Learning and teaching of school algebra

2.1 Introduction
The purpose of this chapter is to present the study’s theoretical framework and the researcher’s perspective on early algebra learning. Considering that the project covers the educational design of (pre-)algebraic activities for students of age 11 to 13, we do not elaborate on the entire domains of algebra and algebra education. Instead we confine ourselves to a discussion of different conceptions, teaching approaches and typical learning difficulties of early school algebra. The emphasis of the second part of the chapter is on the troublesome transition from arithmetic to early algebra, in particular the cognitive obstacles that students encounter when they attempt to symbolize word problems and solve equations, including what is referred to as the cognitive gap or didactical cut. Since there is no consensus amongst researchers on what algebra is or how it should be taught and learned, section 2.7 describes which standpoint has been taken in this study.

2.2 Traditional school algebra
Algebra is known to be a major stumbling block in school mathematics, both in the past and at present. Historical studies on the developments of algebra education in the twentieth century show that the algebra studied in secondary school has not changed much over the years. Unintentionally algebra has functioned as a means of selecting the more capable learners – the ‘happy few’ who understand and enjoy the powers of algebra – from the rest, who experience and remember it as an elusive interplay of letters and numbers. Problems with algebra can be ascribed to external factors like the teaching approach and a poor image, but also to intrinsic difficulties of the topic, which will be described in section 2.4. Researchers have reported that grown-ups often have a negative image of school algebra, and many students can make no sense of it. There is a plausible explanation for this. Traditional school algebra is primarily a very rigid, abstract branch of mathematics, having few interfaces with the real world. It is often presented to students as a pre-determined and fixed mathematical topic with strict rules, leaving no room for own input. Traditional instruction begins with the syntactic rules of algebra, presenting students with a given symbolic language which they do not relate to. Students are expected to master the skills of symbolic manipulation, before learning about the purpose and the use of algebra. In other words, the mathematical context is taken as the starting-point, while the applications of algebra (like problem solving or generalizing relations) come in second place. Students are given little opportunity to find out the powers and possibilities of algebra for themselves. One can imagine that an average or below-average learner finds little satisfaction in practicing mathematics without a purpose or a meaning. Another characteristic of the traditional ap-
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proach is the rapid formalization of algebraic syntax. School algebra has always had a highly structural character, where algebraic expressions are conceived as objects rather than computations or procedures to be carried out. The procedural (or operational) aspects of algebra, which are more closely related to the arithmetical background that early algebra learners have, are usually cast aside soon after the introduction. This procedural-operational duality of algebra is discussed in more detail in section 2.4 and section 2.6.

Even though we all have an immediate idea what students learn when they learn school algebra, it is not an easy task to give a cast-iron definition. In an attempt to capture ‘school algebra’ in one sentence, we might suggest it is the mathematical domain dealing with (general) relationships between quantities on a symbolic level. Still, this description does not do justice to the multiple roles and utilities of algebra. Typical topics of school algebra include simplifying algebraic expressions, the properties of number systems, linear and quadratic equations in one unknown, systems of equations in two unknowns, symbolic representations and graphs of different kinds of functions (linear, quadratic, exponential, logarithmic, trigonometric), and sequences and series. In most of the core activities we find aspects of algebraic thinking (mental processes like reasoning with unknowns, generalizing and formalizing relations between magnitudes and developing the concept ‘variable’) and algebraic symbolizing (symbol manipulation on paper). Generally it is agreed that students must acquire both competencies in order to have full algebraic understanding.

2.3 Approaches to algebra

In the last two decades, the growing interest in algebra learning and teaching has instigated an international discussion on what we believe (school) algebra to be and what we believe it should be. Contemporary researchers have identified kernal characteristics of algebraic reasoning and algebraic language – such as generalizing, formalizing and symbolizing – which are related to different aspects of algebra (Kieran, 1989, 1990, 1992; Filloy & Rojano, 1989; Sfard, 1991, 1995; Sfard & Linchevski, 1994; Herscovics, 1989; Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; Bednarz, Kieran & Lee, 1996; Kaput, 1998). A few months ago the Twelfth ICMI study ‘The Future of the Teaching and Learning of Algebra’ raised issues like ‘why algebra?’, ‘approaches to algebra’, ‘language aspects of algebra’, ‘early algebra education’, ‘technological environments’, and more. Meanwhile it has become clear that there is no agreement on what algebra is or what it should be; each classification has its strong and weak points. Therefore, instead of trying to establish what algebra is, one might consider algebra in terms of its roles in different areas of application instead.

Bednarz et al. (1996) distinguish four principal trends in current research and curriculum development of school algebra: generalizing, problem solving, modeling and functions. These different roles of algebra can be associated with the various ways
in which the authors conceive algebra, and which characteristics of algebraic thinking they believe ought to be developed in order to find algebra meaningful. A fifth perspective presented by Bednarz et al. is the historical one, not as an alternative way to introduce algebra at school but as a valuable pedagogical tool for teachers and educational researchers.

The same researchers recognize that the classification is oversimplified and incomplete, and that various approaches have not yet been adequately researched: “The separation into four approaches to ‘beginning algebra’ is artificial; all four components are needed in any algebra program. (...) Some other possible approaches have probably been omitted.” (ibid., p. 325). Still, Bednarz et al. observe that their classification has helped to structurize their discussion on essential issues of school algebra.

Some years earlier, Usiskin (1988) proposed a slightly different categorization of perceptions of algebra: as generalized arithmetic, as a study of procedures for solving problems, as a study of relationships among quantities (including modeling and functions) and as a study of structures. In each of these approaches to algebra Usiskin identifies different roles of the letter symbols: pattern generalizer, unknown, argument or parameter, or arbitrary object respectively. One might argue that this list is not complete; other meanings of the concept of variable that are mentioned regularly are those of placeholder (a symbol in an arithmetical open sentence such as $3 + \bullet = 5$), letter not evaluated (like π and e) and label (letter to abbreviate an object, or a unit of measurement). A variable that varies (as argument or parameter) is considered to be of a higher level of formality than the variable as generalized number or unknown, which is again more formal than the placeholder; at the top end we find the arbitrary symbol. This subtle variation of meanings of letters has been identified as one of the major obstacles in learning algebra (see also section 2.4).

A number of other characterizations of algebra can be found in the literature. For instance, the National Council of Teachers of Mathematics (1997) identifies four themes for school algebra: functions and relations, modeling, structure, and language and representation. Kaput (1998) has listed five forms of algebraic reasoning: generalizing and formalizing, algebra as syntactically-guided manipulation, algebra as the study of structures, algebra as the study of functions, relations and joint variation, and algebra as a modeling language.

In the present study we do not take an explicit position on what is the best classification of algebra. It is only relevant that we recognize which aspects of algebra are relevant for the proposed learning program. The algebraic activities that we have developed can be described as ‘advanced arithmetic’, with a large component of problem solving and studying relations (see also section 2.7). We have no clear preference for one classification or the other; it is only for the practical reason of having a framework that we have made a choice. The overviews of perceptions of algebra (below) and its typical learning obstacles (in section 2.4) are based on contributions
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**learning algebra through generalizing**

If algebra is construed as a product of generalizing activities, its main purpose is to grasp generality, for instance by expressing the properties of numbers. The Oxford Dictionary exemplifies this perspective by defining algebra as the ‘study of the properties of numbers using general numbers’. Algebraic skills are directed at translating and generalizing given relationships among numbers. This approach to algebra stands a better chance if the learner’s intuitive base for the structure of algebra is already nourished in arithmetical activities. For example, Booth (1984) has suggested that if a student is to perceive an expression like \( a + b \) as an object in algebra, he or she must be able to view the sum \( 5 + 8 \) as an object in arithmetic, rather than as a procedure leading to the outcome 13.

**algebra as a problem solving tool**

Problem solving by constructing and solving equations is not only an historical route into algebra, it has also become a core activity in every algebra curriculum. Translating word problems into equations involves the fundamental issue of transition from arithmetic to algebra, in terms of symbolism as well as reasoning. According to Bell (1996), problem solving seen in a wider sense means ‘exploring problems in an open way, extending and developing them in the search for more results and more general ones.’ Bell sees algebra not as a separate branch of mathematics but as an integrated strand, wherever its symbolism, concept and methods are appropriate.

**learning algebra through modeling**

A modeling approach to learning algebra is based on the conception that students need to become flexible in the description and interpretation of phenomena in the world around them. It includes constructing meaning for various representations (graph, table, formula) and transforming one kind of representation into another. Note that these models are not intended to emerge as constructions of students’ own mathematical activity, but as given, pre-determined symbolic forms. The modeling approach has at least two factors in common with the functional approach to algebra. First, modeling is based on expressing relations between varying quantities, and second, it contributes to developing the student’s sense of what a variable is.

**functional approach to algebra**

Calculators and computers lead to new possibilities in studying relations between two sets of numbers. Computers may be used, for example, to test whether a certain function is hidden behind the structure of a set of numerical data. Various types of
functions and the concept of variable may be investigated in this respect. Bednarz et al. (1996) describe two projects of introducing algebra which are based on this principle.

2.4 Typical learning difficulties

The present study is concerned with students in the age of 11 to 13 years, for which reason we focus on the early learning of algebra. The introduction to algebra usually involves the study of algebraic expressions, equations, equation solving, variables and formulas. According to Kieran (1989, 1992), students’ learning difficulties are centered on the meaning of letters, the change from arithmetical to algebraic conventions, and the recognition and use of structure. Some of these problems are amplified by teaching approaches: often the structural character of school algebra is emphasized, whilst procedural interpretations would be more accessible for children (Kieran, 1990, 1992; Sfard & Linchevski, 1994). A more detailed account of the first two categories – meaning of letters and arithmetical versus algebraic conventions – with respect to equation solving is given in section 2.6.1, where we discuss the relation between arithmetic and algebra and the discontinuities between them. In the present section we describe two more general ontological difficulties of algebra: operational (or procedural) and structural (relational) modes of thinking, and problem solving.

Sfard and Linchevski (Sfard, 1991; Sfard & Linchevski, 1994) suggest that problems encountered in learning algebra can be partly ascribed to the nature of algebraic concepts. According to Sfard (1991) there are two fundamentally different ways to conceive mathematical notions: operationally (as processes) and structurally (as objects). Students struggle to acquire a structural conception of algebra, which is fundamentally different from an arithmetical perspective (see also section 2.6.1). To illustrate the operational conception, Sfard and Linchevski (1994) explain that an algebraic expression like $3(x + 5) + 1$ can be seen as a description of a computational process. It is a sequence of instructions: add 5 to a certain given number, multiply the result by three and then add 1. From another perspective, the expression can also be viewed as the product of the computation, representing a certain number (which at this time cannot be specified). In yet another setting $3(x + 5) + 1$ can behave as a function; instead of representing a fixed number, it reflects a change. And at a very simple, superficial level we can even say the expression is a meaningless string of symbols. As an algebraic object, it can be manipulated and combined with other symbolic expressions. The three latter conceptions – as a computational product, a function and a symbolic string – all reflect a structural understanding of algebra. In fact, Sfard and Linchevski argue that these four different notions of an algebraic expression represent different phases in the individual learning of algebra, based on logical, historical, and ontological analyses.

In addition Sfard (1995) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She claims that the syncopated
stage of algebra (where unknown quantities are represented by abbreviations or letters) is linked to an operational conception of algebra, whereas the *symbolic* stage of algebra (where letters stand for given as well as unknown quantities) corresponds with a structural conception (see section 3.3 in for more information on the terms ‘syncopated’ and ‘symbolic’).

In Kieran’s reviews (1989, 1990) we read that studies on this issue go back a few decades. Matz and Davis, for example, did research in the 1970’s on students’ interpretation of the expression $x + 3$. Students see it as a procedure of adding 3 to $x$, whereas in algebra it represents both the procedure of adding 3 to $x$ and the object $x + 3$. In other words, in algebra the distinction between the process and the object is often not clear. Matz and Davis call this difficulty the ‘process-product dilemma’.

Freudenthal (1983) illustrated the difference between a procedural (in terms of Sfard, ‘operational’) and a static (corresponding with Sfard’s term ‘structural’) outlook by comparing language use and meaning:

> A powerful device – this formal substitution. It is a pity that it is not as formal as one is inclined to believe, and this is one of the difficulties, perhaps the main difficulty, in learning the language of algebra. On the one hand the learner is made to believe that algebraic transformations take place purely formally, on the other hand if he has to perform them, he is expected to understand their meaning. (...) The learner is expected to read formulae with understanding. He is allowed to pronounce:

\[
a + b, \quad a - b, \quad ab, \quad a^2
\]

as

\[
a \text{ plus } b, \quad a \text{ minus } b, \quad a \text{ times } b, \quad a \text{ square}.
\]

Yet he has to understand it as

\[
\text{sum of } a \text{ and } b, \quad \text{difference of } a \text{ and } b, \quad \text{product of } a \text{ and } b, \quad \text{square of } a.
\]

The action suggested by the plus, minus, times, square and the linear reading order must be disregarded. The algebraic expressions are to be interpreted statically if the formal substitution is to function formally indeed (Freudenthal, 1983, p. 483-484).

Sfard (1987) observes that students are better at writing their solution procedures for solving equations verbally than they are at constructing and manipulating symbolic equations. She therefore proposes to foster students’ understanding of processes and algorithms before moving on to the structural perspective. One way of doing this is to incorporate computer programming.

After a while learners of algebra can become quite skilled at performing algorithmic procedures (expanding brackets, solving a system of equations), and yet they fail at problem solving. Professional algebraists seem to forget the catch: that rote skills do not help students in getting started. Da Rocha Falcão (1995) suggests that the difficulty is contained in the difference in approach to problem solving. Arithmetical
Typical learning difficulties

problems can be solved directly, if necessary with intermediate answers. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved.

To illustrate the direct, arithmetical approach and the indirect, algebraic approach let us consider the following example of the task ‘guess my number’:

\begin{itemize}
\item \textit{Student 1}: Think of a number, multiply it by 3 and add 5 to it. What is the outcome?
\item \textit{Student 2}: 32.
\item \textit{Student 1}: Then the number you thought of is 9.
\end{itemize}

The most obvious arithmetical solution procedure to follow is to undo the chain of operations: 32 minus 5 gives 27, and 27 divided by three gives 9. It is a direct approach because it works towards the solution right from the start. The algebraic method for solving this problem is to represent the initial number by \( x \), construct the equation \( 3x + 5 = 32 \) and then solve the equation for \( x \). This approach is indirect: the problem is translated first from a dynamic description ‘do this, do that’ to a static, symbolic representation, before moving onto the actual solution procedure. Mason (1996, p. 23) formulates the difference as follows (comments by the researcher between brackets): “Arithmetic proceeds directly from the known (32, in this case) to the unknown (9) using known computations (the inverse operations of ‘times 3’ and ‘plus 5’); algebra proceeds indirectly from the unknown (\( x \) in our example), via the known (the operations ‘times 3’ and ‘plus 5’), to equations and inequalities which can then be solved using established techniques.” Learning difficulties related specifically to equation solving are discussed in more detail in section 2.6.1 which deals with the transition from arithmetic to algebra.

In order for algebra to be appreciated, its superiority to arithmetic needs to be (made) apparent. It is common to introduce students to equation solving using linear equations in one unknown like the example above, as an alternative to the arithmetical procedure. The algebra expert (teacher, text book author) finds this approach suitable because each step in the solution process can be verified by the arithmetical counterpart (the inverse operation). But in the eye of the learner it is not a logical or natural method; after all, the arithmetical approach is easier and works just as well! Ideally, learners should experience the value and purpose of algebra from the start— for example in situations where arithmetic and common sense no longer comply— but without being forced to a formal level prematurely. In our opinion purpose is unmistakably joint with meaning. Classroom experiments have shown that algebraic competence depends on the ability to give meaning to equations (Abels, 1994; Van Reeuwijk, 1995, 1996). When equations emerge through a good understanding of the underlying relations – when they make sense to the learner – students have been found to be more successful at solving them as well.
2.5 Symbolizing

As we have already said before, the symbolic language of algebra requires students to learn to look at symbolic expressions in a new way. In traditional teaching approaches algebraic expressions like formulas, equations or arithmetical identities are presented to students as ready-made artifacts. The meanings of the symbols are fixed in a rigid framework of conventions. In reaction to this ‘anti-didactical inversion’ (Freudenthal, 1973), we advocate a teaching approach which begins with what the learner already knows and does. In other words, algebra learning and teaching should be based on problem situations leading to symbolizing instead of starting with a ready-made symbolic language. In this section we describe some aspects of how symbolizing and meaning may develop in the proposed early algebra program.

In recent years, research discussions on symbolizing and modeling show a change in ideas of how symbols and models (also called ‘manipulatives’) may be used to support the development of mathematical concepts (see, for example, Gravemeijer & Terwel, 2000). Where models and symbols were previously introduced by teachers as ready-made tools with a pre-determined meaning, intended to make abstract mathematics more accessible, they are now seen as products of students’ own mathematical activities. The corresponding teaching approach is based on the belief that symbolizing and meaning develop interactively as students engage in reflexive discourse:

The basic idea is that forms of symbolization (in schemes, diagrams, models or even verbal terms) emerge in the context of activities that require the availability of such symbolic tools, and that the functional requirements of these activities stimulate the improvement of the children’s way of symbolizing (Gravemeijer & Terwel, 2000, p. 2).

This dynamic view of symbolizing and modeling has called for another way of speaking about symbolizations. Terms like ‘symbols’ and ‘referents’ – connected to the static, representational view of symbolizations – have been replaced by notions like ‘sign’ and ‘inscription’. A sign consists of a pair signifier-sgnified, of which the signified plays a dynamic part in the constitution of new signs. In the so-called ‘chain of signification’ a certain sign combination becomes the signified of the succeeding sign, so that the meaning of the original sign changes. It is during this dynamic process, where signs and meanings change and produce new signs, that mathematical concepts are developed.

In an exposition on the interaction between mathematical discourse and mathematical objects, Sfard (2000) also considers the interplay between symbols (signifiers) and their objects (signifieds). In her conception, signifiers must come before their signifieds, since ‘one simply cannot speak about the object represented by a symbol before the symbol enters the language and becomes a fully fledged element of the discourse’. Sfard observes an inherent circularity in mathematical discourse: the
construction of signifieds relies on talking about their signifiers, while the signifiers themselves obtain their meaning from mathematical discourse. In other words, we have a seemingly paradoxical situation: symbols become meaningful by using them, but how can a symbol be used before it is meaningful? Sfard conjectures that when a new signifier is introduced it does not have a signified yet. It is semantically ‘empty’, and its meaning develops gradually in mathematical activity. In such a way the apparently vicious circle of mathematical discourse and mathematical objects fuels their simultaneous development. Furthermore, Sfard and Linchevski (1994) argue that symbols seem to be a necessary but not sufficient condition for acquiring a structural mode of thinking:

It is true that as long as algebraic ideas are dressed in words and in words only, it is difficult to imagine the more advanced structural approach, where the computational processes are considered in their totality from a higher point of view, and where operational and structural slants meet in the same representations. To put it differently, words are not manipulable in the way symbols are. It is this manipulability which makes it possible for algebraic concepts to have the object-like quality (Sfard & Linchevski, 1994, p. 93).

The current perspective on symbolizing as a dynamic process – the interplay between the development of mathematical meaning and symbol use – implies that instructional design should provide opportunities for students to develop their own sense-making symbolism. A teacher-guided mathematical discourse on the meaning, advantages and shortcomings of these symbolic constructs can result in a mutually accepted (pre-)algebraic symbol system as the basis for further algebra learning. Two cases of informal symbolizing and the development of meaning which are relevant for this study are described below, followed by a brief description of some common models for solving equations.

2.5.1 Symbolizing and schematizing

Let us turn for a moment to the realistic instructional theory of Realistic Mathematics Education (see also section 4.3.1). Point of departure is Freudenthal’s notion of mathematics as an activity of organizing or mathematizing. In this activity, symbolizing is developed as a personally meaningful and convenient problem solving tool. This notion of symbolizing as a tool for mathematical reasoning serves to explain the use of the terms ‘symbolizing’ and ‘schematizing’ in this study.

‘Symbolizing’ and ‘schematizing’ are sometimes both seen as symbolizing activities. Symbolizing in the narrow sense of the word refers to the construction and use of conventional mathematical symbols: numbers, letters, operators, expressions, and so on. Symbolizing in the wider sense of the word refers to the use of material or visual representations such as drawings, notations, diagrams, tables, or concrete, context-bound marks. In order to make this distinction, the latter conception of sym-
bolizing will be named ‘schematizing’. And when the representation at hand is more than just a calculational tool, we say that a student uses ‘schematizing as a problem solving tool’. For example, figure 2.1 shows a solution to the task: ‘How many quarters and dimes do you get for a coin worth 2.5 guilders?’ The table can inspire the student to use a systematic approach – repeated exchange of 2 quarters for 5 dimes – in order to find all possible combinations. In this way a schematic representation like a table can give meaning to a problem solving strategy.

![Figure 2.1: Table as a problem solving tool](image)

2.5.2 Symbolizing equations

The ideas proposed by Sfard (2000) can be linked to the present study by considering the specific case of algebraic symbolizing in the experimental program. In the first few lessons on equation solving students are confronted with different types of symbols: drawings, pictograms, abbreviations (letter combinations) and unknowns, not necessarily in this order.

![Figure 2.2: Two combinations of umbrellas and hats](image)

These symbols suggest a gradual withdrawal from contextual meaning. Let us consider the problem in figure 2.2 taken from the *Mathematics in Context* unit *Compar-
ing Quantities (Mathematics in Context Development Team, 1998, p. 16) as an example. The drawing represents an embedded system of equations in two unknowns: the price of an umbrella and the price of a hat. Note that the visual representation means that the problem situation is already organized. One can imagine that if the problem had been represented as a description, a student might have chosen to organize the information in a drawing in a similar way. The pictures have a direct reference to the objects they stand for: the (price of a) hat and the (price of an) umbrella.

At the informal level we accept that students say ‘2 umbrellas and 1 hat cost 80 dollars’.

At this stage we do not aim to hear the formal, mathematical expression ‘the sum of twice the price of an umbrella and the price of a hat is 80’. The symbols are meaningful, but they are not yet tied to the formal signified. Abbreviations, too, may reflect an informal level of understanding of the signified: in the system of equations

\[
\begin{align*}
2 \text{ um} + 1 \text{ ha} & = 80 \\
1 \text{ um} + 2 \text{ ha} & = 76
\end{align*}
\]

the letters \( \text{ha} \) and \( \text{um} \) are used as labels. The link between abbreviations and the context can easily be reconstructed because the abbreviations refer directly to the situational objects: hats and umbrellas. At a formal level, in the system

\[
\begin{align*}
2 u + h & = 80 \\
u + 2h & = 76
\end{align*}
\]

the unknowns \( h \) and \( u \) are signifiers for the mathematical objects (signifieds) ‘price of a hat’ and ‘price of an umbrella’. The letters are no longer labels but magnitudes, in fact they are determinate unknowns. The transition from the conception ‘2 umbrella’s plus 1 hat equal 80 dollars’ to the more formal conception will need considerable attention. As we see, early algebraic symbolizing can be meaningful for students from the start and the relation signifier-signified can develop quite naturally over time. The teacher should accommodate a gradual shift from an informal to a more formal conception of an unknown (ending with the concept of variable) in classroom discourse.

<table>
<thead>
<tr>
<th>WORDS</th>
<th>PICTURES</th>
<th>SIMPLIFIED PICTURES</th>
<th>SHORTHAND</th>
</tr>
</thead>
<tbody>
<tr>
<td>Think of a number</td>
<td>( \bullet )</td>
<td>( \bullet )</td>
<td>( x )</td>
</tr>
<tr>
<td>Add 3</td>
<td>( \bullet \bullet \bullet )</td>
<td>( \bullet +3 )</td>
<td>( x+3 )</td>
</tr>
<tr>
<td>Double</td>
<td>( \bullet \bullet \bullet \bullet \bullet )</td>
<td>( 2 \bullet +6 )</td>
<td>( 2x+6 )</td>
</tr>
<tr>
<td>Take away 4</td>
<td>( \bullet \bullet \bullet \bullet \bullet )</td>
<td>( 2 \bullet +2 )</td>
<td>( 2x+2 )</td>
</tr>
<tr>
<td>Divide by 2</td>
<td>( \bullet )</td>
<td>( \bullet +1 )</td>
<td>( x+1 )</td>
</tr>
<tr>
<td>Take away original number</td>
<td>( \bullet )</td>
<td>( \square )</td>
<td>( \square )</td>
</tr>
</tbody>
</table>

figure 2.3: gradual steps of symbolization
Figure 2.3 shows another example of progressive symbolizing, where informal symbols can have a contextual meaning at first and a more formal (abstract) meaning later on (Sawyer, 1964, p. 73).

### 2.5.3 Models for equation solving

The teaching of equation solving often involves the use of pre-designed models, intended to make the abstract symbolic equation more accessible. Some models serve to visualize the situation (symbolically or schematically), while others take a purely numerical approach. Such models contain a component of *translation*, where objects and operations in abstract situations are given meaning at a more concrete level. It is important that this translation operates in two directions, that students can identify operations and objects at both the concrete and the abstract levels. A second component in modeling concerns the gradual detachment from the context-bound semantics of the model. Filloy and Sutherland (1996) remark that “(...) fixation on the model can delay the construction of an algebraic syntax since this requires breaking away from the semantics of the concrete model” (ibid., p. 150). We describe five of these manipulatives: the balance model, the geometrical model, the arithmetical model, the notebook model and the linear model. Each model has its advantages and disadvantages; the perfect one is yet to be discovered.

#### balance model

The classical balance model is based on the concept of equal weights on both sides of the scale. For instance, in the equation $3x + 12 = 5x + 8$ the left-hand side of the scale holds 3 elements of weight $x$ and 12 unit weights, while the right hand side holds 5 elements of weight $x$ and 8 unit weights. The weight $x$ can be determined by cancelling equal weights on both sides. The advantage of this model is that it has a meaning in everyday life situations, and students can make a mental image of the balance very easily. Moreover, the balance emphasizes the static character of the equation; the concept of equivalence remains in the foreground as the solution procedures are carried out. The major limitation of the balance model – of all physical and visual models for that matter – is the restrictions of its applicability to equations involving negative terms and negative solutions.

#### geometrical model

The advantage of the geometrical or area model lies in its visualization and concrete meaning (area) for symbolic expressions. This model can be used for linear equations of the form $ax + b = cx$ (a rectangle of length $a$ and width $x$ and a rectangle of area $b$ added together have the same area as a rectangle of length $c$ and width $x$) but also for quadratic equations. Figure 2.4 shows the geometrical representation of the expression $(a + 5)(a + 2)$ and the corresponding multiplication table as it used in
some Dutch mathematics text books. The area model, too, has a few disadvantages. Just like the balance model, the concrete meaning of the area model is limited to positive magnitudes. In addition, it may not be suitable for students who have a weak geometrical foundation. After all, basic geometrical concepts like area and perimeter continue to be an obstacle for many mathematics students. Furthermore, Filloy and Sutherland (1996) observe that automation in both the balance and the area model lead to errors typically associated with algebraic syntax, such as adding and subtracting coefficients of different degree.

![Figure 2.4: Area model and multiplication table](image)

**arithmetical model**

The arithmetical model employs arithmetical identities as precursors of symbolic equations. The identity $3 \times 2 + 12 = 5 \times 2 + 8$, for example, can be used to construct the equations $3 \times ? + 12 = 5 \times ? + 8$, or $3 \times 2 + \bullet = 5 \times 2 + 8$, etcetera. If so desired, the question mark or dot (or any other symbol) can eventually be replaced by a letter symbol to introduce the concept of unknown. After students have seen where an equation might come from and what the solution looks like, one proceeds to teaching the solution method. In order to make the symbolic manipulations meaningful, each step in the solution procedure is demonstrated for the arithmetical identity as well. The solution (the number 2 in the example above) is marked to make it ‘hidden’. Although the arithmetical model does not have the advantage of physical or visual affinity, it makes good use of the arithmetical pre-knowledge that students have and it can be applied to any type of equation.

**notebook model**

The notebook model supports one of the strategies for solving systems of equations developed for the algebra strand in the *Mathematics in Context* project. Figure 2.5 shows how a realistic context of ordering drinks and food in a restaurant is translated into a mathematical representation of different combinations. The notebook model resembles a matrix where the entrees in each row represent the number of items for that particular combination. Matrix equation solving in itself is of course not an innovative approach; from the influential text *Nine Chapters on the*
Mathematical Art composed at the time of the Han Dynasty (206 BC to 220 AD) we know that ancient Chinese civilizations used a similar approach.

Another possibility for visualizing linear equations is based on representing an unknown quantity or magnitude by a line, or a strip where only the length of the strip matters. We call this the linear model. Let us consider the following problem posed by Sawyer, who already suggested using visual representations of equations decades ago:

A man has two sons. The sons are twins; they are the same height. If we add the man's height to the height of 1 son, we get 10 feet. The total height of the man and the 2 sons is 14 feet. What are the heights of the man and his sons? (Sawyer, 1964, p. 40).

The objective of visualization is to organize and thereby clarify the problem situation. Figure 2.6 I illustrates how the description can be transformed into a picture representation, which can then be simplified to a more schematic representation (part II). The visual forms are more accessible for reasoning than the story; at a glance the two human towers are compared to find that each son has to be 4 feet tall. The schematic drawing has the advantages of less work and general applicability, but when it becomes context-free it also loses its meaning. Moreover, students may not accept that an unknown length can be drawn as if it were known. It is therefore important to let students decide their own pace of learning. If at a later stage the student is capable of schematizing a new type of problem in a similar way, we can say
that the schematic drawing has become a model. What started as a model of a given problem situation has become a model for reasoning about a new family of problems.

Perhaps after a few more problems a student will suggest using a simple table (figure 2.6 III) or even mathematical symbolism closer to a system of two equations (part IV). Otherwise, when the time is ready, the teacher may guide the classroom discussion towards a symbolic representation. In chapter 5 and chapter 6 we describe how the linear model has functioned in the learning strand designed for this project.

2.6 Arithmetic and algebra

In section 2.3 we mentioned four different approaches to the teaching and learning of early algebra: generalizing, problem solving, modeling and functions. It is a classification which compares four kernal activities in mathematics, each leading to algebraic learning. Shortcomings like oversimplification and incompleteness have been recognized. Some readers might suggest classifying algebra in terms of its interfaces with other mathematical terrains instead. For instance, algebra can also arise from mathematical activities in geometry (see section 2.8). In this section we discuss
algebra from an arithmetical perspective. The relationship between arithmetic and algebra not only sheds light on some typical learning difficulties of algebra, but it also shows why an approach to early algebra based on arithmetic is a suitable one for this study.

2.6.1 A dual relationship

We have already mentioned in section 2.4 that a number of learning difficulties of early algebra can be ascribed to the different natures of arithmetic and algebra. We can identify differences regarding the interpretation of letters, symbols, expressions and the concept of equivalence. For instance, in arithmetic letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. And in the case of solving linear equations in one unknown there is said to be a discrepancy known as the **cognitive gap** (Herscovics & Linchevski, 1994) or **didactical cut** (Filloy & Rojano, 1989), referring to students’ inability to operate with or on the unknown. In this study we use the terms ‘cognitive breach’, ‘cognitive break’, ‘rupture’ and ‘gap’ to describe the collection of learning difficulties caused by discrepancies between arithmetic and algebra. If in any situation we use the term ‘cognitive gap’, it is to be understood in this broad sense and not in the specific way Herscovics and Linchevski use it.

According to Sfard and Linchevski (1994) the rupture between arithmetic and algebra is an ontogenetic gap caused by the operational-structural duality of mathematical concepts. In the transfer from an arithmetical to an algebraic conception students need to learn that processes can be seen as objects; they must acquire a dual process-product perception. Sfard (1991) proposes a ‘theory of reification’ according to which the development of mathematical concepts occurs in 3 phases: interiorisation, condensation, reification. These phases form a hierarchy of perspectives where processes on objects become objects on their own, which can in turn be part of a process at a higher level. It is a theory which resembles Freudenthal’s vision on levels of learning (see section 4.3.1). There is evidence that this process of reification is difficult to achieve, not in the least because reification and advanced interiorisation appear to be locked in a ‘vicious circle’. On the one hand the ability to perform basic algebraic algorithms is needed to get a feeling for the objects involved, on the other these same objects are needed to gain full technical competence, giving meaning to the algorithms and making it easier to remember them. In section 2.5 we already described how symbolizing and schematizing activities play a role in the reification process. In the case of algebra, Sfard and Linchevski (1994) connect the difficult progression from an operational conception (arithmetic) to a structural one (algebra) with the gap in the process of reification (the ‘vicious circle’).

We want to remark here that in this book we may use the terms operational’ (or procedural) and ‘structural’ when in fact we mean only to distinguish between their ‘dynamic’ and ‘static’ natures respectively. The terms ‘static’ does not include the no-
tion of ‘perception as an object’, so whereas the process-product duality of the former qualification indicates a difference in conceptual level, the dynamic-static duality does not. For all the students who participated in the classroom experiments we can say that our reference to an operational or a structural conception is confined to the dynamic-static distinction. The students have not had enough time to actually develop a structural notion of algebra.

Gray and Tall (1994) have suggested the notion of ‘procept’ and ‘proceptual thinking’ as an intermediate phase between the operational and the structural level. The procept, intended to build bridges across the ‘proceptual divide’, consists of three components: a process which produces a mathematical object, and a symbol to represent either of these. Gray and Tall remark that learners who are able to see symbols as objects and use these symbols to produce new mathematical ideas can formalize their thinking, while students who continue to think in terms of processes are likely to remain at an operational level of thinking.

Decades ago Freudenthal already pointed out that inconsistencies between arithmetic and algebra can cause great difficulties in early algebra learning. He observes that the difficulty of algebraic language is often underestimated and certainly not self-explanatory: “Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory” (Freudenthal, 1962, p. 35). He then says:

The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic $3 + 4$ means a problem. It has to be interpreted as a command: add 4 to 3. In algebra $3 + 4$ means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. $a + b$ cannot easily be interpreted as a problem (Freudenthal, 1962, p. 35).

The two interpretations (arithmetical and algebraic) of the sum $3 + 4$ in the citation above correspond with a procedural (operational) and a structural perception respectively.

In spite of the increase in information available from research on the arithmetic-algebra duality, and perhaps also because of it, the demarcation line between arithmetic and algebra is not clear. In this study a magnifying glass was taken to hand, so to speak, to contrast the characteristics of both. We realize that naming the differences in extreme terms might be dangerous. Some descriptions are self-evident, while others are certainly subject to debate. Moreover, the list is probably not complete. However, we feel the end justifies the means because, for two reasons, table 2.1 has been an effective tool. First, the demarcation has provided ideas for constructing bridges between arithmetic and algebra. And second, it has simplified the identification of solution strategies as ‘arithmetical’, ‘pre-algebraic’ or ‘algebraic’ during the analysis of student work. In the next few paragraphs we clarify some of
the characteristics of arithmetic and algebra and connect them with typical learning difficulties of algebra. The left-hand column in table 2.1 contains eleven arithmetical characteristics, opposed to eleven algebraic characteristics in the right hand column. The middle column represents the transition zone between arithmetic and algebra, the contents of which – the numbers 1 through 11 – will be elaborated in thematic sections below. In each case we discuss the characteristic at hand and give suggestions for an intermediate, pre-algebraic approach. For example, the characteristics 1, 2, and 3 deal with generalization, the theme for the first section.

<table>
<thead>
<tr>
<th>arithmetic</th>
<th>pre-algebra</th>
<th>algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>general aim: to find a numerical</td>
<td>1</td>
<td>general aim: to generalize and symbolize methods of problem solving</td>
</tr>
<tr>
<td>solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>generalization of specific number</td>
<td>2</td>
<td>generalization of relations between numbers, reduction to uniformity</td>
</tr>
<tr>
<td>situations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>table as a calculational tool</td>
<td>3</td>
<td>table as a problem solving tool</td>
</tr>
<tr>
<td>manipulation of fixed numbers</td>
<td>4</td>
<td>manipulation of variables</td>
</tr>
<tr>
<td>letters are measurement labels or</td>
<td>5</td>
<td>letters are variables or unknowns</td>
</tr>
<tr>
<td>abbreviations of an object</td>
<td></td>
<td></td>
</tr>
<tr>
<td>symbolic expressions represent</td>
<td>6</td>
<td>symbolic expressions are seen as</td>
</tr>
<tr>
<td>processes</td>
<td></td>
<td>products and processes</td>
</tr>
<tr>
<td>operations refer to actions</td>
<td>7</td>
<td>operations are autonomic objects</td>
</tr>
<tr>
<td>equal-sign announces a result</td>
<td>8</td>
<td>equal-sign represents equivalence</td>
</tr>
<tr>
<td>reasoning with known quantities</td>
<td>9</td>
<td>reasoning with unknowns</td>
</tr>
<tr>
<td>unknowns as end-point</td>
<td>10</td>
<td>unknowns as starting-point</td>
</tr>
<tr>
<td>linear problems in one unknown</td>
<td>11</td>
<td>problems with multiple unknowns: systems of equations</td>
</tr>
</tbody>
</table>

**table 2.1: characteristics of arithmetic and algebra**

**generalization (1, 2, 3)**

Solving problems in arithmetic is primarily directed at finding numerical solutions in specific situations. The objective of algebra, on the other hand, is usually to discover and express generality of method, looking beyond specificness. Generalization requires the learner to recognize common factors on the one hand and unique characteristics on the other. For example, equation solving is not useful if each new problem requires a new approach. The strength of equation solving is its general applicability: define the unknown(s), describe the relation(s) between the quantities, and solve the problem with algebraic means. Algebra also constitutes the reduction to uniformity; in contemporary mathematics this is done with symbolic language. At times students carry out activities of generalizing in arithmetic. Generalization of number situations helps students to develop abstract notions of numbers, like the decontextualization of fractions. In doing so students internalize and reify the fraction
concept. Algebra, on the other hand, pursues the generalization of relations *between* numbers or methods of manipulating numbers; not the numbers but the *relations* (methods) are the objects of generalization. Generalized relations in turn enable extrapolations and predictions about new situations, broadening the horizon even further. We can illustrate this difference by considering the role of a tabular representation in arithmetic and in algebra. In arithmetic the table is seen as a calculational tool, to support the calculation of ratios or to organize and structurize information. In algebra the table has a purpose in solving problems, for instance to investigate patterns. Table 2.2 can help a student to recognize the relationship between \( n \) and \( A \) and describe this relationship in general terms.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>( \text{number} \times (\text{number} + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>30</td>
<td>42</td>
<td>( \text{number} \times (\text{number} + 1) )</td>
</tr>
</tbody>
</table>

**Table 2.2: From a pattern to a general expression**

In the proposed learning strand the tabular representation is used to solve problems like ‘two numbers added together make 120, while the difference between them is 38’ (see table 2.3). It is very natural for students to use a trial-and-error approach. The table helps students to structurize their attempts, like starting with a difference of zero and then increasing it symmetrically. In both examples the purpose of the tabular representation is to facilitate the acquisition of a general method.

<table>
<thead>
<tr>
<th>( \text{first number} )</th>
<th>60</th>
<th>70</th>
<th>75</th>
<th>79</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{second number} )</td>
<td>60</td>
<td>50</td>
<td>45</td>
<td>41</td>
</tr>
<tr>
<td>( \text{difference} )</td>
<td>0</td>
<td>20</td>
<td>30</td>
<td>38</td>
</tr>
</tbody>
</table>

**Table 2.3: Using the pattern to solve the problem**

**Meaning of letters (4, 5)**

We have already mentioned that letters can have different meanings and functions in algebra. Early in the process of learning of symbolic algebra – the study of algebraic expressions, equations, equation solving, and formulas – letters usually represent general numbers, unknown numbers, arguments or variables (see also Küchemann, 1978; Usiskin, 1988). According to Kieran (1989, 1990), students have been found to be confused by the different ways that a single letter can be used, leading to incorrect interpretations. Moreover, learners may be reluctant to accept the idea that numbers can be represented by letters or that the expression \( x + 3 \) can be a final answer (‘cognitive readiness’). Another common difficulty of calculating symbolic expressions in algebra is related to a conflict with the positional system. In algebra the term \( 6x \) when \( x = 3 \) is evaluated by calculating \( 6 \times 3 \), which clashes with the ar-
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The numerical meaning of the digits 6 and 3 in the number 63. When students encounter letters in arithmetic, the role of these letters is very different from algebraic letters (variables). In the Dutch arithmetic curriculum there are only a few instances where letters refer to generalized numbers, for example in the formula for the area of a rectangle $A = l \times w$. In arithmetic a letter usually represents a label for measuring ($m$ for meters), counting ($p$ for points) or currency ($f$ for guilders), or it can be an arbitrary label to abbreviate a word. In each case the letter refers to the measurement unit or object directly. In algebra letters can have a second meaning, namely the number of meters, points or guilders. If algebra learners continue to interpret letters as labels instead of variables, they are bound to make what is known as the ‘reversal error’ in the ‘student-professor problem’:

Write an equation using the variables $S$ (the number of students) and $P$ (the number of professors) to represent the statement ‘There are six times as many students as professors at this university’ (Rosnick, 1981).

Students who write $6S = P$ instead of $S = 6P$ interpret their expression as ‘6 students for every professor’. Herscovics (1989) suggests that this error is caused by the interference of natural language and algebra.

In the experimental learning strand we tackle this problem by confronting students with different meanings of (word) variables in the context of barter trading. For example, students are required to switch from a statement on value of goods (‘value of a cabbage = 3 × value of an apple’) to a table with numbers of goods, and vice versa, followed by a task on completing a table with numbers of goods given a barter expression like $1a = 2b$ (1 apple for 2 bananas) where the letters are labels. Subsequently, students do an activity with arrow diagrams involving letters where students have to determine the meaning of the letters (representing either the number of items or the value of an item).

**conception of symbolic expressions (6, 7, 8)**

As we discussed earlier, in arithmetic students conceive operations as a command to perform an action (addition, multiplication, etcetera). The operation is only the means to an end: finding a numerical outcome. An operation viewed algebraically, on the other hand, is an autonomic object and the outcome is the expression itself; the operation cannot be carried out, so to speak. For example, an expression like ‘$5 + 3$’ is an open-ended action in arithmetic but in algebra it is a valid, finished product. Along the same lines of reasoning we can say that the arithmetical meaning of the equal-sign is to announce the numerical outcome of a calculation, while the algebraic, relational conception is to depict a state of equivalence. The former viewpoint agrees with a dynamic, procedural conception of operations and expressions, whereas the latter viewpoint fits a static or – at a formal level – structural per-
ception. With these roles of operations and the equal-sign in mind, symbolic expressions can be viewed as commands for action or as static descriptions.

Filloy and Rojano (1989) as well as Herscovics and Linchevski (1994) point out a break in the development of operating on the unknown in an equation. Herscovics and Linchevski describe the cognitive gap as “students’ inability to spontaneously operate with or on the unknown” (1994, p. 59). They object to the definition of didactical cut given by Filloy and Rojano (1989) for restricting the problem to mathematical characteristics, with no eye for the role of solution procedures. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal-sign changes from announcing a result to stating an equivalence. For example, if we symbolize the statement “Jenny is 5 years old, and she is 2 years older than her little brother” in the exact same order, we get the equation \( 5 = x + 2 \). The symbolic expression does not resemble the arithmetical interpretation ‘5 minus 2 gives the little brother’s age’ at all. Furthermore, if the unknown appears on both sides of the equal-sign instead of one side, as in \( 10 - 3x = x + 2 \), the equation can no longer be solved arithmetically (i.e. by inverting the operations on the coefficients one by one). Instead the student is required to treat the unknown quantity as if it were a known number. In other words, operating with or on the unknown requires another notion of equivalence as well as the ability to treat an unknown number as if it were known.

Other cognitive obstacles related to manipulating symbolic expressions and solving equations reported by researchers are:

– recognizing equivalence of expressions;
– handling minus signs in an equation: Linchevski and Herscovics (1996) have found that mixed terms in a symbolic expression become detached from the operations: \( 3x - 5x + 7x \) is interpreted as \( 3x - 12x \), and \( 3x + 2 - 8x \) as \( 11x + 2 \);
– combining like terms, i.e. terms of the same dimension;
– misunderstanding the syntax of expressions can cause the so-called ‘reversal error’ – also referred to as the student-professor problem – where the relation between two quantities is interpreted the wrong way around;
– making formal manipulations meaningful and purposeful;
– most models for solving equations fail to accommodate the transfer from an informal to a formal conception of equations.

The pre-algebra instructional materials designed for this study deal with just a few of these obstacles because linear equations appear only in context-bound form in the secondary school units. Recognition of equivalent expressions, combining like terms and symbolic manipulations are embedded in situations of fair trade, where purpose and meaning are ensured. In the case of solving iconic systems of equations the informal strategies of repeated exchange (see figure 2.1) and making new combinations (see figure 2.5) provide a natural intermediate transition phase from arith-
metric (trial-and-error strategies) to algebra (elimination of one unknown by equalizing coefficients). These early algebra activities can be formalized in a context-free, symbolic environment at a later stage.

**Problem solving: reasoning with (un)knowns (9, 10, 11)**

In arithmetic children reason with and about fixed numbers, mostly in specific, context-bound situations. High ability students may think of numbers as abstract objects, and reason about their properties. Algebraic reasoning involves variables and unknowns instead, and symbolic notation appears to be an additional cognitive obstacle for the novice learner.

Arithmetical unknowns are symbolized by dots, question marks or geometric figures (little squares or circles), or implicitly by ‘stains’ (drawn to ‘hide’ the unknown number). The unknown value can be recovered with arithmetical means like calculating in reverse order or trial-and-error strategies; the unknown is not involved in the calculations. In algebra, too, the sole purpose of the unknown is to be revealed.

But in algebraic applications the unknown is the starting-point of the solution process, in which the symbol itself is the object of manipulation. We have already mentioned before that these different approaches to problem solving – straight-to-the-point inversion in arithmetic, round-about way of constructing an equation first in algebra – can cause great difficulty. It has been found that children have trouble recognizing the structure of the problem as they try to represent the problem symbolically. They can recognize the solution procedure (for example, inverse calculation) but they cannot reason with the unknowns themselves. Moreover, the informal arithmetical approaches do not go hand in hand with algebraic methods; as they learn algebra, students tend to forget their informal knowledge and with it they lose their framework of meaning.

The historical development of algebra implies that reasoning with unknowns is found to be more natural to novice learners than symbolizing it. The experimental learning strand therefore aims to stimulate the dual development of symbolizing and reasoning with unknowns using students’ free productions wherever possible. This is done by offering students various approaches to symbolizing problem situations (tables, abbreviations, pictures, diagrams) and by letting them switch between these different forms of representation to become familiar with them. Subsequent activities provoke students to use some of these representations as tools for mathematical reasoning. We anticipate that the role of the teacher is very important in this process because students might invent notations which are not compatible with algebraic conventions.

### 2.6.2 Accesses to algebra in the Dutch arithmetic curriculum

The dual relationship between arithmetic and algebra offers various opportunities
for pre-algebraic activities in arithmetic at elementary school. For instance, a solid foundation of number sense is prerequisite for developing an understanding of number properties (algebra as generalized arithmetic). Ratio tables are a suitable setting to study simple number patterns and come to a general formulation. Thirdly, inverting operations in activities like ‘guess my number’ — where one student has a number in mind, another student names a string of operations to be carried out, the first student gives the outcome and the other student then determines the initial number — helps to prepare students for ‘undoing’ linear equations of the form $ax + b = c$ in early algebra.

In the Dutch teaching units *Wis en Reken* (Boswinkel et al., 1997) a first impulse has been given to integrate pre-algebra activities in the elementary school curriculum. Some tasks in grade 6 are based on student materials from the *Mathematics in Context* unit *Comparing Quantities*, like story problems on barter trade (substitution of trade relations) and embedded systems of equations such as in figure 2.8. The objective of these tasks is to develop reasoning strategies with which unknown quantities (in the form of concrete objects) can be manipulated and their values be recovered.

Another example of pre-algebraic activity in grade 6 is making the procedures themselves the objects of study. Students are challenged to shorten a string of operations represented by ‘calculating machines’ by combining additions and subtractions into one. For example, the additions ‘$+ 5$’ and ‘$+ 0.25$’ and the subtraction ‘$– 7$’ can be replaced by the subtraction ‘$– 1.75$’.

Two activities for students in grade 4 are based on repeating the same calculational procedures for a range of numbers. The first consists of a list of pies and their prices, and students are asked to determine the price of half a pie. The procedure is as follows: round off the price to the next whole number, divide by 2 and add one guilder. This activity can lead to a general formula for finding the price of half a pie of any kind, and perhaps it can be extended to new situations. However, the teacher guide does not mention generalization as one of the goals of the task, from which we deduce that early algebra is not an explicit part of the curriculum. The second activity we mention here deals with proportions between kites and within kites. Children measure the dimensions of a series of similar kites drawn on the work sheet. One of the kites does not fit on the page entirely, so its dimensions can only be determined by finding the ratio. The students are then asked to complete a table in which some of the measurements are given. The kite’s dimensions are represented by the letters $a$ through $e$, as shown in figure 2.7, which lead quite naturally to remarks like ‘$a$ is always twice as much as $b$’ (internal proportion). Some values in the table can only be found using the internal proportions. A formulation in general terms could be a suitable extension of the task, although presenting the letters ready-made to the students has already enervated the first step of this process. Again there is no indication of an algebraic intention in the teacher guide.
The examples above illustrate that some curriculum developers have included a few isolated activities which facilitate the development of algebraic reasoning and symbolizing, but not as an explicit learning goal. It is therefore difficult to predict to what extent teachers in elementary school currently make use of accesses to algebraic thinking in arithmetic.

2.7 Pre-algebra: on the way to algebra

In this study we have decided to restrict ‘school algebra’ to linear relationships and solving equations in one or two unknowns, in particular the transition from descriptions to (semi-)symbolic representations. The proposed learning strand corresponds most with the problem solving perspective of algebra, but it also includes generalizing and modeling activities. Although the learning strand is arithmetically inclined, it does not really fit the definition ‘algebra as generalized arithmetic’ because it does not aim to generalize number properties (commutativity, distributivity etc.). Instead, algebra and arithmetic are considered to have a dual relationship: algebra has its roots in arithmetic and depends on a strong arithmetical foundation, while arithmetic has ample opportunities for symbolizing, generalizing and algebraic reasoning.

From this perspective we propose to use the term ‘pre-algebra’ as the transition zone of informal explorative activity from arithmetic into early algebra. Pre-algebra involves algebraic thinking and informal symbolizing in an arithmetical setting, broadening and strengthening the arithmetical foundations needed for equation solving. For instance, there are indications that poor number sense and little insight in number relations can cause problems in the early learning of algebra with respect to precedence and inversion of operations.
It is therefore important to determine what makes a mathematical problem or activity algebraic, arithmetical or pre-algebraic. The differences between arithmetic and algebra in table 2.1 help to get a clearer view, suggesting opportunities for intermediate, pre-algebraic conceptions for most issues. However, we believe it is not the nature of the task but the nature of the solution method that matters. The problem in figure 2.8, taken from the first unit in the experimental learning strand, will help to clarify this idea. The picture shows two combinations of candy with two different total prices. There is no algebraic symbolism or other algebraic representation involved. But since the price of a candy bar and that of a magic ball are unknown, the drawings represent an informal system of simultaneous equations. Still, many mathematicians will probably hesitate to call it an algebraic problem.

Let us now consider a few solution methods. A student may solve the task by trial-and-error or trial-and-adjustment, substituting numerical values for each bar and magic ball. This kind of approach is of a primitive, arithmetical level. Another student might compare the two combinations and observe that changing a bar for a magic ball brings down the price by 20 cents. Two more exchanges will result in a combination with only magic balls. This kind of reasoning involves comparing known quantities and continuing the pattern; it certainly has an algebraic tendency. Yet another learner might make new combinations by adding, multiplying and/or subtracting them, until one of the unknowns is eliminated. If these combinations were written in a symbolic form, the solution method would surely be considered algebraic. In other words, the nature of the task – algebraic or otherwise – cannot be seen separate from the solution strategy applied. Similarly we can trace back the beginnings of algebra to ancient Egypt, where an algebraic perception of the unknown (treating it as a known number) accentuates algebraic method, while the problems themselves – written in words – are hardly algebraic from the modern perspective (see also section 3.3). Alternatively many problems we would nowadays solve using an algebraic method were tackled successfully with arithmetic for many centuries.
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We might even say that algebraic problems do not exist; we can only speak of algebraic methods or solutions.

Our design process should therefore be directed at creating tasks which facilitate the development of (pre-)algebraic methods, so that the cognitive break between arithmetic and algebra may be (partly) overcome. But towards what kind of algebraic competence do we aim to work with our experimental pre-algebra learning strand? And which algebraic skills and insights do we deem accessible from an arithmetical problem solving perspective? Our overview of differences between arithmetic and algebra has accentuated a number of territories which deserve special care and attention: generalizing, meaning of letters, symbolic expressions, and reasoning with unknowns. In addition to algebraic computational skills and an algebraic way of reasoning, a student should also develop an algebraic attitude. Flexible use of problem solving strategies and feeling confident to reason about unknown quantities are two characteristics of such an attitude. The process of designing the learning strand is described in chapter 5.

We also wish to find out how algebraic notation and mathematical abstraction are related. Sfard’s theory of reification is based on the idea that an operational conception of a notion precedes a structural perception:

> It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. In other words, we have good reason to expect that in the process of concept formation, operational conceptions would precede the structural (Sfard, 1991, p. 10).

Sfard conjectures that this is basically true for both the historical development and for the development of the individual learner, and gives the following historical example:

> (...) the science of computation, known today under its relatively new name ‘algebra’, has retained a distinctly operational character for thousands of years. The so-called ‘rhetorical’ algebra, which preceded the syncopated and symbolic algebras (the last developed not before the 16th century!) dealt with computational processes as such, while the only kind of abstract objects permitted in the discourse were numbers. Even most complex sequences of numerical operations were presented by help of verbal prescriptions, which bore distinctly sequential character and did not stimulate condensation and reification (Sfard, 1991, p. 23-24).

However, this point of view has also been contradicted. Radford (1997) observes that the categorization rhetoric – syncopated – symbolic is the result of our modern conception of how algebra developed, and that it is often mistaken for a gradation of mathematical abstraction. When the development of algebra is seen from a sociocultural perspective, instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations
of writing and lack of book printing quite naturally led to abbreviations and contractions of words. In other words, we would like to determine whether or not a student’s progressive formalization of notations is accompanied by a process of abstraction.

2.8 Algebra from a geometrical perspective

Even though generalized arithmetic and problem solving are more frequently taken as the starting-point for early algebra learning – in particular for simplifying symbolic expressions and solving equations – geometrical visualizations form a regular part of many algebra school books. One of the common topics in early algebra is recognizing and continuing a pattern to deduce a general formula. An example of such a task is shown in figure 2.9. Adding up the number of dots in the figures on the top row gives you the triangular numbers 1, 3, 6 and 10 ($n = 1, 2, 3$ and 4 respectively). The learner may be asked to a) continue the sequence by drawing the fifth and the sixth triangular number, and b) give a general formula to describe the $n^{th}$ triangular number.

![figure 2.9: continuing triangular numbers](image)

Geometrical visualization can help to deduce a general formula, by changing the triangular figure into a rectangular one. In the second row of figure 2.9 we have drawn the number 10 again, adding crosses to complete it to a 5 by 4 rectangle. The number of dots $N$ – which is the triangular number itself – of the $4^{th}$ triangular number is found to be $N = \frac{1}{2} \times 5 \times 4 = 10$. If we extend this method to the $n^{th}$ figure, the $n^{th}$ number can be described by the expression $N = \frac{1}{2} n (n + 1)$. As shown by this example, geometrical figures can support students in shifting from specificness to generality, while at the same time there is the arithmetical component of number relations. In other words, this kind of activity is suitable for a combined arithmetical-geometrical approach to early algebra.

And yet for this project we have chosen not to include geometrical patterns in the student materials, for two reasons. The first reason is related to content. Constructing general expressions involving varying quantities is not a deliberate part of the learning trajectory (which does not mean it cannot occur spontaneously). The core of the program involves a lower level of concept of the letter symbol, such as the letter as
unknown (in equations) and the letter as a abbreviation of an object. For instance, a trade expression like $2m = 3a + 4b$ symbolizes the act of trading 2 melons fairly for 3 apples and 4 bananas, and the letters $m$, $a$ and $b$ act as arbitrary labels for concrete objects. Second, it has been our intention to choose an approach to early algebra which can be considered new and innovative in The Netherlands. From this point of view, recognizing and continuing patterns of geometrical figures is not so appropriate because it is quite a common activity in most Dutch algebra textbooks.

**content of the experimental pre-algebra strand**

When asked to give an example where geometry can support equation solving, the area model for quadratic equations comes to mind quite readily. However, quadratic equations do not play a role in this study because they are too advanced for our target group. The emphasis of the learning strand lies on developing tools for comparing quantities: a description, a picture, a diagram or symbolism. The contexts which have been chosen do not lead naturally to a geometrical perspective. Early instructional experiments in the study and personal teaching experience have shown that schematic and symbolic representations like a table, a diagram, a pictogram and abbreviations are more accessible and more natural to pre-algebra students than a geometric shape. Beginning algebra books in the Netherlands show the same preference, using ‘calculating machines’, arrow language, pictures and symbols to visualize relations between quantities. The *Mathematics in Context* instructional units also work with tree diagrams.

The empty number line (or any other linear model) can be used as a geometric model for studying linear relations, which is also done in this study. Yet, it is a type of representation that learners do not propose themselves. Teaching practice indicates that students associate geometrical forms such as a rectangle with perimeter and area problems, and the (empty) number line with basic arithmetic.

**geometrical models not so suitable**

Research results suggest that geometric models may not be appropriate for representing relations and solving equations in one unknown. Kieran (1989) describes that students do not profit from visualizing linear equations of the form $ax + b = cx$ using rectangles. Early results in this study indicate that representing a relation such as $A = 3 + B$ using two rectangular bars is not feasible because younger students are reluctant to draw an indeterminate magnitude. These intermediate results have strengthened our decision to emphasize the arithmetical accesses to early algebra rather than the geometrical ones, although we have maintained the rectangular bar as a model in one of the student units (see also chapter 5). Another drawback of geometrical models is that their generality is limited: sometimes magnitudes and algebraic expressions do not have geometrical representations, and dimensional considerations cause restrictions.
historical development of algebra

The third reason for not taking a geometrical approach to equation solving is found in the history of algebra. From its beginning until the sixteenth century, algebra existed as an advanced form of arithmetical problem solving. With the exception of Diophantus and a few others, algebraic problems were stated and solved in natural language (the phase of rhetoric algebra, see also section 3.3). The solutions to these problems were not accompanied by any kind of explanation, and the rhetorical notation held back the development of a more generalized formulation. There was some visualization in Babylonian, Greek, Indian, and Arabic cultures, but it referred to quadratic and higher order equations, whereas for our present purposes we limit ourselves to linear problems and systems of linear equations. The integration of algebra and geometry came with Descartes in the seventeenth century, but his approach is out of range of this study because the level of symbolic algebra it requires is too high. The historical development of algebra, therefore, does not argue for a geometrical approach of solving linear equations either.

2.9 Early algebra in the Dutch curriculum

The experimental pre-algebra learning strand designed for students in grade 6 and grade 7 is intended as a series of lessons which is complementary to the national curriculum. It does not require specific pre-knowledge, nor does it replace any particular part of the early algebra strand in the regular program. We do foresee that giving more priority to informal methods and symbolism will help to prepare students for their first encounter with algebra as it is currently taught in Dutch schools. Since the national algebra curriculum has not influenced the content of the experimental teaching materials, we assume that a brief description of the early algebra program in the Netherlands will suffice.

Two decades ago, the algebra working group of the W12-16 project designed a new algebra program for the first three years in Dutch secondary schools (Algebragroep W12-16, 1990, 1991; W12-16 C.O.W., 1992), although an approach to algebra based on different types of representations is not entirely innovative because it has been suggested previously (Janvier, 1978; Goddijn, 1978). The team deliberately chose to develop algebra from a user’s perspective, for which reason important choices were made regarding mathematical content. First, the learning strand concentrates on interpreting rather than manipulating algebraic expressions. As a result algebraic techniques are subservient to studying relations and solving problems; they are not a goal in itself. Second, the problems are situated in realistic contexts as much as possible. Third, the developers emphasize the acquisition of a wide variety of techniques instead of an in depth study of only a few techniques. And finally, different algebraic concepts are developed simultaneously rather than stacking them in a linear order. Students develop algebraic conceptions and skills very gradually from
concrete situations by connecting different forms of representation: descriptions of situations, tables, graphs and formulas. Word formulas are used extensively before moving on to formal symbolic expressions because, since word formulas are situated in a context, they enable students to reason and manipulate with understanding. Variables are primarily treated as varying quantities; the concept of ‘unknown’ appears when formulas are transformed into equations or when students are asked to determine the point of intersection of two graphs. All in all the mathematical content of the program shows an integration of early algebra and early analysis based on graphical interpretation and meaning, and symbolic manipulations are kept to a minimum.

In the first three years of secondary school, equation solving is not an end goal. Equations are used to study graphical relations or to find a number value in a formula. Equation solving techniques are based on natural strategies like undoing a string of calculations or clenching in the solution by successive bisection of the interval. Formal equation solving, the construction of equations from word problems – which is a very important part of early algebra historically – has been postponed to the higher grades. Meaning and understanding form its foundations: “(...) we think that techniques dealing with ‘manipulating’ succeed directly from ‘interpreting correctly’, and that these techniques will in turn support interpreting” (W12-16 C.O.W., 1992, p. 12, transl.). For more information on the W12-16 algebra program, see also Van Reeuwijk (in press).

The algebra learning strand proposed by the working group has not been implemented nationwide because some essential ideas were not adopted into the national curriculum. Important elements like ‘growth and order of magnitude’, successive bisection and developing general solution strategies instead of specific techniques appear in the national program only sporadically. Also the proposed attention for the structure of formulas has been reduced to a minimum. The W12-16 team developed a learning strand which differentiates between high and average ability students, but since educational authorities decided on one national curriculum for all students aged 12 to 16, what has remained is no more than a diminished version.

2.10 Conclusion
The teaching and learning of school algebra has become a world wide topic of interest over the last few years. An animated discussion on what algebra is and what it should be indicates there is no consensus amongst researchers in the field, resulting in a number of different approaches to how algebra should be learned and taught in school. Still, one matter most people agree on is that students are known to struggle with the structural aspects of algebra. Especially the change from a procedural way of thinking in arithmetic to a structural perspective in algebra causes a rupture in the learner’s development.

The objective of the project is to find ways to overcome the gap between arithmetic
and algebra. We attempt to break through the ‘vicious circle’ of interiorisation and reification (Sfard, 1991) by a connected development of skills and concepts. Students will be guided to develop an informal, pre-algebraic concept of problem solving (arithmetical methods) first, followed by pre-algebraic skills (symbolizing, reasoning), to end with formalizing their skills to a level of algebraic conception (equation solving).