

An example of loop quantization

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1 Motivation

The background for the investigations to be discussed below is provided by recent studies of canonical quantum gravity in 3+1 dimensions. However, related issues arise in other physical theories that can be formulated on spaces of connections and are invariant under a corresponding group of gauge transformations. We will be interested in the so-called loop approach to the quantization of gravity, and will have a closer look at the (2+1)-dimensional theory, in the hope of gaining further understanding of the general approach.

Let us summarize in a nutshell the ideas that have gone into proposals for a quantization program for general relativity.

- * The starting point is Ashtekar's reformulation of 3+1-dimensional Hamiltonian gravity in terms of Yang-Mills variables, namely, an $sl(2, \mathbb{C})$ -valued pair (A, E) of a connection one-form and its conjugate momentum [1].
- * On the phase space spanned by these variables, define Wilson loop variables $T^0(\gamma) = \text{Tr P exp } \oint_{\gamma} A$, and momentum-dependent generalizations which together form a closed Poisson bracket algebra of loop functions.
- * Next, "quantize" this classical structure by finding representations of the Wilson loop algebra on spaces of wave functions that are themselves labelled by spatial loops γ .
- * Rewrite the Hamiltonian in terms of loop variables, regularize it appropriately and look for solutions of the Wheeler-DeWitt equation, i.e. wave functions that are annihilated by the quantum Hamiltonian operator.

Some excitement was generated when in 1990, Rovelli and Smolin claimed that - by working roughly along the lines just described - they had indeed been able to find solutions to the non-perturbative quantum theory [2]. Ever since then, a lot of work has gone into understanding which precise assumptions have led to this conclusion, and to which extent the derivation is unique and can be justified both physically and mathematically. Of central importance in this is the algebraic structure on the set of generalized Wilson loops and what can be said about its representation theory. Given the infinite dimensionality of the problem, these are of course very non-trivial questions, which therefore have led to the study of simpler systems where similar algebraic structures occur. One such model system is given by the theory of (2+1)-dimensional gravity, which also may be formulated as a

theory on connection space, and which will be the subject of the remainder of this paper.

2 Introducing 2+1 gravity

The action for 3-dimensional gravity is the Riemann-Hilbert action for the Lorentz metric 3g ,

$$S[{}^3g] = \int_M d^3x \sqrt{-{}^3g} R[{}^3g], \quad (2.1)$$

whose form is identical to that of the four-dimensional theory. There is however an alternative action principle leading to essentially equivalent equations of motion, which depends on a co-triad e and an $SO(2,1)$ -connection A , namely,

$$S[e, A] = \int_M e \wedge F[A], \quad (2.2)$$

where F denotes the curvature of A . It is in this form that the theory is most obviously exactly soluble [3], although a similar result could be established subsequently in the metric formulation too [4]. We will be interested in the case where the three-dimensional manifold M is a product $M = \mathbb{R} \times \Sigma^g$, with Σ^g a compact, oriented Riemann surface of genus g . The theory is topological in the sense that it does not allow for local field excitations, which leads to drastic simplifications compared with the four-dimensional theory. It becomes non-trivial, with a finite-dimensional physical phase space, through the introduction of a non-trivial topology for M .

One may now perform a Legendre transformation which brings the theory of equation (2.2) into the form of a first-class constrained system with conjugate variable pairs $(A_a^i(x), \tilde{E}_i^a(x))$, and constraints

$$\begin{aligned} \mathcal{D}_a \tilde{E}_i^a &= 0 \\ F_{ab\ i} &= 0, \end{aligned} \quad (2.3)$$

see, for example [5]. Here, a is the spatial and i the internal $SO(2,1)$ -index. The first three constraints are the usual Gauss law conditions associated with the invariance under local $SO(2,1)$ -rotations, and the remaining three are flatness constraints on the components of the spatial curvature F , and contain the three diffeomorphism constraints of the theory. - For computational simplicity it is often convenient to work in the two-dimensional representation of the gauge group $PSU(1,1)$ (i.e. $SU(1,1)$ with opposite points identified) instead of $SO(2,1)$.

The constraints (2.3) can be solved explicitly which (for $g > 1$) leads to a reduced phase space that is the cotangent bundle over the well-known

Teichmüller space $\mathcal{T}(\Sigma^g)$, which will be the subject of the next section. This latter space is contractible and diffeomorphic to \mathbb{R}^{6g-6} . The case $g = 1$ is degenerate from a mathematical point of view; its reduced configuration space is diffeomorphic to \mathbb{R}^2 .

3 The loop formulation

One way of describing points of $\mathcal{T}(\Sigma^g)$ is as follows. Consider the $2g$ generators of the homotopy group $\pi_1(\Sigma^g)$ of the Riemann surface. With each handle of Σ^g we associate a pair (α_i, β_i) of generators, where α_i goes around the i 'th hole and β_i around the i 'th handle. These generators have to obey the fundamental relation

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = 1. \quad (3.1)$$

We define the holonomy of an element $\gamma \in \pi_1(\Sigma^g)$ as the integral $U_\gamma = \text{P exp} \oint_\gamma A$. The integral does not depend on the representative in the homotopy class of γ because the physical connections are all flat. A point in the Teichmüller space $\mathcal{T}(\Sigma^g)$ is uniquely labelled by the values of the $2g$ holonomy matrices $U_{\alpha_i}, U_{\beta_i} \in PSU(1, 1)$ modulo gauge transformations

$$(U_{\alpha_1}, \dots, U_{\beta_g}) \xrightarrow{g} g^{-1}(x_0) \cdot (U_{\alpha_1}, \dots, U_{\beta_g}) \cdot g(x_0) \quad (3.2)$$

at the common, fixed base point $x_0 \in \Sigma^g$ of the generators, and subject to

- a) $U_{\alpha_1} U_{\beta_1} U_{\alpha_1}^{-1} U_{\beta_1}^{-1} \dots U_{\alpha_g}^{-1} U_{\beta_g}^{-1} = \mathbb{1}$, and
- b) $1 - T^0(\gamma)^2 < 0$, $\forall \gamma \in \{\alpha_i, \beta_i\}$ and $T^0(\gamma) := \frac{1}{2} \text{Tr } U_\gamma$.

The first condition is a direct consequence of (3.1), and the second condition picks out (gauge-invariantly) the sector where the holonomy matrices U correspond to boosts about space-like axes [5]. Taking these constraints into account, the counting of the degrees of freedom, $2g \times 3 - 3 - 3 = 6g - 6$, agrees with the dimensionality of Teichmüller space.

One gets rid of the remaining gauge covariance (3.2) of the holonomies by taking traces, $T^0(\gamma) := \frac{1}{2} \text{Tr } U_\gamma$. Doing this for any element of the homotopy group, one obtains an overcomplete set $\{T^0(\gamma), \gamma \in \pi_1(\Sigma^g)\}$ of Wilson loop observables for 2+1 gravity. (Note that just considering Wilson loops of the $2g$ homotopy generators does not lead to a complete set of observables.) Likewise, generalized, momentum-dependent loop variables can be introduced. Thus the kinematical set-up is very much like that for both 3+1 gravity and $SU(2)$ Yang-Mills theory in a canonical loop formulation (see [6] for a review). The difference between those theories lies in the way the physical degrees of freedom are imbedded into the initial configuration (or phase) space, given in terms of the overcomplete set of loop variables. To identify the physical degrees of freedom, one has to solve the so-called

Mandelstam constraints associated with the given gauge group and representation, and also take care of certain inequalities that may exist among the loop variables [7].

Usually there are topological obstructions which prevent us from finding a good global solution to the Mandelstam constraints (and other constraints that come with the theory), i.e. a set of independent Wilson loops that parametrize the physical configuration space globally. It turns out however that the structure of 2+1 gravity is sufficiently simple to allow us to find such solutions explicitly.

We will make use of a particular parametrization for the Teichmüller spaces $\mathcal{T}(\Sigma^g)$, in terms of the so-called Fenchel-Nielsen coordinates [8]. They are associated with the Riemann surface Σ^g equipped with a metric h of constant negative scalar curvature -1 . The surface is cut along $3g-3$ simple (non-intersecting) closed geodesics into $2g-2$ "pairs of pants", so that each pants is a piece of the surface, with three circular boundary components.

We associate with the i 'th cut a pair (l_i, τ_i) of variables in $\mathbb{R}^+ \times \mathbb{R}$, where l_i measures the length of the boundary component (with respect to h) and τ_i the angle of the relative twist between the two pants that meet along the boundary. The Fenchel-Nielsen coordinates parametrize Teichmüller space globally, leading to the identification $\mathcal{T}(\Sigma^g) \simeq (\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$. A natural symplectic form ω exists on $\mathcal{T}(\Sigma^g)$, the so-called Weil-Petersson form, for which the Fenchel-Nielsen coordinates are canonical pairs, i.e. $\omega = \sum_i dl_i \wedge d\tau_i$. Note that this symplectic form has nothing to do with the canonical symplectic structure on the phase space $T^*\mathcal{T}(\Sigma^g)$ of 2+1 gravity, which is $\Omega = \sum_i dl_i \wedge dp_{l_i} + \sum_i d\tau_i \wedge dp_{\tau_i}$.

The existence of an explicit parametrization of the reduced configuration space simplifies calculations enormously, and allows us to set up a quantization of the reduced system (as opposed to solving the constraints à la Dirac *after* quantizing the original system). Our next step will be to rewrite the Wilson loops associated to certain homotopy group elements in terms of the Fenchel-Nielsen coordinates. This can be done using a result of Okai [9] who constructed an explicit cross section of the principal bundle

$$\text{Hom}(\pi_1(\Sigma^g), PSU(1, 1))^{e=2g-2} \quad (3.3)$$

↓

$$\text{Hom}(\pi_1(\Sigma^g), PSU(1, 1))^{e=2g-2} / PSU(1, 1) = \mathcal{T}_g,$$

thus giving explicit expressions $U_{\alpha_i}, U_{\beta_i}$ for the holonomy matrices in terms of the Teichmüller parameters. The superscript $e = 2g - 2$ in (3.3) denotes a technical condition on the space of all homomorphisms which selects exactly the sector we are interested in here. Obviously we can then write all of the gauge-invariant loop observables as $T_{(l_i, \tau_i)}^0(\gamma) = \frac{1}{2} \text{Tr} U_\gamma(l, \tau)$, and it turns out that a general Wilson loop is a rational function of the hyperbolic functions $\sinh l_i, \sinh \tau_i, \cosh l_i$ and $\cosh \tau_i$.

Our aim is to construct a loop representation for 2+1 gravity which mimics the corresponding construction for the (3+1)-dimensional theory. The transition from the connection to the loop representation is usually achieved by defining a so-called loop transform between the Hilbert spaces of the two formulations. For example, the loop transform for Yang-Mills theory is given by

$$\tilde{\psi}(\alpha) := \int_{\mathcal{A}/\mathcal{G}} dV([A]) T_A^0(\alpha) \psi([A]). \quad (3.4)$$

The wave functions ψ of the connection representation depend only on the gauge equivalence class $[A]$ of A , and the wave functions $\tilde{\psi}$ in the loop representation on spatial loops α . The transform is to be thought of as a non-linear analogue of a Fourier-type transform. It remains a heuristic device as long as one does not specify an appropriate gauge-invariant measure dV on the connection space. This so far has not been achieved for either gauge theory or general relativity, but in the present case we are more fortunate and can give the explicit form of the loop transform as

$$\tilde{\psi}(\gamma) := \int_{\mathcal{T}(\Sigma^g)} dV(l, \tau) T_{(l, \tau)}^0(\gamma) \psi(l, \tau), \quad (3.5)$$

where $\gamma \in \pi_1(\Sigma^g)$ and ψ is a square-integrable function in $L^2(\mathcal{T}(\Sigma^g), dV)$. A natural volume element dV on Teichmüller space is the Liouville volume element associated with the symplectic form ω , i.e. $dV = \omega^{3g-3} = dl_1 \wedge d\tau_1 \wedge \dots \wedge dl_{3g-3}$. However, one quickly realizes that with this choice of volume element the integration in (3.5) diverges for general Wilson loops $T^0(\gamma)$. (This problem was first noticed for the $g = 1$ case by Marolf [10].) This happens because $T^0(\gamma)$ depends on hyperbolic sines and cosines which diverge for large values of the Fenchel-Nielsen coordinates l_i and τ_i . This of course could only have occurred because the gauge group for 2+1 gravity is the non-compact group $SO(2, 1)$ and the Wilson loops themselves are not bounded functions. A similar problem is not present for $SU(2)$, say, where we have $-1 \leq T^0(\gamma) \leq 1$.

One way of dealing with this problem has been suggested by Ashtekar and the author and consists in modifying the measure in such a way that the integration over Teichmüller space in (3.5) converges for arbitrary Wilson loops [11]. We multiply the volume element by a function $m(l, \tau)$ which is subject to a number of conditions. It turns out that the construction of the loop transform requires m to be of the form $m = \exp \sum_i b_i T^0(\gamma_i)$ of an exponential of a linear combination of some set of Wilson loops $T^0(\gamma_i)$.

Hence the task is to first find functions m that give an appropriate damping behaviour and then establish how the modified measure affects the construction of the loop representation. For the simplest case of $g = 1$ both of these issues were investigated in [11]. One possibility to define m in this

case is to set $m = \exp -p(T^0(\vec{q}_1) + T^0(\vec{q}_2))$, where $p > 0$ and \vec{q}_1 and \vec{q}_2 are two linearly independent vectors in \mathbb{Z}^2 (homotopy elements for $g = 1$ are labelled by a pair of integers). As a result one obtains modified expressions for the Wilson loop operators in the loop representation, namely,

$$\begin{aligned} (\hat{T}^0(\vec{k})\tilde{\psi})(\vec{n}) &= \frac{1}{2}(\tilde{\psi}(\vec{n} + \vec{k}) + \tilde{\psi}(\vec{n} - \vec{k})) \\ (\hat{T}^1(\vec{k})\tilde{\psi})(\vec{n}) &= -\frac{i\hbar}{2}(\vec{k} \times \vec{n})(\tilde{\psi}(\vec{k} + \vec{n}) - \tilde{\psi}(\vec{k} - \vec{n})) + \\ &+ \frac{ip\hbar}{4} \sum_{i=1,2} (\vec{k} \times \vec{q}_i)(\tilde{\psi}(\vec{k} + \vec{n} + \vec{q}_i) - \tilde{\psi}(\vec{k} + \vec{n} - \vec{q}_i) + \\ &\tilde{\psi}(\vec{k} - \vec{n} + \vec{q}_i) - \tilde{\psi}(\vec{k} - \vec{n} - \vec{q}_i)). \end{aligned} \quad (3.6)$$

What is new compared to the usual loop representation is the term proportional to p in the expression for the Wilson loop momentum \hat{T}^1 . It is the loop analogue of the divergence term that has to be added to the expression for \hat{T}^1 in the connection representation, in order to make it self-adjoint with respect to the measure $m dV$. (Note that the commutator algebra of the loop operators is still exactly the same as in the representation with $m = 1$!)

One may be tempted to take the limit as $p \rightarrow 0$ of (3.6) and obtain the expressions of the standard loop representation. However, this limit is not well-defined, which becomes apparent when calculating the scalar products of basic wave functions, like for instance

$$\langle \tilde{T}^0(\vec{k}), \tilde{T}^0(\vec{n}) \rangle = \int dV e^{-p(T^0(\vec{q}_1) + T^0(\vec{q}_2))} T^0(\vec{k}) T^0(\vec{n}). \quad (3.7)$$

We calculated the general scalar product (3.7) for the case $\vec{q}_1 = (1, 0)$, $\vec{q}_2 = (0, 1)$ and found that the results depend on the modified Bessel functions $K_n(p)$ and inverse powers p^{-n} , and therefore diverge as $p \rightarrow 0$. Also, the explicit form of the scalar product is much more complicated than the usual one, which is essentially proportional to $\delta_{\vec{k}, \vec{n}}$.

For the higher-genus case, which is algebraically more complicated, we have not yet established suitable damping factors. However, this should be reasonably straightforward, given the recently obtained solution for a set of complete and independent loop invariants for arbitrary genus g [12].

4 Conclusions

We have learned from the preceding discussion that difficulties may arise in the construction of the loop representation whenever the gauge group

under consideration is non-compact. In 2+1 gravity and for $g=1$, we dealt with this problem by introducing an appropriate volume element on the space of connections modulo gauge transformations to make the integration in the loop transform convergent. This in turn led to a new loop representation with unusual expressions for the Wilson loop operators and the scalar product. For the higher-genus case we expect similar results to hold.

This immediately raises the question of whether a similar phenomenon occurs in 3+1 dimensions, where with $SL(2, \mathbb{C})$ we also have a non-compact gauge group. This issue is much harder to address, because of the infinite dimensionality of the theory and the fact that in principle there do not exist good global coordinates on the reduced configuration space. In one sense the (3+1)-dimensional theory may however be simpler, since its gauge group is the complexification of a compact gauge group, which is not true for $SU(1, 1)$. So far our knowledge about possible measures for quantum gravity is very limited, although extensive investigations are under way. We may eventually be able to classify inequivalent loop representations, and relate them to classes of suitable measures on the connection space, along the lines proposed above for the "toy model" of 2+1 gravity.

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