Decay of spin-polarized atomic hydrogen in the presence of a Bose condensate

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We study the decay of magnetically trapped doubly spin-polarized atomic hydrogen in the density and temperature regime of the Bose-Einstein phase transition and calculate the temperature and magnetic field dependence of all relevant relaxation rate constants $G_{dd\rightarrow\kappa\lambda}$. At the transition temperature T_c the rate constants have a discontinuous second derivative with respect to T. Below T_c they show a pronounced decrease relative to the extrapolated above-T_c values, the observation of which is a clear signal for the experimental realization of Bose-Einstein condensation.

I. INTRODUCTION

The possibility of trapping and subsequent cooling of neutral atoms down to temperatures in the range of $1-100\ \mu$K has given rise to the opportunity, at least in principle, to study the behavior of ultracold gases in the low-temperature and high-density regime where collective quantum effects due to degeneracy are expected to show up. The first experiments in this context used sodium atoms and a magnetic[1] or optical[2] trap. However, in the cases the achieved densities were far too low to observe any deviations from classical statistics.

From the point of view of achieving the Bose-Einstein transition, magnetically trapped doubly spin-polarized atomic hydrogen ($H\uparrow\Uparrow$) seems to be much more promising,[3] primarily since its atomic mass is much smaller and the attractive well due to the van der Waals interaction between the atoms is very shallow. At temperatures of the order of 10 μK the required densities are so low that only two-body collisions affect the decay of the gas. Hence the three-body recombination process which dominates in the compression experiments with $H\downarrow\Downarrow$ in liquid-helium-covered sample cells[4,5] is avoided.

Recently Hess *et al.*[6] and van Roijen *et al.*[7] reported on the successful trapping of hydrogen atoms at temperatures of about 1 and 100 mK, respectively. A very good quantitative agreement was found with our theoretical predictions[8] for the decay of the gas, both with respect to the magnitude[6] and the magnetic field and temperature dependence.[7] However, these calculations assume a Maxwell-Boltzmann velocity distribution in the gas and thus apply only to the nondegenerate regime in which the above-mentioned experiments are carried out. In this paper we will relax this condition and study the dominant relaxation processes in $H\uparrow\Uparrow$ in a wide temperature range and especially at temperatures below the critical temperature T_c.

To this end we consider in Sec. II the eigenstates of N weakly interacting hydrogen atoms with (electron and proton) spin degrees of freedom by an extension of the ξ method developed by Lee and Yang.[9] We derive the free energy of the system from which all thermodynamical equilibrium quantities, and in particular the condensate fraction for the $|\uparrow\Uparrow\rangle$ spin state, can be found. In Sec. III we use this formalism to calculate all electron-electron dipolar decay rates $G_{dd\rightarrow\kappa\lambda}$ (the hyperfine states of the 1s ground state of atomic hydrogen are denoted by $|a\rangle$, $|b\rangle$, $|c\rangle$, and $|d\rangle$ in order of increasing energy) as a function of T and applied magnetic field B and compare with results obtained earlier. We will also show that the dominant temperature dependence for sufficiently exothermic reactions can be found by an extension of the correlator discussed by Kagan *et al.*[10] to the case of two-body relaxation processes. Section IV will give some concluding remarks.

II. ATOMIC HYDROGEN AS A WEAKLY INTERACTING BOSE GAS

We consider a homogeneous system of N hydrogen atoms with internal degrees of freedom corresponding to the hyperfine states $|\alpha\rangle$ and the energy eigenvalues ε_α. In addition, we use a quantization volume V and periodic boundary condition to facilitate the treatment of the condensate. Our final results will be presented in the thermodynamical limit ($N, V\rightarrow\infty$ and $n=N/V$ constant). The normalized and discrete one-atom states with momentum $\hbar\mathbf{k}$ are denoted by $|\mathbf{k}\alpha\rangle$ and have an energy

$$\varepsilon^0_{\mathbf{k}\alpha}=\frac{\hbar^2\mathbf{k}^2}{2m_H}+\varepsilon_\alpha\ , \tag{1}$$

where m_H is the mass of the hydrogen atom.

The phenomenon of Bose-Einstein condensation in the case of magnetically trapped atomic hydrogen is associated with a macroscopic occupation of the state $|0d\rangle$. Although $|0d\rangle$ is not the one-atom ground state the condensation occurs into this state, because of a spontaneously developing nonequilibrium situation in spin space: Only the doubly polarized $|d\rangle=|\uparrow\Uparrow\rangle$ state remains populated, since the "high-field seeking" a and b atoms are not trapped in a minimum B field, while the "low-field seeking" c atoms are rapidly removed from the trap due to exchange relaxation processes.[11,8]

The details of the macroscopic occupation depend to a great extent on the interaction between the particles. In the following we show that the deviations from the ideal Bose gas are characterized by the parameters $(na^3)^{1/4}$, $(na\Lambda^2)^{1/2}$ and their ratio $(a/n\Lambda^4)^{1/4}$. Here a is the (triplet) scattering length which characterizes the strength of the interaction and Λ denotes the thermal de Broglie wavelength

$$\Lambda(T)=\left[\frac{2\pi\hbar^2}{m_H k_B T}\right]^{1/2}, \tag{2}$$

with k_B Boltzmann's constant. For atomic hydrogen with a density of 10^{14} cm^{-3} and a temperature equal to the critical temperature $T_c \simeq 34\ \mu$K these parameters are small. We therefore neglect first and higher orders in these parameters but take all orders of the degeneracy parameter $n\Lambda^3$ into account, since we want to consider the regime where Bose-Einstein condensation takes place ($n\Lambda^3 \simeq 1$).

The decay of the fully (doubly) polarized gas is due to the weak electron-electron dipolar interaction, which can be treated as a perturbation. This section is concerned with the associated "zeroth-order" problem, in which the central (singlet or triplet) interaction is taken into account. In Sec. III we then turn to the influence of the dipolar interaction.

A. Hamiltonian and eigenstates

At the low temperatures ($a/\Lambda \ll 1$) in which we are interested, it is permitted to use the pseudopotential method[12] and replace the real potential $V_{\alpha'\beta',\alpha\beta}(r)$, which depends on the internal states involved in the interaction, by a pseudopotential $v_{\alpha'\beta',\alpha\beta}\delta(\mathbf{r})$ with the same scattering properties for low energies. This requires $v_{\alpha'\beta',\alpha\beta}$ to be equal to an (on-shell) two-body T-matrix[13] element at zero energy, which in turn can be expressed in singlet and triplet scattering lengths $a^{(S)}$ if we use the very accurate "degenerate-internal-states" approximation.[8] Denoting the projection operators on the part of spin space with total electron spin equal to S by $\mathcal{P}^{(S)}$, the result is given by

$$v_{\alpha'\beta',\alpha\beta}=\frac{4\pi\hbar^2}{m_H}\left\langle\alpha'\beta'\left|\sum_S a^{(S)}\mathcal{P}^{(S)}\right|\alpha\beta\right\rangle . \tag{3}$$

In particular, the important quantity $v_{dd,dd}$ is equal to $4\pi\hbar^2 a^{(1)}/m_H \equiv 4\pi\hbar^2 a/m_H$.

In second quantization and with the convenient notation $v^{id}_{\alpha'\beta',\alpha\beta}=v_{\alpha'\beta',\alpha\beta}+v_{\alpha'\beta',\beta\alpha}$ the Hamiltonian of the system takes the form

$$H=\sum_\alpha\sum_{\mathbf{k}}\varepsilon^0_{\mathbf{k}\alpha}a^\dagger_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha} +\frac{1}{4V}\sum_{\alpha\beta\alpha'\beta'}v^{id}_{\alpha'\beta',\alpha\beta}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}}a^\dagger_{\mathbf{k}+\mathbf{q}\alpha'}a^\dagger_{\mathbf{k}'-\mathbf{q}\beta}a_{\mathbf{k}\alpha}a_{\mathbf{k}'\beta} , \tag{4}$$

with the creation (annihilation) operators $a^\dagger_{k\alpha}$ ($a_{\mathbf{k}\alpha}$) obeying the usual commutation relations. Following the discussion by Lee and Yang[9] for a population of the $|\mathbf{0}d\rangle$ state by a finite fraction ξ of the total number of particles N, we consider a free state $|\{N^0_{\mathbf{k}\alpha}\}\rangle$ with a set of occupation numbers such that

$$N^0_{0d}=\xi N ,$$
$$N^0_{\mathbf{k}d}=O\left[\frac{N^{1/3}(na^3)^{1/6}}{na\Lambda^2}\right] \text{ if } \mathbf{k}\neq\mathbf{0} , \tag{5}$$
$$N^0_{\mathbf{k}\alpha}=O(1) \text{ if } \alpha\neq d ,$$

and

$$\sum_{\mathbf{k}\neq\mathbf{0}}N^0_{\mathbf{k}d}+\sum_{\alpha\neq d}\sum_{\mathbf{k}}N^0_{\mathbf{k}\alpha}=(1-\xi)N . \tag{6}$$

The last line of Eq. (5) reflects the nonequilibrium situation in spin space as mentioned previously: A gas of d atoms can only decay by dipolar relaxation and produce atoms in an $|a\rangle$, $|b\rangle$, or $|c\rangle$ state, which are then rapidly removed from the trapping region.

Including the interaction the free state $|\{N^0_{\mathbf{k}\alpha}\}\rangle$ will evolve into a state of $\{N^0_{\mathbf{k}\alpha}\}$ quasiparticles which can be seen as a superposition of free states with occupation numbers $N_{\mathbf{k}\alpha}$ differing from $N^0_{\mathbf{k}\alpha}$ by $\delta_{\mathbf{k}\alpha}$:

$$N_{\mathbf{k}\alpha}=N^0_{\mathbf{k}\alpha}+\delta_{\mathbf{k}\alpha} . \tag{7}$$

A posteriori it is possible to show that the expectation value of $\delta_{\mathbf{k}\alpha}$ in the resulting state of quasiparticles obeys

$$\langle\delta_{\mathbf{k}d}\rangle=O\left[\frac{N^{2/3}(na^3)^{1/3}}{na\Lambda^2}\right] \text{ if } \mathbf{k}\neq\mathbf{0} ,$$
$$\langle\delta_{\mathbf{k}\alpha}\rangle=0 \text{ if } \alpha\neq d , \tag{8}$$

while the fluctuations of the condensate fraction satisfy

$$\left\langle\left[\frac{1}{N}\sum_{\mathbf{k}\neq\mathbf{0}}\delta_{\mathbf{k}d}\right]^2\right\rangle=O\left[\frac{(na^3)^{1/2}}{na\Lambda^2}\right] . \tag{9}$$

From these estimates and the selection rule $\Delta M_F=0$ for the central interaction, implying that

$$v^{id}_{dd,\alpha\beta}=v^{id}_{\alpha\beta,dd}=v^{id}_{dd,dd}\delta_{\alpha d}\delta_{\beta d}$$

and

$$v^{id}_{\alpha d,\beta d}=v^{id}_{\alpha d,\alpha d}\delta_{\alpha\beta} ,$$

we find that the only terms in the Hamiltonian contributing in the thermodynamical limit are

$$H=\sum_\alpha\sum_{\mathbf{k}}\varepsilon^0_{\mathbf{k}\alpha}a^\dagger_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+\frac{v^{id}_{dd,dd}}{4V}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}}a^\dagger_{\mathbf{k}+\mathbf{q}d}a^\dagger_{\mathbf{k}'-\mathbf{q}d}a_{\mathbf{k}d}a_{\mathbf{k}'d} +\sum_{\alpha\neq d}\frac{v^{id}_{\alpha d,\alpha d}}{4V}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}}a^\dagger_{\mathbf{k}+\mathbf{q}\alpha}a^\dagger_{\mathbf{k}+\mathbf{q}d}a_{\mathbf{k}\alpha}a_{\mathbf{k}'d} . \tag{10}$$

In the nondiagonal part of this Hamiltonian, coupling the single-particle states, we neglect all but the terms of highest order in the number of condensate particles and in view of Eq. (9) also the fluctuations of the condensate fraction. This leads to

$$H^{\rm ND}\simeq\frac{\xi n}{4}v^{id}_{dd,dd}\sum_{\mathbf{k}\neq\mathbf{0}}(a^\dagger_{\mathbf{k}d}a^\dagger_{-\mathbf{k}d}+a_{\mathbf{k}d}a_{-\mathbf{k}d}) . \tag{11}$$

The diagonal part is rewritten in terms of the occupation numbers $N_{\mathbf{k}\alpha}$ with the result

$$H^D=\frac{v^{id}_{dd,dd}}{4V}\left[2\left[\sum_{\mathbf{k}}N_{\mathbf{k}d}\right]^2-\sum_{\mathbf{k}}N^2_{\mathbf{k}d}-\sum_{\mathbf{k}}N_{\mathbf{k}d}\right]+\sum_{\alpha\neq d}\frac{v^{id}_{\alpha d,\alpha d}}{V}\sum_{\mathbf{k}}\left[\sum_{\mathbf{k}'}N_{\mathbf{k}'d}\right]N_{\mathbf{k}\alpha}\ . \tag{12}$$

The last term reduces in thermodynamical limit to the Hartree-Fock correction to the energy of the α atoms due to the interaction with the total set of d atoms, i.e.,

$$\sum_{\alpha\neq d}\sum_{\mathbf{k}} n v^{id}_{\alpha d,\alpha d}N_{\mathbf{k}\alpha}\ .$$

Furthermore, using Eqs. (5)–(7) the remaining part of H^D can be written as

$$\frac{v^{id}_{dd,dd}}{4V}\left[N^2[1+(1-\xi)^2]+2\xi N\sum_{\mathbf{k}\neq 0}N_{\mathbf{k}d}-2(2-\xi)N\sum_{\alpha\neq d}\sum_{\mathbf{k}}N_{\mathbf{k}\alpha}\right]+W\ ,$$

where W equals

$$\frac{v^{id}_{dd,dd}}{4V}\left[-N-\sum_{\mathbf{k}\neq 0}N^2_{\mathbf{k}d}-\left[\sum_{\mathbf{k}\neq 0}\delta_{\mathbf{k}d}\right]^2+\sum_{\alpha\neq d}\sum_{\mathbf{k}}N_{\mathbf{k}\alpha}+2\left[\sum_{\alpha\neq d}\sum_{\mathbf{k}}N_{\mathbf{k}\alpha}\right]^2\right]\ .$$

This contribution can be neglected since explicit calculation shows that in addition to Eq. (9) we have

$$\sum_{\mathbf{k}\neq 0}\langle N^2_{\mathbf{k}d}\rangle=O\left[\frac{N^{4/3}(na^3)^{2/3}}{(na\Lambda^2)^2}\right]\ . \tag{13}$$

Therefore, in the thermodynamical limit, W leads to negligible corrections of order $N(na^3)^{1/2}k_BT$ to the energy eigenvalues (and thus the free energy).

Collecting the remaining results, we find that the Hamiltonian of the gas can be approximated to a high degree of accuracy by

$$H\simeq N[1+(1-\xi)^2]\frac{nv^{id}_{dd,dd}}{4}+\sum_{\mathbf{k}\neq 0}\left[\varepsilon^0_{\mathbf{k}d}+\frac{\xi}{2}nv^{id}_{dd,dd}\right]a^\dagger_{\mathbf{k}d}a_{\mathbf{k}d}+\frac{\xi}{4}nv^{id}_{dd,dd}\sum_{\mathbf{k}\neq 0}\left[a^\dagger_{\mathbf{k}d}a^\dagger_{-\mathbf{k}d}+a_{\mathbf{k}d}a_{-\mathbf{k}d}\right]+\sum_{\alpha\neq d}\sum_{\mathbf{k}}[\varepsilon^0_{\mathbf{k}\alpha}+nv^{id}_{\alpha d,\alpha d}-(1-\xi/2)nv^{id}_{dd,dd}]a^\dagger_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}\ , \tag{14}$$

where we have taken ε^0_{0d} as our zero-point of energy. This Hamiltonian can be diagonalized by means of a Bogoliubov transformation[14] of the operators $a^\dagger_{\mathbf{k}d}$ and $a_{\mathbf{k}d}$ to creation ($b^\dagger_{\mathbf{k}d}$) and annihilation ($b_{-\mathbf{k}d}$) operators of $|d\rangle$-state quasiparticles. To lowest order in $(na^3\xi^5)^{1/2}$ the result is written as

$$H\equiv N[1+(1-\xi)^2]\frac{nv^{id}_{dd,dd}}{4}+\sum_{\mathbf{k}\neq 0}\varepsilon_{\mathbf{k}d}b^\dagger_{\mathbf{k}d}b_{\mathbf{k}d}+\sum_{\alpha\neq d}\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}\alpha}a^\dagger_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}\ , \tag{15}$$

i.e., a Hamiltonian for a system of independent (quasi)particles with quasiparticle energies

$$\varepsilon_{\mathbf{k}d}=\varepsilon^0_{\mathbf{k}d}\left[1+\frac{n\xi v^{id}_{dd,dd}}{\varepsilon^o_{\mathbf{k}d}}\right]^{1/2}\ . \tag{16}$$

We remind the reader of the property that the dispersion relation is phononlike,

$$\varepsilon_{\mathbf{k}d}\simeq\hbar k(n\xi v^{id}_{dd,dd}/2m_H)^{1/2}\ ,$$

for values of k small compared to $(na)^{1/2}$ and particlelike,

$$\varepsilon_{\mathbf{k}d}\simeq\varepsilon^0_{\mathbf{k}d}+n\xi v^{id}_{dd,dd}/2\ ,$$

for $k\gg(na)^{1/2}$.

In the case of atomic hydrogen it is important to note that the densities and temperatures near the Bose-Einstein transition are such that $na\Lambda^2$ is a small quantity. This means that the relevant thermal momenta are always large compared to $\hbar(na)^{1/2}$. Therefore the dispersion relation is well approximated by its particlelike limit. This approximation is identical with a Hartree-Fock or mean-field treatment of the gas, taking the macroscopic occupation of the one-atom $|0d\rangle$ state into account.[15] Notwithstanding the validity of the particlelike limit in our circumstances, it is of interest to impose this limitation only in the final stage of calculating rate constants in Sec. II B: We then find more explicitly the region of applicability of the results.

B. Thermal average

The previous derivation allows us to calculate the transition probabilities Γ for the relaxation processes. To find the rate constants we need to perform a thermal average over the possible initial states, without $\alpha\neq d$ atoms, at a certain temperature. In principle this would imply a summation of the form

$$\frac{1}{Z}\sum_{\xi=0(1/N)1}\sum_{\{N^0_{\mathbf{k}d}\}}{}' e^{-\beta E(\xi,\{N^0_{\mathbf{k}d}\})}\Gamma(\xi,\{N^0_{\mathbf{k}d}\})\ , \tag{17}$$

with $\beta=(k_BT)^{-1}$, $E(\xi,\{N^0_{\mathbf{k}d}\})\equiv E_0(\xi)+E_{qp}(\xi,\{N^0_{\mathbf{k}d}\})$ the eigenvalues of the Hamiltonian (15) and the sum over states Z as a normalization factor. The prime in the second summation sign indicates that the summations over ξ and $\{N^0_{\mathbf{k}d}\}$ are not independent because the occupation numbers $N^0_{\mathbf{k}d}$ correspond not only to the number of quasiparticles with momentum $\hbar\mathbf{k}$, but also to the number of (real) particles with the same momentum from which this state evolved by inclusion of the interaction. Therefore they must satisfy the condition

$$N_{qp} \equiv \sum_{\mathbf{k}\neq 0} N^0_{\mathbf{k}d} = (1-\xi)N \tag{18}$$

for the total number of quasiparticles.

It turns out that thermal averages of the type (17) are conveniently carried out by deleting the subsidiary condition using a grand canonical ensemble with an associated chemical potential $\mu(\xi)$ and by limiting oneself to one "most probable" value $\bar{\xi}$ to ξ, depending on temperature.[9,16]

In the thermodynamical limit $\mu(\xi)$ and the corresponding fugacity $\zeta(\xi) \equiv e^{\beta\mu(\xi)}$ follow from

$$(1-\xi)n = \frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{1}{\zeta^{-1}(\xi)e^{\beta\epsilon_{\mathbf{k}d}(\xi)}-1}, \tag{19}$$

while $\bar{\xi}$ is the value of ξ for which the free energy per particle

$$f(\xi,\zeta(\xi)) = \frac{E_0(\xi)}{N} + \frac{k_BT}{(2\pi)^3}\frac{1}{n}\int d\mathbf{k}\ln[1-\zeta(\xi)e^{-\beta\epsilon_{\mathbf{k}d}(\xi)}] + (1-\xi)\ln\zeta(\xi) . \tag{20}$$

is minimal.

Since Eq. (19) is equivalent to $\partial f(\xi,\zeta)/\partial\zeta = 0$, we have

$$\frac{df(\xi,\zeta(\xi))}{d\xi} = \left.\frac{\partial f(\xi,\zeta)}{\partial \xi}\right|_\zeta . \tag{21}$$

Therefore, using the expressions in Ref. 9 and expanding both Eqs. (19) and (21) in powers of $(na\Lambda^2\xi)^{1/2}$ we obtain the coupled set of equations

$$\begin{aligned} (1-\xi)n &= \Lambda^{-3}[g_{3/2}(\zeta) + O((na\Lambda^2\xi)^{1/2})] , \\ \frac{df(\xi,\zeta)}{d\xi} &= \Lambda^{-2}[-4\pi\ln(\zeta) - O((na^3\xi)^{1/2})] , \end{aligned} \tag{22}$$

from which we can deduce $\bar{\xi}$ and $\bar{\zeta} \equiv \zeta(\bar{\xi})$. In Eq. (22) we made use of the well-known Bose functions $g_n(\zeta)$ (Ref. 17) defined by

$$g_n(\zeta) = \sum_{i=1}^{\infty}\frac{\zeta^i}{i^n} = \frac{1}{\Gamma(n)}\int_0^\infty dy\frac{y^{n-1}}{\zeta^{-1}e^y - 1} . \tag{23}$$

Especially those with $n = 1, \frac{3}{2}, 2$, and $\frac{5}{2}$ play an important role in Sec. III and are shown in Fig. 1.

The solution of Eq. (22) is fundamentally different below and above a critical temperature T_c, deviating only slightly (for atomic hydrogen $\Lambda \simeq a$ would correspond to a temperature of about 600 K) from the critical temperature T_0 of an ideal Bose gas:

$$\begin{aligned} T_c &= T_0\left[1 + O\left[\frac{a}{\Lambda}\right]\right] , \\ T_0 &= \frac{2\pi\hbar^2}{m_H k_B}\left[\frac{n}{g_{3/2}(1)}\right]^{2/3} = \frac{2\pi\hbar^2}{m_H k_B}\left[\frac{n}{2.612}\right]^{2/3} . \end{aligned} \tag{24}$$

In the nondegenerate case $T > T_c$ the derivative $df/d\xi$ is always greater than zero and thus $f(\xi,\zeta(\xi))$ has a minimum on the ζ axis, i.e.,

$$\begin{aligned} \bar{\xi} &= 0 , \\ n\Lambda^3 &= g_{3/2}(\bar{\zeta}) . \end{aligned} \tag{25}$$

However, in the degenerate case $T < T_c$ a solution of $df/d\xi = 0$ exists and $\bar{\xi}$ becomes different from zero. It turns out that $\bar{\xi}$ and $\bar{\zeta}$ now obey

$$\begin{aligned} (1-\bar{\xi})n &= \Lambda^{-3}[g_{3/2}(1) - O((na^3\bar{\xi})^{1/4}) + O((na\Lambda^2\bar{\xi})^{1/2})] , \\ \bar{\zeta} &= 1 - O((na^3\bar{\xi})^{1/2}) , \end{aligned} \tag{26}$$

because

$$g_{3/2}(\zeta) \underset{\zeta\to 1}{\sim} g_{3/2}(1) - 2\pi^{1/2}(1-\zeta)^{1/2} .$$

(Refs. 17 and 18).

Omitting the higher-order terms the solution of Eqs. (25) and (26) is shown in Fig. 2, in which we present $\bar{\xi}$ and $\bar{\zeta}$ as functions of T/T_c. In particular, we recover the well-known (ideal Bose gas) power law for the number of particles in the condensate:

$$N_{0d} = \bar{\xi}N\left[1 - \left[\frac{T}{T_c}\right]^{3/2}\right] . \tag{27}$$

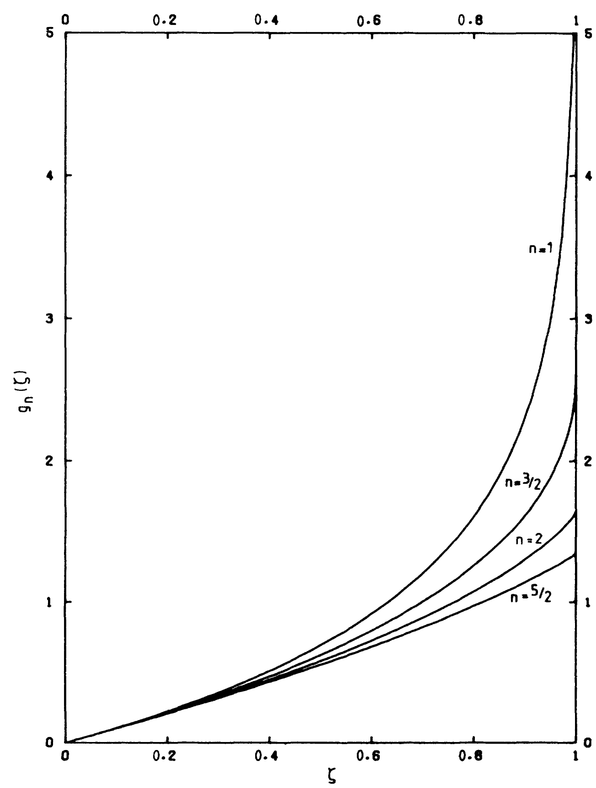

FIG. 1. Bose functions $g_n(\zeta)$ for $n = 1, \frac{3}{2}, 2$, and $\frac{5}{2}$.

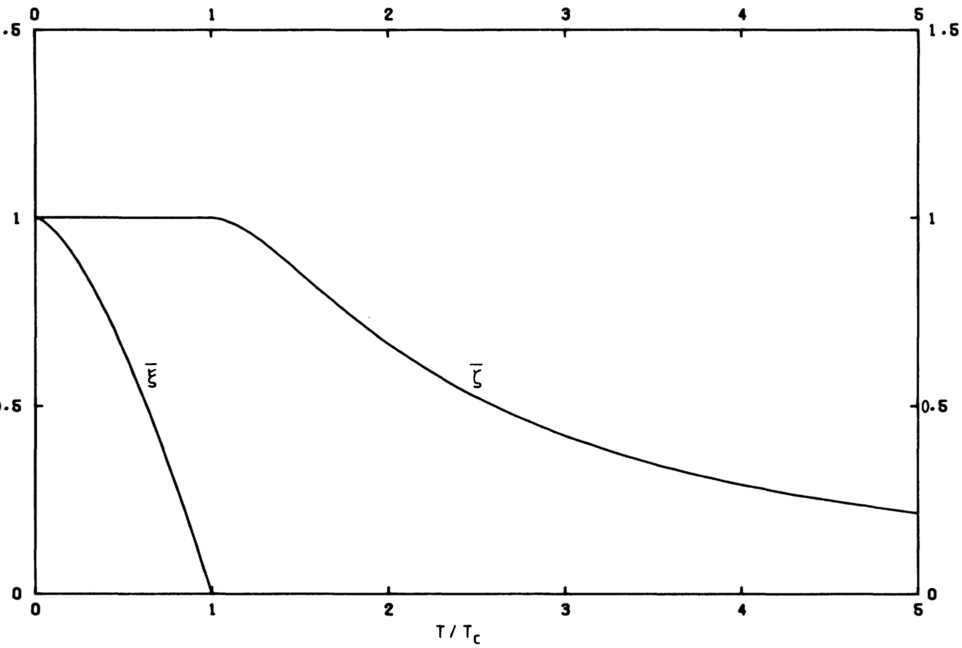

FIG. 2. Condensate fraction $\bar{\xi}$ and fugacity $\bar{\zeta}$ as functions of temperature.

Apparently, we find that a Bose gas behaves almost ideally if both $(na^3)^{1/4} \ll 1$ and $(na\Lambda^2)^{1/2} \ll 1$. However, as Eq. (9) explicitly shows, also the ratio of these parameters must be small compared to 1, for the ξ method to be self-consistent and the fluctuations of the condensate fraction to be negligible. This additional condition, which is not mentioned in Ref. 9, strongly restricts the region of applicability, but is still satisfied in the hydrogen case. Note that the smallness of $(a/n\Lambda^4)^{1/4}$ together with $(na^3)^{1/4} \ll 1$ implies $a/\Lambda \ll 1$.

The main conclusion of this section is that the thermal averages of the decay probabilities of interest can be carried out conveniently with a fixed-ξ $(=\bar{\xi})$ grand canonical ensemble. Although the summation over ξ has thus disappeared from Eq. (17), it is of importance to stress that this refers only to the initial states. A summation over final-state ξ values still occurs in the expression for the transition probability Γ. We come back to this in Sec. III.

III. DIPOLAR RELAXATION IN H↑‡

The dominant decay mechanism in H↑‡ is due to the magnetic dipolar interaction between the electron spins of the hydrogen atoms. This interaction is not invariant under spatial rotations and can be written as[8,19]

$$V^d(\mathbf{r}) = -\frac{\mu_0\mu_e^2}{4\pi r^3}\left[\frac{4\pi}{5}\right]^{1/2}\sum_{\mu} Y_{2\mu}^*(\hat{\mathbf{r}})\Sigma_{2\mu}^{ee} . \quad (28)$$

In terms of the total electron and proton spin states $|SM_SIM_I\rangle$ the matrix elements of the tensor operator Σ_2^{ee} of rank 2 are proportional to a Clebsch-Gordan coefficient

$$\langle S'M_{S'}I'M_{I'}|\Sigma_{2\mu}^{ee}|SM_SIM_I\rangle = 2\sqrt{10}(SM_S2\mu|S'M_{S'})\delta_{S,1}\delta_{S',1}\delta_{I,I'}\delta_{M_I,M_{I'}} , \quad (29)$$

from which the matrix elements in the basis $\{|\alpha\beta\rangle\}$ of hyperfine states can easily calculated. We also need the Fourier transform of the spin matrix elements of $V^d(\mathbf{r})$:

$$V_{\alpha'\beta',\alpha\beta}^d(\mathbf{q}) = \frac{\mu_0\mu_e^2}{3}\left[\frac{4\pi}{5}\right]^{1/2}\sum_{\mu} Y_{2\mu}^*(\hat{\mathbf{q}})\langle\alpha'\beta'|\Sigma_{2\mu}^{ee}|\alpha\beta\rangle , \quad (30)$$

where the summation over μ reduces to only one term with μ equal to ΔM_F, i.e., the change of the total (electron) spin projection on the direction of the magnetic field.

As mentioned previously we treat the effect of the weak dipole interaction using first-order perturbation theory. The probability (per second) for a transition from a $|\xi_i,\{N_{\mathbf{k}d}\}_i\rangle$ initial state of the doubly polarized gas to a final state $|\xi_f,\{N_{\mathbf{k}\alpha}\}_f\rangle$ with only one or two atoms having an internal state different from $|d\rangle$ obeys Fermi's golden rule,

$$\Gamma_{dd\to\kappa\lambda}(\xi_i,\{N_{\mathbf{k}d}\}_i) = \frac{2\pi}{\hbar}\sum_{\xi_f}\sum_{\{N_{\mathbf{k}\alpha}\}_f} \text{“}\delta\text{”}(E(\xi_f,\{N_{\mathbf{k}\alpha}\}_f) - E(\xi_i,\{N_{\mathbf{k}d}\}_i))\left|\left\langle\xi_f,\{N_{\mathbf{k}\alpha}\}_f\left|\sum_{i<j}V_{ij}^d\right|\xi_i,\{N_{\mathbf{k}d}\}_i\right\rangle\right|^2 . \quad (31)$$

The quotation marks indicate that the energy-conserving function "δ" becomes equal to Dirac's δ function only in the thermodynamical limit. The rate constants $G_{dd\to\kappa\lambda}(B,T)$ are found by thermal averaging over all possible initial states. As explained in Sec. II B this leads to

$$G_{dd\to\kappa\lambda}(B,T)=\frac{1}{n^2V}\sum_{\{N_{\mathbf{k}d}\}_i}\Gamma_{dd\to\kappa\lambda}(\bar{\xi}_i,\{N_{\mathbf{k}d}\}_i)\frac{e^{-\beta[E(\bar{\xi}_i,\{N_{\mathbf{k}d}\}_i)-\mu(\bar{\xi}_i)N_{qp}(\{N_{\mathbf{k}d}\}_i)]}}{Z_{gr}(\bar{\xi}_i)} . \tag{32}$$

Note the different roles played by ξ_i and ξ_f: although we restrict ourselves to one ξ_i, ξ_f can still have various values, depending on the number of condensate particles participating in the collision.

We proceed to calculate $G_{dd\to\kappa\lambda}$ for all five possible processes that are allowed by the selection rules [cf. Eq. (29)] of the electron-electron dipole interaction: $dd\to aa$, $dd\to ac$, $dd\to cc$, $dd\to ad$, and $dd\to cd$. Denoting the magnetic-field-dependent change of internal energy in the reaction $dd\to\kappa\lambda$ by $\Delta_{\kappa\lambda}$ we will give in Sec. III A an expansion of $G_{dd\to\kappa\lambda}$ in terms of $k_BT/\Delta_{\kappa\lambda}$, which for most practical purposes can be considered as a small quantity. However, near the center of the trap the magnetic field strength is of the order of a few gauss, leading to small energy splittings $\Delta_{cd}\simeq\mu_eB$ and $\Delta_{cc}\simeq 2\mu_eB$. With this application in mind it is also interesting to look at small $\Delta_{\kappa\lambda}$ and even at the extreme case $\Delta_{\kappa\lambda}=0$. To this end we consider in Sec. III B the transitions $dd\to cc$ and $dd\to cd$ at zero magnetic field.

A. The case $\Delta_{\kappa\lambda}\gg k_BT$

To evaluate the quantities $\Gamma_{dd\to\kappa\lambda}$ and $G_{dd\to\kappa\lambda}$ we will apply in this and Sec. III B the particlelike (Hartree-Fock) approximation to the dispersion relation of the quasiparticles,

$$\varepsilon_{\mathbf{k}d}\simeq\varepsilon^0_{\mathbf{k}d}+\frac{n\xi v^{id}_{dd,dd}}{2}=\varepsilon^0_{\mathbf{k}d}+k_BT\,O(na\Lambda^2\xi) , \tag{33}$$

leading to a trivial Bogoliubov transformation: The quasiparticle operators $b^\dagger_{\mathbf{k}d}$ and $b_{-\mathbf{k}d}$ are equalt to $a^\dagger_{\mathbf{k}d}$ and $a_{-\mathbf{k}d}$, respectively. Furthermore, in the nondiagonal matrix element of Eq. (31) we again neglect the fluctuations of the condensate fraction, which amounts to the replacement of $a^\dagger_{0d}$ and a_{0d} by the c number $\sqrt{\xi N}$.

We then find, for a specific process $dd\to\kappa\lambda$ ($\kappa\neq d$),

$$\begin{aligned}\Gamma_{dd\to\kappa\lambda}(\xi,\{N_{\mathbf{k}d}\})=\frac{2\pi}{\hbar}\Bigg\{&\frac{\xi^2N^2}{V^2}\frac{1}{(1+\delta_{\kappa\lambda})}\sum_{\mathbf{q}}|V^d_{\kappa\lambda,dd}(\mathbf{q})|^2(N_{-\mathbf{q}d}\delta_{\lambda d}+1)\text{“}\delta\text{”}(\Delta^{(2)}_{\kappa\lambda}E)\\&+\frac{\xi N}{V^2}\frac{1}{(1+\delta_{\kappa\lambda})}\sum_{\mathbf{k}\neq0}\sum_{\mathbf{q}}|V^d_{\kappa\lambda,dd}(\mathbf{q})+V^d_{\kappa\lambda,dd}(\mathbf{q}+\mathbf{k})|^2(N_{-\mathbf{q}d}\delta_{\lambda d}+1)N_{\mathbf{k}d}\text{“}\delta\text{”}(\Delta^{(1)}_{\kappa\lambda}E)\\&+\frac{1}{V^2}\frac{1}{2(1+\delta_{\kappa\lambda})}\sum_{\mathbf{k},\mathbf{k}'\neq0}\sum_{\mathbf{q}}|V^d_{\kappa\lambda,dd}(\mathbf{q})+V^d_{\kappa\lambda,dd}(\mathbf{q}+\mathbf{k}-\mathbf{k}')|^2\\&\qquad\times(N_{\mathbf{k}'-\mathbf{q}d}\delta_{\lambda d}+1)N_{\mathbf{k}d}N_{\mathbf{k}'d}\text{“}\delta\text{”}(\Delta^{(0)}_{\kappa\lambda}E)\Bigg\} ,\end{aligned} \tag{34}$$

where the factors $(1+\delta_{\kappa\lambda})^{-1}$ and $[2(1+\delta_{\kappa\lambda})]^{-1}$ assure that the incoherent summations over final momenta are effectively over distinguishable states. The three contributions to $\Gamma_{dd\to\kappa\lambda}$ correspond to a collision between two condensate particles, between one condensate and one noncondensate particle, and between two noncondensate particles. The values of ξ_f involved are equal to ξ_i-2/N, ξ_i-1/N, and ξ_i, respectively. Furthermore, the energy differences are

$$\begin{aligned}\Delta^{(2)}_{\kappa\lambda}E&=\frac{\hbar^2\mathbf{q}^2}{m_H}-\Delta_{\kappa\lambda}+k_BT\,O(na\Lambda^2) ,\\ \Delta^{(1)}_{\kappa\lambda}E&=\frac{\hbar^2(\mathbf{k}+\mathbf{q})^2}{2m_H}+\frac{\hbar^2\mathbf{q}^2}{2m_H}-\frac{\hbar^2\mathbf{k}^2}{2m_H}-\Delta_{\kappa\lambda}+k_BT\,O(na\Lambda^2) ,\\ \Delta^{(0)}_{\kappa\lambda}E&=\frac{\hbar^2(\mathbf{k}+\mathbf{q})^2}{2m_H}+\frac{\hbar^2(\mathbf{k}'-\mathbf{q})^2}{2m_H}-\frac{\hbar^2\mathbf{k}^2}{2m_H}-\frac{\hbar^2\mathbf{k}'^2}{2m_H}-\Delta_{\kappa\lambda}+k_BT\,O(na\Lambda^2) .\end{aligned} \tag{35}$$

To obtain Eqs. (34) and (35) we need the amount of energy required to remove a particle from the condensate, which is equal to

$$E\left[\xi-\frac{1}{N},\{N_{\mathbf{k}\alpha}\}\right]-E(\xi,\{N_{\mathbf{k}\alpha}\}) .$$

Using the particlelike limit of the dispersion relation, this difference is calculated to be

$$-\frac{1}{N}\frac{d}{d\xi}E(\xi,\{N_{\mathbf{k}\alpha}\})=\frac{n\xi v^{id}_{dd,dd}}{2}=k_BT\,O(na\Lambda^2\xi) .$$

In addition, scattering processes that create a condensate particle in the case $\lambda=d$ are omitted, since momentum and energy cannot be simultaneously conserved if $\Delta_{\kappa d} \gg k_B T$.

The rate constants are now found from Eq. (34) replacing ξ and $N_{\mathbf{k}d}$ by their thermal averages $\bar{\xi}$ and $\langle N_{\mathbf{k}d}\rangle_{gr}$. The latter is given by the Bose distribution

$$\langle N_{\mathbf{k}d}\rangle_{gr}=\frac{1}{\bar{\xi}^{-1}e^{\beta\varepsilon_{\mathbf{k}d}}-1} . \tag{36}$$

Due to conservation of energy the influence of the antiblocking factors ($\langle N_{\mathbf{k}d}\rangle_{gr}+1$) appearing in the resulting expression for $G_{dd\to\kappa d}$ can also be neglected. The energy $\varepsilon_{\mathbf{k}d}$ of the quasiparticle produced in the process $dd\to\kappa d$ at low temperatures is of the order of the hyperfine splitting $\Delta_{\kappa d}$, leading to the estimate

$$\langle N_{\mathbf{k}d}\rangle_{gr}\simeq \exp(-\beta\Delta_{\kappa d})\ll 1$$

for the average occupation.

We now calculate the three contributions $G^{(n)}_{dd\to\kappa\lambda}$ ($n=0,1,2$ denotes the number of condensate particles involved) to the rate constants by taking the thermodynamical limit and neglecting the $k_B T\, O(na\Lambda^2)$ terms in the energy difference $\Delta^{(n)}_{\kappa\lambda}$. The first line of Eq. (34) is associated with $n=2$ and leads to

$$\begin{aligned}G^{(2)}_{dd\to\kappa\lambda}(B,T)&=\frac{4\pi(\mu_0\mu_e^2)^2}{45}\frac{\bar{\xi}^2}{(2\pi)^2\hbar}\frac{|\langle\kappa\lambda|\Sigma^{ee}_{2\Delta M_F}|dd\rangle|^2}{1+\delta_{\kappa\lambda}}\\&\quad\times\int d\mathbf{q}\,|Y^*_{2\Delta M_F}(\hat{\mathbf{q}})|^2\delta(\Delta^{(2)}_{\kappa\lambda}E)\\&=\frac{\bar{\xi}^2}{2}G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0) .\end{aligned} \tag{37}$$

The quantity $G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0)$ is the $T\to 0$ limit of the rate constant using a Maxwell-Boltzmann velocity distribution for the atoms in the gas and the plane-wave Born approximation (PWBA) to the transition amplitude. It gives an excellent description of the decay of the doubly spin-polarized gas in the temperature regime $T_c \ll T \ll \Delta_{\kappa\lambda}/k_B$ and for magnetic field strengths $B \lesssim 0.1$ T.[8] Explicitly it reads

$$\begin{aligned}G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0)&=\frac{m_H(\mu_0\mu_e^2)^2}{90\pi\hbar^3}\left[\frac{m_H\Delta_{\kappa\lambda}}{\hbar^2}\right]^{1/2}\\&\quad\times|\langle\{\kappa\lambda\}|\Sigma^{ee}_{2\Delta M_F}|\{dd\}\rangle|^2 ,\end{aligned} \tag{38}$$

introducing the symmetrized and normalized spin states $|\{\alpha\beta\}\rangle$ of Ref. 8. Actually, Eq. (38) is a direct generalization of the expressions found by van den Eijnde[19] and Ruckenstein,[20] who considered the case of $bb\to ab$ relaxation, relevant to the earlier experiments with $H\downarrow\ddagger$.

At zero temperature all particles are in the condensate ($\bar{\xi}=1$) and only collisions with $n=2$ are possible. Hence from Eq. (37) we can already conclude that in this limit all decay rates decrease by a factor of 2. This is easily explained by the following symmetry argument. In the classical regime all occupation numbers are small compared to 1, so that it is highly improbable to find two atoms in the same state $|\mathbf{k}d\rangle$. Therefore, if two atoms (i and j) collide, they are in the normalized symmetric state

$$(|\mathbf{k}d\rangle_i\otimes|\mathbf{k}'d\rangle_j+|\mathbf{k}'d\rangle_i\otimes|\mathbf{k}d\rangle_j)/\sqrt{2} .$$

In the $T\to 0$ limit $\mathbf{k}$ and $\mathbf{k}'$ become zero and this state goes to $\sqrt{2}|0d\rangle_i\otimes|0d\rangle_j$, which results in a transition probability that is a factor of $(\sqrt{2})^2=2$ too large because the correct normalized state would be $|0d\rangle_i\otimes|0d\rangle_j$.

For nonzero temperatures we need to consider also $G^{(1)}$ and $G^{(0)}$. For $G^{(1)}_{dd\to\kappa\lambda}$ we obtain

$$G^{(1)}_{dd\to\kappa\lambda}(B,T)=\frac{4\pi(\mu_0\mu_e^2)^2}{45}\frac{\bar{\xi}}{(2\pi)^5 n\hbar}\frac{|\langle\kappa\lambda|\Sigma^{ee}_{2\Delta M_F}|dd\rangle|^2}{1+\delta_{\kappa\lambda}}\int d\mathbf{k}\int d\mathbf{q}\,|Y^*_{2\Delta M_F}(\mathbf{q})+Y^*_{2\Delta M_F}(\mathbf{q}+\mathbf{k})|^2\langle N_{\mathbf{k}d}\rangle_{gr}\delta(\Delta^{(1)}_{\kappa\lambda}E) . \tag{39}$$

By changing the integration variables $\mathbf{q}$ and $\mathbf{k}$ to $\mathbf{p}+\mathbf{p}'$ and $-2\mathbf{p}'$, respectively, and using the isotropy of the thermally averaged occupations numbers, the double integral can be written as

$$8\int dp\,p^2\int dp'p'^2\langle N_{2p,d}\rangle_{gr}\delta\left[\frac{\hbar^2p'^2}{m_H}-\frac{\hbar^2p^2}{m_H}-\Delta_{\kappa\lambda}\right]\int d\hat{\mathbf{p}}\int d\hat{\mathbf{p}}'|Y^*_{2\Delta M_F}(\mathbf{p}+\mathbf{p}')+Y^*_{2\Delta M_F}(\mathbf{p}-\mathbf{p}')|^2 .$$

In the Appendix we show that the double integral over the directions of $\mathbf{p}$ and $\mathbf{p}'$ equals

$$4\sum_{\substack{ll'\\ \text{even}}}|f_{l'l}(p',p)|^2\underset{p\to 0}{\sim}16\pi\left[1-\frac{p^2}{p'^2}+O\left[\frac{p^4}{p'^4}\right]\right] , \tag{40}$$

where the functions $f_{l'l}(p',p)$ characterized a transition (induced by the dipole interaction) from a two-body state with relative momentum p and relative angular momentum l to a similar state with p' and l'. Collecting these results we find

$$G^{(1)}_{dd\to\kappa\lambda}(B,T)=2\bar{\xi}(1-\bar{\xi})G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0)\left[1-\frac{9}{8}\frac{g_{5/2}(\bar{\xi})}{g_{3/2}(\bar{\xi})}\frac{k_BT}{\Delta_{\kappa\lambda}}+O\left[\left[\frac{k_BT}{\Delta_{\kappa\lambda}}\right]^2\right]\right] . \tag{41}$$

In principle, the argument of the Bose functions should be $\bar{\xi}\exp(-\beta v^{id}_{dd,dd}\bar{\xi}/2)$ instead of $\bar{\xi}$. However, the relative difference of the two is of the order $na\Lambda^2\bar{\xi}$ and negligible.

The evaluation of $G^{(0)}_{dd\to\kappa\lambda}$ proceeds similarly. After taking the thermodynamical limit we transform from initial momenta $\mathbf{k}$ and $\mathbf{k}'$ to center-of-mass and relative momenta $\mathbf{P}$ and $\mathbf{p}$ and write the transferred momentum $\mathbf{q}$ as the difference $\mathbf{p}'-\mathbf{p}$ between final and initial relative momenta. This gives

$$G^{(0)}_{dd\to\kappa\lambda}(B,T)=\frac{4\pi(\mu_0\mu_e^2)^2}{45}\frac{1}{(2\pi)^8 n^2\hbar}\frac{|\langle\kappa\lambda|\Sigma^{ee}_{2\Delta M_F}|dd\rangle|^2}{2(1+\delta_{\kappa\lambda})}$$
$$\times\int d\mathbf{p}\int d\mathbf{p}'\left[\int d\mathbf{P}\langle N_{\mathbf{P}/2+\mathbf{p},d}\rangle_{gr}\langle N_{\mathbf{P}/2-\mathbf{p},d}\rangle_{gr}\right]\delta\left[\frac{\hbar^2p'^2}{m_H}-\frac{\hbar^2p^2}{m_H}-\Delta_{\kappa\lambda}\right]$$
$$\times|Y^*_{2\Delta M_F}(\mathbf{p}+\mathbf{p}')+Y^*_{2\Delta M_F}(\mathbf{p}-\mathbf{p}')|^2 . \quad (42)$$

We note that the function in the first pair of large parentheses, which is up to the factor $[n(2\pi)^3]^{-1}$ equal to the distribution function for the relative momentum $\mathbf{p}$, is independent of the direction of $\mathbf{p}$. Therefore it is possible to first carry out the integrations over $\mathbf{p}'$ and the direction of $\mathbf{p}$ and subsequently use the expansion (40). Finally, reintroducing $\mathbf{k}$ and $\mathbf{k}'$ as integration variables it is possible to express our result in terms of, in particular, the average single-particle kinetic energy. This leads to

$$G^{(0)}_{dd\to\kappa\lambda}(B,T)=(1-\bar{\xi})^2G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0)\left[1-\frac{9}{4}\frac{g_{5/2}(\bar{\xi})}{g_{3/2}(\bar{\xi})}\frac{k_BT}{\Delta_{\kappa\lambda}}+O\left[\left[\frac{k_BT}{\Delta_{\kappa\lambda}}\right]^2\right]\right], \quad (43)$$

if we omit $O(na\Lambda^2\bar{\xi})$ terms in the argument of the Bose functions.

For temperatures above T_c the condensate is absent and only $G^{(0)}_{dd\to\kappa\lambda}$ survives. In particular, in the classical regime $T_c \ll T \ll \Delta_{\kappa\lambda}/k_B$ the fugacity becomes very small ($\bar{\xi}\ll 1$) and we find for the rate constant

$$G_{dd\to\kappa\lambda}(B,T)\equiv G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T)=G^{\mathrm{MB}}_{dd\to\kappa\lambda}(B,T=0)\left[1-\frac{9}{4}\frac{k_BT}{\Delta_{\kappa\lambda}}+O\left[\left[\frac{k_BT}{\Delta_{\kappa\lambda}}\right]^2\right]\right], \quad (44)$$

in complete agreement with the PWBA calculations of van den Eijnde[19] and Ruckenstein.[20] This is not surprising, since for $\bar{\xi}=0$ the quasiparticle Hamiltonian (14) reduces to an independent particle Hamiltonian. The deviation of Eq. (43) from Eq. (44) is due to the fact that the average kinetic energy of an ideal gas of bosons is not equal to $\frac{3}{2}k_BT$, as in the case of distinguishable particles, but equal to

$$\tfrac{3}{2}k_BT[g_{5/2}(\bar{\xi})/g_{3/2}(\bar{\xi})] .$$

In the present practical realizations of a magnetic trap the largest part of the trapping region has such high magnetic field values that $\Delta_{\kappa\lambda}\gg k_BT$ for temperatures around T_c. In these circumstances even the linear terms

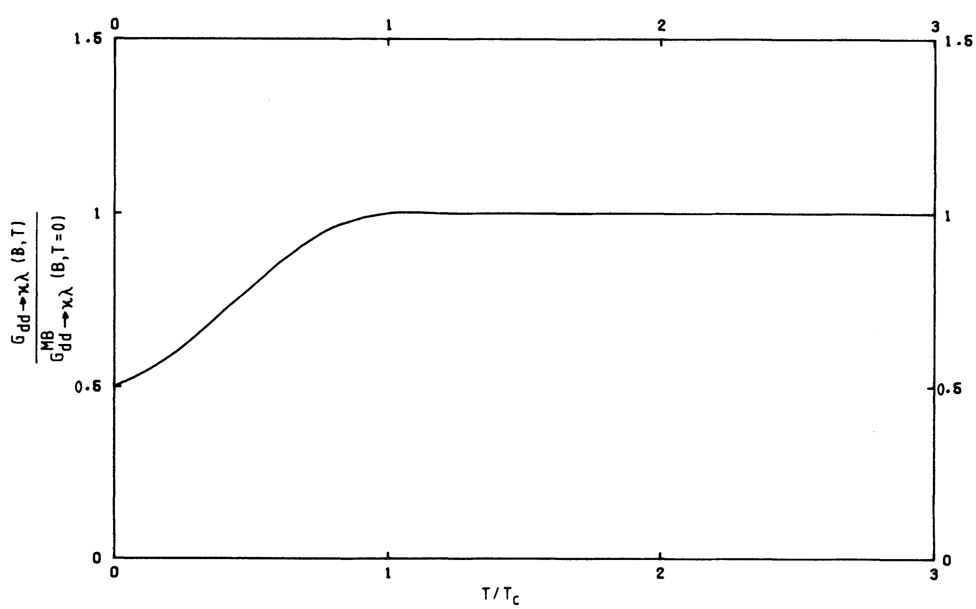

FIG. 3. Temperature dependence of relaxation rates $G_{dd\to\kappa\lambda}$ at large magnetic fields.

in the parameter $k_B T/\Delta_{\kappa\lambda}$ can be neglected and the rate constant reduces to

$$G_{dd\to\kappa\lambda}(B,T)=\tfrac{1}{2}(2-\bar{\xi}^2)G^{\rm MB}_{dd\to\kappa\lambda}(B,T=0) . \quad (45)$$

The temperature dependence of the decay rates is then fully described by the function $(2-\bar{\xi}^2)/2$, which is shown in Fig. 3. Note that here deviations due to the Bose character of the velocity distribution of the hydrogen atoms only appear at temperatures below T_c and are maximal when $T=0$. As mentioned above, at zero temperature the rate of decay is smaller by a factor of 2. From the discussion following Eq. (38) it is clear that for an N-body process the transition probability is reduced by $1/N!$. The corresponding factor of 6 reduction in the case of three-body recombination (the dominant decay channel in $H\downarrow\updownarrow$) was first noticed by Kagan *et al.*[10] Applying their method to the case of two-body relaxation processes, we find that the dominant temperature dependence is given by the correlator K, which in terms of field operators $\Psi(\mathbf{r})$ is equal to

$$K=\frac{1}{2n^2}\langle\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r})\rangle , \quad (46)$$

where $\langle\ \rangle$ denotes a thermal average. Writing the field operator $\Psi(\mathbf{r})$ in the usual form of a sum of a condensate and noncondensate part,

$$\Psi(\mathbf{r})=(\bar{\xi}n)^{1/2}+\Psi'(\mathbf{r}) ,$$

and using the identity

$$(1-\bar{\xi})n=\langle\Psi'^\dagger(\mathbf{r})\Psi'(\mathbf{r})\rangle ,$$

the correlator is calculated to be $\frac{1}{2}(2-\bar{\xi}^2)$. This agrees with our results to lowest order in $k_B T/\Delta_{\kappa\lambda}$. One does not expect to find agreement in higher orders, because Kagan *et al.* assume the classical (Maxwell-Boltzmann) results to be temperature independent. Note also that, contrary to Kagan *et al.*'s three-body rate, our rates have a continuous first derivative at $T=T_c$. The discontinuity appears in our case in the second derivative.

In the center of the trap we have such small magnetic fields that the hyperfine splitting between the $|d\rangle$ and $|c\rangle$ states becomes comparable to the thermal energy. Then the linear terms in $k_B T/\Delta_{\kappa\lambda}$ must be retained and we need

$$G_{dd\to\kappa\lambda}(B,T)=G^{\rm MB}_{dd\to\kappa\lambda}(B,T=0)\left[\tfrac{1}{2}\bar{\xi}^2+2\bar{\xi}(1-\bar{\xi})\left[1-\frac{9}{8}\frac{g_{5/2}(\bar{\xi})}{g_{3/2}(\bar{\xi})}\frac{k_B T}{\Delta_{\kappa\lambda}}+\cdots\right]+(1-\bar{\xi})^2\left[1-\frac{9}{4}\frac{g_{5/2}(\bar{\xi})}{g_{3/2}(\bar{\xi})}\frac{k_B T}{\Delta_{\kappa\lambda}}+\cdots\right]\right] \quad (47)$$

to obtain accurate results. For a realistic situation with a magnetic field of 5 G and a density of 10^{14} cm^{-3} ($T_c\simeq 34$ μK) we give in Fig. 4 the rates $G_{dd\to cc}(B,T)$ and $G^{\rm MB}_{dd\to cc}(B,T)$ as a function of T/T_c. Similar results are valid for the $dd\to cd$ process. In these cases deviations due to the proper use of Bose statistics already appear for temperatures above T_c. Although the effect is small and the decay rates for the processes $dd\to aa$, $dd\to ac$, and

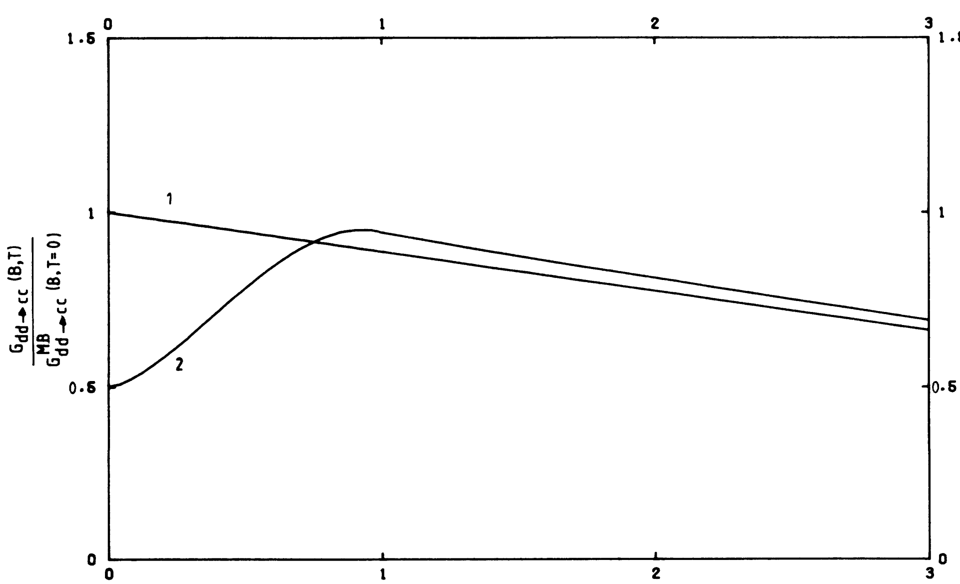

FIG. 4. Temperature dependence of relaxation rate $G_{dd\to cc}$ at a magnetic field of 5 G and a density of 10^{-14} cm^{-3}. Curve 1, without Bose-Einstein condensation and using a Maxwell-Boltzmann velocity distribution. Curve 2, with Bose-Einstein condensation.

$dd \rightarrow ad$ are roughly an order of magnitude larger than those for $dd \rightarrow cc$ and $dd \rightarrow cd$, this precursor effect may prove useful to study the approach to the Bose-Einstein transition.

Finally, in connection with Ref. 20 we would like to point out that Ruckenstein considers $bb \rightarrow ab$ relaxation also at temperatures below T_c. However, he considers the somewhat unrealistic situation of a gas of a anb b atoms ($H\downarrow$) in thermal equilibrium neglecting the important mechanism of preferential recombination leading to the formation of the doubly spin-polarized gas. In this case the Bose condensation takes place in the one-atom ground state, i.e., the state $|0a\rangle$. Therefore Ruckenstein finds that $G_{bb\rightarrow ab}$ consists of two contributions with a different temperature dependence. One is proportional to $\bar{\xi}$ and the other to $1-\bar{\xi}$. It is in this respect important to point out that, as can easily be seen from our physical picture of the decrease of $G_{dd\rightarrow\kappa\lambda}$ by a factor of 2, to lowest order in k_BT/Δ_{ab} the Bose character of the hydrogen atoms does not affect the decay of the gas, since we have

$$\begin{aligned} G_{bb\rightarrow ab}(B,T) &= G^{\rm MB}_{bb\rightarrow ab}(B,T=0)[\bar{\xi}+(1-\bar{\xi})] \\ &= G^{\rm MB}_{bb\rightarrow ab}(B,T=0) \ . \end{aligned}$$

If we do take preferential recombination into account and consider a gas of b atoms, then Eq. (47) is applicable and we find a deviation from classical behavior. Unfortunately, the dominant decay mechanism is now three-body recombination both in the bulk and on the liquid-helium surface, which prevents the accurate determination of the relaxation rates near the phase transition.

B. The case $\Delta_{\kappa\lambda}=0$

Although one avoids zero-field regions experimentally to suppress Majorana spin flips, we consider the process $dd \rightarrow cc$ as an example of what may happen if such regions are present. We can still apply the completely general Eqs. (37), (39), and (42) and neglect the $k_BT\,O(na\Lambda^2)$ terms in $\Delta^{(n)}_{cc}E$, because the relevant thermal energies are of the order k_BT. First of all we conclude that $G^{(2)}_{dd\rightarrow cc}=0$. For the calculation of $G^{(1)}_{dd\rightarrow cc}$ and $G^{(0)}_{dd\rightarrow cc}$ we again need the integral over the directions of $\mathbf{p}$ and $\mathbf{p}'$,

$$\int d\hat{\mathbf{p}} \int d\hat{\mathbf{p}}' |Y^*_{2,-2}(\mathbf{p}+\mathbf{p}')+Y^*_{2,-2}(\mathbf{p}-\mathbf{p}')|^2 \ ,$$

but now with equal magnitudes of the momenta $\mathbf{p}$ and $\mathbf{p}'$. In the Appendix we find that in this situation the integral is equal to 4π. We are thus able to reduce the double momentum integral in Eq. (39) to Bose functions with the result

$$\begin{aligned} G^{(1)}_{dd\rightarrow cc}(B=0,T) &= \frac{4\pi(\mu_0\mu_e^2)^2}{45}\frac{m_H}{8\pi^2\hbar^3}\left[\frac{m_Hk_BT}{2\pi\hbar^2}\right]^{1/2} |\langle cc|\Sigma^{ee}_{2,-2}|dd\rangle|^2 \frac{g_2(\bar{\xi})}{g_{3/2}(\bar{\xi})}\bar{\xi}(1-\bar{\xi}) \\ &\equiv \sqrt{2}\bar{\xi}(1-\bar{\xi})G^{\rm MB}_{dd\rightarrow cc}(B=0,T)\frac{g_2(\bar{\xi})}{g_{3/2}(\bar{\xi})} \ . \end{aligned} \quad (48)$$

Similarly, after a straightforward but elaborate calculation $G^{(0)}_{dd\rightarrow cc}$ becomes

$$\begin{aligned} G^{(0)}_{dd\rightarrow cc}(B=0,T) &= \frac{4\pi(\mu_0\mu_e^2)^2}{45}\frac{m_H}{8(2\pi)^8 n^2\hbar^3} |\langle cc|\Sigma^{ee}_{2,-2}|dd\rangle|^2 \int d\mathbf{k} \int d\mathbf{k}' |\mathbf{k}-\mathbf{k}'| \langle N_{\mathbf{k}d}\rangle_{gr} \langle N_{\mathbf{k}'d}\rangle_{gr} \\ &= 2\sqrt{2}(1-\bar{\xi})^2 G^{\rm MB}_{dd\rightarrow cc}(B=0,T)\frac{h_{5/2,1}(\bar{\xi})+\frac{1}{2}h_{3/2,2}(\bar{\xi})}{g^2_{3/2}(\bar{\xi})} \ , \end{aligned} \quad (49)$$

where we introduced the functions $h_{n,m}(\zeta)$. They are defined by

$$\begin{aligned} h_{n,m}(\zeta) &\equiv \sum_{i=2}^{\infty}\frac{1}{i^n}\left[\sum_{j=1}^{i-1}\frac{1}{j^m}\right]\zeta^i \\ &= \frac{1}{\Gamma(p)}\int_0^\infty dy \frac{y^{n-1}}{\zeta^{-1}e^y-1} g_m(\zeta e^{-y}) \end{aligned} \quad (50)$$

and can easily be calculated numerically. Those occurring in Eq. (49) are shown in Fig. 5. Note for $\zeta\rightarrow 0$ the functions $h_{n,m}(\zeta)$ behave as $2^{-n}\zeta^2$. Consequently, if $T \gg T_c$ the rate $G_{dd\rightarrow cc}(B=0,T)$ reduces to the classical result $G^{\rm MB}_{dd\rightarrow cc}(B=0,T)$. This feature can also be seen in Fig. 6, in which we give both rate constants.

For the process $dd \rightarrow cd$ at zero magnetic field the evaluation of the decay rates requires elaborate numerical calculations, because the antiblocking factors in Eq. (34) are involved in the integral over $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$. In the most difficult case we have a ninefold integral over $\mathbf{P}$, $\mathbf{p}$, and $\mathbf{p}'$ which cannot be handled analytically and may be calculated using Monte Carlo techniques. We therefore do not consider it here.

IV. CONCLUSIONS

We modified the ξ- method of Lee and Yang to analyze the decay of doubly spin-polarized atomic hydrogen in a static magnetic trap. We assumed that the translational degrees of freedom of the atoms are in thermal equilibrium and calculated all dipolar relaxation rates $G_{dd\rightarrow\kappa\lambda}$ as a function of temperature and applied magnetic field. Our results are correct to lowest order in the parameters $(na^3)^{1/4}$, $(na\Lambda^2)^{1/2}$, and $(a/n\Lambda^4)^{1/4}$, which is sufficiently accurate for all practically accessible temperatures and densities. Most importantly we find that the decay probability has a discontinuous second derivative with respect to temperature at $T=T_c$ and decreases well below the transition point by a factor of 2. In addition, we show

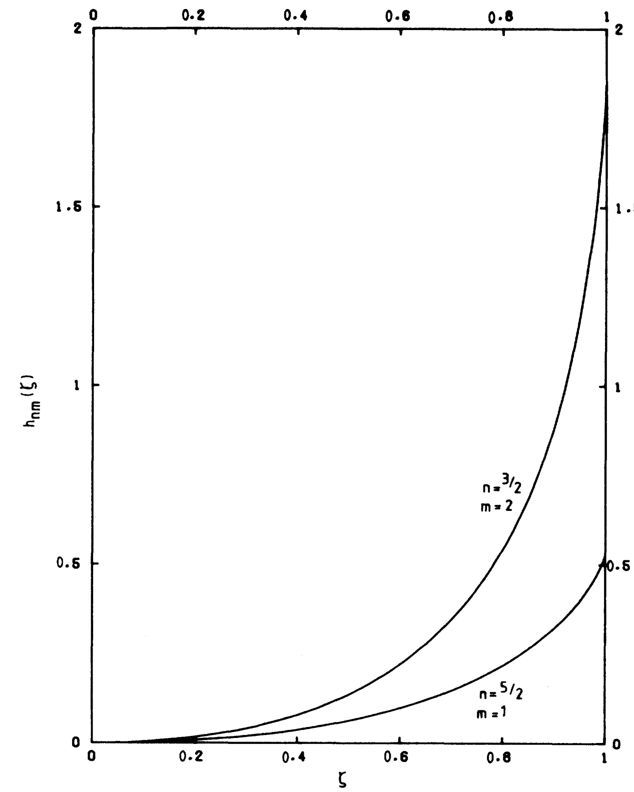

FIG. 5. Functions $h_{n,m}(\zeta)$ for $(n=\frac{5}{2}, m=1)$ and $(n=\frac{3}{2}, m=2)$.

that for sufficiently small magnetic fields also precursor effects occur. All these phenomena may be of importance to studying the approach and realization of the Bose-Einstein transition.

APPENDIX

To calculate the contribution $G^{(0)}_{dd\to\kappa\lambda}$ and $G^{(1)}_{dd\to\kappa\lambda}$ to the decay rates $G_{dd\to\kappa\lambda}$ we have to evaluate the integral

$$\int d\hat{\mathbf{p}} \int d\hat{\mathbf{p}}' |Y_{2\mu}(\mathbf{p}+\mathbf{p}')+Y_{2\mu}(\mathbf{p}-\mathbf{p}')|^2 ,$$

with $\mu=-\Delta M_F$. To this end we expand the spherical harmonic $Y_{2\mu}(\mathbf{p}+\mathbf{p}')$ in terms of the functions $\mathcal{Y}^{LM}_{ll'}(\hat{\mathbf{p}},\hat{\mathbf{p}}')$, which are formed from separate spherical harmonics by angular momentum coupling:

$$\mathcal{Y}^{LM}_{ll'}(\hat{\mathbf{p}},\hat{\mathbf{p}}')=\sum_{mm'}(lml'm'|LM)Y_{lm}(\hat{\mathbf{p}})Y_{l'm'}(\hat{\mathbf{p}}') . \quad \text{(A1)}$$

The desired expansion reads

$$Y_{2\mu}(\mathbf{p}+\mathbf{p}')=\sum_{ll'} f_{l'l}(p',p)\mathcal{Y}^{2\mu}_{ll'}(\hat{\mathbf{p}},\hat{\mathbf{p}}') , \quad \text{(A2)}$$

with the coefficients $f_{l'l}(p',p)$ obeying

$$f_{l'l}(p',p)=\int d\hat{\mathbf{p}} \int d\hat{\mathbf{p}}' \mathcal{Y}^{2\mu *}_{ll'}(\hat{\mathbf{p}},\hat{\mathbf{p}}')Y_{2\mu}(\mathbf{p}+\mathbf{p}') . \quad \text{(A3)}$$

Using the convenient notation $\hat{1}=2l+1$ and the formula[13]

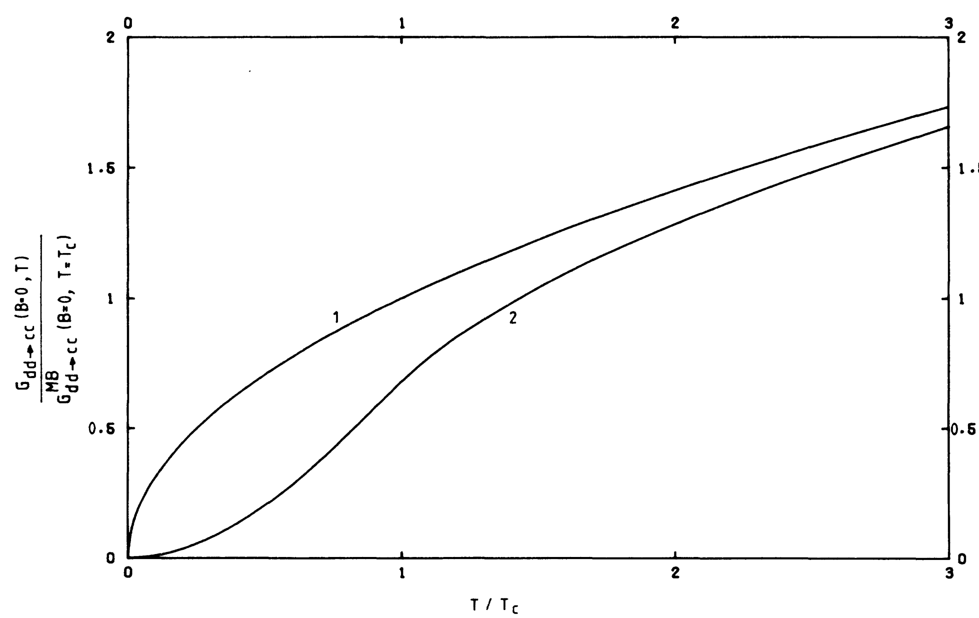

FIG. 6. Temperature dependence of relaxation rate $G_{dd\to cc}$ at zero magnetic field. Curve 1, without Bose-Einstein condensation and using a Maxwell-Boltzmann velocity distribution. Curve 2, with Bose-Einstein condensation.

$$Y_{lm}(\mathbf{p}+\mathbf{p}')=\sum_{l_1+l_2=l}\frac{p^{l_1}p'^{l_2}}{|\mathbf{p}+\mathbf{p}'|^l}\left[\frac{4\pi\hat{\mathbf{l}}!}{\hat{\mathbf{l}}_1!\hat{\mathbf{l}}_2!}\right]^{1/2}\mathcal{Y}^{lm}_{l_1l_2}(\hat{\mathbf{p}},\hat{\mathbf{p}}')\ , \tag{A4}$$

we are able to carry out the integrations over the directions of $\mathbf{p}$ and $\mathbf{p}'$ analytically if we expand also $|\mathbf{p}+\mathbf{p}'|^{-2}$ in terms of spherical harmonics:

$$\frac{1}{|\mathbf{p}+\mathbf{p}'|^2}\frac{2\pi}{pp'}\sum_{l''}(-1)^{l''}Q_{l''}(z)\sum_{m''}Y^*_{l''m''}(\hat{\mathbf{p}})Y_{l''m''}(\hat{\mathbf{p}}')\ , \tag{A5}$$

where $Q_{l''}(z)$ are Legendre functions of the second kind[21] and $z=(p^2+p'^2)/2pp'$. After some Racah algebra we finally obtain

$$f_{l'l}(p',p)=2(30\pi\hat{\mathbf{l}}\hat{\mathbf{l}}')^{1/2}\sum_{l''}\hat{\mathbf{l}}''Q_{l''}(z)\sum_{l_1+l_2=2}\frac{p^{l_1-1}p'^{l_2-1}}{\sqrt{(2l_1)!(2l_2)!}}\begin{bmatrix} l & l_1 & l'' \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} l'' & l_2 & l' \\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix} l & l' & 2 \\ l_2 & l_1 & l''\end{Bmatrix}\ , \tag{A6}$$

and

$$\int d\hat{\mathbf{p}}\int d\hat{\mathbf{p}}'|Y_{2\mu}(\mathbf{p}+\mathbf{p}')+Y_{2\mu}(\mathbf{p}-\mathbf{p}')|^2=4\sum_{\substack{ll'\\ \text{even}}}|f_{l'l}(p',p)|^2\ , \tag{A7}$$

making use of the selection rules of the Wigner $3-j$ and $6-j$ symbols in Eq. (A6).

The special case $p=p'$, needed when $\Delta_{\kappa\lambda}=0$, has to be considered separately because the Legendre functions $Q_{l''}(z)$ diverge logarithmically if $z=1$. However, if we split off the logarithmic part and write $Q_{l''}(z)$ as[21]

$$Q_{l''}(z)=\tfrac{1}{2}P_{l''}(z)\ln\left[\frac{z+1}{z-1}\right]-W_{l''-1}(z)\ , \tag{A8}$$

with $P_{l''}(z)$ a Legendre function of the first kind and

$$W_{l''-1}(z)=\tfrac{1}{2}\int_{-1}^{1}dx\frac{P_{l''}(z)-P_{l''}(x)}{z-x}\ ,$$

we find that the divergent parts cancel. Equation (A6) thus reduces to

$$f_{l'l}(p,p)=-2(30\pi\hat{\mathbf{l}}\hat{\mathbf{l}}')^{1/2}\sum_{l''}\hat{\mathbf{l}}''W_{l''-1}(1)\sum_{l_1+l_2=2}\frac{1}{\sqrt{2l_1)!(2l_2)!}}\begin{bmatrix} l & l_1 & l'' \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} l'' & l_2 & l' \\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix} l & l' & 2 \\ l_2 & l_1 & l''\end{Bmatrix}\ . \tag{A9}$$

Substituting $W_{l''-1}(1)=\sum_{i=1}^{l''}i^{-1}$ we find that

$$\int d\hat{\mathbf{p}}\int d\hat{\mathbf{p}}'|Y_{2\mu}(\mathbf{p}+\mathbf{p}')+Y_{2\mu}(\mathbf{p}-\mathbf{p}')|^2=4\sum_{\substack{l\\ \text{even}}}[|f_{l-2,l}(p,p)|^2+|f_{l,l}(p,p)|^2+|f_{l+2,l}(p,p)|^2]=4\pi\ , \tag{A10}$$

if $p=p'$.

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