2 + x_2 s_1 + x_3$. The fronts that we consider are big wavefronts. The singular points on them that we are considering have a tangent space to the stratum on which they lie of dimension at least one. This means, see the list of examples 6.4 in [**Arn76**], that the time function is a trivial parameter in the unfolding for the big wavefront. Locally the non-degenerate phase function for the big fronts that we are considering can be written:

$$x_0 = F(x, s)$$

where $F(x, s)$ is a versal unfolding from the ADE list.
However there is more than one big wavefront. What we know of the big wavefronts is that their equisingularity strata intersect transversally. Hence we can use these equisingularity strata to define coordinates.
For the description of these singularities the main distinction is the difference $n-l$. Indeed, if $(\mu_1, \mu_2, \cdots, \mu_l)$ is the list of codimensions then we seek μ_i with $1 \leq \mu_i$ and $\sum_{i=1}^{l} \mu_i \leq n+1$. Those μ_i that are 1 correspond to smooth hypersurfaces. This is because the A_1 singularity is just a Morse function and the unfolding is

$$G_1 \colon x_0 = s_1^2 + x_1$$

The equations $G_1 = 0$ and $\mathrm{d}_s G_1 = 0$ imply $x_0 = x_1$. Because of the transversal intersection of the equisingularity manifolds x_1 can be discarded as a coordinate. Every A_1 singularity presents a reduction of n and l by 1.
If $n - l$ is fixed then for arbitrary n a certain number of parts in the partition have to be 1. Let k be the number that is > 1, thus at least 2 . It follows that $2k + (l - k) \leq n + 1$ so that a maximum of $n - l + 1$ codimensions is > 1. The others are 1. In the following list of codimensions we have already eliminated the A_1 possibilities.

$\boxed{n - l = 0}$ If $n = l$ then at most 1 of the μ_i is > 1. So the only case to consider is $l = 1$. We can have only two cases: (1), (2).

$\boxed{n - l = 1}$ At most 2 of the codimensions are > 1. So it is enough to consider $n = 3$, $l = 2$. In addition to the above combinations we will have: $(2, 2)$ and (3).

$\boxed{n - l = 2}$ The relevant dimensions are: $n = 5$, $l = 3$. The new cases are: (4), $(3, 2)$ and $(2, 2, 2)$.

$\boxed{n - l = 3}$ Dimensions: $n = 7$, $l = 4$. New cases: (5), $(4, 2)$, $(3, 3)$, $(3, 2, 2)$, $(2, 2, 2, 2)$

$\boxed{n - l = 4}$ Dimensions: $n = 9$, $l = 5$. New cases: (6), $(5, 2)$, $(4, 3)$, $(4, 2, 2)$, $(3, 3, 2)$, $(3, 2, 2, 2)$ and $(2, 2, 2, 2, 2)$.

For each of the strata there are only a limited number of singularities, from the ADE list. The conflict set has dimension $n-l+1$. The codimension of a singularity on a generic front of dimension $n - l + 1$ is maximally $n - l + 2$. If we look at the above list we see that on the conflict set the codimension can add up to $2(n - l + 1)$.

V.4.7. $n - l = 0$**.** If $n - l = 0$ the singularities of the conflict set are the generic singularities of 2-dimensional fronts. Those are A_1, A_1^2 and A_2. Note the marked difference with the case of symmetry sets. Their list - see p. 168 of [**JB85**] - contains two more normal forms, namely A_3 and A_1^3.
The A_3 is an "endpoint". One can imagine conflict sets where this happens. One could take $M_1 = M_2$, but this is surely no generic situation. The singularity A_1^3 happens when a symmetry set on the big wavefront of a curve M_1 in $\mathbb{R}^2$ gets cut by a smooth big wavefront. In the case of symmetry sets the big wavefronts all come from one curve and there are thus three branches meeting. On the conflict set A_1^3 is the sum of A_1^2 and A_1, thus there will only be two branches meeting.

V.4.8. $n - l = 1$**.** If $n - l = 1$ the codimension can add up to 4. The cases to consider are A_2A_2, $A_1^2A_2$, $A_1^2A_1^2$. All other singularities are just those of generic 2-dimensional fronts. Pictures are partly in [**JB85**]
The A_2A_2 singularity is a generic projection of two transversely intersecting cuspidal edges in $\mathbb{R}^4$. These cuspidal edges can intersect in two ways. This is indicated in figure V.3. One

way is that only a point remains, another way is where the adjacent A_1 strata intersect. To

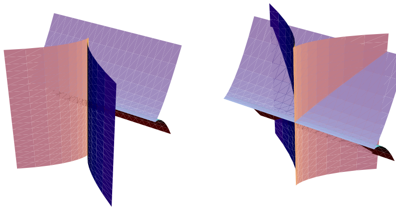

FIGURE V.3. Scheme of the two ways in which A_2 and A_2 can intersect

obtain a picture we will take two copies of our previous example

$$G_1 \colon x_0 = s_1^3 + x_1 s_1 + x_2 \qquad G_2 \colon x_0 = s_2^3 + x_3 s_2 - x_2$$

We take tangent spaces to the strata at $x = 0$. For $F = s^3 + As + B$ the variety determined by $(F, \mathrm{d}_s F) = 0$ is $\frac{A^3}{27} + \frac{B^2}{4} = 0$. Thus the A_1 stratum of G_1 is determined by $B = x_2 - x_0 = 0$. The A_2-stratum is $A = B = 0$, thus $x_2 - x_0 = x_1 = 0$. For G_2 we have the A_1-stratum $x_2 + x_0 = 0$ and the A_2 stratum $x_3 = 0$. At zero these intersect transversally. We project the intersection along the time axis x_0 to $\mathbb{R}^3$. The surface we get is in figure V.4. This

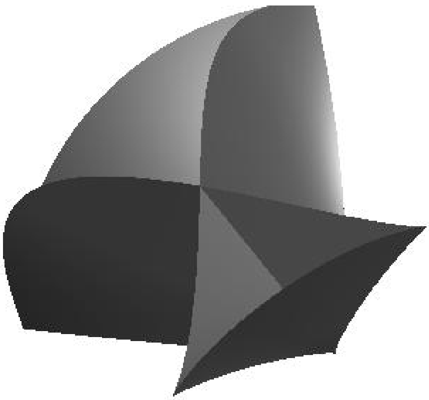

FIGURE V.4. The $A_2 A_2$ surface

surface is also known as D_4^+ if we view it as a metamorphosis of a wavefront in $\mathbb{R}^3$. Recall that a metamorphosis is a one dimensional family of fronts, see[**Arn90**]. The name D_4^+ is chosen because the surface is also obtained with an unfolding

$$G_1 - G_2 = s_1^3 - s_2^3 + x_1 s_1 - x_3 s_2 + 2x_2 \tag{V.24}$$

The unfolding $G_1 - G_2$ is not a versal unfolding. If we want to unfold the D_4^+ germ $s_1^3 + s_2^3$ with a V-versal unfolding we need 4 parameters. In the unfolding $G_1 - G_2$ the term $s_1 s_2$ misses.

The picture D_4^+ is in [**AGZV85**], §22. In [**JB85**] it is on p. 174.

We proceed to discuss the differences between the list of singularities of symmetry sets in

[**JB85**] and our list. Again for symmetry sets the list is larger. There are the endpoints $D_4^{\pm}$. As mentioned in the introduction they do not occur on conflict sets. We do have A_1^4 because we have to consider 1 parameter metamorphoses. On the symmetry set A_1^4 appears as the intersection of $\binom{6}{2}$ planes. On the conflict set two planes are not present. Confirm also the picture 3 in the introduction.
On both the conflict set and the symmetry set we have $A_1^2A_2$. The $A_1^2A^2$ singularity is on the level of big fronts an intersection of a segment of a symmetry set with a cuspidal edge of a big front. On the conflict set it looks as the right hand side of V.3.
The picture the authors of [**JB85**] mention as A_1A_3 we do not have because in our case only a suspension of A_3 can occur on one big wave front. The other big wavefront is a hyperplane that cuts the suspension of A_3 transversely. Such an intersection is a normal swallowtail, which Bruce et. al. mention as A_4.

V.4.9. $n-l=2$**.** If $n-l=2$ we need at least $n=4$ and $l=2$ to obtain an interesting new local model. Indeed the case (4) has A_4 and $D_4^{\pm}$ and suspensions of the cases that occur with $n-l=1$. So the first really new case is $(3,2)$. On this stratum we have amongst others A_3A_2. This is a metamorphosis of a 3-dimensional front. Some sections of this surface are in figure V.5. In one them we actually see a swallowtail meeting a cuspidal edge. For A_3A_2

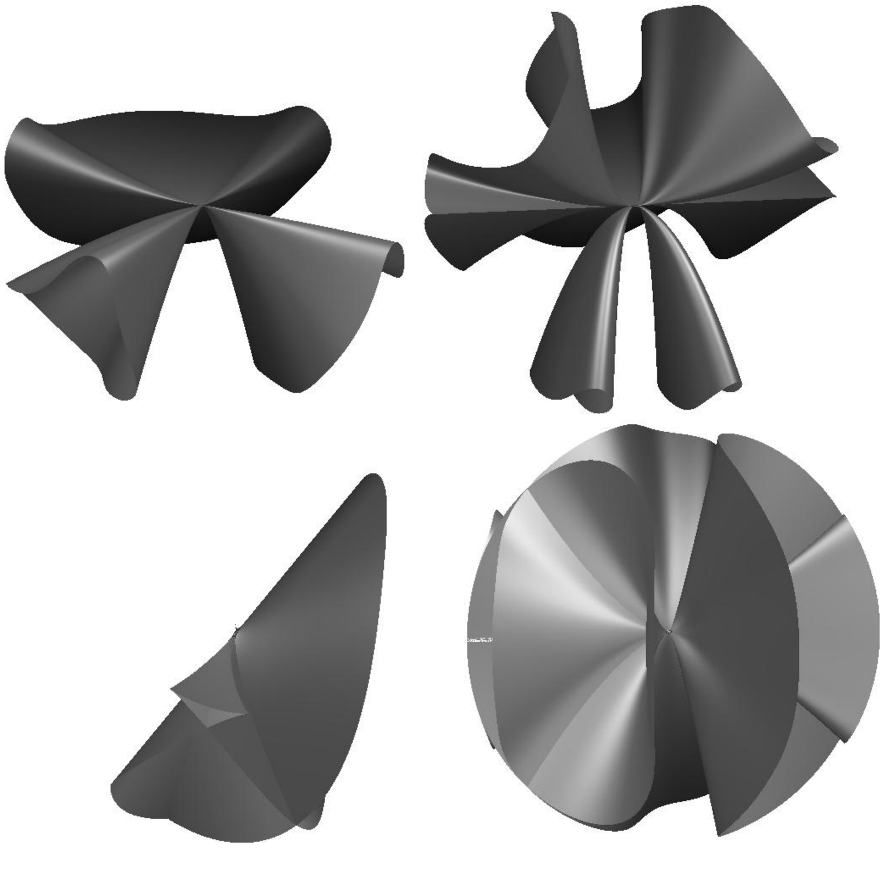

FIGURE V.5. Sections of A_3A_2

we have the same illustration of our main theorem as with A_2A_2. The A_3A_2 singularity can happen when $n-l=2$ thus $n=4$ for the first time.
Suppose we make an A_3 with $x_0 = s_1^4 + x_1 s_1^2 + x_2 s_1 + x_3$. The tangent space to the A_1 stratum of the front is $x_0 = x_3$. For the A_2-stratum and A_1^2 add $x_2 = 0$ and for A_3 add $x_1 = 0$.
We need an A_2 that transversally intersects these. We can choose it to be $x_0 = s_2^3 + x_4 s_2 - x_3$. The unfolding which we obtain is again not versal. It occurs in a 2-parameter family. We see this from the unfolding:

$$s_1^4 + x_1 s_1^2 + x_2 s_1 + 2x_3 - s_2^3 - x_4 s_2 \tag{V.25}$$

A two dimensional family in which this unfolding exists is made by augmenting $x_5 s_1^2 s_2$ and $x_6 s_1 s_2$. Adding $x_5 s_1^2 s_2$ and $x_6 s_1 s_2$ to (V.25) we get an unfolding of an E_6-germ.
The remaining interesting case if $n-l=2$ is $A_2A_2A_2$. Here $n=5$ and the surface itself is three dimensional. To get a universal unfolding we have to find three A_2 unfoldings that lie in general position.
We could take $G_1\colon x_0 = s_1^3 + x_1 s_1 + x_4$ and $G_2\colon x_0 = s_2^3 + x_2 s_2 + x_5$ and $G_3\colon x_0 = s_3^3 + x_3 s_3 - x_5$. We have to verify that at 0 the big wavefronts G_i lie transverse to each other. This is done in the same way as above.
The intersection can be projected along the x_0 axis at first. In the above we have proven that in addition to the time axis there must exist some other direction in which we can project. To determine it we will use the fact that the conflict set in the nice dimensions is locally Whitney stratified. Hence at the singular point we consider we can try to compute the tangent space.
The conflict surface can be parameterized in the following way: $x_1 = -3s_1^2$, $x_2 = -3s_2^2$, $x_3 = -3s_3^2$ and $x_4 - x_0 = 2s_1^3$, $x_5 - x_0 = 2s_2^3$, $-x_5 - x_0 = 2s_3^2$. The projection along the x_0 surface results in $x_5 = s_2^3 - s_3^3$ and $x_4 = 2s_1^3 - s_2^3 - s_3^3$. In 0 the limit of tangent spaces to the $A_1A_1A_1$ stratum is thus δx_1, δx_2 and δx_3. Thus we could take any direction except those to project along.
Let us determine an unfolding. In $\mathbb{R}^n = \mathbb{R}^5$ the unfolding is still an unfolding of a multigerm, that is two copies of the A_2A_2. Two parameters will be missing in this unfolding. We can project still further down to $\mathbb{R}^4$. A generic section and projection is δx_5, in the x_1, x_2, x_3, x_4 space this unfolding is

$$F = 2G_1 - G_2 - G_3 = 2s_1^3 - s_2^3 - s_3^3 + 2x_1 s_1 - x_2 s_2 - x_3 s_3 - 2x_4$$

The germ $2s_1^3 - s_2^3 - s_3^3$ is a T_{333} germ. Thus we have some sort of answer to a question posed by T. C. Wall in [**Wal77**]. He asked for "some geometrical discussion of the higher order singularities". In a way such a geometrical discussion is already present in [**AGZV85**]. Here we obtained an interpretation of T_{333} as a falling together of three cuspidal edges.
All pictures here were obtained with the help of the software [**GPS01**]. Sections of the $A_2A_2A_2$ surface apperently are too singular to be depicted. In the next subsection we will also no longer be able to obtain the right pictures.
The T_{333} germ we get is contained in the list of wavefronts listed in [**AGZV85**], §21.8. The case we have is mentioned as ${}_0P_8^0$. As a wavefront it occurs generically in $\mathbb{R}^7$. Here we encounter it in $\mathbb{R}^4$. We summarize as follows:

PROPOSITION V.20. *If $n-l=2$ then the Legendrian singularities occur in at most 2 parameter families in $\mathbb{R}^4$.*

V.4.10. $n - l > 2$. After the previous longer treatments of examples we will now be brief. If the singularities on the individual wavefronts correspond to germs f_i then the singularity of the conflict set is the germ $\sum f_i$. (In the $A_2A_2A_2$ case the f_i were s_1^3, s_2^3 and s_3^3.)
Our definition of codimension was not the usual one. What most authors call the codimension we will call the multiplicity. It is

$$\dim_{\mathbb{R}} \frac{\mathcal{C}^\infty(s_0)}{\mathcal{C}^\infty(s_0)\left(\frac{\partial f}{\partial s}\right)}$$

All the germs we consider are quasi-homogeneous, hence we have:
LEMMA V.21.

$$\mathrm{codim}(\sum_{i=1}^{l} f_i) = \prod_{i=1}^{l} \mathrm{codim}(f_i)$$

PROOF. As the f_i can be given a normal form where they are quasi-homogeneous. If their weights are $(\alpha_1, \cdots, \alpha_{\mathrm{corank}(f_i)})$ then their multiplicity is

$$\prod_{i=1}^{\mathrm{corank}(f_i)} \left(\frac{1}{\alpha_i} - 1\right)$$

The germ $\sum f_i$ associated to the conflict set thus has as multiplicity the product of the multiplicities of f_i. But in the case of quasi-homogeneous germs the multiplicity equals the codimension (i. e. Milnor and Tjurina number coincide.) $\square$

The general picture sketched above for the examples A_2A_2, A_2A_3 and $A_2A_2A_2$ is that the big wavefronts can be assumed to have $l-1$ equations

$$x_0 = f_1(s_1) + x_1 + R_1(x, s_1)$$
$$\cdots$$
$$x_0 = f_{l-1}(s_{l-1}) + x_{l-1} + R_2(x, s_2)$$
$$x_0 = f_l(s_l) - x_1 + R_l(x, s_l) \tag{V.26}$$

Here the R_i have no x-variables in common. The s_i-variables look like

$$s_i = (s_{i,1}, \cdots, s_{i,\mathrm{corank}(f_i)})$$

In R_i we will meet at least $\mathrm{corank}(f_i)$ different x-variables.
The R_i split up in terms linear and non-linear in s_i. We have for instance:

$$R_1 = x_l s_{1,1} + \cdots + x_{l+\mathrm{corank}(f_1)-1} s_{1,\mathrm{corank}(f_1)} + S_1(x, s_1)$$

The terms linear in s_1 assure that the rank condition is not violated. The term S_1 has none of the x coordinates that appear in the linear terms. The total s_1 degree of the S_1 term is strictly higher than 1, i. e. $S_1(x, s_1)$ contains no terms linear in the s_1. For the other R_i we have a similar normal form. No two of the R_i have any x-variables in common.
A normal form as in (V.26) assures that all the equisingularity manifolds intersect transversely.
The non-versal unfoldings we obtain always contain for all s variables the terms x_is_i. Also they contain the constant term. When studying the versal unfolding one distinguishes between the basis elements e_i and the elements J_i. The J_i are those monomials that do not

affect the multiplicity, their weighted degree is ≥ 1. In our non-versal unfolding the monomials J_i do not occur. The number of monomials $\#J$ is called the inner modality.
If we then carry out $n-l+1$ section and projection steps - as described in section III.2.17 - we get a non-degenerate phase function with germ $\sum f_i$ at $x=0$.
Let us compare the versal unfolding of the germs $\sum f_i$ to the non-versal unfolding we get. We will take the liberty of speaking of "the basis" for a versal unfolding even though there is no unique or canonical basis.
Following §13.2 in [**AGZV85**] the number $\#J$ of monomials J is equal to the modality, for the quasi-homogeneous germs that we have. Hence an upper estimate for the number of parameters necessary for a wavefront to occur in a family is the codimension minus $\#J$. Hence if we know that a conflict set is a front in $\mathbb{R}^{n-l+2}$, the number of parameters to obtain such a front is less or equal to

$$\operatorname{codim}(\sum f_i) - \#J - (n-l+2)$$

V.4.11. $n-l=3$. With $l=2$ and thus $n=5$ we have as a first case the $(4,2)$ stratum. This leads to at least three cases: $(D_4^{\pm}, A_2)$ and (A_4, A_2). We have to study the $D_4^{\pm}$ versal unfolding. Unfoldings for the umbilics $D_4^{\pm}$ are:

$$D_4^+ \quad : \quad 0 = s_1^3 + s_2^3 + As_1s_2 + Bs_1 + Cs_2 + D$$

and

$$D_4^- \quad : \quad 0 = s_1^3 - 3s_1s_2^2 + A(s_1^2+s_2^2) + Bs_1 + Cs_2 + D$$

The limit of the tangent planes at $0 \in \mathbb{R}^4$ to the wavefronts is $D=0$. Big wavefronts we could choose in order to have transversal intersections of strata are

$$\begin{aligned} D_4^- \quad &: \quad x_0 = s_1^3 - 3s_1s_2^2 + x_1(s_1^2+s_2^2) + x_2s_1 + x_3s_2 + x_4 \\ A_2 \quad &: \quad x_0 = s_3^3 + x_5s_3 - x_4 \end{aligned}$$

The germ $s_1^3 - 3s_1s_2^2 - s_3^3$ has codimension 8. The corresponding conflict set is a hypersurface in $\mathbb{R}^5$.
The number of J-polynomials is in $(D_4^{\pm}A_2)$-cases 1. We see that the $(D_4^{\pm}A_2)$ arise in 2-parameter families of fronts in $\mathbb{R}^5$. Note that though $D_4^+A_2$ results in the same germ as $A_2A_2A_2$ they have non-isomorphic unfoldings and thus according to theorem V.7 their wavefronts are not diffeomorphic.
On the $(4,2)$-stratum we also have (A_4A_2). The germ we get is $s_1^5 - s_2^3$. This is still a simple singularity, namely E_8. Its codimension is 8. Here there are no J polynomials. We need 3 parameters.
The following case is $(3,3)$. It also occurs in $\mathbb{R}^5$. The germ becomes $s_1^4 + s_2^4$. Its codimension is 9. This is the singularity X_9 as the modality is 1 it will first occur generically as a 7-dimensional front in $\mathbb{R}^8$. Hence we need a 3 parameter family in $\mathbb{R}^5$.
In $\mathbb{R}^6$ we will meet $(3,2,2)$ for the first time. The germ is $s_1^4 + s_2^3 + s_3^3$. It has codimension 12. Its modality is 1. Hence this singularity happens in $12-1-(3+2)=6$ parameter families in $\mathbb{R}^5$.
In $\mathbb{R}^7$ we find a corank 4 singularity $(2,2,2,2)$. It has codimension 16. Hence there are five elements of the basis of a versal unfolding whose weight is equal to or exceeds 1. The monomials J_1 to J_5 are

$$s_1s_2s_3, s_2s_3s_4, s_1s_3s_4, s_1s_2s_4, s_1s_2s_3s_4$$

Hence the modality is 5 and this singularity happens in $16-5-5=6$ parameter families of fronts in $\mathbb{R}^5$.

Proposition V.22. *If $n-l=3$ then singularities of the conflict set happen in at most 6 parameter families of fronts in $\mathbb{R}^5$.*

V.4.12. $n-l=4$**.** The first new case is $(5,2)$. Here we have a corank 3 germ for D_5A_2. We also have corank 2 with A_5A_2. In both cases the modality is 1 and the codimension 10. Hence these occur in 3 parameter families of fronts in $\mathbb{R}^6$. The germ we call D_5A_2 is also known as Q_{10} in the list of Arnold.

Then comes $n=7$ with $(4,2,2)$. We can have three different triples of strata: $A_4A_2A_2$ and $D_4^{\pm}A_2A_2$. The codimension here is 16. The germ $A_4A_2A_2$ has modality 2. The germ $D_4^{\pm}A_2A_2$ has modality 5. Hence we expect $A_4A_2A_2$ in 8 parameter families of fronts in $\mathbb{R}^6$ and for $D_4^{\pm}A_2A_2$ we need 5 parameters in $\mathbb{R}^6$.

A different corank 4 case comes with $n=8$ and $(3,2,2,2)$. This has codimension 24. It is of weighted degree 1 with weights $(\alpha_1,\cdots,\alpha_4)=(1/4,1/3,1/3,1/3)$. The basis of the local algebra contains 6 monomials that are of weighted degree ≥ 1. Hence we see that this singularity happens in 12 parameter families of fronts in $\mathbb{R}^6$.

Finally, there is the most singular one, which has codimension 32 with $n=9$ and $l-2=5$ We calculate the stratum for which the multiplicity remains constant. This consist of 16 basis vectors in the local algebra. In this case $n-l=4$ and $n-l+2=6$ so that another 6 parameters are missing.

Proposition V.23. *If $n-l=4$ then the singularity of the conflict set appear in at most 12-parameter families in $\mathbb{R}^6$.*

List of notations

H	Homogeneous Hamiltonian, page 35
J	Complexification mapping, page 44
L^h	Lifted conflict set, see equation (III.16), page 44
$M_1, \cdots, M_l$	l hypersurfaces in an ambient manifold, page 1
M_c	Conflict set in $\mathbb{R}^n$, page 1
M_i^h	Big wavefront in $\mathbb{R}^n \times \mathbb{R}$, page 7
$N^*M_i^h$	Big wavefront in $T^*(X \times \mathbb{R})$., page 40
Q_f	Local algebra, page 73
$T^*_\Delta(X)^l$	Diagonal in the fiber., page 41
$T^*_{(1,1)}\mathbb{R}^{2n} \setminus 0$	Diagonal in the base, see equation (IV.1), page 55
X	Ambient manifold of dimension n in which the hypersurfaces M_i reside, page 35
$\mathcal{C}^\infty(S)$	Ring of germs of functions at finite set $S \subset M$, page 72
$\mathcal{C}^\infty(s_0)$	Ring of germs of functions at $s_0 \in M$, page 72
$\Sigma(t, M)$	Wavefront at time t emanating from M, page 37
$\bar{x}$	Coordinates (x_0, x) in ambient space $\times\mathbb{R}$, page 21
$\mathbf{I\!I}$	Fundamental two-form, page 1
$\mathbf{I}_n$	$n \times n$ identity matrix., page 49
$\mathbf{K}$	Rank matrix, page 45
$\mathrm{T}f$	Extended tangent space, page 73
$\mathrm{vf}(K)$	Hamiltonian vectorfield, page 35
$d_{i,j}$	Signed distance between curvature (kissing) spheres, page 10

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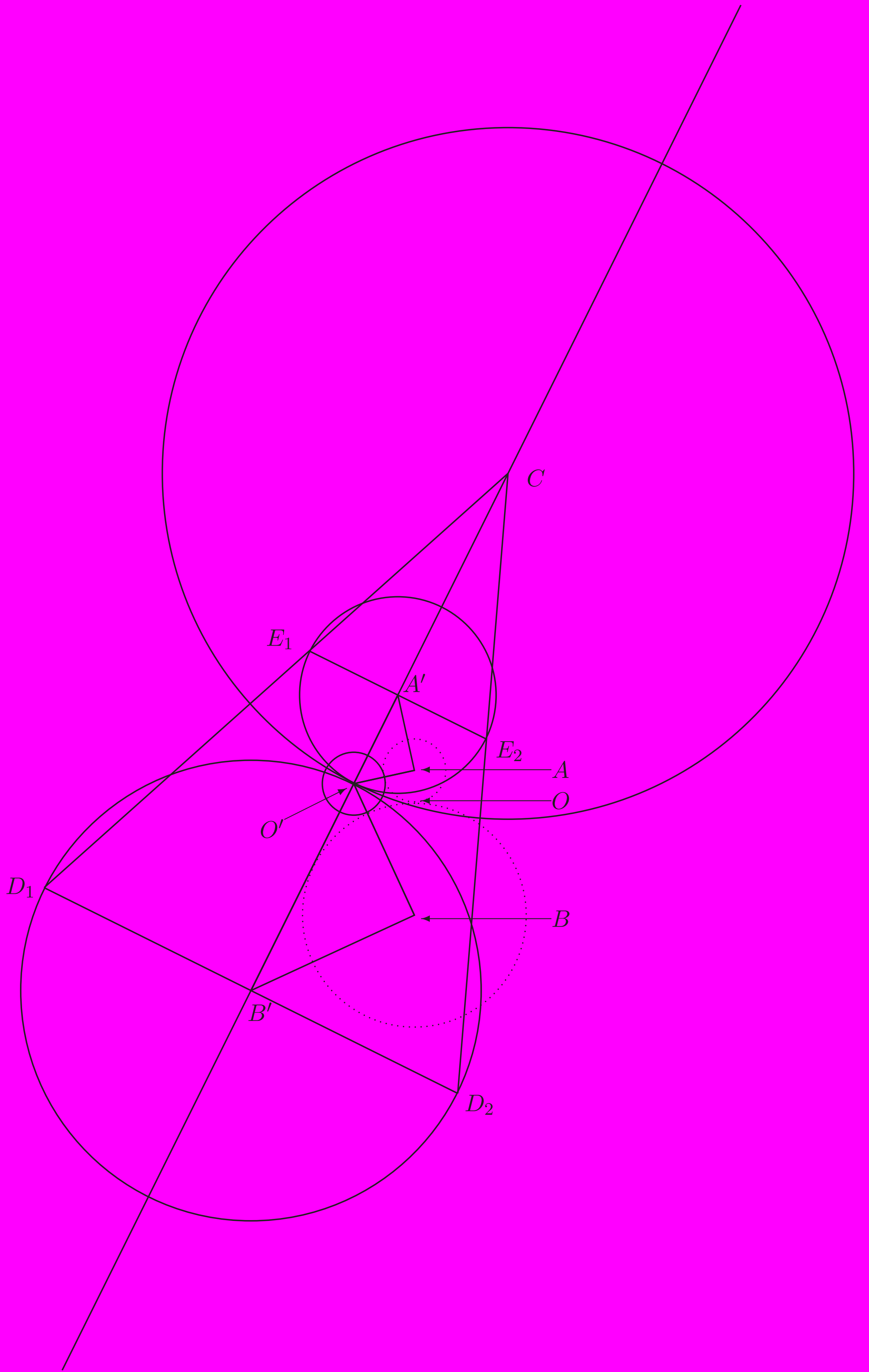
Samenvatting in het nederlands

Wat voorkennis

Stel dat van een oppervlak in de ruimte een golffront komt - het kan om licht gaan, of om geluid. Is het oppervlak een ellipsoide en breidt de golf zich uit naar buiten dan blijft het golffront steeds glad. Volgt de golf zijn weg naar binnen dan ontstaan er interferentie patronen. Het golffront krijgt vanaf een zeker tijdstip zelfdoorsnijdingen en scherpe kanten. Beschouw alle golffronten op alle tijden tezamen. De scherpe kanten van al die golffronten vormen de **caustiek** of de **fokale verzameling** van de ellipsoide. De zelfdoorsnijdingen vormen de symmetrieverzameling.
Het is gemakkelijk in te zien dat de symmetrieverzameling van een parabool een halve lijn is. De caustiek van een parabool is al een gecompliceerder verzameling. Een golffront dat van een parabool vertrekt en naar binnen beweegt is in eerste instantie nog glad. Echter na korte tijd al ontstaat er een knik in het golffront. Die knik gaat over in een zelfdoorsnijding en twee scherpe kanten. De ontwikkeling van het naar binnen bewegende golffront is dus:  . De scherpe kanten laten een spoor na dat zelf ook weer een scherpe kant heeft.
Een scherpe kant van een oppervlak of een kromme heet ook wel een singulier punt of een **singulariteit**. Zelfdoorsnijdingen heten ook wel **multi-singulariteiten**. De singulariteit op de caustiek en die op het golffront komende van een parabool heet een **cusp**, of spits. Al deze cuspen kunnen glad afgebeeld worden op de kromme $y^3 = x^2$.
Neem een willekeurig oppervlak en beschouw een golffront dat zich uitbreidt: hoe kunnen de singuliere punten eruit gaan zien? Het antwoord op deze vraag is eenvoudig: hoe je maar wilt. Dat is als volgt in te zien.
Neem een willekeurig vreemd oppervlak met allerhande zelfdoorsnijdingen en scherpe kanten. Plaats op ieder punt van dat oppervlak een bol met straal 1. De gezamenlijke rand, of omhullende van al die bollen, is ook weer een oppervlak. Laat nu van de omhullende een golffront vertrekken. Dan is de golf na tijd 1 weer terug op het willekeurig gekozen oppervlak, met de willekeurige scherpe kanten.
Met wat meer moeite kan min of meer hetzelfde aangetoond worden voor de symmetrieverzameling en de focale verzameling. Ieder willekeurig gekozen oppervlak met ingewikkelde scherpe kanten en zelfdoorsnijdingen kan optreden als symmetrieverzameling of caustiek. Er lijkt geen enkele beperking te zijn. Het is des te verbazingwekkender dat, mits de dimensie van de ruimte waarin de golf zich voortplant kleiner dan 7 is, er in redelijk sterke zin slechts een stuk of tien soorten singulariteiten op golffronten bestaan.
Voor dit sterke resultaat over golffronten is het nodig een precies begrip te hebben van wat een golffront is. Ieder gollfront beweegt zich voort. Op ieder punt van een golffront is er dus een goed gedefinieerde richting, die niet aan het gollffront mag raken. Aan de andere kant is er de ruimte waarin het golffront zich voortplant, uitbreidt. Voeg nu aan ieder punt van deze ruimte alle richtingen toe waarin een golffront zich kan voortplanten. De nieuwe ruimte is de eenheidslengte coraakbundel. Een golffront kan gekarakteriseerd worden als de projectie van een bepaald type glad oppervlak in die eenheidslengte coraakbundel. Deze speciale gladde oppervlakken in de eenheidslengte coraakbundel heten Legendre variëteiten. Op de omslag van dit proefschrift staat een Legendre-variëteit voor de spits.


C
E_1
A'
E_2
A
O
O'
D_1
B
B'
D_2

Conflictverzamelingen

Een conflictverzameling is de verzameling van punten die op gelijke afstand liggen van een aantal gegeven oppervlakken. Conflictverzamelingen in het vlak zijn het gemakkelijkst voor te stellen. De conflictverzameling van twee lijnen bestaat uit weer twee lijnen. De conflictverzameling van een cirkel en een lijn is een parabool, de conflictverzameling van twee cirkels is een aantal hyperbolen en/of ellipsen. Bekijk figuur I.1 voor wat voorbeelden.
Laat nu n de dimensie zijn van de ruimte X waarin gladde hyperoppervlakken M_1 tot M_l liggen. Neem tevens voor het gemak aan dat de ruimte waarin de oppervlakken liggen niet gekromd is. Noem de conflictverzameling van M_1 tot M_l in X M_c.
In het algemeen bestaan conflictverzamelingen uit een aantal oppervlakken. Voor ieder punt q van de conflictverzameling ligt er een aantal punten op M_j die basispunten heten. De normaal vanuit een basispunt op M_j loopt door het punt q op de conflictverzameling. Voor ieder basispunt op ieder van de M_j is die afstand gelijk.
Het is ook mogelijk algemener afstandsfuncties toe te laten. Dan staat bijvoorbeeld de afstand tot q vanaf basispunten op M_j als $\frac{\lambda_j}{\lambda_i}$ tot de afstand vanaf basispunten op M_i. In formules uitgedrukt:

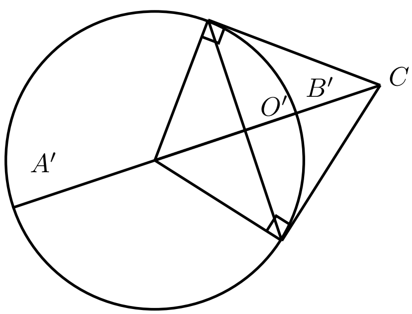
$$\frac{\lambda_j}{\lambda_i}\mathrm{afst}(p_i, q) = \mathrm{afst}(p_j, q)$$

waar p_i een basispunt op M_i is en p_j een basispunt op M_j. Door deze verhoudingen te varieren ontstaat er een hele familie conflictverzamelingen.
Het voornaamste resultaat van dit proefschrift beschrijft de aard van een conflictverzameling van niet al te speciale oppervlakken. Het blijkt dat gegeven afstandsfuncties en "generieke" basis oppervlakken M_i conflictverzamelingen de bovengenoemde "Legendre"-eigenschap, kararakteristiek voor golffronten, hebben en dat als het verschil $n - l$ tussen de dimensie n van de omhullende ruimte X en het aantal oppervlakken l niet meer dan 4 is er op zulke generieke conflictverzameling op gladde equivalentie na slechts eindig veel verschillende singulariteiten bestaan. We bewijzen dus een analogon van de bekende classificatiestellingen voor golffronten, brandpuntverzamelingen en symmetrieverzamelingen.
In het vlak komt in bovenstaande familie van conflictverzamelingen slechts in geïsoleerde gevallen een niet Legendre punt voor.
Naast dit grotere resultaat worden er in dit proefschrift tal van kleinere en eenvoudiger dingen bewezen. In hoofdstuk 4 komen allerlei generalisaties en variaties op het begrip conflictverzameling aan bod. In de hoofdstukken 1 en 2 worden een aantal krommingsformules bewezen.

Twee meetkundige constructies m.b.t. tot conflictverzamelingen

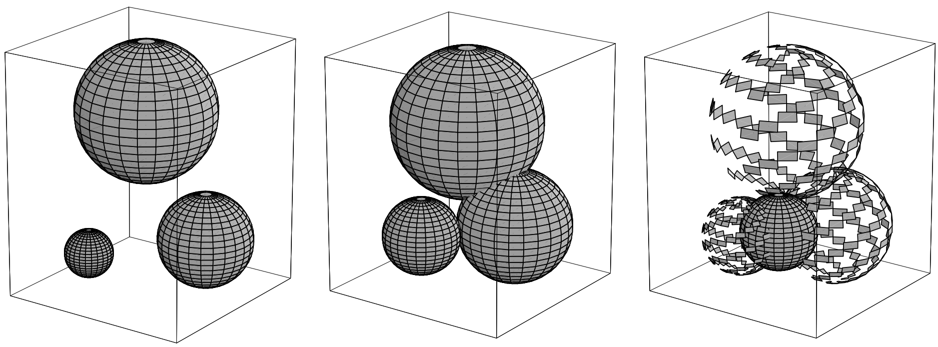
Om na het tohu-bohu van de vorige paragraaf de niet-ingewijden toch nog een beetje een idee te geven van de zaken die in dit proefschrift aan de orde komen noem ik nu hier nog twee constructies, die beide iets zeggen over de krommingsformules van de eerste twee hoofdstukken.
Uit de geometrische optica is een formule bekend die beschrijft waar het brandpunt van een in een spiegel gereflecteerde stralenbundel komt te liggen, indien gegeven zijn de hoek van inval van de stralenbundel en de kromming van de spiegel.
De transformatie van een stralenbundel in zijn gereflecteerde is iets wat vaker bestudeerd wordt in de wiskunde. Meetkundig gezien is er weinig verschil tussen een lichtstraal die een spiegel raakt en een biljartbal die de rand van de biljarttafel raakt. Zo bestaat er een

uitgebreide hoeveelheid wiskundige theorie over kaatsende biljartballen en de baan die ze afleggen op een biljarttafel.
Om de baan van de gereflecteerde lichtstraal te bepalen construeert men het virtuele beeld van de lichtbron achter de spiegel. Dit virtuele beeld geniet in de wiskunde enige bekendheid als de "orthomtic". In een omgekeerde wereld is de spiegel de conflictverzameling van de lichtbron en het virtuele beeld. De formule uit de geometrisch optica blijkt dus iets te zeggen over de kromming van conflictverzamelingen.
Een maat voor de kromming van een kromme is één gedeeld door de straal van de best rakende cirkel. Een cirkel met straal 2 heeft bijvoorbeeld een constante kromming van een half. We gaan nu uitgaande van de lichtbron en het virtuele beeld de best rakende cirkel aan de spiegel ofwel de conflictverzameling construeren, m.a.w. uitgaande van twee objecten gaan we de best rakende cirkel aan de conflictverzameling construeren.
Dat blijkt te kunnen met een klassieke constructie, die van de harmonische dubbelverhouding. Hoe dat werkt is te zien in de figuur op de achterkant van dit boekje. Het punt O' ligt op de conflictverzameling van de twee gestippelde cirkels want er is een cirkel met middelpunt O' die raakt aan beide gestippelde cirkels. De normaal aan de conflictverzameling is de lijn die de hoek tussen de twee normalen vanuit A en B naar O' in tweeën deelt.
Laat nu de punten A en B neer op de normaal aan de conflictverzameling als in de figuur aangegeven. Dan ontstaan er op de normaal aan de conflictverzameling drie punten. Het vierde punt C is nu het unieke punt zodanig dat de paren (A', B') en (O', C) de harmonische dubbelverhouding hebben, i.e. $A'C/A'O' = -B'C/B'O'$. De cirkel met middelpunt C door O' blijkt de best rakende cirkel aan de conflictverzameling te zijn, zoals gegeven door genoemde formule uit de geometrische optica. In het eerste hoofdstuk van dit proefschrift staan formules die het algemene n-dimensionale geval behandelen.
Het punt C kan ook geconstrueerd worden met behulp van **inversie**. Neem de cirkel c door A' en B' met middelpunt op de lijn $A'B'$. Inversie door een cirkel of door een bolschil is de afbeelding die alles binnenstebuiten keert: het middelpunt gaat naar oneindig, de cirkel of bolschil zelf blijft op zijn plaats en andere punten worden afgebeeld als in onderstaande figuur.



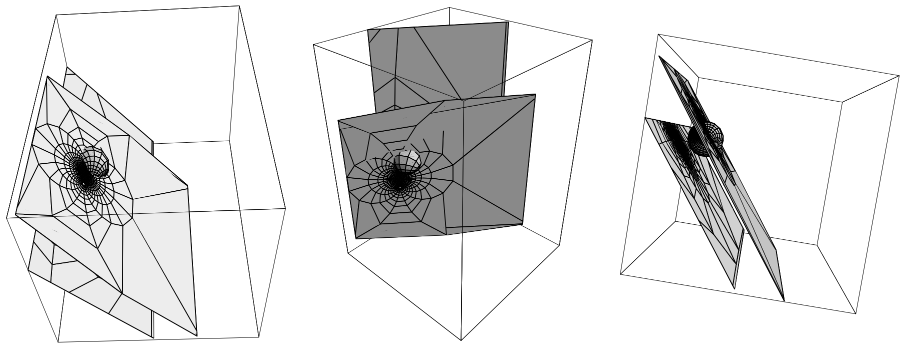
De definitie van inversie: het beeld onder inversie van O' is C en viceversa

De tweede constructie die aan bod komt gaat meer over het behandelde in hoofdstuk 2. In dat hoofdstuk staan formules die de best rakende bol vinden in het geval dat er drie basis oppervlakken in $\mathbb{R}^3$ zijn. In de figuur .1 zien we de eenvoudigste constellatie met drie basisoppervlakken. In dit geval blijkt het mogelijk om met inversie in te zien wat de conflictverzameling is. Laat de drie bollen uitdijen tot een punt waar twee van de drie bollen raken. In dat punt waar twee van de drie bollen raken plaatsen we een vierde bolschil waardoor



FIGUUR .1. Drie bollen: wat is de conflictverzameling?

we de drie bollen inverteren. Als we dat doen ontstaat figuur .2. De inversie afbeelding is



FIGUUR .2. Dezelfde drie bollen, nu geïnverteerd

een gladde één-op-één afbeelding buiten het centrum van de bolschil waardoor de inversie plaatsvindt. Dat betekent in het bijzonder dat de beelden van elkaar rakende objecten elkaar raken. Een vijfde bolschil die raakt aan de drie bolschillen van figuur .1 heeft als beeld dus een bolschil die raakt aan de twee vlakken en de bol in figuur .2. Maar het middelpunt van de vijfde bolschil is een punt van de conflictverzameling. Dus is het beeld onder inversie van de conflictverzameling een cirkel die zweeft tussen de twee vlakken en draait om de bolschil van figuur .2. De conflict verzameling zelf is dus ook een kromme die in een vlak ligt, voor een plaatje zie figuur II.2.

Dankwoord

Ten eerste wil ik mijn promotor D. Siersma bedanken voor de gelegenheid die hij mij heeft geboden dit proefschrift te schrijven. Dit proefschrift was er alsnog nooit gekomen, ofwel de inhoud was een stuk magerder geweest als ik niet zo af en toe een duwtje had gekregen van Hans Duistermaat. Hij wist mijn misschien te wilde ideeën vaster vorm te geven.

I also wish to thank the other members of the reading committee: O. Diekmann, S. Janeczko, G. Vegter and C.T.C. Wall. I hope they see that their comments have been important in the step from draft to final version.

Avec ceci je voudrais remercier les gens qui ont écrit les lettres de recommendation qui ont convaincu Dirk qu'il devait m'embaucher: Jan Wiegerinck , A.M. Chollet - j'èspere que vous avez quand meme mis le fromage au frigo - et J. Bochnak.

From all the people I met during the last years at conferences I remember maybe with most fondness Ian Porteous. I hear he is still going strong and I hope he will keep doing so for a lot of years. Let me also not forget to thank the people in Moscow for their hospitality.

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Michiel, kijk! Je staat in een boekje, daar waar je vader had moeten staan.

Souki, entre les lignes t'as ajouté beaucoup.

Curriculum vitae

Martijn van Manen werd op 23 juli 1969 geboren in Groningen. Vanaf 1981 volgde hij VWO, eerst op de Marianum scholengemeenschap in Groenlo en vanaf 1983 op het Hervormd Lyceum Zuid in Amsterdam. In 1987 verkreeg hij zijn VWO diploma van deze laatste instelling. In datzelfde jaar ging hij natuurkunde studeren aan de Universiteit van Amsterdam. In 1988 behaalde hij zijn propaedeuse natuurkunde. Na wat vakken wijsbegeerte en nog wat doctoraal vakken natuurkunde behaalde hij uiteindelijk in september 1995 zijn doctoraaldiploma wiskunde.

Gedurende zijn studietijd was hij lid van verscheidene besturen en commissies. Hij was onder andere voorzitter van de studievereniging WEIS en lid van het faculteitsbestuur. In het studiejaar 1993-1994 volgde hij een aantal vakken maitrise en behaalde hij het diplôme d'études approfondies mathématiques pures aan de USTL te Lille, Frankrijk.

Na zijn afstuderen werkte hij bij verscheidene IT bedrijven, waaronder PAC Greenware in Den Haag. In oktober 1997 werd hij AIO bij het Mathematisch instituut aan de Universiteit Utrecht. Het onderzoek sinds die tijd uitgevoerd resulteerde in dit proefschrift. De resultaten van dit onderzoek werden tevens gepresenteerd op verscheidene conferenties in Cambridge, Liverpool, Moskou en Warschau.