CHAPTER 2

Global Bifurcations and Chaotic solutions of an Autoparametric System in 1:1 Internal Resonance with Parametric Excitation

ABSTRACT. We analytically study aspects of local dynamics and global dynamics of system presented in [1]. The method of averaging is used to yield a set of autonomous equation of the approximation to the response of the system. We use two different methods to study this averaged system. First, the center manifold theory is used to derive a codimension two bifurcation equation. The results we found in this equation are related to local dynamics in full system. Second, we use a global perturbation technique developed by Kovacic and Wiggins [2] to analyze the parameter range for which a Šilnikov type homoclinic orbit exists. This orbit gives rise to a well-described chaotic dynamics. We finally combine these results and draw conclusions for the full averaged system.

1. Introduction

This chapter contains a further analysis of the system first presented in Chapter 1. There we considered an autoparametric system where the oscillator is excited parametrically:

\[
\begin{align*}
x'' + k_1 x' + q_1^2 x + a p(\tau) x + f(x, y) &= 0 \\
y'' + k_2 y' + q_2^2 y + g(x, y) &= 0
\end{align*}
\]

(1.1)

where \( f(x, y) = c_1 x y^2 + d_1 x^2, g(x, y) = d_2 y^3 + c_2 x^2 y, \) and \( p(\tau) = \cos 2\tau. \) The natural frequencies \( q_1 \) and \( q_2 \) are both close to 1, so there exists a 1 : 1 internal resonance as well as a 1 : 2 resonance with the external excitation. The nonlinear terms can be chosen more general. However, an averaging procedure will be used to study (1.1) and the indicated terms are the only ones that give a contribution, therefore there is no loss of generality in the choice of nonlinearity.

In Chapter 1 we studied the behavior of a stable periodic solution \( x(\tau) \) of \( x'' + k_1 x' + q_1^2 x + f(x, 0) = 0. \) Various types of bifurcation of this solution were analysed. Also, numerical simulation suggested the existence of non-trivial solutions which were either periodic, quasi-periodic or chaotic.

The aim of this chapter is to show the existence of these non-trivial solution in a more rigorous, analytical way. To this end we combine the analysis of a codimension two bifurcation with the application of a generalized Melnikov method to yield a full picture of the dynamics of (1.1). The results of this theoretical analysis, in particular
concerning the existence of chaotic solutions, show a remarkable degree of agreement with the numerical results.

2. The averaged System in Action Angle Variables

Writing \( q_1^2 = 1 + \varepsilon \sigma_1, \quad q_2^2 = 1 + \varepsilon \sigma_2 \), scaling \( k_i = \varepsilon \tilde{k}_i, \quad a = \varepsilon \tilde{a} \), \( x = \sqrt{\varepsilon \tilde{x}} \), and \( y = \sqrt{\varepsilon \tilde{y}} \), then dropping the tildes. We transform \( x = u_1 \cos \tau + v_1 \sin \tau, \quad y = u_2 \cos \tau + v_2 \sin \tau \), perform an averaging procedure, then rescale \( \tau = \varepsilon \tau \), to arrive at:

\[
\begin{align*}
  u' &= -k_1 u_1 + (\sigma_1 - \frac{1}{2} a) v_1 + v_1 (u_1^2 + v_1^2) + \frac{1}{4} c_1 u_2^2 v_1 + \frac{3}{4} c_1 v_2^2 v_1 + \frac{1}{2} c_1 u_2 v_2 u_1 \\
  v' &= -k_1 v_1 - (\sigma_1 + \frac{1}{2} a) u_1 - u_1 (u_1^2 + v_1^2) - \frac{3}{4} c_1 u_2^2 u_1 - \frac{1}{4} c_1 v_2^2 u_1 - \frac{1}{2} c_1 u_2 v_2 u_1 \\
  u_2' &= -k_2 u_2 + \sigma_2 v_2 + v_2 (u_2^2 + v_2^2) + \frac{1}{4} c_2 u_1^2 v_2 + \frac{3}{4} c_2 v_1^2 v_2 + \frac{1}{2} c_2 u_1 v_1 u_2 \\
  v_2' &= -k_2 v_2 - \sigma_2 u_2 - u_2 (u_2^2 + v_2^2) - \frac{3}{4} c_2 u_1^2 u_2 - \frac{1}{4} c_2 v_1^2 u_2 - \frac{1}{2} c_2 u_1 v_1 u_2
\] 
\]

for \( d_i = \frac{4}{3}, \quad i = 1, 2 \), see Chapter 1 for details. In the sequel a different formulation of (2.1) will often be used. Transforming system (2.1) by using the action-angle variables

\[
(2.2) \quad u_i = -\sqrt{2R_i} \cos \theta_i \quad \text{and} \quad v_i = \sqrt{2R_i} \sin \theta_i, \quad i = 1, 2
\]

yields

\[
\begin{align*}
  R_1' &= -2k_1 R_1 + c_1 R_1 R_2 \sin 2(\theta_1 - \theta_2) + a R_1 \sin 2 \theta_1 \\
  R_1 \theta_1' &= \sigma_1 R_1 + \frac{1}{2} a R_1 \cos 2 \theta_1 + 2R_1^2 + c_1 R_1 R_2 + \frac{1}{2} c_1 R_1 R_2 \cos 2(\theta_1 - \theta_2) \\
  R_2' &= -2k_2 R_2 - c_2 R_1 R_2 \sin 2(\theta_1 - \theta_2) \\
  R_2 \theta_2' &= \sigma_2 R_2 + \frac{1}{2} c_2 R_1 R_2 \cos 2(\theta_1 - \theta_2) + 2R_2^2 + c_2 R_1 R_2
\] 
\]

3. Analysis of a Codimension Two Bifurcation

In this section we will perform a bifurcation analysis of (2.1), to show the existence of periodic and quasi-periodic solutions of (1.1) as well as solutions that are homoclinic to a periodic solution. We rewrite (2.1) as

\[
(3.1) \quad X' = F(X)
\]

with \( X = (u_1 \quad v_2 \quad u_2 \quad v_2)^T \) and note that the equation has a fixed point \( X_o = (u_o, \quad v_o, \quad 0, \quad 0)^T \) where

\[
\begin{align*}
  u_o &= \pm \frac{R_o (\sigma_1 - \frac{1}{2} a + R_o^2)}{\sqrt{(\sigma_1 - \frac{1}{2} a + R_o^2)^2 + k_1^2}} \quad (3.2) \\
  v_o &= \pm \frac{R_o k_1}{\sqrt{(\sigma_1 - \frac{1}{2} a + R_o^2)^2 + k_1^2}}
\] 
\]
and

\[ R^2 = -\sigma_1 + \sqrt{\frac{1}{4} a^2 - k_1^2} \quad \text{and} \quad u_0^2 + v_0^2 = R^2 \]  

(3.3)

This corresponds to a semi-trivial solution of system (1.1). In Chapter 1, the stability analysis of this solution was given, see Figure 1 and 2. It was found that when \( a = 2k_1 \), and

\[ \sigma_2 = -\frac{1}{2} c_2 R^2 \pm \sqrt{\frac{1}{16} c_2^2 R^4 - k_2^2} \quad \text{for} \quad R^2 \geq \frac{k_2}{c_2}, \]  

(3.4)

system (2.1), linearized near the fixed point, has a double eigenvalue zero, which corresponds to a codimension two bifurcation.
Figure 2. A partial bifurcation diagram of system (2.1) in the \((\sigma_1, \sigma_2)\)-plane for fixed \(k_1 = k_2 = c_1 = 1, \ c_2 = -1, \ a = 2.1\). Points A and C represent branching points of \(X^\circ\). Points B, D, and E, respectively, represent Hopf points and a limit point of a non-trivial solution, respectively. The curves of the Hopf bifurcation and the saddle-node bifurcation are obtained by numerical simulation.

3.1. Derivation of the bifurcation equation. A transformation of (3.1), using \(X = X^\circ + Z\), leads to

\[ Z' = F(X^\circ + Z) = G(Z). \]  

Note that equation (3.5) is still invariant under \(S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\). In Chapter 2 it was shown that the linear part of \(G(Z)\) has the form \(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\), where the \(2 \times 2\) matrices \(A_1\) and \(A_2\) depend on the bifurcation parameters \(\sigma_1, a,\) and \(\sigma_2\). The bifurcation we want to study occurs when both \(A_1\) and \(A_2\) have one zero eigenvalue. This double zero eigenvalue bifurcation is not equivalent with the standard Bogdanov-Takens bifurcation because of the invariance under \(S\).

Rather, the bifurcation equation in this case is equivalent to the one found in the case of a fold-Hopf bifurcation (one zero eigenvalue and a pair of imaginary eigenvalues), see [6]. For details on the derivation of this bifurcation, see [6] and [9]. Among the several equivalent forms of the bifurcation equations, we use the following:
where \( \mu \) and \( \lambda \) are bifurcation parameters, and \( r \) constant with \( r \neq 0 \).

### 3.2. Analysis of the bifurcation equation

In this section we study the bifurcation equation (3.6) for the case \(-1 < r < 0\). The results of the analysis can be stated as follow: If \( r < 0 \) there is a neighborhood \( U \) of \((x, y) = (0, 0)\) and a neighborhood \( V \) of \((\mu, \lambda) = (0, 0)\) such that \( V \) is divided into regions as shown in Figure 3. The curves \( C_1, C_2, \) and \( C_3 \) are given by:

\[
C_1 : \quad \lambda = \sqrt{-\frac{\mu}{r}}
\]
\[
C_2 : \quad \lambda = \frac{r - 1}{2} \left( -1 + \sqrt{1 - \frac{4\mu}{r}} \right)
\]
\[
C_3 : \quad \lambda = -\sqrt{-\frac{\mu}{r}}
\]

where \( \mu \geq 0 \).

**Figure 3.** The bifurcation diagram of equation (3.6) in the \((\mu, \lambda)\)-plane for \( r = -\frac{1}{2} \).

Curves \( C_1 \) and \( C_3 \) represent branch points (pitchfork bifurcations) and curve \( C_2 \) represents a Hopf bifurcation. In \( U \) there are three fixed points. Two of them
with $y = 0$ exist for $\mu > 0$ and are given by $X_{1,2} = (\mp \sqrt{-\frac{1}{r}} \mu, 0)$. These fixed points appear via a generic fold (saddle-node) bifurcation on the line $\mu = 0$. Crossing this line for $\lambda < 0$ gives rise to a stable node and a saddle.

The fixed points $X_{1,2}$ can bifurcate at curve $C_3$. At this bifurcation curve the non-trivial fixed point $X_3 = (x, y) = \left( \frac{1}{2r} (1 + \sqrt{1 - 4r(\mu + \lambda)}), \sqrt{x - \lambda} \right)$ appears. Note that since equation (3.6) has the symmetry $y \rightarrow -y$, the fixed points come in pairs with respect to the $x$-axis. The fixed point $X_3$ is a stable node. It has two purely imaginary eigenvalues for parameter values on curve $C_2$. Therefore, crossing this curve a Hopf bifurcation takes place. Since this bifurcation is supercritical, a stable periodic orbit exists for nearby parameters value in region IIIa (see Figure 4).

Under parameter variation this periodic orbit can approach a heteroclinic cycle at curve $C_2^*$. This curve is given by

$$C_2^* : \lambda = -2\mu$$

To obtain (3.7), we introduce the scaling

$$x \rightarrow \alpha u, \quad y \rightarrow \alpha v, \quad \mu = \alpha^2 a, \quad \lambda = \alpha^2 b,$$

and the time scaling $\tau \rightarrow -\alpha \tau$ with $a, b$ of $O(1)$ and $\alpha < 0$. We fix $a = 1$ corresponding to $\mu \geq 0$ and $r = -\frac{1}{2}$, so that equation (3.6) becomes:

$$u' = 1 - \frac{1}{2} u^2 - v^2$$
$$v' = -\epsilon bv + uv - \alpha v^3$$

If we let $\alpha \rightarrow 0$ with $b \neq 0$ fixed, equation (3.8) becomes an integrable Hamiltonian system

$$u' = 1 - \frac{1}{2} u^2 - v^2$$
$$v' = uv$$

with Hamiltonian

$$H(u, v) = v - \frac{1}{2} u^2 v - \frac{1}{3} v^3,$$

Note that the closed orbits and the saddle connection correspond to the level curve $H(u, v) = 0$.

To find a saddle connection for values $\alpha \neq 0$ a Melnikov method can be used, see Chow and Hale [10]. We find that for $b = -2$, the approximate bifurcation curve is given in (3.7).

Now the neighborhood $V$ is divided into regions as shown in Figure 4. The flow of equation (3.6) in the $(x, y)$-plane is also depicted in this figure for values of parameters in each region.
3.3. Relation to the full system. We can use the bifurcation diagram for equation (3.6) to reconstruct the diagram of the full system (2.1). This yields $\lambda$ and $\mu$ as functions of $(a, \sigma_1, \sigma_2)$. A straight forward calculation gives

\begin{equation}
\mu = b_1(a - 2k_1) \quad \text{and} \quad \lambda = b_2(-\sigma_2 + b_3 + \frac{1}{2}c_2\sigma_1) + b_4(a - 2k_1)
\end{equation}

where

\begin{equation}
\begin{aligned}
b_1 &= \frac{-\sigma_1 b_4^2}{c_1^2 k_2^3(2b_4 + \frac{1}{4}c_2\sigma_1)} , & b_2 &= \frac{-b_4^2}{c_1 k_2^2(2b_4 + \frac{1}{4}c_2\sigma_1)} , \\
b_3 &= \frac{-b_4 b_5}{4c_1 k_2 r k_4^2(2b_4 + \frac{1}{4}c_2\sigma_1)} , & b_4 &= \sqrt{\frac{c_2^2\sigma_1^2}{2\sigma_1^2} - k_2^2} , \\
b_5 &= k_2^2 + \sigma_1^2 , \quad \text{and} \quad r = -\frac{4k_2\sigma_1}{c_2 k_1(2b_4 + \frac{1}{4}c_2\sigma_1)}
\end{aligned}
\end{equation}

The stable fixed point $X_1$ in (3.6) corresponds with the stable semi-trivial solution $X_\circ$ of the full system (2.1). Thus, the branching curve $C_3$ corresponds to the branching curve (3.4) for the full system (2.1). The fixed point $P_3$ in (3.6) corresponds to the non-trivial solution in (2.1). Therefore, the Hopf curve $C_2$ in (3.6) will correspond with the Hopf curve in (2.1).

In Figure 5 the bifurcation diagram of system (2.1) in the $(\sigma_1, \sigma_2)$-plane is shown. The branching curve of the semi-trivial solution $X_\circ$ is represented by curve $C_4$ and $C_7$. Crossing the curve $C_4$ from outside, a non-trivial solution appears. This solution
undergoes a Hopf bifurcation on the curve $C_5$ and then a heteroclinic bifurcation on the curve $C_6$.

The curves $C_4$ and $C_7$ are obtained from equation (3.4) by varying the parameter $\sigma_1$ and fixing the other parameters. The curves $C_5$ and $C_6$ are given by

\begin{align}
C_5 : \sigma_2 &= b_4 + \frac{1}{2} c_2 \sigma_1 + \frac{1}{b_2} \left( r - \frac{1}{r} b_1 - b_3 \right) (a - 2k_1) \\
C_6 : \sigma_2 &= b_4 + \frac{1}{2} c_2 \sigma_1 + \frac{1}{b_2} (2b_1 - b_3) (a - 2k_1)
\end{align}

(3.13)

where both of curves $C_5$ and $C_6$ are defined for $\sigma_1 < \frac{4k_2}{c_2}$ and $\sigma_1$ is negative.

As illustrated in section (3.2) the analysis result shows that there is at most one stable periodic orbit of (2.1) in the narrow area between curves $C_5$ and $C_6$. In a numerical simulation shown in Chapter 1, this stable periodic solution underwent period doubling bifurcations, led to a chaotic solution. In the next section we are going to study the appearance of chaotic dynamics analytically.

4. Analytical study of chaotic solution by using a generalized Melnikov method

By using a generalized version of Melnikov’s method, we show that for certain values of the parameters, the averaged system (2.3) contains a heteroclinic cycle
with properties similar to that of a homoclinic orbit of Šilnikov-type. The existence of this heteroclinic cycle implies the existence of chaotic dynamics.

The method used here is based on Wiggins [3] and is similar to the one used in Feng and Sethna [4]. However, in [4] only non-dissipative, in fact Hamiltonian, perturbations are considered. In Tien and Namachchivaya [5] and in Zhang and Liu [12], a system closely resembling (2.1) is studied using the same methods. Unfortunately, both papers contain the same systematic error.

To apply the method, a rescaling of the parameters is needed, which leads to a four-dimensional system where the unperturbed part is Hamiltonian and integrable. The unperturbed system possesses a two-dimensional invariant manifold \( M \), consisting of two components \( M_1 \) and \( M_2 \) (see Figure 8). Both \( M_1 \) and \( M_2 \) have three-dimensional stable and unstable manifolds. The two components of \( M \) survive, as do their invariant manifolds, when the perturbation is added. The perturbed invariant manifolds will be denoted by \( M_{1\varepsilon} \) and \( M_{2\varepsilon} \). Within \( M_{1\varepsilon} \) and \( M_{2\varepsilon} \) we identify two fixed points, \( p_{1\varepsilon} \) and \( p_{2\varepsilon} \). The phase-diagram in the invariant manifold \( M_{1\varepsilon} \), near \( p_{1\varepsilon} \) is shown in Figure 6 (c). In this figure we indicate a subset of \( M_{1\varepsilon} \), close to \( p_{1\varepsilon} \) denoted by \( A_{1\varepsilon} \). It also has a 3-dimensional stable manifold \( W_s(A_{1\varepsilon}) \). Orbits in \( W_u(A_{1\varepsilon}) \) are in the domain attraction of \( p_{1\varepsilon} \), provided a phase-shift condition is satisfied. Conversely, the phase space in \( M_{2\varepsilon} \) near \( p_{2\varepsilon} \) also looks like Figure 6 (c). The fixed point \( p_{2\varepsilon} \) has a 1-dimensional unstable manifold. By using a Melnikov method, a range of parameters can be found for which \( W_u(p_{2\varepsilon}) \subset W_s(A_{1\varepsilon}) \), see Figure 10. Therefore, there exists an orbit from \( p_{2\varepsilon} \) to \( p_{1\varepsilon} \). It will be shown that there also exists an orbit back from \( p_{1\varepsilon} \) to \( p_{2\varepsilon} \), see Figure 11. This Šilnikov-type heteroclinic orbit is associated with chaotic dynamics, see [13].

4.1. The General Theory. In order to use the method in [2], we consider the system in the form

\[
X' = J \frac{\partial H_o}{\partial X} + \varepsilon g^X(X, P_2, Q_2, \mu) \\
P'_2 = \varepsilon g^{P_2}(X, P_2, Q_2, \mu) \\
Q'_2 = \frac{\partial H_o}{\partial P_2} + \varepsilon g^{Q_2}(P, P_2, Q_2, \mu)
\]

(4.1)

where \( X = (P_1, Q_1) \) and \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). All functions are differentiable on the domains of interest and \( \mu \) is a vector of parameters. We assume that \( H_o(X, P_2) \) is independent of \( Q_2 \), so that the system (4.1) is Hamiltonian for \( \varepsilon = 0 \).

The main point of this method is to use the simple structure of the unperturbed system (4.1) to build up a picture of the geometry in the full four dimensional phase space. We use the following assumption on the system (4.1), when \( \varepsilon = 0 \).

**Assumption 1.** There exists an interval \([P_{21}, P_{22}]\) such that for all \( P_2 \in [P_{21}, P_{22}] \), the equation \( X' = J \frac{\partial H_o}{\partial X} \) has two hyperbolic fixed points, \( p_1 \) and \( p_2 \), connected through a heteroclinic cycle. See Figure 7.

Assumption 1 implies that in the full four dimensional phase space the set
\[ M = \{(X, P_2, Q_2) | X = p_1, P_{21} \leq P_2 \leq X_{22}\} \cup \]
\[ \{(X, P_2, Q_2) | X = p_2, P_{21} \leq P_2 \leq X_{22}\} \]
\[ = M_1 \cup M_2 \]

is a two dimensional, invariant manifold consisting of two components, see Figure 8. By using the persistence theorem (see Fenichel [7]), we can show that \( M \) persists under small perturbations as a locally invariant manifold \( M_\epsilon \) with a boundary. \( M \) has three dimensional stable and unstable manifolds which we denote as \( W^s(M) \) and \( W^u(M) \), respectively. Moreover, the manifolds \( W^s(M) \) and \( W^u(M) \) also persist as locally invariant manifolds \( W^s(M_\epsilon) \) and \( W^u(M_\epsilon) \).

The next step is to study the unperturbed system restricted to \( M \). In this manifold the unperturbed system is given by

\[ P_2' = 0 \]
\[ Q_2' = \frac{\partial H_\epsilon}{\partial P_2}, \quad P_{21} \leq P_2 \leq P_{22} \]

We refer to a value of \( P_2 \) for which \( Q_2' = 0 \) as a resonant \( P_2 \) value. We make the following

**Assumption 2.** There exists a value of \( P_2 \in [P_{21}, P_{22}] \), denoted \( P_2^r \) at which \( \frac{\partial H_\epsilon}{\partial P_2} = 0 \).

Next we will study the dynamics of the system on a component of \( M \), namely \( M_{1\epsilon} \) in an \( O(\sqrt{\epsilon}) \) neighbourhood near the resonance \( P_2^r \), by introducing the following change of coordinates

\[ P_2 = P_2^r + \sqrt{\epsilon} \tilde{P} \]
\[ Q_2 = Q_2. \]

Localizing the system on \( M_{1\epsilon} \) near the resonance \( P_2^r \), gives a system in \((\tilde{P}, Q_2)\)-coordinates as a one-degree of freedom Hamiltonian system at \( \epsilon = 0 \). The integrable Hamiltonian structure at leading order is typical near resonance, and it is useful for understanding the qualitative (as well as the quantitative) structure of the dynamics near the resonance on \( M_{1\epsilon} \).
We denote the annulus centered at $P_2 = P_2^r$ as

$$A_{1\varepsilon} = \{(X, \tilde{P}, Q_2)|X = (P_2^r, Q_{1s}), |\tilde{P}| < C\}, \quad A_{1\varepsilon} \subset M_{1\varepsilon}$$

where $C > 0$ is some constant and chosen sufficiently large such that the annulus contains the unperturbed homoclinic orbit. The three dimensional stable manifold of $A_{1\varepsilon}$ is denoted by $W^s(A_{1\varepsilon})$, where $W^s(A_{1\varepsilon}) \subset W^s(M_{1\varepsilon})$.

We assume that on $M_{1\varepsilon}$ the leading order system is Hamiltonian in the coordinates $(\tilde{P}, Q_2)$, and furthermore has the following property:

**Assumption 3.** For $\mu = \mu_\circ$ there exists $Q_{2e}(\mu_\circ)$ and $Q_{2s}(\mu_\circ)$ such that $q_1 = (\tilde{P}, Q_2) = (0, Q_{2s}(\mu_\circ))$ is a saddle type fixed point and $p_1 = (\tilde{P}, Q_2) = (0, Q_{2e}(\mu_\circ))$ is a center type fixed point. Moreover, $q_1$ is connected to itself by a homoclinic orbit and $p_1$ is the only fixed point inside this homoclinic orbit.

Figure 6 (a) and (b) illustrate the conditions described in assumptions 2 and 3.

The next order of perturbation of the vector field on $M_{1\varepsilon}$ contains dissipative terms and yields the phase-portrait pictured in Figure 6 (c). In particular $p_{1\varepsilon}$ has become a hyperbolic fixed point.

By using a higher dimensional Melnikov method, it will be shown that for certain values of the parameters, the one-dimensional unstable manifold of a fixed point contained in $M_{2\varepsilon}$, which we denote by $p_{2\varepsilon}$, intersects the stable manifold of $A_{1\varepsilon}$. When a certain phase-shift condition is satisfied, the unstable manifold of $p_{2\varepsilon}$ will fall in the basin of attraction of $p_{1\varepsilon}$, leading to a heteroclinic connection. The heteroclinic cycle is completed by showing that there exists a connection back from $p_{1\varepsilon}$ to $p_{2\varepsilon}$. Again, using a Melnikov method, it will be shown that the heteroclinic
connection from $p_1$ and $p_2$ is unbroken when the perturbation is added, see Figure 10 and Figure 11. In the following sections the details of this analysis will be given.

4.2. **Transformation to Hamiltonian Coordinates.** We transform system (2.3) into system (4.1) by introducing the following transformations

$$
\begin{align*}
\tilde{Q}_1 &= 2(\theta_1 - \theta_2), \\
\tilde{Q}_2 &= 2\theta_2, \\
\tilde{P}_1 &= -c_2 R_1, \\
\tilde{P}_2 &= \tilde{P}_1 - c_1 R_2,
\end{align*}
$$

(4.5)

Note that, because $R_1 \geq 0$ and $R_2 \geq 0$, we are only interested in the case when $c_2 < 0$ and $c_1 > 0$ in the area of phase-space where $\tilde{P}_2 \leq \tilde{P}_1$. In particular, the hyper-plane $\tilde{P}_2 = \tilde{P}_1$ corresponds to the invariant space $R_2 = 0$.

Rescaling system variables as $\tilde{P}_{1,2} \rightarrow \epsilon P_{1,2}$, $\tilde{Q}_{1,2} \rightarrow -2Q_{1,2}$, $k_{1,2} \rightarrow \epsilon^2 \tilde{k}_{1,2}$, $\sigma_{1,2} \rightarrow \epsilon \sigma_{1,2}$, $a \rightarrow \epsilon^2 \tilde{a}$, and $\tau \rightarrow 2\epsilon \tilde{\tau}$, system (2.3) then becomes

$$
\begin{align*}
P'_1 &= -2 \frac{\partial H_o}{\partial Q_1} + \epsilon \left(-4k_1 P_1 - \frac{\partial H_1}{\partial Q_1}\right) \\
Q'_1 &= \frac{\partial H_o}{\partial P_1} - \epsilon \frac{\partial H_1}{\partial P_1} \\
P'_2 &= \epsilon \left(-2 P_1 + 4 \tilde{k}_2 P_2\right) \\
Q'_2 &= \frac{\partial H_o}{\partial P_2}
\end{align*}
$$

(4.6)

where

$$
\begin{align*}
H_o &= -\sigma P_1 - 2\sigma_2 P_2 - \tilde{c}_1 P_1 P_2 + \frac{1}{2} \tilde{c}_3 P^2 + P_1(P_2 - P_1)(\tilde{c}_2 + \cos 2Q_1) \\
H_1 &= -\tilde{a} P_1 \cos 2(Q_1 + Q_2)
\end{align*}
$$

(4.7)

and $\sigma = 2(\sigma_1 - \sigma_2)$, $\tilde{c}_1 = \frac{2}{c_1} - \frac{2}{c_2}$, $\tilde{c}_2 = 2 - \frac{2}{c_1} - \frac{2}{c_2}$, $\tilde{c}_3 = \frac{4}{c_1}$, and $\tilde{k} = 2(\tilde{k}_2 - \tilde{k}_1)$.

It is clear that for $\tilde{k}_1 = \tilde{k}_2 = 0$, system (4.6) is in canonical form, with Hamiltonian $H = H_o + \epsilon H_1$. Note that $\frac{\partial H}{\partial P_2} = 0$.

4.3. **Analysis of the Unperturbed System.** In this subsection we study the dynamic of the unperturbed ($\epsilon = 0$) system (4.6). It is integrable, since it possesses the independent integrals $H_o$ and $P_2$. We will first study the equations for $P_1$ and $Q_1$, taking $P_2$ as a constant.

$$
\begin{align*}
P'_1 &= 2P_1(P_2 - P_1) \sin 2Q_1 \\
Q'_1 &= -\sigma - \tilde{c}_1 P_2 - (2P_1 - P_2)(\tilde{c}_2 + \cos 2Q_1)
\end{align*}
$$

(4.8) (4.9)

We are only interested in studying the dynamics of these equations in the range $0 < P_2 \leq P_1$ and $0 < Q_1 < \pi$, since the equations are $\pi$-periodic in $Q_1$. One set of fixed points is given by $P_1 = P_2$ and $Q_1$ a solution of

$$
\cos 2Q_1 = -\frac{1}{P_2}(\sigma + \tilde{c}_1 P_2 + \tilde{c}_2 P_2)
$$

(4.10)
This yields solution $Q_{1s}$ and $\pi - Q_{1s}$. A simple stability analysis shows that these points are of saddle type. Note that these points are connected through a heteroclinic orbit on the invariant line $P_1 = P_2$. We also note that this invariant line $P_1 = P_2$ corresponds, in the original coordinates, with the invariant space $R_2 = 0$, i.e. $y = 0$. Therefore, these two fixed points correspond to semi-trivial solutions. Another fixed point is given by $Q_1 = \frac{\pi}{2}$ and $P_1 = P_1 = -\frac{\sigma + (\vec{c}_2 - \vec{c}_1 - 1)P_2}{2(\vec{c}_2 - 1)}$. From the condition that $P_1 \geq P_2$, it follows that such a $P_1$ only exists for $P_{21} < P_2 < P_{22}$, with $P_{21} = -\sigma/(3 - \frac{4}{\vec{c}_2})$ and $P_{22} = -\sigma/(1 - \frac{4}{\vec{c}_2})$. This fixed point is a center-point, and in the original coordinates it represents a non-trivial periodic solution.

The orbits in the $(P_1, Q_1)$-plane are the level curves of the unperturbed Hamiltonian $H_o$ restricted to the plane. The orbits through the saddle points $(P_1, Q_1) = (P_2, Q_{1s})$ and $(P_1, Q_1) = (P_2, \pi - Q_{1s})$ can be found by solving $H_o(P_1, Q_1) - H_o(P_2, Q_{1s}) = 0$ for $P_1$. We then have

$$\text{orbit } A' : P_1 = P_2$$

$$\text{orbit } A : P_1 = -\frac{\sigma + \vec{c}_1 P_2}{\vec{c}_2 + \cos 2Q_1}$$

These expressions for the heteroclinic orbits will be used later, when we apply the Melnikov method. The phase-portrait in the $(P_1, Q_1)$-plane is shown in Figure 7. Using Figure 7, we can get an impression of the dynamics of the unperturbed system in the full, four-dimensional, phase-space. Since the 2-dimensional phase-space for $P_1$ and $Q_1$ is qualitatively the same for all $P_{21} < P_2 < P_{22}$ and the equation for $Q_2$ is decoupled, we can picture the phase-space as in Figure 8.

It is clear that the set $M = M_1 \cup M_2$, defined in (4.2), is invariant in system (4.6) for $\varepsilon = 0$.

The existence of the heteroclinic orbits joining $M_1$ and $M_2$ implies the non-traversed intersection of the three-dimensional stable manifold $W^s(M)$ and the three-dimensional unstable manifold $W^u(M)$ along a three-dimensional heteroclinic manifold $\Gamma$ (Figure 8), where

$$\Gamma \equiv W^s(M) \cap W^u(M)$$

$$= \{(P_1, Q_1, P_2, Q_2) | H_o(P_1, Q_1, P_2) - H_o(P_2, Q_{1s}, P_2) = 0\}$$

The trajectories in $\Gamma$ approach a trajectory in $M$ asymptotically as $\tau \to \pm \infty$.

The dynamics of the unperturbed system restricted to $M_1$ is given by

$$P'_2 = 0$$

$$Q'_2 = -2\sigma_1 - \beta P_2$$

where $\beta = 2\vec{c}_1 - \vec{c}_3$, see Figure 6.(a). Since $Q_2$ is a $\pi$-periodic coordinate, the phase-space is the cylinder obtained by identifying the edges $Q_2 = 0$ and $Q_2 = \pi$. The phase-space therefore consists of a collection of invariant circles. However, $Q'_2 = 0$ for $P_2^r = \frac{\sigma_1}{\beta}$. Therefore for $P_2^r = \frac{\sigma_1}{\beta}$ (the resonant value) we have a circle of
fixed points. We denote the annulus centered at \( P_2 = P_2^r \) as \( A_1 \). In next section we study the perturbed system of (4.13) in \( A_{1\varepsilon} \).

Figure 7. The phase-portrait of the unperturbed system (5) in the \((Q_1, P_1)\)-plane, for values \( c_1 = 1, c_2 = -1, \sigma_1 = -8, \sigma_2 = 5.3, \) and \( P_2 = 4 \).

Figure 8. The unperturbed system and manifold \( M \) in the \((P_1, Q_1, P_2)\)-space.

4.4. Structure of the Perturbed System in \( A_{1\varepsilon} \). We have already defined the perturbed system in the \((P_2, Q_2)\) coordinates by restricting the system to \( M_1 \). We now want to study the dynamics of the perturbed system restricted to \( M_{1\varepsilon} \) near the resonance \( P_2 = P_2^r \). For this purpose we will change variables in order to
derive a simpler equation which can describe the dynamics in the neighbourhood of the resonance. Let \( P_2 = P_2^r + \sqrt{\varepsilon}P \) and \( \tau = \sqrt{\varepsilon}\tau \), we then have

\[
\begin{align*}
P' &= -4\tilde{k}_1 P_2^r - 2\tilde{a} P_2^r \sin 2(Q_1 + Q_2) + \sqrt{\varepsilon}(-4\tilde{k}_1 P - 2\tilde{a} P \sin 2(Q_1 + Q_2)) + \mathcal{O}(\varepsilon) \\
Q'_2 &= -\bar{\beta} P + \mathcal{O}(\varepsilon)
\end{align*}
\]

where \( \bar{\beta} = \frac{4}{c_1} \).

After an appropriate time-scaling, equation (4.13) can be written as:

\[
\dot{Q}_2 + \frac{\tilde{a}}{2\tilde{k}_1} \sin 2(Q_2 + Q_1) + 1 = \mathcal{O}(\sqrt{\varepsilon})
\]

The unperturbed system is Hamiltonian, and for \( \tilde{a} > 2\tilde{k}_1 \) its phase-portrait has the familiar “fish” shape, see Figure 9. In the sequel, the distance \( Q_{2n} - Q_{2s} \) will be important. We note that this distance only depends on \( \frac{\tilde{a}}{2\tilde{k}_1} \).

\[ \text{Figure 9. The homoclinic orbit of system (4.14) in the (P, Q_2)-plane, for values } \tilde{k}_1 = \tilde{k}_2 = 1, \tilde{a} = 2.1, c_1 = 1, c_2 = -1, \tilde{\sigma}_1 = -8, \tilde{\sigma}_2 = 5.3, \text{ and } P_2 = 4. \]

When the perturbation terms of order \( \sqrt{\varepsilon} \) are taken into account, the perturbed \( p_1 \varepsilon \) becomes a sink due to the \( \mathcal{O}(\sqrt{\varepsilon}) \) terms. Moreover, the homoclinic orbit breaks with a branch of the unstable manifold of \( q_1 \varepsilon \) falling into \( p_1 \varepsilon \), see Figure 6 (c).
4.5. Melnikov functions and phase-shift. In order to show that $W^u(p_{2\varepsilon})$ and $W^s(A_{1\varepsilon})$ intersect, we use a version of the Melnikov method. For details on this method see Wiggins [3] and Kovačić and Wiggins [2]. A function $\mathcal{M}(\mu)$ can be defined, which measures the distance from $W^u(p_{2\varepsilon})$ to $W^s(A_{1\varepsilon})$, and we need to solve the equation $\mathcal{M}(\mu) = 0$. The function $\mathcal{M}(\mu)$ is a line integral, which is to be
evaluated on a solution which connects \( p_2 \) to \( p_1 \), in the unperturbed system. If we make the natural choice that \( Q_1(0) = \frac{\pi}{2} \), for this unperturbed heteroclinic connection (see Figure 7) then also \( P_1(0) \) and \( P_2(0) \) are fixed. However, we are still free to chose the initial condition for \( Q_2 \), namely \( Q_2(0) = Q_{20} \). We will therefore consider \( M(\mu) \) to be a function of \( \mu = (\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2, Q_{20}) \). In the Appendix it will be shown that \( M(\mu) \) is of the form.

\[
M(\mu) = M_1(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2) + \sin 2Q_{20}M_2(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2),
\]

therefore, given values of \( \bar{a}, \bar{\sigma}_1, \) and \( \bar{\sigma}_2 \), a solution for \( M(\mu) = 0 \) will exist if

\[
\frac{M_2(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2)}{M_1(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2)} > 1
\]

The condition (4.17) is not enough to ensure a heteroclinic connection from \( p_{2\varepsilon} \) and \( p_{1\varepsilon} \). It is also necessary that \( W^u(p_{2\varepsilon}) \) fall in the domain of attraction of \( p_{1\varepsilon} \). In particular this means that

\[
Q_{2s} < Q_{2\varepsilon} + \Delta Q_2 + m\pi < Q_{2n}, \quad m \in \mathbb{Z},
\]

where \( \Delta Q_2 = Q_2(+\infty) - Q_2(-\infty) \), see Figure 9. This is because \( Q_{2s} \) and \( Q_{2n} \) are, to order \( \sqrt{\varepsilon} \), the boundaries of attraction in the \( Q_2 \) direction of \( p_{2\varepsilon} \).

Finally, the heteroclinic cycle is completed by noting that a Melnikov function measuring the distance from \( W^up_{1\varepsilon} \) and \( W^s(A_{2\varepsilon}) \), see Figure 10, is identically zero (see Appendix), which implies that the heteroclinic connection which exists back from \( p_1 \) to \( p_2 \), is not broken by the perturbation, see Figure 11.

5. Results

A careful analysis of the condition (4.18) shows that it can be reduced to the form

\[
C_1(\bar{a}) \leq \frac{\bar{\sigma}_1}{\bar{\sigma}_2} \leq C_2(\bar{a})
\]

To compare the results of the analysis of this chapter with the results of the previous chapter we have taken \( a = 2.1 \), calculated \( C_1(\bar{a}) \) and \( C_2(\bar{a}) \), and checked that condition (4.17) was satisfied. For the value \( \sigma_1 = -8 \) this yields that for \( 5.193 < \bar{\sigma}_2 < 5.372 \), we have the existence of a Šilnikov type heteroclinic cycle. Such a cycle implies the existence of chaotic dynamics (see Bykov [13]). These values compare well with the numerically found values \( 5.250 < \sigma_2 < 5.319 \).
5. Global Bifurcations and Chaotic Solutions of an Autoparametric System

![Parameter diagram of system (2.1) in the ($\sigma_1, \sigma_2$)-plane for values $\bar{k}_1 = \bar{k}_2 = 1$, $c_1 = 1$, $c_2 = -1$, and $\bar{a} = 2.1$. Lines $L_1$ and $L_6$ are represented the branching lines of the semi-trivial solution $X_0$ of system (2.1). Hopf bifurcation line is indicated by line $L_2$ where a saddle connection by line $L_3$. The lines $L_4$ and $L_5$ are obtained from the condition (4.18).](image)

Figure 12.

6. Conclusion

In this paper, a codimension two bifurcation and global bifurcations of system (1.1) have been studied, complementing the results of Chapter 2 and giving a broad picture of the dynamics. This picture is represented by Figure 12. This figure should be compared to Figure 5. Note that the rescaling in Section 4 has “blown up” $\sigma_1$ and $\sigma_2$ with respect to $a$, $k_1$, and $k_2$. In other words, if in Figure 5 the parameters $a$, $k_1$, and $k_2$ were to be rescaled, then the resulting figure would look much like Figure 2.

On line $L_1$ and $L_5$, the semi-trivial solution undergoes a period-doubling. On line $L_2$, a Hopf bifurcation occurs in the averaged system, leading to an invariant torus in the original coordinates. In between the lines $L_3$ and $L_4$, chaotic solutions can occur.

7. Acknowledgments

The research was conducted in the department of Mathematics of the University of Utrecht. It is supported by PGSM Project of Indonesia and CICAT TU Delft.
The author, S. Fatimah, is on leave from Mathematics department of the University of Education Indonesia, UPI, Bandung of Indonesia.

8. Appendix

Melnikov Function

In calculating the Melnikov function it will be important to have forms for $P_1$, $Q_1$ and $Q_2$ as functions of time $\tau$. We substitute equation (4.11) into equation (4.9) and integrate. Orbit $A$, $Q_1(\tau)$ can implicitly be written as

\begin{equation}
\tanh(e_A \tau) = \frac{\sin 2Q_{1s} \sin 2Q_1(\tau)}{1 - \cos 2Q_{1s} \cos 2Q_1(\tau)}
\end{equation}

where $e_A = -P_2 |\sin 2Q_{1s}|$, and the expressions for $\cos 2Q_1(\tau)$ is

\begin{equation}
\cos 2Q_1(\tau) = \frac{\cos 2Q_{1s} \cosh(e_A \tau) - 1}{\cosh(e_A \tau) - \cos 2Q_{1s}}
\end{equation}

Substituting equation (8.2) into (4.11), we have the explicit form for $P_1$ as function of time $\tau$. The expression is

\begin{equation}
P_1(\tau) = P_2 \frac{\cosh(e_A \tau) - \cos 2Q_{1s}}{\cosh(e_A \tau) + \cos(f_A)}
\end{equation}

where

\begin{equation}
\cos(f_A) = -\frac{1 + \tilde{c}_2 \cos 2Q_{1s}}{\tilde{c}_2 + \cos 2Q_{1s}} < 1
\end{equation}

where $\tilde{c}_2 > 0$. Finally, to calculate $Q_2$ as function of time $\tau$, we substitute equation (4.11) into (4.13), yield

\begin{equation}
Q'_2 = c_A P_2 + \tilde{c}_1 (P_2 - P_1)
\end{equation}

where $c_A = -\frac{2\sigma_1}{P_2} + \tilde{c}_3 - 2\tilde{c}_1$. After substituting equation (8.3)-(8.4) into (8.5), we thus have

\begin{equation}
Q'_2 = c_A P_2 - \tilde{c}_1 P_2 \frac{\cos 2Q_{1s} + \cos(f_A)}{\cosh(e_A \tau) + \cos(f_A)}
\end{equation}

On integrating equation (8.6) we obtain

\begin{equation}
Q_2(\tau) = c_A P_2 \tau - g_A \tan^{-1} \left[ \tan \left( \frac{f_A}{2} \right) \tanh \left( \frac{e_A \tau}{2} \right) \right]
\end{equation}

where

\begin{equation}
g_A = 2\tilde{c}_1 \frac{\cos(f_A) + \cos 2Q_{1s}}{|\sin 2Q_{1s} \sin(f_A)|},
\end{equation}

note that from equation (4.11), $P_1(\tau)$ a constant for orbit $A'$. 
By letting $P_2 = P_2^r$, we compute the phase shift $\Delta Q_2$ of orbits which are asymptotic to points on the circle of fixed points as $\tau \to \pm \infty$. From equation (8.7), we have

\begin{equation}
\Delta Q_2 = Q_2(+\infty) - Q_2(-\infty) = g_A |f_A|
\end{equation}

We now consider system (4.6). It is in the form of (1.1) in Wiggins [3]. The Melnikov function is

\begin{equation}
\frac{\partial H_0 \partial H_1}{\partial P_1 \partial Q_1} - \frac{\partial H_0 \partial H_1}{\partial Q_1 \partial P_1} - 4\tilde{k}_1 P_1 \frac{\partial H_0}{\partial P_1} + \left( \frac{\partial H_0}{\partial P_2}(P_1, Q_1, P_2) - \frac{\partial H_0}{\partial P_2}(P_1, Q_{1s}, P_2) \right) \left( -\frac{\partial H_1}{\partial Q_2} + 2\kappa P_1 - 4\tilde{k}_2 P_2 \right)
\end{equation}

This Melnikov function integrand can be simplified by using the chain rule gives

\begin{equation}
\frac{dH_1}{d\tau} = \frac{\partial H_0 \partial H_1}{\partial Q_1 \partial P_1} - \frac{\partial H_0 \partial H_1}{\partial P_1 \partial Q_1} - \frac{\partial H_0 \partial H_1}{\partial P_2 \partial Q_2}
\end{equation}

where we have used the fact that for $\varepsilon = 0$, $P_2' = 0$. The Melnikov function thus simplifies to

\begin{equation}
\mathcal{M}(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2) = \int_{-\infty}^{+\infty} \left( -\frac{dH_1}{d\tau} + 4\tilde{k}_1 P_1 Q_1' + (-2\kappa P_1 + 4\tilde{k}_2 P_2') \right) - 4\bar{a}P_1 \sin 2(Q_1 + Q_2)Q_2' d\tau
\end{equation}

We now integrate around the unperturbed heteroclinic orbit at $P_2 = P_2^r$ that approaches $p_0$ asymptotically as $\tau \to -\infty$.

It is clear that the first term in (8.12) can be integrated to give

\begin{equation}
\mathcal{I}_1 = -\int_{-\infty}^{+\infty} \frac{dH_1}{d\tau} = -\bar{a}P_2^r \cos 2(Q_1(\tau) + Q_2(\tau))|_{-\infty}^{+\infty}
\end{equation}

where $\mu = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{a}, Q_{20})$. The Melnikov function is evaluated on the orbit emanating from the center fixed point $p_2$, at the resonance value ($P = 0$).

The second term in (8.12) can be integrated by using the relation in equation (4.11) to obtain

\begin{equation}
\mathcal{I}_2 = \int_{-\infty}^{+\infty} 4\tilde{k}_1 P_1 Q_1' d\tau = -8\tilde{k}_1(\bar{\sigma}_1 + \bar{\sigma}_2 P_2^r) \tan^{-1}\left( \frac{\sqrt{\bar{c}_2^2 - 1}}{\sqrt{\bar{c}_2^2 - 1}} \right) |\tan Q_{1s}|
\end{equation}

and the third term in (8.12) can be integrated as
\[
\int_{-\infty}^{+\infty} -2\kappa P_1 + 4\bar{k}_2 P_2 \tau - 4\bar{a} P_1 \sin 2(Q_1 + Q_2) Q_2' d\tau = \\
(8.15)
\]

\[
4\bar{k}_2 P_2 \tau \Delta Q_2 + 2\kappa (\sigma + c_1 \bar{P}_2 \tau) \int_{-\infty}^{+\infty} \frac{Q_2' d\tau}{\bar{c}_2 + \cos 2Q_1(\tau)} \\
- 4\bar{a} (\sigma + \bar{c}_1 P_2 \tau) \int_{-\infty}^{+\infty} \frac{\sin 2(Q_1(\tau) + Q_2(\tau))}{\bar{c}_2 + \cos 2Q_1(\tau)} Q_2' d\tau
\]

The first and the second integrands in (8.15) can further be simplified as

\[
I_3 = \int_{-\infty}^{+\infty} \frac{Q_2 d\tau}{\bar{c}_2 + \cos 2Q_1(\tau)} \\
(8.16)
I_4 = \int_{-\infty}^{+\infty} \frac{\sin 2(Q_1(\tau) + Q_2(\tau))}{\bar{c}_2 + \cos 2Q_1(\tau)} Q_2' d\tau \\
= \sin Q_{20} \int_{-\infty}^{+\infty} \frac{\cos 2(Q_1(\tau) + Q_2(\tau))}{[\bar{c}_2 + \cos 2Q_1(\tau)]^{\frac{1}{2}}} \bar{c}_1 [P_2' - P_1(\tau)] d\tau
\]

From equations (8.13)- (8.16) the Melnikov function can be written as

\[
M(\mu) = M_1(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2) + \sin Q_{20} M_2(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2)
\]

where \(M_1 = I_1 + I_2 + I_3\) and \(M_2(\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2)\) comes from \(I_4\).
Bibliography