On the construction and stationary distributions of some spatial queueing and particle systems

Over de constructie en stationaire verdelingen van enkele ruimtelijke wachtrij- en deeltjessystemen

(met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

Introduction

This chapter is meant to be an informal introduction to this thesis, in such a way that it is hopefully also comprehensible for non-probabilists. In addition to this introduction, every chapter has its own introductory section, in which the results of that chapter will be described in a more mathematical language. Hence the chapters will be self-contained, and sometimes concepts and models explained in this chapter are again introduced in one of the other chapters.

The systems discussed in this thesis are queueing and interacting particle systems. In Section 1.1 a basic queueing system will be introduced, to get some feeling for the subject and to introduce some concepts. Section 1.2 handles the queueing systems in this thesis. In Section 1.3 interacting particle systems are introduced, and finally Section 1.4 focusses on the particle systems in this thesis.

1.1 A basic queueing system

1.1.1 Introduction

Let us consider a shop with only one shopkeeper. Customers, who want to pay for their shopping, have to line up in a queue and are served according to the 'first-come-first-served' principle. This means that if a customer wants to pay and the shopkeeper is not busy at that moment, then this customer can pay immediately. If the shopkeeper is already helping someone else, this customer can pay after all the customers who joined the queue earlier than him have been served.

We are interested in the behaviour of the number of customers in the queue (including the customer who is currently being served). For example, we might want to know the average number of customers in the queue, or what percentage of the time the shopkeeper is busy helping customers. To be able to say something, we should at least know:

- The service times of the customers.
- The process according to which the customers join the queue in the shop (arrival process).

To give an example, suppose that the time between two consecutive customers who are arriving at the check-out (we call this time an inter-arrival time) is exactly two minutes, and that each service lasts exactly one minute. Then the queueing system is completely deterministic (and therefore from our point of view not too interesting): if we know when it started and which time it is now, we can easily decide whether the shopkeeper is busy helping a customer or whether he can drink his coffee.

1.1.2 Modelling with randomness

The deterministic system described in Section 1.1.1 is not a very realistic model for a queue in a shop. We expect the service time of a customer to depend upon how much he wants to buy, how fast he can pay, whether he chats with the shopkeeper etc. We also do not expect all inter-arrival times to be equal.

Since we cannot predict the inter-arrival and service times exactly, we will model them using randomness. This means that we specify for each length of time s which percentage (or fraction) of the inter-arrival or service times is at most s minutes, in the long run.

In this way we get a model for the number of customers in the queue, and in such a model we can answer questions as posed in Section 1.1.1. Of course we hope that the model resembles the real situation, such that conclusions drawn in the model also say something about the real queue in the shop.

1.1.3 Service times

As we mentioned above, if we model the service times as random times, we should for each s specify the fraction of the service times (in the long run) that is at most s minutes. We interpret this fraction as the probability that the service time is not larger than s minutes. We write S for the (random) duration of a service time and $P(S \leq s)$ for the probability that the service does not take more than s minutes. The function specifying $P(S \leq s)$ for each s is called a *probability distribution.*

We shall now give an important example of a probability distribution which is often used to model service times, the so-called *exponential distribution with parameter* μ, where μ is some positive real number. This distribution is characterised by the following two properties:

1. The average duration of a service with this distribution is $\frac{1}{\mu}$ minutes.

2. The probability that a service will take more than k extra minutes, if you know that it lasts already r minutes, is the same as the probability that a service will last at least k minutes in total. This property is called the 'lack of memory' property of the exponential distribution (since knowing how long the service has already lasted, does not affect the distribution of the remaining service time).

These two properties in fact completely determine the probability that S is at most s minutes: it turns out that

$$P(S \leq s) = 1 - e^{-\mu s},$$

which explains the name of this distribution.

Maybe you think that is completely unreasonable to model a service time in such a way that it has the second property, since you think that the remaining service time should be shorter, if you know that the service is already lasting for a long time. If service times are exponentially distributed, in for example a post-office with two counters, the lack of memory property implies that if at one counter a customer has already been served for a while, but at the other counter the service of a new customer just started, both customers have the same probability of finishing first. This might be not so unrealistic at all, if you think of the number of times you chose to line up behind a customer who had already been served for a long time in your local post-office and this turned out to be the wrong choice of queue. Moreover, the lack of memory property makes life mathematically easier.

1.1.4 Arrival process

A process that is often used to model the arrival times of customers is the *Poisson arrival process with parameter* λ, where λ is a positive real number indicating how many customers enter the queue per minute on average. The Poisson process (with parameter λ) is strongly related to the exponential distribution (with parameter λ) as described in Section 1.1.3: in a Poisson process, the first customer arrives after an exponential time and after that, the periods between two consecutive arrivals (inter-arrival times), have again the exponential distribution and do not depend on each other. This means that having information about the length of some preceding inter-arrival times doesn't say anything about the length of any other inter-arrival time. So for example, the fact that the shopkeeper has only seen a very few customers during the last hour, does not guarantee that soon lots of customers will arrive.

Two properties of the Poisson arrival process are:

1. The average number of customers that arrives in a time period of length l equals λl.

2. The numbers of arrivals in two intervals that do not overlap are independent.

1.1.5 Stationarity

Think of a model for a 'first-come-first-served' queue with one server and suppose that the arrival and service processes are specified. We are interested in what happens to the number of customers in the queue in the long run. Can the server have a break now and then, or does the number of customers in the queue get larger and larger as time goes on?

To say a bit more about these questions, let us consider the following model for a queue in a shop with one shopkeeper. Suppose customers join the queue according to a Poisson process with parameter λ, and service times are independent and exponentially distributed with parameter μ. Recall that this implies that the average inter-arrival time is $\frac{1}{\lambda}$ and the mean service time $\frac{1}{\mu}$. The shopkeeper serves the customers in the order of their arrivals, and the poor man does not have any coffee breaks as long as there are still customers in the queue.

In Figure 1.1 and Figure 1.2 you see two simulations of this model. In Figure 1.1 we have taken $\lambda = \frac{1}{5}$ and $\mu = 1$, in Figure 1.2 $\lambda = \frac{1}{5}$ and $\mu = \frac{1}{10}$. The crosses on the time-axis denote the moments at which a customer arrived, the little circles denote the moments at which a service ended.

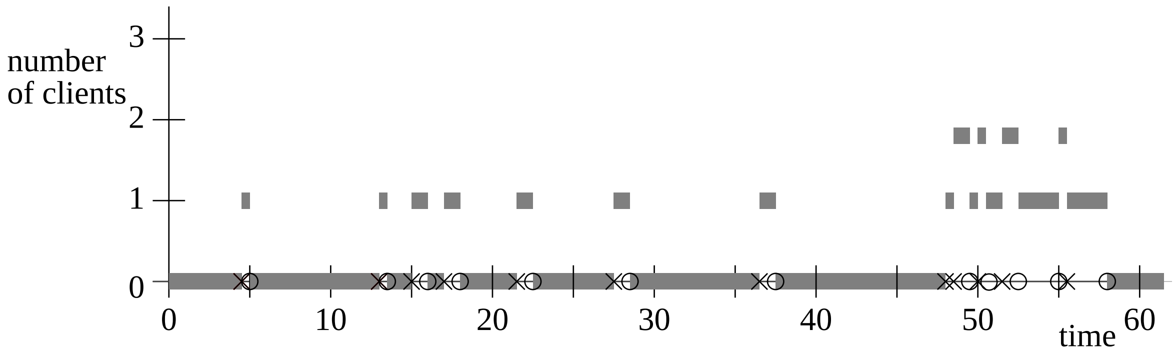

Figure 1.1: Queueing process with $\lambda = \frac{1}{5}$ and $\mu = 1$

Let $Q(t)$ be the number of customers in the queue at time t (including the possible customer who is currently being served). The number of customers $Q(t)$ is random, that is to say, for each non-negative number a there is a certain probability that $Q(t) = a$. We write $P(Q(t) = a)$ and

$$P(Q(t) = 0) + P(Q(t) = 1) + \cdots = 1,$$

for all t. You can interpret this as follows. If you see a lot of these queues, say you look through the window of this shop every day at time t, then in the long run, the fraction of the days that you see a customers in the queue is $P(Q(t) = a)$.

Let us pretend now that the shop is open forever. We let the time t go to infinity and we want to know whether the server can handle all the customers. It will not surprise you that the answer of this question depends on the values of λ and μ.

If $\lambda > \mu$, then the average inter-arrival time is shorter than the average service time, and the number of customers in the queue will tend to infinity. In this case we call the system *unstable* (see also Figure 1.2).

If $\lambda < \mu$ (so then the average of the inter-arrival times is larger than the average time needed for a service), then, if we let t go to infinity, the probability of seeing a customers does not change much anymore, but tends to, say, $P(a)$. The $P(a)$ sum up to one again. Imagine that the system starts at time 0 with a random number of customers in the queue, where the probability that this number of customers equals a is $P(a)$ for all a. In that case, we have that the probability that we see a customers at time t is also $P(a)$, for all times t. We call this the stationary distribution of the system. So if the system is in its stationary distribution, for example the probability of seeing 3 customers in the queue does not change in time. This does not mean that the number of customers does not change, as there are still customers joining the queue

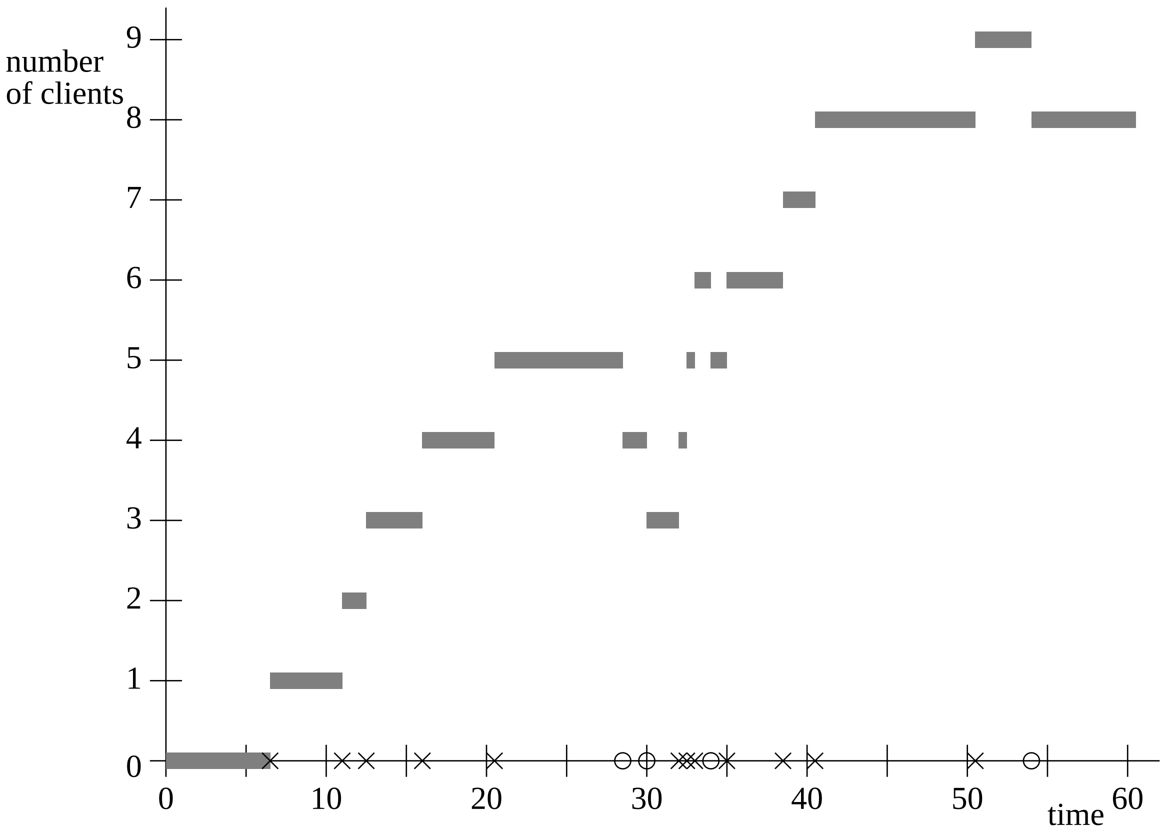

Figure 1.2: Queueing process with $\lambda = \frac{1}{5}$ and $\mu = \frac{1}{10}$

and customers leaving, it is just that the probabilities are constant. If a system has a stationary distribution, we call it *stable*.

For $\lambda < \mu$, the stationary distribution of the system can be computed. It turns out that the (stationary) probability that there are a customers in the queue is

$$P(a) = \left(\frac{\lambda}{\mu}\right)^a \left(1 - \frac{\lambda}{\mu}\right).$$

In the stationary distribution, the average number of customers in the queue equals $\frac{\lambda}{\mu-\lambda}$. To see whether the shopkeeper has often time to drink his coffee, we consider the so called 'busy periods' of the shopkeeper, which are the maximal periods in which he was serving continuously. So his first busy period starts when the first customer asks for service, and it ends at the first time that there are again no customers in the queue and the shopkeeper is not helping a customer. In this queue (with $\lambda < \mu$), the average length of a busy period of the shopkeeper is $\frac{1}{\mu-\lambda}$ minutes. The average length of a coffee-break between two busy periods is, by the lack of memory property, equal to the average inter-arrival time, so $\frac{1}{\lambda}$ minutes.

For $\lambda = \mu$, a stationary distribution as above does not exist and we call the queue unstable in this case.

1.2 Queueing systems in this thesis

In Chapter 2 and Chapter 3 we consider some queueing systems on a circle. The similarity between these systems and the systems discussed in Section 1.1 is that these systems are also queues with one server. In contrast to the systems in Section 1.1, customers do not simply line up, but choose a position on a circle. The order in which the customers are served is not 'first-come-first-served', but depends on the place where the customer is waiting. In Chapter 2 two new stability proofs for such systems are given. These results were already known, but the proofs we present are new, shorter, more natural and more transparent than the existing proofs. Chapter 3 deals with approximations of one of these systems. Both chapters are based on joint work with Ronald Meester.

We will now describe the systems involved and indicate which results are obtained. We do not pretend that the systems model a real-life situation, but the following description hopefully makes clear what is going on.

1.2.1 The non-greedy system

Imagine that we are in a canteen. After people have eaten their lunch, they have to put their tray on a circular conveyor belt (the trays will play the role of the customers). Assume that the belt is empty at time 0. The arrival process of the trays is a Poisson process with parameter λ, and the location where a tray is placed is selected completely randomly, independently of the position of the other trays on the conveyor belt. We assume that trays are infinitely thin, so that there are no physical problems to put a tray at the selected place. Suppose the conveyor belt is rotating in a fixed direction and with a fixed speed. There is a poor skivvy standing at a fixed position at the conveyor belt, stopping the belt each time that a tray arrives at his position. He removes the tray from the conveyor belt and puts the dishes in the dishwasher. After that, he starts the conveyor belt again, and it rotates in the same direction as before. Suppose the time it takes to clear up a tray has a certain probability distribution (which is the same for all trays). Suppose further that it takes on average $\frac{1}{\mu}$ minutes to clear up a tray. For instance, the time it takes to clear up a tray can be exponentially distributed with parameter μ. This system is called the non-greedy system, since the direction of the conveyor belt is not influenced by the locations of the trays.

We are interested in the question of whether this system has a stationary distribution. In this case, if we want to prove stability of the system, it suffices to show that the average length of the busy periods of the skivvy are finite.

It might surprise you, that the (un)stability of the non-greedy system only depends on the parameter values λ and μ and not on the speed of the conveyor belt. The parameter values for which the system is stable are in fact exactly the same as for the queue described in Section 1.1.5: the non-greedy system is stable precisely when λ is smaller than μ, which was proved in Kroese and Schmidt (1992, 1993). An intuitive explanation for this fact (also given by Kroese and Schmidt (1993)) is that when there are more trays on the conveyor belt, the time needed to get to the next tray is smaller.

We give a new proof of this stability result, which is very simple and uses the average time the belt rotates between the cleaning of two consecutive trays. We call this the average rotation time. The idea of the proof is as follows. We saw that the queue in Section 1.1.5, in which there were no rotation times at all, is unstable for $\lambda \geq \mu$. This implies that the non-greedy system (with rotation times) must also be unstable for $\lambda \geq \mu$. So we only need to consider the case $\lambda < \mu$. We shall *assume* that the system is unstable for $\lambda < \mu$ and then try to reach a contradiction. If $\lambda < \mu$ and the system is unstable, we obtain that the average rotation time should have some positive value. But if the average rotation time is positive, there can be no accumulation of customers and the system must be stable. So the assumptions that $\lambda < \mu$ and that the non-greedy is unstable contradict each other and therefore we conclude that the non-greedy system must be stable for $\lambda < \mu$.

1.2.2 The greedy system

Let us consider a slight modification of the non-greedy system. In this system, the conveyor belt can rotate in both directions. The arrival process and the times needed to clear up the trays are distributed in the same way as above. However, the skivvy is greedy to get the work done, in the sense that he always lets the conveyor belt rotate in the direction of the nearest tray. It is possible that he changes the direction in which the belt rotates when a tray is added, if this new tray is nearest to the skivvy. We expect this greedy system to be stable under the same conditions, but surprisingly this has not been proved until now. We still hope that considering the average rotation time carefully might help to give a stability proof for this system.

1.2.3 The greedy system with a finite number of queueing positions

The idea of the stability proof of the non-greedy system described above does apply to modifications of the greedy system. In these systems the arrival process of the trays and the time needed to clear up a tray have the same distributions as before, but now a certain (arbitrary but fixed) number of stacks is fixed on the conveyor belt and people have to put their tray in one of these stacks. They choose a stack independently of the state of the system and every stack is chosen with equal probability. We suppose that each stack has an infinite capacity. The skivvy makes the conveyor belt rotate according to the greedy strategy: always in the direction of the nearest stack in which one or more trays need to be cleared up. If there are two of these stacks, he flips a fair coin to consider which direction he takes. When a stack arrives at the skivvy, he clears up all the trays in that stack. This system is also stable exactly for λ smaller than μ, which was proved by Foss and Last (1996). We will give a new proof of this result. The proof is in the same spirit as the stability proof of the non-greedy system.

1.2.4 Approximations of the non-greedy system

In Chapter 3, we consider variants of the non-greedy system, in which there is a fixed maximal number of trays allowed on the conveyor belt - let us denote this number by k. These systems behave almost the same as the non-greedy system, but there is one restriction. Whenever someone wants to put his tray on the conveyor belt and there are already k trays on the belt, he is not allowed to do so, but has to clear up his tray by himself.

We show that if we send k to infinity, the stationary distributions of these systems with finite capacity tend to the stationary distribution of the non-greedy system, in which there is no bound on the number of trays. This does not seem surprising, but it takes a lot of effort to prove this. The idea of the proof is not difficult. We shall construct the non-greedy system and a non-greedy system with finite capacity, in such a way that they look exactly the same for a very long time (if k is large). Then there is a period in which the systems look different, but if there are no customers in the non-greedy system, then also the system with finite capacity will be empty, and then again a period starts in which the systems look the same for a long time. We show that if k tends to infinity, the probability that the system with finite capacity looks the same as the non-greedy system tends to one, which is exactly what we want.

1.3 Interacting particle systems

In this section we will first give a very general description of an interacting particle system. Then we consider an example: the so-called contact process. We explain the problem of constructing interacting particle systems and indicate what kind of results one seeks for.

1.3.1 General description

Interacting particle systems are systems which consist of a grid with a (possibly infinite) number of particles. As time goes on, particles might be removed from or added to the grid, move to some other place in the grid, etc. The way in which this happens is random, and may depend on the state of the system. For example a particle might consider the position of its neighbours to decide whether it wants to move to another position. The fact that the behaviour of a certain particle can be influenced by (some of) the other particles in the system, explains the term *interacting* particle system. We consider an example.

1.3.2 The contact process

Imagine a city with one street with infinitely many houses, which are numbered

$$\cdots, -2, -1, 0, 1, 2, \cdots.$$

In each house lives a family. There is an outbreak of an infectious disease and at time 0, each family may or may not be infected. The infected families play the role of the particles in the general description. If the family at number i is infected, we will denote this by $h(i) = 1$, if this family is not infected, we write $h(i) = 0$.

Now the state (i.e. infected or healthy) of each family can change repeatedly. A family which is infected by the disease becomes healthy after an exponential time with parameter 1. A family which is not infected, can become infected if at least one of their two nearest neighbours is infected. To know whether (and when) this family becomes infected, take an exponential period with parameter λ if only one of these neighbours is infected, and with parameter 2λ if both neighbours are infected. If that period ends at a time before the state of one of the nearest neighbours has changed, this family becomes infected after the end of that exponential period. Otherwise, after the change of state of one of the neighbours, the process continues as above.

1.3.3 Construction problems

Perhaps you are happy with the above description of the contact process, but one has to be a bit careful when it comes to constructing such a process. It is not a priori clear that this can always be done. To explain what problems may arise, suppose that at time 0 infinitely many families are infected, and that there are also infinitely many families which are healthy. Observe also that for each positive time t, each infected family has a certain positive probability of recovering between time 0 and time t. This means that at each time, a certain fraction of the families recovered (and some of them became even infected again...). These are still infinitely many families.

Suppose we consider a family which is not infected, and we want to know the probability that this family becomes infected during the next week. To compute this probability, we should know when their neighbours are ill during that week, but that depends on the neighbours of the neighbours etc. This might cause a problem, as what happens to our family could depend on what happens to infinitely many other families.

Nevertheless it is possible to construct a system that fits the description of the contact process. We denote the state of the system at time 0 by

$$\cdots, h(-2), h(-1), h(0), h(1), h(2), \cdots$$

where each $h(i)$ is 1 or 0 depending on whether the family is ill or not. We will suppose that only the families in the houses numbered from $-n$ up to n can change their states as described earlier. We call this process the n-process. There is no problem in constructing these n-processes, since there are only finitely many houses where something can change. Then it can be shown that if we make the block of houses where things can happen larger and larger, the n-processes converge to a limiting process. This will be the contact process.

Suppose that that the family at number 23 is not infected, and we want to know the probability that this family becomes infected during the next week. We look at whether this family becomes infected in the n-process. Either the family will become infected for all large n, or the family will stay free of infection for all large n. The probability that that family becomes infected the the n-processes will therefore converge to the probability that this family is infected in the 'real' contact process, if we take n larger and larger.

In general, when we give an informal description of the behaviour of the particles in a certain system, we should make clear that that system can be constructed.

1.3.4 Stationary distributions

As in queueing systems, also in interacting particle systems one can ask whether there exist stationary distributions of a system.

We will explain what we mean by this in the example of the contact process. The state of the contact process at time t is random, and one could consider e.g. the probability that the families at the houses 34 and -89 are both infected at time t. We could ask whether this probability (and other probabilities) converge to a constant when time goes on. In other words, is there a stationary distribution of the process? Such a stationary distribution should then specify which houses are infected with what probabilities. Moreover, it should be the case that if we start at time 0 with a random state in which the probabilities that houses are infected are the same as in this stationary distribution, these probabilities stay the same for all t.

We will not go into the details of the stationary distributions of the contact process. We just notice that e.g. the distribution in which all families are not infected is stationary: if at time 0 nobody is infected, this stays the same for all t. There are also stationary distributions in which a fraction of the families is infected. To which stationary distribution the distribution of the system converges, as time tends to infinity, may depend on the state of the system at time 0 or the parameter λ. For details on the contact process, see Durrett (1995) and Liggett (1985, 1999).

Generally, when we have constructed a certain particle system, we will ask whether there is a stationary distribution and whether this stationary distribution is unique. We are also interested in properties of these stationary distributions, for example the mean number of particles in a certain part of the grid.

1.4 Particle systems in this thesis

In this section we describe the particle systems in this thesis and the results we obtain. The particle system discussed in Chapter 4 can be interpreted as a rather exotic queueing system, although that is not the way it is presented there. The system in Chapter 5 can be interpreted as a bricklayer process. Chapter 4 is based on joint work with Ronald Meester and Chapter 5 on joint work with Lorna Booth.

1.4.1 A supermarket with infinitely many check-outs

We will first give an informal description of the particle system in Chapter 4. Suppose you are in an infinitely large supermarket. There is also an infinite row of check-outs, which does not end, no matter whether you look to the left or to the right. The check-outs are numbered ..., -2, -1, 0, 1, 2, ...

At each check-out, there are customers who want to pay for their shopping, arriving according to a Poisson process with parameter λ. These arrival processes are independent of each other. Suppose that the times needed to help a customer are independent and exponentially distributed with parameter μ.

Until now, we have just described an infinite row of queues and, without any interaction, we know how each of these rows behaves, namely as the queues in Section 1.1.5.

We now let these queues interact in the following way. If a customer wants to pay his shopping but there is already a customer at his check-out, he walks to the nearest empty check-out on the left. We assume that the customer arrives at this check-out instantaneously (so he can walk at infinite speed).

As indicated in Section 1.3.3, we must give a construction of the system described above. A way to do this, is to consider first modifications of this system, in which there are still infinitely many check outs, but in which there are only arriving customers at the check-outs numbered from $-n$ to n. We call such a system an n-system. It is possible that a customer is served at a check-out to the left of $-n$, but there is no arrival stream associated to those check-outs.

The n-systems are easy to construct; there are only finitely many changes in a finite time.

We use the same (infinite) sequence of arrival processes to compute the state of every n-system at a certain time t. Of course, for every n-system, only $2n + 1$ of the arrival streams are used. So for example if a customer arrives at check-out number 16, we suppose it arrives there in all n-systems with $n \geq 16$. We let corresponding customers have the same service times in all n-systems. It turns out, that the limit of these n-systems exists for n tending to infinity. We define the state of our queueing system at time t to be equal to this limit.

We will prove that, as we let t tend to infinity, the probability distribution of the state of the system at time t converges to a stationary distribution. This stationary distribution is unique: given λ and μ, there is only one stationary distribution, and the distribution of the system converges to this stationary distribution, no matter in which configuration the system started. Remember that if the system starts according to this stationary distribution, it will have that distribution forever, so the distribution of the system does not vary in time in that case. Again, in general this does not mean that the system itself does not change in time, it is just that for every event the probability of seeing that event is constant in time.

For $\lambda \geq \mu$, in the stationary distribution, every check-out is occupied (so in this case the state of the system does not change), but for $\lambda < \mu$, the probability that a check-out is occupied will converge to $\frac{\lambda}{\mu}$. This means that in the stationary distribution, a fraction $\frac{\lambda}{\mu}$ of the check-outs is occupied. So there is a positive fraction of unoccupied check-outs for exactly those parameter values for which the queueing system described in Section 1.1.5 is stable, which is intuitive, since in this case the average inter-arrival times are larger than the average service times.

1.4.2 The bricklayer process

Chapter 5 deals with the construction of a bricklayer process. This process was introduced and analysed by Balàzs (2001), but not constructed. We will construct this process, the construction proceeds again via a limit of simpler processes.

The bricklayer process can informally be described as follows. Imagine an infinitely long wall, built of bricks as in Figure 1.3. We are only interested in the height differences in the wall, which we denote by $\tilde{\omega}(i)$ for all integers i. If you make a walk along the top of this wall from the right to the left, the difference is positive if you go up and negative

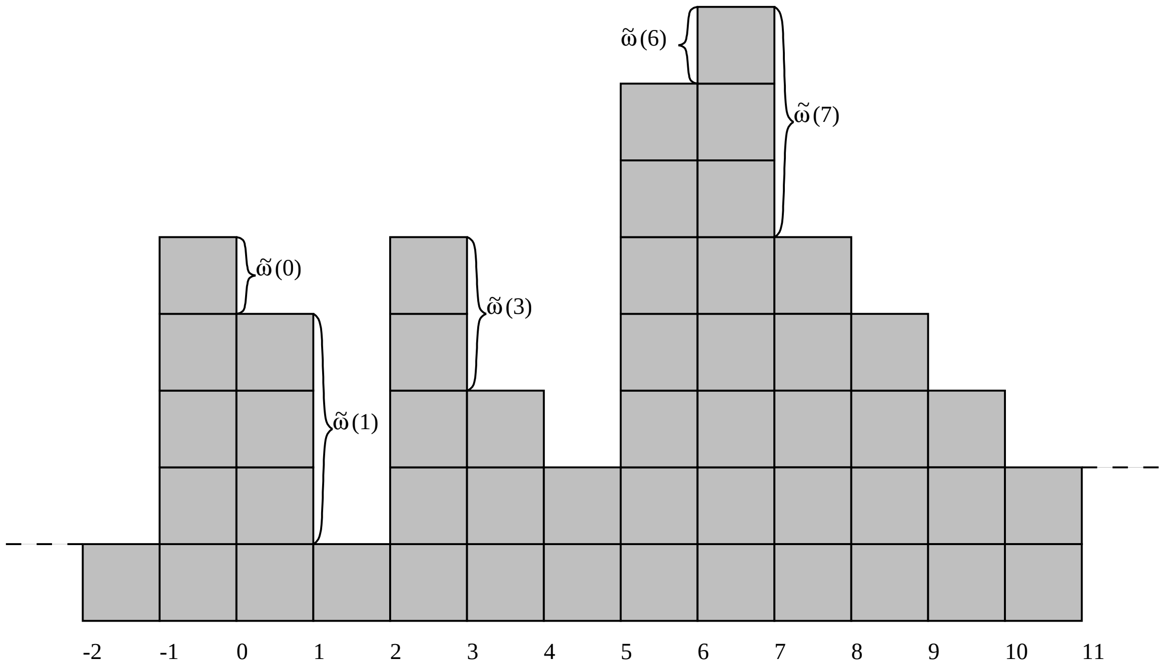

Figure 1.3: Bricklayer process

if you go down. For example in Figure 1.3, $\tilde{\omega}(0) = 1$, $\tilde{\omega}(1) = 3$, $\tilde{\omega}(3) = 2$, $\tilde{\omega}(6) = -1$ and $\tilde{\omega}(7) = 3$. We pretend that there is a bricklayer standing at each point i. These bricklayers add bricks to the wall. Each bricklayer considers the height difference at his position, and after a time which is exponentially distributed, he puts a new brick on the wall, either to his right or to his left. The parameter of this exponential distribution depends on the height difference at his position.

For example consider the situation in the left side of Figure 1.4 and suppose the bricklayer at position 23 decides to lay a brick to his right. Before this brick is laid, we have $\tilde{\omega}(22) = -1$, $\tilde{\omega}(23) = 2$, $\tilde{\omega}(24) = -1$ and $\tilde{\omega}(25) = 1$. After the new brick is laid, the wall looks as in the right side of Figure 1.4 and $\tilde{\omega}(22) = -1$, $\tilde{\omega}(23) = 1$, $\tilde{\omega}(24) = 0$ and $\tilde{\omega}(25) = 1$. So each new brick changes two height differences.

The construction we give is only valid if the parameter values of the exponential times obey certain conditions, so the construction problem of this process is still partly open. In the construction, we first allow only the bricklayers at the positions from $-n$ up to n to lay bricks. Then we make the region in which bricklayers lay bricks larger and larger, and we show that the limit of these processes exists. This limit is the bricklayer process.

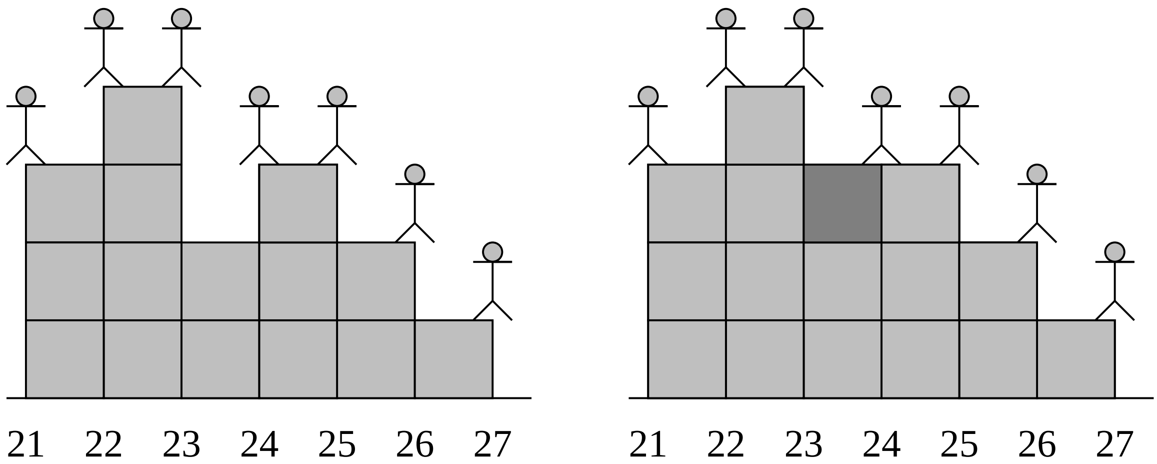

Figure 1.4: Adding a brick

Chapter 2

Stability of certain queueing systems on a circle

2.1 Introduction and results

In this chapter we give new stability proofs for two queueing systems on a circle. The stability results were obtained earlier, but for several reasons we think it is interesting to have new proofs. One reason is that we think they are more intuitive than the earlier proofs, but of course one can argue about this. More importantly, there are queueing systems which one expects to be stable, but for which the stability question is still open. An example of such a system is the continuous greedy system described below. In fact we hope that the idea of our proofs will eventually be the key to proving the stability of the continuous greedy system, but until now we have not succeeded in doing so.

Consider the following continuous 'non-greedy' queueing system. Customers arrive on the perimeter of a circle with circumference one, according to a Poisson process with parameter $\lambda > 0$ (we assume that the system is empty at time 0). Each customer chooses a waiting position on the circle uniformly, independently of the state of the system and of each other. A server is travelling clockwise along the circle at constant speed, and without loss of generality we assume that the server travels at speed one. When the server encounters a customer, he stops and serves that customer; after the service he continues his journey (in the same direction as before). Unless stated otherwise, we assume that the server does not travel if there are no customers present on the circle, but this is of course arbitrary. Service times are independent and identically distributed (i.i.d.) with finite mean μ^{-1}.

A variant of this system is the continuous 'greedy' queueing system, in which the arrival process is the same as above, but now the server travels at constant speed in the direction of the nearest customer on the circle. This means that arriving new customers can make the server change his direction. When the server encounters a customer, he stops and serves that customer. Service times are i.i.d. with finite mean μ^{-1} and after the service the server travels on, again in the direction of the nearest customer. Light-traffic results (i.e. results for systems in which λ is relatively small) for the continuous greedy system can be found in Kroese and Schmidt (1994). In that article, first-and second-order Taylor expansions are derived for the expected configuration of the customers, the mean queue length etc. They also show that, in light traffic, the greedy queueing system is more efficient than the non-greedy system.

We say that a system is *stable* if the expected length of a busy period is finite. In Section 2.2 we give a new and elementary proof of the following result, which was also obtained in Kroese and Schmidt (1992, 1993).

Theorem 2.1.1 *The continuous non-greedy queueing system described above is stable if and only if* $\lambda < \mu$.

Observe that an ordinary $M/G/1$ system (in which the distribution of the number of customers in the system is the same as in a continuous greedy or non-greedy system in which the server travels at infinite speed) is also stable if and only if $\lambda < \mu$, so the speed of the server does not affect the parameter values for which the continuous non-greedy queueing system is stable. The continuous greedy system is believed to be stable under the same conditions, but this has not yet been proved.

Our proof of Theorem 2.1.1 is based on the average travel time between customers and our strategy will be as follows. Suppose that $\lambda < \mu$. For the system to be unstable, the average travel time must be positive, otherwise we can essentially compare with an ordinary M/G/1 system. But if the average travel time is positive, then there can be no accumulation of customers, and this implies that the system must be stable.

In Section 2.3 we show that the idea of the stability proof, as presented for the continuous non-greedy queueing system, is also applicable to a discrete greedy system on a circle, in which the server always travels to the nearest customer. As for Theorem 2.1.1, the following result (Theorem 2.1.2) was obtained earlier, this time in Foss and Last (1996).

We first describe the discrete greedy system. Consider a system with k waiting stations, which are numbered $1, \ldots, k$. The stations are located at equal distances on a circle with circumference 1, so the distance of a station to the two nearest stations is k^{-1}. Each station has an infinite waiting capacity. Customers enter the system according to a Poisson process with parameter $\lambda > 0$. Each customer joins the queue at one of the stations, where the choice of station is independent of the current state of the system and each station has probability k^{-1} to be chosen. Service times are i.i.d. with expectation $\mu^{-1} > 0$. A server is travelling along the circle at constant speed, always in the direction of the nearest non-empty station. If there are two nearest non-empty stations, he chooses one of them, by tossing a fair coin. Without loss of generality we assume that the server travels at speed 1. When he arrives at a station with waiting customers, he stops and serves all customers at this station until the station is empty. When the station is empty, he finds the nearest non-empty station, and starts walking in that direction. It is possible that the server changes direction during a walk due to an arrival of a new customer at a station which is nearer than the station to which the server was travelling originally. The server does not travel when no customer is present in the system.

Results on similar systems dealing with stability of polling systems with state dependent travelling strategies can be found in for instance Foss and Last (1996, 1998) and Schassberger (1995). We shall prove the following stability theorem:

Theorem 2.1.2 *The discrete greedy queueing system described above is stable if and only if* $\lambda < \mu$.

The idea of the proof is the same as the idea of the stability proof for the continuous non-greedy queueing system described earlier. Suppose $\lambda < \mu$. For the system to be unstable, the average travel time must again be positive. But if the average travel time is positive, it is easy to show that it regularly happens that the server visits all stations in the order $(1, 2, \dots, k, 1)$, i.e. the server makes a complete tour along all stations. This essentially implies that the number of customers on the circle can not become too large and this quickly leads to a stability proof.

2.2 Stability of the continuous non-greedy system

In this section we make the idea, described in the introduction, rigorous and start with some notation. The number of customers that has arrived in the system until time t is denoted by $A(t)$, the length of the i^{th} service in the system by S_i. The amount of time used for serving until time t is denoted by $S(t)$, the amount of time used for travelling by $W(t)$, and $Z(t)$ denotes the amount of time until time t that the system was empty. Note that

$$S(t) + W(t) + Z(t) = t. \tag{2.1}$$

The travel time of the server between the $(i-1)^{th}$ service and the i^{th} service is denoted by W_i. Finally, the travelling of the server between two consecutive services is called a journey.

Lemma 2.2.1 *Let* $\lambda < \mu$ *and suppose that the system is not stable. Then there exists* $\epsilon_0 > 0$ *such that with probability one,*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i > \epsilon_0.$$

Proof: We prove the contrapositive. Fix some $0 < \delta < 1$, and let W_δ be the event that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i < \delta$. Suppose that $P(W_\delta) > 0$. The strong law of large numbers tells us that a.s. for t large enough we have

$$A(t) \leq t(\lambda + \delta). \tag{2.2}$$

Also with probability one, for n large enough we have

$$\frac{1}{n}\sum_{i=1}^{n} S_i \leq \frac{1}{\mu} + \delta. \tag{2.3}$$

Combining (2.2) and (2.3), we see that for some $\kappa > 0$ (independent of δ), we have for t large enough,

$$S(t) \leq t\,(\lambda + \delta)\left(\frac{1}{\mu} + \delta\right) \leq t\left(\frac{\lambda}{\mu} + \kappa\delta\right). \tag{2.4}$$

If W_δ occurs, we know that for n large enough

$$\frac{1}{n}\sum_{i=1}^{n} W_i < 2\delta, \tag{2.5}$$

and since the number of journeys the server has made up to time t is bounded above by the total number of customers arrived by time t, we conclude from (2.2), (2.4) and (2.5) that for some $\kappa' > 0$ (again independent of δ), we have with probability one for that for t large enough,

$$S(t) + W(t) \leq t\left(\frac{\lambda}{\mu} + \kappa'\delta\right).$$

Now take δ so small that $\frac{\lambda}{\mu} + \kappa'\delta < 1$. For these values of δ we have, using (2.1), that if W_δ occurs,

$$\liminf_{t\to\infty} \frac{Z(t)}{t} > 0.$$

We conclude that if $P(W_\delta) > 0$ for *any* small enough δ, the empty state is positive recurrent with positive probability and therefore also positive recurrent almost surely. This is the contrapositive of what we wanted to prove and therefore we are done. □

Proof of Theorem 2.1.1: Clearly the system cannot be stable if $\lambda \geq \mu$. Next suppose that $\lambda < \mu$ and that the system is unstable. From Lemma 2.2.1 we obtain $\epsilon_0 > 0$ such that with probability one,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} W_i > \epsilon_0.$$

This implies that there almost surely exists a (random) sequence $L_1 < L_2 < \cdots$ such that for all j,

$$\frac{1}{L_j}\sum_{i=1}^{L_j} W_i > \frac{1}{2}\epsilon_0. \tag{2.6}$$

Denote the time at which the i^{th} service starts by T_i. Let us mark the position of the server at time 0 by $\star$. It follows from (2.6) that the number of times (after time 0) that the server has been in $\star$ until T_{L_j} is at least $\lfloor \frac{1}{2}\epsilon_0 L_j \rfloor$. Let M_i be the number of customers on the circle at the moment the server is in $\star$ for the i^{th} time after time 0. Since all $M_1 + \cdots + M_i$ customers must have been served by the time the server reaches $\star$ for the $(i+1)^{th}$ time, we see that

$$\sum_{i=1}^{\lfloor \frac{1}{2}\epsilon_0 L_j \rfloor - 1} M_i < L_j.$$

Hence there exists a constant $C > 0$, such that for all large j we have

$$\frac{1}{\lfloor \frac{1}{2}\epsilon_0 L_j \rfloor - 1}\sum_{i=1}^{\lfloor \frac{1}{2}\epsilon_0 L_j \rfloor - 1} M_i < C. \tag{2.7}$$

From (2.7) it follows that at least half of the M_i's in the sum are smaller than $2C$. We conclude that there exists a positive constant γ, such that for all large j the following statement is true:

Statement 2.2.2 *The number of times before T_{L_j} that the server has been in $\star$, while at the same time the corresponding M_i is at most $2C$, is at least γL_j.*

Each time this happens, there is a uniform positive lower bound on the probability that all (at most) $2C$ customers are served before a new one arrives, and this lower bound does not depend on the past of the process. That is to say that there is another positive constant γ' such that for all large j the following statement is true:

Statement 2.2.3 *The number of time intervals before T_{L_j} during which the system was empty is at least $\gamma' L_j$.*

To complete the proof, we observe that there exists almost surely a positive constant K, such that

$$T_i \leq Ki, \text{ for all large } i.$$

This bound follows from the observation that $T_{i+1} - T_i$ is dominated by the sum of a service time, an inter-arrival time and 1 (the maximal travel time), all independent of each other, and independent for different values of i. Hence the number of time intervals until KL_j during which the system was empty is at least $\gamma' L_j$, for j large, that is, a number linear in time. This implies that the expected time between two 'empty intervals' cannot be infinite and we are done. □

2.3 Stability of the discrete greedy queueing system

Before we prove Theorem 2.1.2 we start with the following lemma.

Lemma 2.3.1 *Let $M \in \mathbb{R}$ and let $x_1, x_2, \ldots$ be a sequence with $0 \leq x_i \leq M$ for all i and let $\delta > 0$. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \geq \delta,$$

implies that there exists a sequence $K_1 < K_2 < \cdots$ such that

$$\frac{1}{K_j} \sum_{i=1}^{K_j} \mathbf{1}_{\{x_i > 0\}} \geq \frac{\delta}{2M}, \text{ for } j = 1, 2, \ldots$$

Proof: Suppose

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \geq \delta,$$

then there exists a sequence $K_1 < K_2 < \cdots$ such that

$$\frac{1}{K_j} \sum_{i=1}^{K_j} x_i \geq \frac{\delta}{2}.$$

Since $0 \leq x_i \leq M$ for all x_i, we can conclude that for all j, the fraction of the x_i's, $i \leq K_j$, which are positive must be at least $\frac{\delta}{2M}$. □

Proof of Theorem 2.1.2: If $\lambda \geq \mu$ it is obvious that the system is not stable (we can compare to a M/G/1 system again). So we have to prove that $\lambda < \mu$ implies stability of the system. The idea of the proof is very much the same as for the proof of Theorem 2.1.1, only the details are a bit trickier.

Suppose $\lambda < \mu$ and suppose that the system is not stable. Let W_i be the travel time of the server between the $(i-1)^{th}$ service and the i^{th} service in the system. Lemma 2.2.1 applies also to this system, giving an $\epsilon_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} W_i \geq \epsilon_0, \tag{2.8}$$

with probability 1.

Before we continue, it pays to indicate the difference between the proof for the non-greedy system as described in Section 2.2 and the current proof. In the non-greedy system we could conclude from (2.8) that the server travelled around the whole circle regularly, which implied that the number of customers on the circle could not get too large. In the current case the server can change direction and it is not immediately clear from (2.8) that the server visits all stations regularly. We shall show that this is the case nevertheless and once we have proved that, we can finish the proof in a same way as the proof of Theorem 2.1.1.

To prove that the server visits all stations regularly, we start by proving that the server regularly starts a walk of positive length from (say) the first station. From the fact that with probability 1, $0 \leq W_i \leq \frac{1}{2}$, (2.8) and Lemma 2.3.1 we conclude that with probability 1, there exists a random sequence $L_1 < L_2 < \cdots$ such that for all j,

$$\frac{1}{L_j} \sum_{i=1}^{L_j} \mathbf{1}_{\{W_i > 0\}} \geq \epsilon_0. \tag{2.9}$$

Define for $l = 1, \ldots, k$,

$$A_i^l = \begin{cases} 1 & \text{if } W_i > 0 \text{ and the } i^{th} \text{ walk starts at station } l, \\ 0 & \text{otherwise.} \end{cases}$$

Since there are only finitely many stations, we claim that there exist an l, a $\delta > 0$ and a subsequence $L_1' < L_2' < \cdots$ such that

$$\frac{1}{L_j'} \sum_{i=1}^{L_j'} A_i^l \geq \delta. \tag{2.10}$$

To see this, note that $\mathbf{1}_{\{W_i > 0\}} = \sum_{l=1}^{k} A_i^l$. From (2.9) we find that

$$\frac{1}{L_j} \sum_{i=1}^{L_j} \sum_{l=1}^{k} A_i^l \geq \epsilon_0.$$

Interchanging the summation order yields for all j the existence of $l_0(j)$ such that

$$\frac{1}{L_j}\sum_{i=0}^{L_j} A_i^{l_0(j)} \geq \frac{\epsilon_0}{k}.$$

One of the l's must be equal to infinitely many $l_0(j)$'s and hence there exists an l, such that

$$\frac{1}{L_j}\sum_{i=1}^{L_j} A_i^l \geq \frac{\epsilon_0}{k} \quad \text{i.o.}$$

This proves (2.10) and without loss of generality we assume that (2.10) is the case for $l = 1$.

Next we define B_i as follows: $B_i = 1$ if $A_i^1 = 1$ and in addition, the server does not return to station 1 before he has visited all other stations. In all other cases, B_i is defined to be 0. For instance, B_i is equal to 1 if $A_i^1 = 1$ and the server chooses stations $2, 3, \ldots, k$ (in that order) which is possible if we 'make sure' that customers are present at the appropriate stations at the appropriate times. This can be arranged by 'letting customers arrive' at all stations after the end of the $(i-1)^{th}$ service together with certain choices of the server about the next direction to go to. This makes it clear that the conditional probability that $B_i = 1$, given $A_i^1 = 1$ and the complete history of the process until the end of the $(i-1)^{th}$ service is uniformly bounded away from zero. This implies that for all large n we have, for some $\eta > 0$,

$$\eta \leq \frac{\sum_{i=1}^n B_i}{\sum_{i=1}^n A_i^1} \leq 1. \tag{2.11}$$

In particular, for j large enough, we have from (2.10) and (2.11) that

$$\frac{1}{L'_j}\sum_{i=1}^{L'_j} B_i \geq \delta\eta. \tag{2.12}$$

Let N_i be the total number of customers present at stations $2, \ldots, k$ after the $(i-1)^{th}$ service if $A_i^1 = 1$; $N_i := 0$, otherwise. The remark above concerning the uniform lower bound on the conditional probability for B_i to be 1 implies that we also have for all C and n large,

$$\eta \leq \frac{\sum_{i=1}^n \mathbf{1}_{\{B_i=1\}}\mathbf{1}_{\{N_i>C\}}}{\sum_{i=1}^n \mathbf{1}_{\{A_i^1=1\}}\mathbf{1}_{\{N_i>C\}}} \leq 1. \tag{2.13}$$

We claim the following:

Statement 2.3.2 *There exists a C such that with probability one, for L'_j large enough, we have*

$$\frac{\sum_{i=1}^{L'_j} \mathbf{1}_{\{A_i^1=1\}}\mathbf{1}_{\{N_i\leq C\}}}{\sum_{i=1}^{L'_j} \mathbf{1}_{\{A_i^1=1\}}} \geq \frac{1}{2}. \tag{2.14}$$

To see this, we assume that the converse of Statement 2.3.2 is true, and derive a contradiction. This converse is the following statement: For all C, there is positive probability that there is a subsequence L'_{j_k} such that for all k,

$$\frac{\sum_{i=1}^{L'_{j_k}} \mathbf{1}_{\{A_i^1=1\}}\mathbf{1}_{\{N_i>C\}}}{\sum_{i=1}^{L'_{j_k}} \mathbf{1}_{\{A_i^1=1\}}} \geq \frac{1}{2}. \tag{2.15}$$

If (2.15) were true, (2.11) and (2.13) would give that for some positive β, the following statement is true:

Statement 2.3.3 *For all C, there is a positive probability that there is a subsequence L'_{j_k} such that for all k,*

$$\frac{\sum_{i=1}^{L'_{j_k}} \mathbf{1}_{\{B_i=1\}}\mathbf{1}_{\{N_i>C\}}}{\sum_{i=1}^{L'_{j_k}} \mathbf{1}_{\{B_i=1\}}} \geq \beta.$$

We claim that Statement 2.3.3 contradicts (2.12). To see this, just note that (2.12) implies that for large k, the number of indices $i \leq L'_{j_k}$ for which $B_i = 1$ is at least $\delta\eta L'_{j_k}$. Since none of the N_i customers in Statement 2.3.3 are at station 1, they can not contribute to

$$\sum_{j=1}^{L'_{j_k}} B_j.$$

Therefore Statement 2.3.3 tells us that the number of indices $i \leq L'_{j_k}$ for which $B_i = 1$ does *not* occur, is at least $(\delta\eta L'_{j_k} - 1)C\beta$. (We subtract one since after the last occurrence of $B_i = 1$ before L'_{j_k}, it is not clear that all N_i customers really count.) These two estimates are incompatible for large C.

Now we finish the argument as in the proof for the non-greedy system in Section 2.2. Statement 2.3.2 together with (2.10) yield that there exists a C such that for j large enough,

$$\frac{1}{L'_j}\sum_{i=1}^{L'_j} \mathbf{1}_{\{A_i^1=1\}}\mathbf{1}_{\{N_i\leq C\}} \geq \frac{\delta}{2}. \tag{2.16}$$

Let T_i be the time at which the i^{th} service starts. We can conclude from (2.16) that there exists a positive constant γ, such that the following statement is true:

Statement 2.3.4 *The number of times before $T_{L'_j}$ that the server has been at station 1 starting a walk of positive length, while at the same time the total number of customers at stations $2, \ldots, k$ was at most C, is at least $\gamma L'_j$.*

Each time this happens, there is a uniform positive lower bound on the probability that all (at most) C customers are served before a new one arrives, and this lower bound does not depend on the past of the process. So there is another positive constant γ' such that for all j large enough the following statement is true:

Statement 2.3.5 *The number of time intervals before $T_{L'_j}$ during which the system was empty is at least $\gamma' L'_j$.*

Observe that, as in the non-greedy case, there exists a constant K such that

$$T_i \leq Ki, \text{ for all large } i,$$

since in this case, the difference $T_i - T_{i+1}$ is dominated by the sum of a service time, an inter-arrival time and $\frac{1}{2}$, the maximal travel time. These are all independent of i and each other. We can now finish our argument in the same way as in the non-greedy case. $\square$

Chapter 3

Weakly convergent approximations of the non-greedy queueing system

3.1 Introduction and result

In this chapter we consider the continuous non-greedy queueing system as described in Section 2.1. We call this system the original system. We also consider a system that strongly resembles the original system, but with one extra restriction: at most k customers are allowed on the circle. Those customers who arrive at a moment that there are already k customers present on the circle are sent away, and do not return. We call this system the *k-system*. We will prove that the k-systems are approximations of the original system in some sense.

To describe the result precisely, we have to say a few words about weak convergence of random counting measures. We first need some notation. Denote the circumference of the circle by C, and suppose that the server is always at a fixed position on the circle. This is equivalent to supposing that the server stands still and the customers move towards him. For two points x and y on the circle, we define $d(x, y)$ to be the (shortest) distance between these points, measured along the perimeter of the circle. Observe that $d : C \times C \to [0, \frac{1}{2}]$ is a metric. The space C is a complete separable metric space, which we equip with the Borel σ-algebra $\mathcal{B}$.

We define X_t to be the random counting measure on C, corresponding to the customers who are waiting (or being served) on the circle at time t in the original system. Similarly, let X_t^k be the random counting measure on C, corresponding to the customers who are waiting (or being served) on the circle at time t in the k-system.

When $\lambda < \mu$, it follows from Theorem 2.1.1 that X_t is a regenerative process, with regeneration periods that have absolutely continuous distributions and finite expectations. Hence, as in Kroese and Schmidt (1992, 1993), X_t converges in distribution to a limiting random counting measure X, when $t \to \infty$. Similarly, X_t^k converges in distribution to a limiting random counting measure X^k.

We shall prove the following result, which might appear obvious, but which seems surprisingly difficult to prove.

Theorem 3.1.1 *Let $\lambda < \mu$. Then X^k converges weakly to X, when $k \to \infty$.*

A few words of explanation are appropriate here. In connection with weak convergence of random counting measures we recall Theorem 9.1.VI in Daley and Vere-Jones (1988), which says that weak convergence of random counting measures in the appropriate setting is equivalent to convergence of finite dimensional distributions of continuity sets, i.e. sets whose boundary has probability zero to contain points under the limiting counting measure. This means that we need only show that the appropriate finite dimensional distributions converge weakly.

The result is proved by making a coupling of the original system and the k-system. This is described in Section 3.2. In the coupling, we start in two empty systems. Both systems behave identically until the moment that there are $k+1$ customers present in the original system. From that moment the systems are not identical, but, in this coupling, it is the case that when the original system is empty the k-system is also empty. So after the original system becomes empty, both systems are identical for a long time again (if k is large), until in the original system the level of $k+1$ customers is reached. In Section 3.3 we consider time periods, during which the level of the number of customers in the original system has been larger than k and since then has has not achieved the zero level again. We show that the stationary probability of being in such a period tends to zero, as $k \to \infty$. This will suffice to prove the result.

3.2 Coupling of the original system and the k-system

Let $\lambda < \mu$. We construct a coupling of the k-system and the original system. In this coupling we assume, contrary to what we said in Section 2.1, that the servers do continue to travel when no customers are present on the circle, but this does not make a real difference. At time 0 both systems are empty. Customers arrive and depart in the original system as described in Section 2.1. In the k-system, we let customers arrive at exactly the same moments as in the original system. Of course some of them are sent away, because there are already k customers present in the system at their arrival. We call customers that arrive at the same time in both systems *corresponding customers*. The arrival location of the customers in the k-system is chosen such that at the moment of arrival, the distance between the server and the arriving customer in the k-system equals the distance between the server and the corresponding customer in the original system. The service time of a customer in the k-system is equal to the service time of the corresponding customer in the original system. In the coupling, we denote the random counting measure on C corresponding to the customers in the original system by Y_t. The random counting measure on C corresponding to the customers in the k-system is denoted by Y_t^k. The pair $(Y_t^k, Y_t)_{(t\geq 0)}$ is a coupling of $X_t^k{}_{(t\geq 0)}$ and $X_{t(t\geq 0)}$. In this coupling, the following lemma holds, the proof of which is surprisingly lengthy.

Lemma 3.2.1 *In the coupling described above, $Y_t(C) = 0$ implies that $Y_t^k(C) = 0$, for all k.*

Proof: The proof is by induction. Observe both systems from the first moment that a customer is sent away from the k-system until the next moment that the original system is empty again. During this time interval, let l denote the number of customers that arrived in the original system at a moment that there were less than k customers in the k-system (so the corresponding customer was not sent away in the k-system). Let n be the number of customers that arrived in the original system at a moment that there were k customers present in the k-system (so the corresponding customers were sent away in the k-system). We call the latter customers *additional customers* since they are present in the original system, but have no corresponding customers in the k-system.

We use the following notation:

- $U(t)$ is the distance that the server has travelled in the original system until time t.
- $U_k(t)$ is the distance that the server has travelled in the k-system until time t.
- T is the first moment at which the original system is empty, after a customer has been sent away from the k-system.
- S_i is the service time of the i^{th} additional customer.
- S_i^* is the service time of the i^{th} customer who arrives *and* takes his place in the k-system. By definition of the coupling, S_i^* is then also the service time for the corresponding customer in the original system.
- $S_k(t)$ is the total time used for serving in the k-system, until time t.

We shall show that for all $t \leq T$,

$$U_k(t) - U(t) \leq S_1 + \cdots + S_n. \tag{3.1}$$

We claim that (3.1) implies that the k-system is empty at time T. To see this note that, by definition, the original system is empty at time T, so

$$T = U(T) + S_1^* + S_2^* + \cdots + S_l^* + S_1 + S_2 + \cdots + S_n. \tag{3.2}$$

Since $T = U_k(T) + S_k(T)$, we conclude from (3.2) that

$$U_k(T) - U(T) = S_1^* + S_2^* + \cdots + S_l^* + S_1 + S_2 + \cdots + S_n - S_k(T). \tag{3.3}$$

From $U_k(T) - U(T) \leq S_1 + \cdots + S_n$ and (3.3) we find

$$S_1^* + S_2^* + \cdots + S_l^* + S_1 + S_2 + \cdots + S_n - S_k(T) \leq S_1 + \cdots + S_n. \tag{3.4}$$

This implies that

$$S_k(T) \geq S_1^* + S_2^* + \cdots + S_l^*. \tag{3.5}$$

Since the server in the k-system cannot have served more customers than the customers who have arrived in the k-system,

$$S_k(T) \leq S_1^* + S_2^* + \cdots + S_l^*. \tag{3.6}$$

From (3.5) and (3.6) we conclude that

$$S_k(T) = S_1^* + S_2^* + \cdots + S_l^*,$$

which implies that at time T all customers that have arrived in the k-system have been served, so that the k-system is empty at time T. It therefore suffices to prove (3.1).

We first prove that (3.1) holds for all realisations of the two systems in which $n = 1$ and $l = 0$. Next we prove that if we assume that (3.1) is true for all realisations of the coupling in which $n = 1$ and $l = q$, then (3.1) must be true for all realisations of the coupling in which $n = 1$ and $l = q + 1$. Finally we show that if (3.1) is true for all realisations of the coupling in which $n \leq p$ and l is arbitrary, (3.1) is true for all realisations in which $n = p + 1$ and l is arbitrary.

In the case that $n = 1$ and $l = 0$, one customer has been sent away in the k-system, and after that no other customers arrived until the original system became empty. To prove that for $t \leq T$, $U_k(t) - U(t) \leq S_1$, we distinguish between two possibilities:

- The additional customer is the last customer served before the original system is empty.

  In this case, until the server arrives at the additional customer in the original system, both servers are at the same position on the circles. Observe that the k-system is empty at the moment that the server starts the service of the additional customer. During this service, the server in the original system stands still, while the server in the k-system travels on. Therefore, when the server serves the additional customer, the difference $U_k(t) - U(t)$ grows from zero to S_1. After the service of the additional customer the original system is empty, so that we are at time T. So (3.1) holds in this case.

- The additional customer waits between customers who arrived earlier.

  This implies that, until in the original system the additional customer gets served, both servers are at exactly the same location on the circles. Therefore, during that period, $U_k(t) - U(t) = 0$. During the service of the additional customer, $U_k(t) - U(t) \leq S_1$, since the server in the original system does not move for a period of length S_1. After the service of the additional customer, $U(t) = U_k(t - S_1)$. Observe now that the difference $U_k(t) - U_k(t - S_1)$ is at most S_1 (this can happen if the server of the k-system travels the whole period from time $t - S_1$ until time t). So (3.1) holds also in this case.

This proves (3.1) for all realisations of the coupling in which $n = 1$ and $l = 0$.

Next, suppose that, for all realisations of the coupling in which $n = 1$ and $l = q$, $U_k(t) - U(t) \leq S_1$, $\forall t \leq T$ (induction hypothesis). We want to prove that this implies that also $U_k(t) - U(t) \leq S_1$, $\forall t \leq T$, when we have a realisation of the coupling in which $n = 1$ and $l = q + 1$. We can prove this by looking at the last corresponding customers who arrived in both systems, before T. This is the $(q + 1)^{st}$ customer that entered the systems after the additional customer arrived in the original system (that is why we shall call this customer the $(q + 1)^{st}$ customer). Now we compare the positions of the servers with the position of the servers in the realisation of the systems in which these $(q+1)^{st}$ customers do not arrive (but the other customers arrive at the same times and places and have the same service times). As long as the services of these corresponding $(q+1)^{st}$ customers have not started, we see no difference. In such a realisation, without the $(q+1)^{st}$ customers, the number of customers that arrive after the additional customer but before time T equals q, so we know by the induction hypothesis that as long as the services of these corresponding customers have not started, $U_k(t) - U(t) \leq S_1$.

Now we distinguish between four cases for the positions where the $(q+1)^{st}$ customers are situated on the circles with respect to the other customers in both systems:

1. The $(q+1)^{st}$ customer is the last customer served before time T, in both systems.
2. In both systems, the $(q+1)^{st}$ customer is not the last customer served before time T.
3. The $(q+1)^{st}$ customer is the last customer served before time T in the k-system, but is not the last customer served before time T in the original system.
4. The $(q+1)^{st}$ customer is the last customer served before time T in the original system, but is not the last customer served before time T in the k-system.

Case 1. The induction hypothesis implies that had the $(q+1)^{st}$ customer not been present, the k-system would not empty later than the original system. Therefore, the server in the k-system starts its journey to the $(q+1)^{st}$ customer not later than the server in the original system. In other words, until the server in the k-system starts travelling to his $(q+1)^{st}$ customer, $U_k(t) - U(t) \leq S_1$.

We first explain that it is not possible that the difference $U_k(t) - U(t)$ exceeds the level S_1 before the server in the k-system arrives at his $(q+1)^{st}$ customer. If the difference $U_k(t) - U(t)$ equals S_1, then the server in the k-system must have travelled S_1 more than the server of the original system, and have served all customers in the k-system but the last one. So at that moment, the server in the original system would also have served all corresponding customers, and since he has been serving for S_1 longer than the server in the k-system, he has also served the additional customer in his system. So if the difference $U_k(t) - U(t)$ is equal to S_1 and the server in the k-system is travelling, the server in the original system must be travelling too, so that the difference $U_k(t) - U(t)$ cannot grow.

We show now that the server in the k-system arrives earlier at his last customer than the server in the original system arrives at the corresponding customer. Consider the distance which the server in the k-system has travelled at the moment that he reaches his last customer. By the observation above, this distance is at most S_1 larger than the distance which the server in the original system has travelled at the moment that he reaches his last customer. So when the server in the k-system starts serving the last customer, he used at most S_1 more time for travelling than the server in the original system. Since the server in the original system required S_1 more time for serving his additional customer, he cannot arrive earlier at his last customer.

During the service of the $(q+1)^{st}$ customer in the k-system, the difference $U_k(t) - U(t)$ cannot increase, since the server in the k-system does not move.

After the server in the k-system has served the last customer, the difference $U_k(t) - U(t)$ cannot become larger than S_1 either. Since if at a certain moment T^*, $U_k(T^*) - U(T^*)$ would be equal to S_1, we would have

$$T^* = S_1^* + \cdots + S_{q+1}^* + U_k(T^*),$$

because the server in the k-system has served all customers. Since $U_k(T^*) - U(T^*) = S_1$ we have:

$$T^* = S_1^* + \cdots + S_{q+1}^* + U(T^*) + S_1,$$

so that at that moment the server in the original system has also served all customers.

Case 2. We have that $U_k(t) - U(t) \leq S_1$, until one of the servers reaches the $(q+1)^{st}$ customer, according to the induction hypothesis. Now we distinguish between two possibilities:

- The $(q+1)^{st}$ customer is served earlier in the original system than in the k-system.

  In this case, the difference $U_k(t) - U(t)$ becomes larger than it would have been without the $(q+1)^{st}$ customer present.

  Until the $(q+1)^{st}$ customer gets served in the k-system, it is impossible that $U_k(t) - U(t) > S_1$. Since if $U_k(t) - U(t)$ would equal S_1, the server in the k-system must have arrived at the $(q+1)^{st}$ customer, because at the moment that the $(q+1)^{st}$ customer enters the systems, the server in the k-system had not travelled more than a distance S_1 extra compared to the server of the original system, according to the induction hypothesis.

  During the service of the $(q+1)^{st}$ customer in the k-system, the difference $U_k(t) - U(t)$ cannot become larger, since the server in the k-system stands still.

  After this service, both servers have served the $(q+1)^{st}$ customer and the difference $U_k(t) - U(t)$ cannot grow too large either, since both servers have not moved for the same extra time S_{q+1}^*. This means that the difference $U_k(t) - U(t)$ is what $U_k(t - S_{q+1}^*) - U(t - S_{q+1}^*)$ would be in systems where the $(q+1)^{st}$ customers did never arrive, which is not larger than S_1 by the induction hypothesis.

- The $(q+1)^{st}$ customer is served earlier in the k-system than in the original system.

  Observe that $U_k(t) - U(t)$ becomes smaller than it would be in the case that the $(q+1)^{st}$ customer would not be present. When the $(q+1)^{st}$ customer is served in the original system, the difference $U_k(t) - U(t)$ grows again, but it cannot grow larger than S_1. Since after the $(q+1)^{st}$ customer is served in the original system, both servers stood still for the same (extra) time and continue as if the $(q+1)^{st}$ customers had never been present.

Case 3. As long as the $(q+1)^{st}$ customers are not served, $U_k(t) - U(t)$ cannot become larger than S_1, according to the induction hypothesis. Again, there are two possibilities.

- The $(q+1)^{st}$ customer is served earlier in the original system than in the k-system.

  During the service of the $(q+1)^{st}$ customer in the original system $U_k(t) - U(t)$ grows larger.

As long as the $(q+1)^{st}$ customer has not been served in the k-system, $U_k(t) - U(t)$ cannot achieve the value S_1, since, according to the induction hypothesis, $U_k(t) - U(t) \leq S_1$ at the moment that the $(q+1)^{st}$ customers arrived.

After the $(q+1)^{st}$ customer is served in the k-system the difference cannot become larger than S_1 either, since again, both servers stood still for the same time.

- The $(q+1)^{st}$ customer is served earlier in the k-system than in the original system.

  In this case, the server in the k-system must have served all other customers before he travels to his $(q+1)^{st}$ customer.

  During the journey to the $(q+1)^{st}$ customer in the k-system, the difference $U_k(t) - U(t)$ cannot grow larger than S_1. Suppose that would be the case, then the server in the original system could have served all customers in his system, if he had left out the service of the $(q+1)^{st}$ customer. However the $(q+1)^{st}$ customer is not the last customer to be served in the original system, so he would have come in for his turn already. This contradicts the assumption that the $(q+1)^{st}$ customer is served earlier in the k-system than in the original system.

  Also when the server has finished the service of the $(q+1)^{st}$ customer, $U_k(t) - U(t)$ cannot get larger than S_1. Suppose that at a certain moment T^* the difference is equal to S_1. Then

$$T^* = S_1^* + \cdots + S_{q+1}^* + U(T^*) + S_1,$$

  so that at T^* all customers are served in the original system.

Case 4. It is impossible that the $(q+1)^{st}$ customer is served earlier in the original system than in the k-system. When the server in the original system is finished with the other q customers and the additional customer, the server in the k-system could also have been, had he left out the service of the $(q+1)^{st}$ customer. Since at the moment that the $(q+1)^{st}$ customers arrived, $U_k(t) - U(t) \leq S_1$, the server in the k-system must arrive at the $(q+1)^{st}$ customer earlier than the server in the original system.

As long as the server in the original system has not reached the $(q+1)^{st}$ customer, $U_k(t) - U(t) \leq S_1$, according to the induction hypothesis. The difference $U_k(t) - U(t)$ cannot grow larger than S_1 during the service of the last customer in the original system either. If that would be the case, the server in the k-system had served all customers, because the difference between the distances, that the servers have travelled until they reach the last customer in their systems, is not larger than S_1 according to the induction hypothesis. So there would exist a time $T^* < T$, with

$$T^* = S_1^* + \cdots + S_{q+1}^* + U(T^*) + S_1.$$

So at T^* all customers in the original system would have been served, which contradicts the assumption that $T^* < T$.

Finally, we must show that if (3.1) holds for all realisations of the coupling in which $n \leq p$ and l is arbitrary, it holds for all realisations in which $n = p + 1$ and l is arbitrary. Observe that as long as the $(p+1)^{st}$ additional customer has not arrived yet, the difference $U_k(t) - U(t)$ does not grow larger than $S_1 + \cdots + S_p$ according to the induction hypothesis. Then look at the number of customers r that arrives after the $(p+1)^{st}$ additional customer in the original system. Inductively we can prove (in the same way as above) that for all $r \geq 0$ and $t \leq T$: $U_k(t) - U(t) \leq S_1 + \cdots + S_{p+1}$. This proves Lemma 3.2.1. □

3.3 Proof of Theorem 3.3.1

As mentioned before, to prove that the random counting measures X^k converge weakly to the random counting measure X, it suffices to show that the finite dimensional distributions converge weakly. That is to say that we have to prove that for all n and for all sets $D_1, D_2, \ldots, D_n$ with D_i an element of the Borel σ-algebra $\mathcal{B}$ and D_i a continuity set for X, the joint distributions of $(X^k(D_1), X^k(D_2), \ldots, X^k(D_n))$ converge weakly to the joint distribution of $(X(D_1), X(D_2), \ldots, X(D_n))$. In fact we shall prove this for measurable sets D_i. Referring to the coupling in the previous section, it suffices to show that for all n, for all $D_1, D_2, \ldots, D_n \in \mathcal{B}$ and all $k_1, k_2, \ldots, k_n \in \mathbb{N}$,

$$\begin{aligned} &\lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n) = \\ &\quad \lim_{t\to\infty} P(Y_t(D_1) = k_1, \ldots, Y_t(D_n) = k_n)). \end{aligned} \tag{3.7}$$

To prove (3.7) we introduce some further notation. Define

$$I_k(t) = \begin{cases} 1 & \text{if } \exists t^* < t : \; Y_{t^*}(C) = k+1 \text{ and } \forall t' \in (t^*, t] : \; Y_{t'}(C) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

So $I_k(t) = 1$ if there was a moment before time t at which there were $k+1$ customers in the original system, and between this moment and time t the original system has not yet been empty. Observe that

$$\begin{aligned} &\lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n) = \\ &\quad \lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n, I_k(t) = 0) + \\ &\quad \lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n, I_k(t) = 1). \end{aligned} \tag{3.8}$$

Lemma 3.2.1 tells us that $I_k(t) = 0$ implies that $Y_t^k(D_i) = Y_t(D_i)$. Hence for all sets D_i, we can rewrite (3.8) as

$$\begin{aligned} &\lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n) = \\ &\quad \lim_{k\to\infty} \lim_{t\to\infty} P(Y_t(D_1) = k_1, \ldots, Y_t(D_n) = k_n, I_k(t) = 0) + \\ &\quad \lim_{k\to\infty} \lim_{t\to\infty} P(Y_t^k(D_1) = k_1, \ldots, Y_t^k(D_n) = k_n, I_k(t) = 1), \end{aligned}$$

from which we conclude that it suffices to prove that

$$\lim_{k\to\infty} \lim_{t\to\infty} P(I_k(t) = 0) = 1. \tag{3.9}$$

Define

$$A_t := \{t^* : t^* \le t, \forall t' \in (t^*, t] : Y_{t'}(C) \ne 0\}$$

and let

$$M(t) = \max_{t^* \in A_t} Y(t^*),$$

so $M(t)$ is the maximum of the number of customers that has been in the original system since the last time before time t that the original system was empty. Since

$$I_k(t) = 1 \Leftrightarrow M(t) \ge k + 1,$$

we find

$$\lim_{k\to\infty} \lim_{t\to\infty} P(I_k(t) = 1) = \lim_{k\to\infty} \lim_{t\to\infty} P(M(t) \ge k + 1) \tag{3.10}$$

and since $M(t)$ has a stationary distribution as $t \to \infty$, we conclude that

$$\lim_{k\to\infty} \lim_{t\to\infty} P(M(t) \ge k + 1) = 0.$$

Together with (3.10), this proves (3.9), so we are done. □

Chapter 4

A long range particle system with unbounded flip rates

4.1 Introduction

Particle systems with local and bounded rates have been studied extensively over the last twenty years or so. Excellent entries to this field are the two books of Liggett (1985, 1999). More recently, there has been a growing interest, especially in the physics literature, in systems with long range dependencies and non-local flip rates. Some of these systems have attracted attention under the name self-organised criticality (Bak (1996), Jensen (1998)). The physical literature emphasises the ‘critical’ behaviour of such systems, that is, power law decay in time and space of various quantities. In the pure mathematical sense, critical classical thermodynamic systems are not very well understood. Sometimes it is even unclear whether a model really exists. This makes it clear that mathematicians have other priorities when it comes to long range interaction particle systems.

The first obstacle for mathematicians is the very construction of such models in infinite volume. The classical construction techniques break down under non-locality. In the cases where an explicit construction can in fact be carried out, mathematicians are primarily interested in stationary distributions and their properties. In the light of the remarks above, it is not surprising that mathematicians try to get a feeling for this new class of models by looking at concrete examples which are simple enough to allow rigorous analysis, but which do have the required non-local flip rates.

In Maes et al. (2000) an infinite volume one-dimensional sandpile model is constructed. The resulting Markov process is not Feller and the only stationary distribution is the trivial one in which the system is completely full. In this chapter we introduce a new long range particle system which can be constructed with similar ideas as in Maes et al. (2000), but which turns out to have a non-trivial stationary distribution, various properties of which can be established.

Informally, our system can be described as follows. The state space is $\Omega = \{0,1\}^{\mathbb{Z}}$. Let $\lambda > 0$, $\mu \geq 0$. Typically, we denote a state of the system by $\eta \in \Omega$. If $\eta(x)$ equals one, it flips to zero at rate μ. If $\eta(x)$ equals zero, it flips to one at rate λ times one plus the number ones to the right of x, until the next zero. The (random) configuration of the process at time t if the initial state was η is denoted by η_t.

The 'global' reason for studying this system is that it is about the simplest non-local particle system for which we can expect a non-trivial stationary distribution. More specifically, a number of interpretations is possible, and we mention two such interpretations:
1. One can think of a toy model for a sandpile with dissipation. Grains of sand fall down on each site $i \in \mathbb{Z}$ according to a Poisson process with intensity λ. All these Poisson processes are independent of each other. If a grain falls down on some site i at a moment that site i is occupied by another grain, the falling grain slides to the nearest site on the left (i.e. a site with a lower number) where no grain is present. We suppose that the grain arrives at that site instantaneously. Grains of sand disapear independently of each other after an exponentially distributed time with parameter μ.
2. One can also interpret this system as a queueing system with impatient customers, where each site $i \in \mathbb{Z}$ is associated to a Poisson arrival process with intensity λ. There is a server at each site. The arrival processes are independent of each other. If there is an arrival of the Poisson process associated to some site i, we assign a service place to this customer in the following way. If the server at site i is not busy at the moment the customer arrives (i.e. there is no customer present at site i), the customer takes the place at site i. If the server at site i is busy, the customer is not allowed to take the place at site i. He must go to the nearest server on his left who is not busy, and is served there. We assume that customers arrive at their service place instantaneously and that service times are independent and exponentially distributed with parameter μ. After a customer is served, he leaves the system.

Because of the first interpretation, we shall call the system a sandpile model with dissipation (SMD). Because of the dissipation of sand at every site, we do not expect genuine self-organised criticality behaviour (whatever that may be).

As anticipated above, it is not immediately clear that the above description gives rise to a well defined process in infinite volume.

In Section 4.2 we construct a Markov semigroup $S(t)$, which is the semigroup of the SMD. The construction uses the monotonicity of the process and is in the same spirit as the constructions of the one-dimensional sandpile process in Maes et al. (2000) and the long range exclusion processes in Liggett (1980). This construction allows us also to start with a completely full system for example, although the informal description breaks down in that case.

In Section 4.3 we will show that the function $E(\eta_t(0))$ is not for all initial $\eta \in \Omega$ right continuous. Hence for some initial configurations the process has no right-continuous paths, in contrast to the processes described in Liggett (1985). Nevertheless, we will prove that there is, for some 'special' functions and configurations, a relation between the Markov semigroup of the SMD and its formal generator.

To describe the result of Section 4.4, we need some more notation. Define $T : \Omega \to \Omega$ to be the left shift on Ω, that is

$$T\eta(x) = \eta(x+1), \text{ for all } x \in \mathbb{Z}.$$

A spatially stationary measure ν on Ω is said to be *strongly mixing* if, for all measurable

$A, B \subset \Omega$,

$$\lim_{k\to\infty} \nu(T^k A \cap B) = \nu(A)\nu(B).$$

Let $\nu S(t)$ be the distribution of the SMD at time t if its initial configuration has distribution ν. We will prove the following result.

Theorem 4.1.1 *Let $\lambda > 0$ and $\mu \geq 0$ be given and let ν be a probability measure on Ω. Then the weak limit $\nu_\infty = \lim_{t\to\infty} \nu S(t)$ exists and is independent of ν. The measure ν_∞ is strongly mixing, with $\nu_\infty(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}$.*

When we think of our interpretation of the system as a queueing system, we see that the system has a non-trivial stationary distribution for exactly those parameter values λ and μ for which a $M(\lambda)/M(\mu)/1$ queueing system is stable. At first sight this might be surprising, since there is no 'waiting room' available in the SMD. On the other hand, when $\lambda < \mu$, there is globally enough service capacity. Generally speaking, it seems reasonable that, if one allows interactions between queues, the time of the servers can be used more efficiently, which decreases the waiting time (in this case, the waiting time even reduces to 0).

Since ν_∞ is a strongly mixing measure with marginals given by

$$\nu_\infty(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\},$$

the question arises whether, for $\lambda < \mu$, ν_∞ could be a product measure. In Section 4.5 we show that this is not the case.

Finally, in Section 4.6 we show that ν_∞ has positive correlations and that, if the process starts from ν_∞, the correlation between the initial state at site 0 and the state of the process at site 0 at time t decays exponentially.

4.2 Construction of the SMD and notation

Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the state space of the SMD. The space Ω is equipped with the product topology and the Borel σ-algebra $\mathcal{B}$, and is a compact metric space. Initial configurations will be denoted by $\eta, \xi \in \Omega$ and the (random) configuration of the system at time t if the initial configuration was η or ξ will be denoted by η_t or ξ_t respectively.

In this section we shall define a Markov semigroup $S(t)$ acting on bounded measurable functions $f : \Omega \to \mathbb{R}$, which will be the semigroup of the SMD:

$$S(t)f(\eta) := E(f(\eta_t)).$$

We start with some notation. We use the following metric $d : \Omega \times \Omega \to \mathbb{R}$,

$$d(\eta, \xi) := \begin{cases} (\sup\{j \in \mathbb{N} : \forall i \in [-j, j], \eta(i) = \xi(i)\})^{-1} & \\ \qquad\qquad\qquad\qquad\qquad \text{if } \eta(i) = \xi(i) \text{ for } i \in [-1, 1], & \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} & \end{cases}$$

When we write $\eta \leq \xi$, we mean that $\eta(i) \leq \xi(i)$ for all $i \in \mathbb{Z}$. Let $\mathcal{M}$ be the space of bounded Borel measurable increasing functions on Ω and let $C(\Omega)$ be the space of all continuous functions on Ω. We call a site $i \in \mathbb{Z}$ *occupied* in η if and only if $\eta(i) = 1$; we interpret this as the presence of a particle at site i in the SMD. Let Ω_F be the set of all configurations in $\{0,1\}^{\mathbb{Z}}$ which have only finitely many occupied sites. We define $l_\eta(i) \in \mathbb{N}$ to be the number of occupied sites in configuration η to the right of site i until the nearest site to the right of site i that is not occupied:

$$l_\eta(i) := \#\{j \in \mathbb{Z}\colon\ j > i \text{ and for all } i < j' \leq j\colon\ \eta(j') = 1\}.$$

Define for $i \in \mathbb{Z}$, the following flipping transformation T_i, which changes the configuration at site i and leaves all other sites unchanged:

$$T_i(\eta)(x) := \begin{cases} \eta(x) & \text{if } x \neq i, \\ 1 - \eta(x) & \text{if } x = i. \end{cases}$$

We define the formal generator G of the SMD by

$$\begin{aligned} Gf(\eta) \quad := \quad & \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) \left(f\left(T_i(\eta)\right) - f(\eta) \right) \\ & + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta) \right), \end{aligned}$$

where f is a real function on Ω. The formal generator is just a formal sum, and one should not worry about existence at the moment. In Section 4.3, it will turn out that for 'special' functions f and 'special' $\eta \in \Omega$,

$$\lim_{t \downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}$$

exists and is equal to $Gf(\eta)$. We shall now outline the construction of the SMD. Although the construction is very similar to the construction of the one-dimensional sandpile process as carried out in Maes et al. (2000) the details are different, and therefore we prefer to include a full construction.

The construction proceeds in five steps. We shall first briefly outline the procedure, and then work out the details. In order to understand the rest of this chapter, omitting the details should not cause much trouble.

Step 1. We define an interacting particle system with state space Ω_F in the following way. Choose $n \in \mathbb{N}$. To each site $i \in \mathbb{Z} \cap [-n, n]$, we associate a Poisson process with parameter $\lambda > 0$; these processes are independent of each other. Particles enter the system according to the following mechanism. If the system is in state $\eta \in \Omega_F$ and a Poisson arrival occurs of the Poisson process associated to site $i \in [-n, n]$ then:

- If $\eta(i) = 0$, $\eta(i)$ changes to 1, i.e. the particle is placed at site i.
- If $\eta(i) = 1$, then the particle is placed at the nearest site with a number smaller than i which is not occupied.

Particles leave the system, independently of each other and of the arrival processes, after a period which is exponentially distributed with parameter $\mu \geq 0$. We call the

Markov process described above the n-process (because of the restriction on the arrival processes). The associated semigroup is denoted by $S_n(t)$ and is defined for bounded measurable functions on Ω_F. The state of the n-process at time t if its initial state was $\eta \in \Omega_F$ is denoted by $\eta_{n,t}$.

Step 2. We observe that the n-process (defined on Ω_F) is monotone, i.e. for $\eta \leq \xi$ and $f \in \mathcal{M}$,

$$S_n(t)f(\eta) \leq S_n(t)f(\xi).$$

Step 3. The monotonicity of the n-process on Ω_F makes it possible to extend the n-process to a process with state space Ω in the following way. For $f \in \mathcal{M}$, the semigroup of the extension of the n-process is given by

$$S_n(t)f(\eta) := \lim_{\xi \in \Omega_F,\ \xi \uparrow \eta} S_n(t)f(\xi),$$

which is well defined.

Step 4. We observe that the semigroups $S_n(t)$ are monotone in n, i.e. for $\eta \in \Omega$, $f \in \mathcal{M}$,

$$S_n(t)f(\eta) \leq S_{n+1}f(\eta).$$

Step 5. Since the semigroups $S_n(t)$ are monotone in n, we can define a 'limiting' process with semigroup $S(t)$, which is for $f \in \mathcal{M}$ and $\eta \in \Omega$ defined by

$$S(t)f(\eta) := \lim_{n \uparrow \infty} S_n(t)f(\eta).$$

Observe that it suffices to define $S(t)$ only for $f \in \mathcal{M}$, since the distribution of the 'limiting' process at time t is completely determined by the outcomes of $S(t)f(\eta)$ for $f \in \mathcal{M}$. Finally we define the SMD to be the Markov process that corresponds to the semigroup $S(t)$.

We give the details of the construction outlined above:

Step 1. We must show that $\eta \in \Omega_F$ implies that

$$P(\eta_{n,s} \in \Omega_F,\ \forall s \leq t) = 1,$$

for all t. This is obvious, since the total arrival rate in this process equals $\lambda(2n+1)$.

We can compute the generator of the n-process. Define $l^n_\eta(i)$ to be the number of occupied sites in $[-n, n]$ to the right of site i, until the nearest site to the right of site i that is not occupied:

$$l^n_\eta(i) := \#\{j \in \mathbb{Z} \cap [-n, n]\colon\ j > i \text{ and for all } i < j' \leq j\colon\ \eta(j') = 1\}$$

and let f be a measurable bounded function on Ω_F, $\eta \in \Omega_F$. The generator of the n-process is given by

$$G_n f(\eta) \quad = \quad \lim_{t \downarrow 0} \frac{S_n(t)f(\eta) - f(\eta)}{t}$$

$$\begin{aligned} = & \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda l_\eta^n(i) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ & + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ & + \sum_{i=-n}^{n} \mathbf{1}_{\{\eta(i)=0\}} \lambda \left(f\left(T_i(\eta)\right) - f(\eta)\right). \end{aligned}$$

Step 2. We prove that the n-process (with state space Ω_F) is monotone. This is done by showing that there exists a coupling

$$\left(\hat{\xi}_{n,t}, \hat{\eta}_{n,t}\right)_{(t\geq 0)}$$

of the processes $\xi_{n,t_{(t\geq 0)}}$ and $\eta_{n,t_{(t\geq 0)}}$, which has the property that for $\xi \leq \eta$,

$$P\left(\hat{\xi}_{n,t} \leq \hat{\eta_{n,t}} \text{ for all } t\right) = 1.$$

This coupling is defined as follows. We use for both processes (with initial configurations ξ and η) the same sequence of Poisson arrival processes, and if both processes have a customer at the same site, we let these corresponding customers leave at the same time. This is possible since the exponential distribution has no memory.

Observe that if the starting configurations $\xi, \eta \in \Omega_F$ have the property that if both $\xi \leq \eta$ and $\xi(i) = \eta(i) = 0$, then the flipping rate of $\xi(i)$ is not larger than the flipping rate of $\eta(i)$ since $l_\xi^n(i) \leq l_\eta^n(i)$. Also, if both $\xi \leq \eta$ and $\xi(i) = \eta(i) = 1$, then the flipping rate of $\eta(i)$ is the same as the flipping rate of $\xi(i)$. From this we can conclude that the coupling has the property that for $\xi \leq \eta$, $\hat{\xi}_{n,t} \leq \hat{\eta}_{n,t}$ for all t with probability 1 (see Lindvall (1992), p. 178).

Step 3. Because of the monotonicity of the n-process we can extend the n-process to a process with state space Ω by defining its semigroup (for $f \in \mathcal{M}$) by

$$S_n(t)f(\eta) := \lim_{\xi\in\Omega_F,\ \xi\uparrow\eta} S_n(t)f(\xi)$$

(the fact that $S_n(t)$ is a semigroup follows from the construction). We show that $S_n(t)$ is well defined, that is, we show that the limit of $S_n(t)f(\xi_m)$ is independent of the sequence $(\xi_m)_{m\in\mathbb{N}}$ with elements in Ω_F that increases to η. Suppose that there exist an $\eta \in \Omega$ and two sequences $(\xi_m)_{m\in\mathbb{N}}$ and $(\xi'_m)_{m\in\mathbb{N}}$ with $\xi_m, \xi'_m \in \Omega_F$ for all $m \in \mathbb{N}$, $\xi_m \uparrow \eta$, $\xi'_m \uparrow \eta$ and

$$\lim_{m\to\infty} S_n(t)f(\xi_m) \neq \lim_{m\to\infty} S_n(t)f(\xi'_m).$$

Without loss of generality we may assume that

$$l_2 := \lim_{m\to\infty} S_n(t)f(\xi'_m) > \lim_{m\to\infty} S_n(t)f(\xi_m) =: l_1.$$

Let $\epsilon := \frac{1}{2}(l_2 - l_1)$. Then there exists an $N \in \mathbb{N}$ such that for all $m > N$,

$$S_n(t)f(\xi_m) \in [l_1 - \epsilon, l_1]$$

and there exists an $N' \in \mathbb{N}$ such that for all $m > N'$,

$$S_n(t)f(\xi'_m) \in [l_2 - \epsilon, l_2].$$

Observe that these intervals are disjoint, which implies that for $m > N$ and $m' > N'$

$$S_n(t)f(\xi'_{m'}) > S_n(t)f(\xi_m). \tag{4.1}$$

Take some number $k' > N'$. Then there exist a $k > N$ with $\xi'_{k'} \leq \xi_k$, so by the monotonicity of the n-process we get that

$$S_n(t)f(\xi'_{k'}) \leq S_n(t)f(\xi_k). \tag{4.2}$$

(4.1) and (4.2) contradict each other, so the assumption that $l_1 \neq l_2$ cannot be correct. This implies that $S_n(t)f(\eta)$ is uniquely defined for all $\eta \in \Omega$ and $f \in \mathcal{M}$.

Step 4. To prove that $S_n(t)$ is monotone in n, we show that there is a coupling

$$\left(\hat{\eta}_{n,t}, \hat{\xi}_{n+1,t}\right)_{(t\geq 0)}$$

of the processes $\eta_{n,t_{(t\geq 0)}}$ and $\xi_{n+1,t_{(t\geq 0)}}$, with the property that if $\xi \leq \eta$, then

$$P\left(\hat{\xi}_{n,t} \leq \hat{\eta}_{n+1,t} \text{ for all } t\right) = 1.$$

This coupling is defined as follows. Use the sequence of Poisson arrival processes of the $(n+1)$-process with initial configuration η also for the n-process with initial configuration ξ. The Poisson arrival streams associated to the sites $-(n+1)$ and $(n+1)$ of the $(n+1)$-process are of course not needed to construct the n-process. Further, if both processes have a customer at the same site, we let these customers depart at the same time. This coupling shows that for $\eta \in \Omega_F$, $f \in \mathcal{M}$ we have that

$$S_n(t)f(\eta) \leq S_{n+1}(t)f(\eta).$$

Now let $\eta \in \Omega, f \in \mathcal{M}$ and take an increasing sequence ξ_k, $\xi_k \in \Omega_F$ with $\xi_k \uparrow \eta$. We get that for all k,

$$S_n(t)f(\xi_k) \leq S_{n+1}(t)f(\xi_k)$$

and taking the limit $k \to \infty$ yields:

$$S_n(t)f(\eta) \leq S_{n+1}(t)f(\eta).$$

Step 5. By monotonicity of the semigroup $S_n(t)$ in n, we can define for $\eta \in \Omega$ and $f \in \mathcal{M}$,

$$S(t)f(\eta) = \lim_{n\to\infty} S_n(t)f(\eta).$$

We know that for all n, $S_n(t)$ is a semigroup on bounded functions on Ω_F. Because of monotonicity this implies that $S(t)$ is also a Markov semigroup on $\mathcal{M}$ (as in Maes et al. (2000)) and we can extend the definition of $S(t)f$ to all bounded Borel measurable functions as described in Liggett (1980). So there exists a unique Markov process η_t such that

$$S(t)f(\eta) = E^{\eta}f(\eta_t);$$

this process is the SMD.

4.3 The relation between $S(t)$ and G

Analogously to the relation between semigroups and generators of Feller processes, one can hope that for continuous f and all $\eta \in \Omega$,

$$\lim_{t \downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t} = Gf(\eta).$$

This is too optimistic to expect, as we see in the next proposition.

Proposition 4.3.1 *There exist $f \in C(\Omega)$ and $\eta \in \Omega$ such that*

$$\liminf_{t \downarrow 0} S(t)f(\eta) \neq f(\eta).$$

Proof: Let η^* be given by

$$\eta^*(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and let η^*_m for $m \in \mathbb{N}$ be defined by

$$\eta^*_m(x) := \begin{cases} \eta^*(x) & \text{if } x \in [-m, m] \cap \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $f : \Omega \to \mathbb{R}$ by $f(\eta) = \eta(0)$ and note that $f(\eta^*) = 0$. We shall show that

$$\liminf_{t \downarrow 0} S(t)f(\eta^*) \geq \frac{\lambda}{\lambda + \mu}.$$

Recall that $\eta^*_{mn,t}$ is the state of the n-process at time t, if the initial configuration was η^*_m. Let $A_{m,n}$ be the event (in the n-process with initial configuration η_m) that during the time interval $[0, t]$ a particle is placed at site 0 and that this particle does not leave before time t. Let $B_{m,n}$ be the event (again in the n-process with initial configuration η_m) that there is an arrival during $[0, t]$ in at least one of the Poisson processes associated to the sites in $[1, n]$ *before* any of the sites in $[1, n]$ becomes unoccupied. Then, for $m \geq n$:

$$\begin{aligned} S_n(t)f(\eta^*_m) &= P(\eta^*_{m_{n,t}}(0) = 1) \\ &\geq P(A_{m,n}) \\ &\geq P(B_{m,n})e^{-\mu t} \\ &= \frac{\lambda n}{\lambda n + \mu n}(1 - e^{-n(\lambda+\mu)t})e^{-\mu t}. \end{aligned}$$

So

$$\begin{aligned} \liminf_{t \downarrow 0} S(t)f(\eta^*) &= \liminf_{t \downarrow 0} \lim_{n \to \infty} \lim_{m \to \infty} S_n(t)f(\eta^*_m) \\ &\geq \lim_{t \downarrow 0} \lim_{n \to \infty} \lim_{m \to \infty} \frac{\lambda n}{\lambda n + \mu n}(1 - e^{-n(\lambda+\mu)t})e^{-\mu t} \\ &= \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

This proves Proposition 4.3.1. $\square$

However, as in the one-dimensional sandpile model in Maes et al. (2000), we do have that for some class of 'nice' functions and configurations:

$$\lim_{t \downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}$$

exists, and is equal to $Gf(\eta)$.

To achieve this, we need the concepts of N-local functions and decent configurations as introduced in Maes et al. (2000). We repeat the definitions here. Let Ω_1 be the set of configurations with an infinite number of unoccupied sites at either side of the origin,

$$\Omega_1 := \{\eta \in \Omega : \sum_{i<0}(1 - \eta(i)) = \sum_{i>0}(1 - \eta(i)) = \infty\}.$$

We write the ordered indices i with $\eta(i) = 0$ as

$$\{\ldots, R_{-1}(\eta), R_0(\eta), R_1(\eta), \ldots\},$$

where

$$R_0(\eta) := \min\{i \geq 0 : \eta(i) = 0\}.$$

Let, for $\eta \in \Omega_1$,

$$I_n(\eta) = (R_{n-1}(\eta), R_n(\eta)] \cap \mathbb{Z},$$

be a random partition of $\mathbb{Z}$ into finite sets. We write

$$K_N(\eta) := \bigcup_{j=-N}^{N} I_j(\eta),$$

and $|\cdot|$ for cardinality. When we write a set as $\{R_{-(N+1)}(\eta), \ldots, R_N(\eta)\}$, we mean *all* indices from $R_{-(N+1)}(\eta)$ up to $R_N(\eta)$. A function $f : \Omega \to \mathbb{R}$ is called *N-local* if for all η, ξ in Ω_1 with

$$K_N(\eta) = K_N(\xi)$$

and

$$\eta(i) = \xi(i), \text{ for all } i \in K_N(\eta) = K_N(\xi),$$

we have

$$f(\eta) = f(\xi).$$

We shall also use this notion for functions which are only defined on a subset of Ω which contains Ω_1. A configuration η is called *decent* if $\eta \in \Omega_1$ and

$$a(\eta) := \limsup_{n \to \infty} \frac{|I_{-n}(\eta)| + \cdots + |I_n(\eta)|}{2n + 1} < \infty.$$

If η has a positive density $\rho(\eta)$ of zeroes, then

$$a(\eta) = \frac{1}{\rho(\eta)},$$

and hence η is decent. The set of decent configurations is called Ω_{dec}.

Theorem 4.3.2 *Let $f \in \mathcal{M}$ be N-local for some $N \in \mathbb{N}$ and let $\eta \in \Omega_{dec}$. Then $Gf(\eta)$ is well defined, and for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,*

$$S(t)f(\eta) = \sum_{n=0}^{\infty} \frac{t^n G^n f(\eta)}{n!}$$

and therefore,

$$\lim_{t \downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}$$

exists and is equal to $Gf(\eta)$.

Again, the details of the proof are similar to the proof of the corresponding result in Maes et al. (2000). Nevertheless, we include the proof of Theorem 4.3.2, also because we need a similar result in Section 4.5. It is possible to skip this part and continue reading at the next section. We first need some lemmas.

Lemma 4.3.3 *Let $f : D \subset \Omega \to \mathbb{R}$ be N-local, with $\Omega_1 \subset D$. Then Gf is $(N+1)$-local.*

Proof: We show first that if f is N-local, $Gf(\eta)$ is finite on a subset of Ω which contains Ω_1. Remember that $Gf(\eta)$ was defined by

$$\begin{aligned} Gf(\eta) \quad := \quad & \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ & + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right). \end{aligned}$$

Assume that $\eta \in \Omega_1$ and that f is a N-local function. It follows that for

$$i \in \mathbb{Z} \setminus \{R_{-(N+1)}(\eta), \dots, R_N(\eta)\},$$

$$f(T_i(\eta)) - f(\eta) = 0.$$

This implies that the above sum converges. Let us assume now that f is N-local and show that it follows that Gf is $(N+1)$-local. Assume that $\eta, \xi \in \Omega_1$ with

$$K_{N+1}(\eta) = K_{N+1}(\xi)$$

and

$$\eta(i) = \xi(i) \text{ for all } i \in K_{N+1}(\eta) = K_{N+1}(\xi).$$

We saw already that the sums in $Gf(\eta)$ and $Gf(\xi)$ run over

$$i \in \{R_{-(N+1)}(\eta), \dots, R_N(\eta)\}.$$

Observe that it follows from our assumptions that $f(\eta) = f(\xi)$ and that for

$$i \in \{R_{-(N+1)}(\eta), \dots, R_N(\eta)\},$$

$$\mathbf{1}_{\{\eta(i)=1\}} = \mathbf{1}_{\{\xi(i)=1\}},\ f(T_i(\eta)) = f(T_i(\xi)) \text{ and } l_i(\eta) = l_i(\xi).$$

So $Gf(\eta) = Gf(\xi)$ and we conclude that Gf is $(N+1)$-local. $\square$

Lemma 4.3.4 *Let $f:\Omega\to\mathbb{R}$ be N-local and bounded and let $\eta\in\Omega_1$. Then*

$$|G^n f(\eta)| \leq (2(\lambda+\mu))^n ||f||_\infty (|I_{-(N+n)}(\eta)| + \cdots + |I_{N+n}(\eta)|)^n. \tag{4.3}$$

Proof: We use induction on n. Suppose $f:\Omega\to\mathbb{R}$ is N-local and bounded and $\eta\in\Omega_1$. For $n=1$ we saw in the proof of Lemma 4.3.3 that only the terms where

$$i \in \{R_{-(N+1)}(\eta),\ldots,R_N(\eta)\}$$

contribute to the sum, so

$$\begin{aligned}
|Gf(\eta)| &\leq \sum_{i\in\{R_{-(N+1)}(\eta),\ldots,R_N(\eta)\}} \mathbf{1}_{\{\eta(i)=0\}}\lambda(1+l_\eta(i))|(f(T_i(\eta)-f(\eta))| \\
&\quad + \sum_{i\in\{R_{-(N+1)}(\eta),\ldots,R_N(\eta)\}} \mathbf{1}_{\{\eta(i)=1\}}\mu|\left(f\left(T_i(\eta)\right)-f(\eta)\right)| \\
&\leq 2||f||_\infty\lambda(|I_{-N}(\eta)|+\cdots+|I_{N+1}(\eta)|) \\
&\quad +2||f||_\infty\mu\left(|I_{-N}(\eta)|+\cdots+|I_N(\eta)|\right) \\
&\leq 2(\lambda+\mu)||f||_\infty\left(|I_{-(N+1)}(\eta)|+\cdots+|I_{N+1}(\eta)|\right).
\end{aligned}$$

So for $n=1$, statement (4.3) in Lemma 4.3.4 is true. Assume that we know that (4.3) holds for all $n\leq k$ (induction hypothesis) and consider

$$\begin{aligned}
|G^{k+1}f(\eta)| &= \Bigg|\sum_i \mathbf{1}_{\{\eta(i)=0\}}\lambda(1+l_\eta(i))\left(G^kf\left(T_i(\eta)\right)-G^kf(\eta)\right) \\
&\quad + \sum_i \mathbf{1}_{\{\eta(i)=1\}}\mu\left(G^kf\left(T_i(\eta)\right)-G^kf(\eta)\right)\Bigg|.
\end{aligned}$$

If f is N-local, then G^kf is $(N+k)$-local (this follows from Lemma 4.3.3), so for

$$i\in\mathbb{Z}\setminus\{R_{-(N+k+1)}(\eta),\ldots,R_{N+k}(\eta)\}$$

we have that

$$G^kf\left(T_i(\eta)\right)-G^kf(\eta)=0.$$

From this and the induction hypothesis we conclude that

$$\begin{aligned}
&|G^{k+1}f(\eta)| \leq \\
&\quad \sum_{i\in\{R_{-(N+k+1)}(\eta),\ldots,R_{N+k}(\eta)\}} (1+l_\eta(i))\lambda\mathbf{1}_{\{\eta(i)=0\}}\left(\left|G^kf(\eta)\right|+\left|G^kf(T_i(\eta))\right|\right) \\
&\quad + \sum_{i\in\{R_{-(N+k+1)}(\eta),\ldots,R_{N+k}(\eta)\}} \mu\mathbf{1}_{\{\eta(i)=1\}}\left(\left|G^kf(\eta)\right|+\left|G^kf(T_i(\eta))\right|\right) \\
&\leq [(2(\lambda+\mu))^k||f||_\infty(|I_{-(N+k)}(\eta)|+\cdots+|I_{N+k}(\eta)|)^k + \\
&\quad (2(\lambda+\mu))^k||f||_\infty(|I_{-(N+k+1)}(\eta)|+\cdots+|I_{N+k+1}(\eta)|)^k]\times \\
&\quad [\lambda\left(|I_{-(N+k)}(\eta)|+\cdots+|I_{N+k+1}(\eta)|\right)+ \\
&\quad \mu\left(|I_{-(N+k)}(\eta)|+\cdots+|I_{N+k}(\eta)|\right)] \\
&\leq (2(\lambda+\mu))^{k+1}||f||_\infty(|I_{-(N+k+1)}(\eta)|+\cdots+|I_{N+k+1}(\eta)|)^{k+1}.
\end{aligned}$$

This proves Lemma 4.3.4. Observe that the statement of the lemma also holds for $\eta \in \Omega_F$ and G replaced by G_m, the generator of the m-process on Ω_F. $\square$

Finally we need the following lemma from Maes et al. (2000):

Lemma 4.3.5 *Let* $\{a_n : n \geq 0\}$ *be a sequence of positive real numbers such that* $\limsup_{n\to\infty} a_n/n = a < \infty$. *Then the series* $\sum_{n=0}^{\infty} t^n a_n^n/n!$ *converges for* $|t| < \frac{1}{ae}$.

Proof of Theorem 4.3.2: Let $f \in \mathcal{M}$ be N-local. For $\eta \in \Omega_F$, $f \in \mathcal{M}$ we have that for all t,

$$S_n(t)f(\eta) = \sum_{i=0}^{\infty} \frac{t^i G_n^i f(\eta)}{i!}.$$

So by definition we get that for $\eta \in \Omega$,

$$S(t)f(\eta) = \lim_{n\to\infty} \lim_{\eta' \in \Omega_F, \eta' \uparrow \eta} \sum_{i=0}^{\infty} \frac{t^i G_n^i f(\eta')}{i!}.$$

Suppose now that $\eta \in \Omega_{dec}$. We have from the remark at the end of the proof of Lemma 4.3.4 that when $\eta' \in \Omega_F$, $\eta' \leq \eta$,

$$\begin{aligned} |G_n^i f(\eta')| &\leq (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta')| + \cdots + |I_{N+i}(\eta')|)^i \\ &\leq (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta)| + \cdots + |I_{N+i}(\eta)|)^i. \end{aligned}$$

From Lemma 4.3.5 it follows that for decent configurations η we have, for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,

$$\sum_{i=0}^{\infty} \frac{t^i (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta)| + \cdots + |I_{N+i}(\eta)|)^i}{i!} < \infty,$$

so, using dominated convergence, we obtain that for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,

$$S(t)f(\eta) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{t^i G_n^i f(\eta)}{i!}.$$

We can deal with the limit for $n \to \infty$ in the same way, which leads to the desired result. $\square$

4.4 The stationary distribution of the SMD

In this section we shall prove Theorem 4.1.1. The strategy will roughly be as follows. Proposition 4.4.1 states that if the initial configuration of the SMD is chosen according to either a spatially stationary, ergodic or a strongly mixing measure, then the law of the configuration at time t has the same property. We use this result to show that the stationary distribution of the SMD is strongly mixing.

Lemma 4.4.2 and Lemma 4.4.3 identify a process $\hat{\eta}_t$ such that for all t, $\hat{\eta}_t$ and η_t are identically distributed. In fact, $\hat{\eta}_t$ is a version of η_t, since $\hat{\eta}_t$ and η_t have the same semigroup $S(t)$ and one can show that the process $\hat{\eta}_t$ is a Markov process. Since there is a unique Markov process associated to $S(t)$, this gives that $\hat{\eta}_t$ is a version of η_t. We will not go into this though, since we only need the fact that η_t and $\hat{\eta}_t$ have the same distribution at any fixed time t. The process $\hat{\eta}_t$ has the property that if the initial configuration of the system is chosen according to an ergodic stationary measure with a strictly positive density of empty sites, then there is a strictly positive density of sites that have not been occupied during a small time period, in the process $\hat{\eta}_t$.

We need these results in the proof of Proposition 4.4.4 to obtain a differential equation for the density of occupied sites, which makes it possible to compute the density of occupied sites at time t explicitly, if the starting configuration was chosen according to an ergodic stationary measure with a strictly positive density of empty sites. This is a key ingredient in the proof of Theorem 4.1.1.

We shall first define the process $\hat{\eta}_t$ mentioned above. Let $X_i(t)$, $i \in \mathbb{Z}$ be independent Poisson processes with parameter λ and let $D_i(t)$, $i \in \mathbb{Z}$ be independent Poisson processes with parameter μ. Let, for $\xi \in \Omega_F$, $\hat{\xi}_{n,t}$ be the state of the n-process at time t if the initial configuration is ξ, the arrivals take place according to the Poisson processes $X_i(t)$, and the departures according to the processes $D_i(t)$. Define for $\eta \in \Omega$, $\hat{\eta}_{n,t}$ by

$$\hat{\eta}_{n,t}(i) := \lim_{\xi \in \Omega_F, \xi \uparrow \eta} \hat{\xi}_{n,t}(i),$$

We claim that the above limit is independent of the sequence $\xi \uparrow \eta$. The proof of the claim is omitted, but very similar to the proof of the fact that $S_n(t)$ is well defined (see the details of Step 3 in Section 4.2). Now we define $\hat{\eta}_t$ by

$$\hat{\eta}_t(i) := \lim_{n \to \infty} \hat{\eta}_{n,t}(i).$$

Observe that the finite dimensional distributions of $\hat{\eta}_t$ and η_t are equal, which implies that $\hat{\eta}_t$ and η_t are identically distributed.

Proposition 4.4.1 *If ν is either a spatially stationary measure, an ergodic stationary measure or a strongly mixing measure on Ω, then $\nu S(t)$ has the same property.*

Proof: The idea is that $\nu S(t)$ is a factor of a stationary, ergodic or strongly mixing measure respectively. Since this measure is not the measure ν, we prefer to provide the details.

Suppose that ν is a measure on Ω. Let $\mathcal{P}$ be the space of realisations of Radon counting measures on the real line and let ρ_λ be the product measure on $\mathcal{P}^{\mathbb{Z}}$ whose marginals are Poisson measures with parameter λ.

We define

$$\Omega^* := \Omega \times \mathcal{P}^{\mathbb{Z}} \times \mathcal{P}^{\mathbb{Z}}.$$

and write for each $\omega \in \Omega^*$: $\omega = (\omega_1; \omega_2; \omega_3)$, in which $\omega_1 \in \Omega$, $\omega_2, \omega_3 \in \mathcal{P}^{\mathbb{Z}}$. The k^{th} component of ω_i is denoted by $\omega_i(k)$ for $i \in \{1, 2, 3\}$.

Let the shift operation $S : \Omega^* \to \Omega^*$ be given by:

$$S(\omega_1; \omega_2; \omega_3) = (\bar{\omega_1}; \bar{\omega_2}; \bar{\omega_3}),$$

where $\bar{\omega_i}(k) = \omega_i(k+1)$, for $k \in \mathbb{Z}$, $i \in \{1, 2, 3\}$. Let ρ be the product measure on Ω^* given by $\rho := \nu \times \rho_\lambda \times \rho_\mu$. Let $T : \Omega \to \Omega$ be the left shift on Ω.

Define the function $\psi_t : \Omega^* \to \Omega$ by

$$\psi_t(\omega) := \lim_{n\to\infty} \psi_{n,t}(\omega),$$

here $\psi_{n,t}(\omega)$ is the state of the process $\hat{\omega}_{1_{n,t}}$ if the realisation of the arrival processes is given by ω_2 and the realisation of the (potential) departure times by ω_3. Observe that

$$T\psi_{n-1}(\omega) \leq \psi_{n,t}(S\omega) \leq T\psi_{n+1,t}(\omega),$$

so that

$$\psi_t S = T\psi_t.$$

The function ψ_t is measurable and

$$\rho(\psi_t^{-1}(A)) = \nu S(t)(A), \text{ for all } A \in \mathcal{B}.$$

We conclude that ψ_t is a factor map.

If ν is a spatially stationary measure, then ρ is a spatially stationary measure, and $\nu S(t)$ is also spatially stationary.

If ν is ergodic, it follows from Theorem 6.1 in Petersen (1983) that ρ is an ergodic stationary measure on Ω^*, since ρ_λ and ρ_μ are strongly mixing. Ergodicity of ρ implies that $\nu S(t)$ is also an ergodic stationary measure.

If ν is strongly mixing, ρ is also strongly mixing, since ρ is a product of strongly mixing measures. So in that case, $\nu S(t)$ is a factor of a strongly mixing measure and therefore strongly mixing itself. $\square$

The next lemma gives a condition which ensures the vacancy of a site in the process $\hat{\eta}_{n,s}$ during a period of length t. We need this for the proof of Lemma 4.4.3, which gives that in the process $\hat{\eta}_s$, if we start with a positive density of unoccupied sites, this density remains positive for some amount of time (see also Bouma (1998)).

Lemma 4.4.2 *Let $\eta \in \Omega$ and let $X_k(s)$, $k \in \mathbb{Z}$ be a sequence of independent Poisson arrival processes with parameter λ. Let $\hat{\eta}_{n,s}$ be as defined above. Then*

$$\eta(i) = 0,\ \ X_i(t) = 0$$

and

$$(j-i) - \sum_{k=i+1}^{j} (X_k(t) + \eta(k)) \geq 0,$$

for all $j \in [i+1, n] \cap \mathbb{Z}$ together imply that $\hat{\eta}_{n,s}(i) = 0$, for all $s \leq t$. (Here $[i+1, n] := \emptyset$, for $i \geq n$).

Proof of Lemma 4.4.2: It suffices to prove the lemma for the case $\mu = 0$ (that is the case in which particles do not leave the system), since the state of the process $\hat{\eta}_{n,s}$ with $\mu = 0$ cannot be larger than the state of the process $\hat{\eta}_{n,s}$ where $\mu > 0$, if we use the same sequence of arrival processes in both cases. Furthermore, we shall only consider the case $i = 0$, the general statement can be proved analogously.

So assume that $\mu = 0$ and that, until time t, $\eta(0) = 0$, $X_0(t) = 0$, and

$$j - \sum_{k=1}^{j} (X_k(t) + \eta(k)) \geq 0,$$

for all $j \in [1, n] \cap \mathbb{Z}$. We will show that these conditions ensure that until time t, none of the particles that arrived in the arrival processes associated to sites $1, \ldots, n$ ended up at site 0. Together with the conditions $\eta(0) = 0$ and $X_0(t) = 0$ it follows that for all $s \leq t$, $\hat{\eta}_{n,s}(0) = 0$. We make this precise by an elementary induction argument on n.

For $n = 1$ the statement is true: $\eta(0) = 0$ and $X_0(t) = 0$ imply that site 0 cannot have been occupied by a particle of the arrival process associated to site 0. Site 0 cannot have been occupied by a particle of the Poisson process associated to site 1 either, since

$$1 - \eta(1) - X_1(t) \geq 0$$

implies that until time t, none of the particles of the arrival process associated to site 1 had to go to site 0. So we conclude that

$$\eta(0) = 0,\ X_0(t) = 0 \text{ and } 1 - \eta(1) - X_1(t) \geq 0$$

imply that

$$\hat{\eta}_{1,s}(0) = 0, \text{ for all } s \leq t.$$

Suppose that $\eta(0) = 0$, $X_0(t) = 0$ and

$$j - \sum_{k=1}^{j} X_k(t) - \eta(k) \geq 0$$

for all $j \in [1, n] \cap \mathbb{Z}$, imply that

$$\hat{\eta}_{n,s}(0) = 0, \forall s \leq t$$

(induction hypothesis). Further assume that $\eta(0) = 0$, $X_0(t) = 0$ and that

$$j - \sum_{k=1}^{j} X_k(t) - \eta(k) \geq 0,$$

for all $j \in [i + 1, n + 1] \cap \mathbb{Z}$. We shall show that this implies that $\hat{\eta}_{n+1,t}(0) = 0$ (and hence, since $\mu = 0$, $\hat{\eta}_{n+1,s}(0) = 0$, for all $s \leq t$). We distinguish between two cases:

1. $1 - X_{n+1}(t) - \eta(n + 1) \geq 0$,
2. $1 - X_{n+1}(t) - \eta(n + 1) < 0$.

In the first case, none of the particles that arrived in the Poisson process associated to site $(n + 1)$ until time t, needed to be placed at another site. This implies that $\hat{\eta}_{n+1,t}(0) = 0$ if and only if $\hat{\eta}_{n,t}(0) = 0$, in the n-process with the same sequence of arrival processes. The latter follows from the assumptions and the induction hypothesis.

In the second case, until time t,

$$X_{n+1}(t) + \eta(n+1) - 1$$

of the particles from the Poisson arrival process associated to site $(n+1)$ had to go to another site, since site $(n+1)$ was already occupied at the moment of their arrival. To determine the positions of all particles it makes no difference if we pretend that those particles belonged to the arrival process associated to site n. So we assume that the number of particles that arrived until time t in the arrival process associated to site n was equal to

$$X_n(t) + X_{n+1}(t) + \eta(n+1) - 1.$$

So if in an n-process where the numbers of customers that arrived at sites 0 up to n were equal to

$$X_0(t), \ldots, X_{n-1}(t), X_n(t) + X_{n+1}(t) + \eta(n+1) - 1$$

respectively, we have that $\hat{\eta}_{n,t}(0) = 0$, then also $\hat{\eta}_{n+1,t}(0) = 0$. This follows easily from our assumptions and the induction hypothesis. $\square$

Consider the process $\hat{\eta}_s$. We will call i an *empty point* at time t for configuration η if $\hat{\eta}_s(i) = 0$ for all $s \leq t$. The next lemma implies that if η is chosen according to an ergodic stationary measure ν_0 with $\nu_0(\eta(0) = 0) = \gamma_0$, then empty points at time $\frac{\gamma_0}{2\lambda}$ exist almost surely.

Lemma 4.4.3 *Consider the SMD with $\lambda > 0$ and $\mu = 0$. Let ν_0 be an ergodic stationary measure on Ω and suppose that*

$$\nu_0(\eta(0) = 0) = \gamma_0,$$

for some $\gamma_0 > 0$. Then for $t \leq \frac{\gamma_0}{2\lambda}$,

$$\nu_0 S(t)(\eta(0) = 0) > 0.$$

Proof: Let $X_i(s)$ and $\hat{\eta}_s$ be defined as above. Assume that η is chosen according to ν_0. We call i a *special empty point* at time t for configuration η (s.e.p. for short), if

$$\eta(i) + X_i(t) = 0$$

and

$$(j - i) - \sum_{k=i+1}^{j} (X_k(t) + \eta(k)) \geq 0,$$

for all $j \geq (i+1)$. When i is a s.e.p at time t for configuration η, it follows from Lemma 4.4.2 that $\hat{\eta}_{n,t}(i) = 0$ for all n, which implies that $\hat{\eta}_t(i) = 0$ (and hence i is an empty point at time t).

We shall prove that for $t \leq \frac{\gamma_0}{2\lambda}$, and η chosen according to ν_0, the probability that 0 is a s.e.p. at time t is positive. We denote this probability by

$$P_{\nu_0}(0 \text{ is s.e.p.}).$$

Define for $l \in \mathbb{N}$,

$$H_l(t) := \sum_{i=0}^{l}(1 - X_i(t) - \eta(i)).$$

Observe that $P_{\nu_0}(0 \text{ is s.e.p.}) = P_{\nu_0}(H_n(t) > 0, \forall n \geq 0)$. Since for $t \leq \frac{\gamma_0}{2\lambda}$,

$$E(1 - X_i(t) - \eta(i)) \geq \frac{\gamma_0}{2} > 0,$$

and $X_i(t) + \eta(i)$ is an ergodic stationary sequence, we know that

$$P_{\nu_0}(H_n(t) = 0 \text{ i.o.}) = 0,$$

which implies that $P_{\nu_0}(H_n(t) > 0, \forall n \geq 0) > 0$, so $P_{\nu_0}(0 \text{ is s.e.p.}) > 0$. This implies that $\nu_0 S(t)(\eta(0) = 0) > 0$, for all $t \leq \frac{\gamma_0}{2\lambda}$. □

We write

$$\beta_t := \int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\, d\nu,$$

so β_t is the probability that site 0 is occupied at time t, if the initial configuration has distribution ν.

Proposition 4.4.4 *Let ν be an ergodic stationary measure on Ω, with $\nu(\eta(0) = 0) = \gamma > 0$, and let $t \leq \frac{\gamma}{4\lambda}$. Then*

$$\frac{d}{dt}\beta_t = \lambda - \mu\beta_t.$$

Hence, for $\mu = 0$,

$$\beta_t = 1 - \gamma + \lambda t,$$

and for $\mu > 0$,

$$\beta_t = \frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}.$$

Proof: Let ν be given as above. Let $t \leq \frac{\gamma}{4\lambda}$ be fixed, and consider $h > 0$ so small that $t + h < \frac{\gamma}{3\lambda}$. Let $X_i(t)$, $D_i(t)$ and $\hat{\eta}_t$ be as defined earlier. Let χ be a realisation of the Poisson arrival and departure processes. (Formally, this means that χ is an element of the product space of state spaces of Radon counting measures on the real line, but we do not bother about these details.) We call the corresponding probability measure σ. We consider the process $\hat{\eta}_t$, for $t \geq 0$. Define $Y_i(t)(\chi, \eta)$ to be the number of particles that were placed at site i between time 0 and time t, and $Z_i(t)(\chi, \eta)$ to be the number of particles that left site i, if the initial configuration of the process was η and the realisation of the arrival and departure processes was χ. We write $X_i(t)(\chi)$ instead of $X_i(t)$ for the sake of clarity.

We will prove that, for $t \leq \frac{\gamma}{4\lambda}$, $\frac{d}{dt}\beta_t$ exists, and that

$$\frac{d}{dt}\beta_t = \lambda - \mu\beta_t. \tag{4.4}$$

Consider

$$\beta_{t+h} - \beta_t = \int_\Omega S(t+h)\mathbf{1}_{\{\eta(0)=1\}}\, d\nu - \int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}\, d\nu.$$

Observe that

$$\mathbf{1}_{\{\hat{\eta}_{t+h}(0)=1\}} = \mathbf{1}_{\{\hat{\eta}_t(0)=1\}} + Y_0(t+h)(\cdot,\eta) - Y_0(t)(\cdot,\eta) - Z_0(t+h)(\cdot,\eta) + Z_0(t)(\cdot,\eta).$$

Taking expectations with respect to the arrival and departure processes and then integrating with respect to ν yields:

$$\begin{aligned}\int_\Omega S(t+h)\mathbf{1}_{\{\eta(0)=1\}}\,d\nu = \\ &\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}\,d\nu + \int_\Omega EY_0(t+h)(\cdot,\eta)\,d\nu - \int_\Omega EY_0(t)(\cdot,\eta)\,d\nu \\ &- \int_\Omega EZ_0(t+h)(\cdot,\eta)\,d\nu + \int_\Omega EZ_0(t)(\cdot,\eta)\,d\nu. \end{aligned} \tag{4.5}$$

We claim that, for $t \le \frac{\gamma}{4\lambda}$,

$$\lim_{h\downarrow 0}\frac{1}{h}\left(\int_\Omega EY_0(t+h)(\cdot,\eta)\,d\nu - \int_\Omega EY_0(t)(\cdot,\eta)\,d\nu\right) = \lambda, \tag{4.6}$$

and

$$\lim_{h\downarrow 0}\frac{1}{h}\left(\int_\Omega EZ_0(t+h)(\cdot,\eta)\,d\nu - \int_\Omega EZ_0(t)(\cdot,\eta)\,d\nu\right) = \mu\beta_t. \tag{4.7}$$

(The existence of the limits is part of the claim.) Existence of the above limits, together with (4.5), (4.6) and (4.7) gives that $\frac{d}{dt}\beta_t$ exists and leads to (4.4). To prove (4.6), we first examine

$$\int_\Omega EY_0(t+h)(\cdot,\eta)\,d\nu.$$

Let $m,n\in\mathbb{N}$. By ergodicity,

$$\int_\Omega EY_0(t+h)(\cdot,\eta)\,d\nu = \lim_{m,n\to\infty}\frac{1}{m+n}\sum_{i=-m}^{n} Y_i(t+h)(\chi,\eta), \tag{4.8}$$

$(\sigma\times\nu)$-a.s. Now we use the empty points as defined before Lemma 4.4.3. Suppose that $\cdots < L_{-1}(\chi,\eta) < L_0(\chi,\eta) < L_1(\chi,\eta) < \cdots$ are the empty points at time $\frac{\gamma}{3\lambda}$ (when the initial configuration was η and the realisation of the arrival and departure processes was χ). We define $L_0(\chi,\eta)$ to be the first empty point which is positive. We will often write L_i instead of $L_i(\chi,\eta)$.

Then, for positive k and j,

$$\sum_{i=L_{-k}}^{L_j} Y_i(t+h)(\chi,\eta) = \sum_{i=L_{-k}}^{L_j} X_i(t+h)(\chi).$$

Since the $X_i(t+h)$ are independent Poisson processes with mean $\lambda(t+h)$, we have that

$$\lim_{k,j\to\infty}\frac{1}{-L_{-k}+L_j}\sum_{i=L_{-k}}^{L_j} X_i(t+h)(\chi) = \lambda(t+h),$$

σ-almost surely. It follows from Lemma 4.4.3 that L_i's exist $(\sigma \times \nu)$-almost surely, so

$$(\sigma \times \nu) \left(\lim_{k,j\to\infty} \frac{1}{-L_{-k} + L_j} \sum_{i=L_{-k}}^{L_j} Y_i(t+h)(\chi, \eta) = \lambda(t+h) \right) = 1,$$

and together with (4.8) we conclude that

$$\int_\Omega EY_0(t+h)(\cdot, \eta)\, d\nu = \lambda(t+h).$$

Analogously,

$$\int_\Omega EY_0(t)(\cdot, \eta)\, d\nu = \lambda t.$$

These two equalities easily lead to (4.6). We already see, that in case $\mu = 0$, for $t \le \frac{\gamma}{4\lambda}$, $\frac{d}{dt}\beta_t$ exists and equals λ, which gives that for $\mu = 0$ and $t \le \frac{\gamma}{4\lambda}$,

$$\beta_t = 1 - \gamma + \lambda t. \tag{4.9}$$

We finally prove (4.7). Recall that

$$\int_\Omega (EZ_0(t+h)(\cdot, \eta) - EZ_0(t)(\cdot, \eta))\; d\nu$$

is the expected number of particles that left from site 0 between time t and time $t+h$, if the initial configuration of the process has distribution ν. This is bounded from below by the probability that there was a particle present at site zero at time t, and this particle left between time t and time $t+h$. On the other side, this expectation is bounded from above by the probability that there ever was a particle at site zero between time t and time $t+h$ times the expected number of departures in the associated Poisson process D_0 during the same period.

We get from (4.9) that if there are no departures after time t (i.e. $\mu = 0$), then for h small enough, $\beta_{t+h} = \beta_t + \lambda h$. For $\mu \neq 0$, the probability that site 0 is occupied at some time between t and $t+h$ can only be smaller, so the above observations lead to

$$\frac{1}{h}\beta_t(\mu h + o(h)) \le \frac{1}{h}\int_\Omega (EZ_0(t+h)(\cdot, \eta) - EZ_0(t)(\cdot, \eta))\; d\nu \le \frac{1}{h}(\beta_t + \lambda h)\mu h.$$

Taking the limit $h \to 0$ proves (4.7). So

$$\frac{d}{dt}\beta_t = \lambda - \mu\beta_t, \text{ for } t \le \tfrac{\gamma}{4\lambda}.$$

Solving this equation yields that for $\mu > 0$,

$$\beta_t = \frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t},$$

which proves the proposition. $\square$

In the last proposition we only computed β_t for t small. In fact we can use the differential equation found above to give an expression for β_t, which holds for all $t \ge 0$.

Proposition 4.4.5 *Let ν be an ergodic stationary measure on Ω with*

$$\nu(\eta(0) = 0) = \gamma > 0.$$

If $\lambda > 0, \mu > 0$, then for $t \geq 0$ we have

$$\beta_t = \nu S(t)(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}, 1\right\}.$$

If $\lambda > 0, \mu = 0$, then for $t \geq 0$,

$$\nu S(t)(\eta(0) = 1) = \min\{1 - \gamma + \lambda t, 1\}.$$

Proof: Let $\lambda > 0$, $\mu > 0$ be given and let ν be as in the proposition. Write $t^* = \frac{\gamma}{4\lambda}$. We already know from Proposition 4.4.4 that the statement of the proposition is true for $t \leq t^*$, and from Proposition 4.4.1 and Lemma 4.4.3 that $\nu S(t^*)$ is an ergodic stationary measure with

$$\nu S(t^*)(\eta(0) = 0) = 1 - \beta_{t^*} > 0.$$

This means that the differential equation which we derived in the proof of Proposition 4.4.4 also holds for $t \in \left[t^*, t^* + \frac{1-\beta_{t^*}}{4\lambda}\right]$, and that the expression for β_t in Proposition 4.4.4 is also true for $t \in \left[t^*, t^* + \frac{1-\beta_{t^*}}{4\lambda}\right]$. Applying the same trick again and again leads to the conclusion that

$$\beta_t = \frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t},$$

for all t for which this expression smaller than 1. When $\lambda \leq \mu$, this is the case for all t and we are done. When $\lambda > \mu$, we have in this way that for

$$t < \frac{\log(\gamma - 1 + \frac{\lambda}{\mu}) - \log(\frac{\lambda}{\mu} - 1)}{\mu} := T(\lambda),$$

$$\beta_t = \frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}.$$

We claim that $\beta_t = 1$ for all $t \geq T(\lambda)$. To achieve this, we use the monotonicity of the process in the parameter λ (which can easily be proved using the basic coupling (as defined in Lindvall (1992) p. 177) for the n-processes on Ω_F and taking limits). If we consider β_t as a function of λ, we have that for $\lambda \leq \lambda'$, $\beta_t(\lambda) \leq \beta_t(\lambda')$. For $\alpha < 1$, we claim that it is impossible that $\beta_t = \alpha$ for some $t \geq T(\lambda)$, since there exists a unique $\lambda'' < \lambda$ such that the process with parameters λ'' and μ has $\beta_t(\lambda'') = \frac{1+\alpha}{2}$. So we have that

$$\nu S(t)\,(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu} + (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}, 1\right\}.$$

The proof for the case $\lambda > 0,\ \mu = 0$ proceeds analogously. □

Proof of Theorem 4.1.1: It follows immediately from Proposition 4.4.5 that the theorem is true when $\mu = 0$, hence we suppose that $\mu > 0$. Let η_0 be the configuration in which all sites are unoccupied and η_1 be the configuration in which all sites are

occupied. Let the measures δ_0 and δ_1 be defined by $\delta_0(\{\eta_0\}) = 1$ and $\delta_1(\{\eta_1\}) = 1$. The proof is based on the following observations:
Observation 1:
By monotonicity of the process, for $f \in \mathcal{M}$ and η arbitrary,

$$S(t)f(\eta_1) \geq S(t)f(\eta) \geq S(t)f(\eta_0),$$

so for all ν we have,

$$\delta_0 S(t) \leq \nu S(t) \leq \delta_1 S(t).$$

Observation 2:
The weak limits $\lim_{n\to\infty} \delta_0 S(t)$ and $\lim_{n\to\infty} \delta_1 S(t)$ exist and are spatially stationary measures. Existence follows from the fact that $\delta_0 S(t)$ is increasing in t and $\delta_1 S(t)$ is decreasing in t. We see this as follows: Since $\eta_{0_\epsilon} \geq \eta_0$ for all ϵ, we get that for $f \in \mathcal{M}$:

$$S(t+\epsilon)f(\eta_0) = S(t)S(\epsilon)f(\eta_0) \geq S(t)f(\eta_0).$$

So $\delta_0 S(t) \leq \delta_0 S(t+\epsilon)$. Similarly, since for all ϵ $\eta_{1_\epsilon} \leq \eta_1$,

$$S(t+\epsilon)f(\eta_1) = S(t)S(\epsilon)f(\eta_1) \leq S(t)f(\eta_1),$$

$\delta_1 S(t) \geq \delta_1 S(t+\epsilon)$. We conclude that the weak limits $\lim_{t\to\infty} \delta_0 S(t)$ and $\lim_{t\to\infty} \delta_1 S(t)$ exist and denote the limiting measures by ν_0 and ν_1 respectively. Since by Proposition 4.4.1, $\delta_0 S(t)$ and $\delta_1 S(t)$ are spatially stationary measures for all t, ν_0 and ν_1 are also spatially stationary measures.
Observation 3: We claim that

$$\nu_1(\eta(0) = 1) = \nu_0(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}.$$

To see this, use Proposition 4.4.5, to obtain

$$\nu_0(\eta(0) = 1) = \lim_{t\to\infty} \delta_0 S(t)(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}.$$

For ν_1, things are a bit more subtle. We do have that

$$\nu_1(\eta(0) = 1) = \lim_{t\to\infty} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1), \tag{4.10}$$

but since δ_1 does not satisfy the assumptions of Proposition 4.4.5 we cannot use this proposition directly as was the case for ν_0. We resolve this by approximating δ_1 by appropriate Bernoulli measures. Let δ_p be the Bernoulli measure on Ω, with for all $x \in \mathbb{Z}$

$$\delta_p(\eta(x) = 1) = p,$$

and let ξ_m be defined by

$$\xi_m(x) =:= \begin{cases} 1 & \text{if } x \in [-m, m] \\ 0 & \text{otherwise.} \end{cases}$$

We claim that

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p = S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1). \tag{4.11}$$

To prove (4.11), observe that

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p \le S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1),$$

so we only need to prove that

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p \ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1).$$

By definition and by monotonicity,

$$\begin{aligned} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1) &= \lim_{n\to\infty}\lim_{m\to\infty} S_n(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \\ &= \lim_{m\to\infty}\lim_{n\to\infty} S_n(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \\ &= \lim_{m\to\infty} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \end{aligned}$$

Let $\epsilon > 0$ and let

$$p(\epsilon, m) := (1-\epsilon)^{\frac{1}{2m+1}}.$$

Then

$$\delta_{p(\epsilon,m)}(\eta(-m)=1,\ldots,\eta(m)=1) = 1-\epsilon$$

and

$$\begin{aligned} &\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_{p(\epsilon,m)} = \\ &\quad \int_{\{\eta:\eta\ge\xi_m\}} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_{p(\epsilon,m)} + \\ &\quad \int_{\Omega\setminus\{\eta:\eta\ge\xi_m\}} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_{p(\epsilon,m)} \\ &\quad \ge (1-\epsilon)S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m), \end{aligned}$$

where the last inequality holds because of the monotonicity of the process. So we get that for all m,

$$\begin{aligned} \lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p &= \lim_{\epsilon\downarrow 0}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_{p(\epsilon,m)} \\ &\ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m). \end{aligned}$$

Sending $m\to\infty$ leads to

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p \ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1)$$

and (4.11) is proved. Putting (4.10), (4.11) and Proposition 4.4.5 together yields that

$$\begin{aligned}
\nu_1(\eta(0)=1) &= \lim_{t\to\infty}\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)\,d\delta_p \\
&= \lim_{t\to\infty}\lim_{p\uparrow 1}\min\left\{(1-(1-p)-\frac{\lambda}{\mu})e^{-\mu t}+\frac{\lambda}{\mu},1\right\} \\
&= \lim_{t\to\infty}\min\left\{(1-\frac{\lambda}{\mu})e^{-\mu t}+\frac{\lambda}{\mu},1\right\} \\
&= \min\left\{\frac{\lambda}{\mu},1\right\}.
\end{aligned}$$

Conclusion: From Observation 1 and Observation 2 we conclude that

$$\nu_0 = \lim_{t\to\infty}\delta_0 S(t) \le \lim_{t\to\infty}\delta_1 S(t) = \nu_1,$$

with ν_0 and ν_1 spatially stationary measures. If we combine this with Observation 3 and Corollary 2.8 (p. 75) of Liggett (1985), we get that $\nu_0 = \nu_1$. So the process has a unique invariant measure $\nu_\infty = \lim_{t\to\infty}\nu S(t)$, which equals ν_0 and ν_1, and which is spatially stationary.

Finally we show that ν_∞ is strongly mixing. Observe that we cannot find this measure as a factor of a strongly mixing measure as in the proof of Proposition 4.4.1. We use the monotonicity of the process and the fact that $\delta_0 S(t)$ and $\delta_1 S(t)$ are strongly mixing measures for all t, by Proposition 4.4.1. To show that ν_∞ is strongly mixing, it suffices to show that for all cylinder sets E and F for which $\mathbf{1}_E, \mathbf{1}_F \in \mathcal{M}$,

$$\lim_{k\to\infty}\nu_\infty(T^k E\cap F) = \nu_\infty(E)\nu_\infty(F), \tag{4.12}$$

since the collection of cylinders described above is closed under intersections and generates $\mathcal{B}$. By monotonicity, for E and F as above,

$$\delta_0 S(t)(T^k E\cap F) \le \nu_\infty(T^k E\cap F) \le \delta_1 S(t)(T^k E\cap F).$$

Taking limits for $k\to\infty$ and using the fact that both $\delta_0 S(t)$ and $\delta_1 S(t)$ are mixing yields that

$$\delta_0 S(t)(E)\delta_0 S(t)(F) \le \lim_{k\to\infty}\nu_\infty(T^k E\cap F) \le \delta_1 S(t)(E)\delta_1 S(t)(F).$$

Now let $t\to\infty$, which leads to

$$\nu_\infty(E)\nu_\infty(F) \le \lim_{k\to\infty}\nu_\infty(T^k E\cap F) \le \nu_\infty(E)\nu_\infty(F),$$

and proves (4.12). We conclude that ν_∞ is strongly mixing.

It follows from Observation 3 that for $\lambda<\mu$,

$$\nu_\infty(\eta(0)=1) = \frac{\lambda}{\mu},$$

and that for $\lambda\ge\mu$, ν_∞ is degenerate at $\{1\}^{\mathbb{Z}}$. This proves the theorem. □

4.5 The measure ν_∞ is not a product measure

In this section we assume that $\lambda < \mu$, since for the case $\lambda \geq \mu$, ν_∞ is degenerate. In Section 4.4 we saw that ν_∞ is strongly mixing. Before trying to discover more properties of the stationary distribution of the SMD, the question arises whether ν_∞ could just be a product measure. The answer to this question is no, as is stated in the following proposition. The proof is more subtle than one might expect.

Recall that for $p \in [0,1]$, δ_p was defined to be the product measure on Ω with $\delta_p(\eta(0) = 1) = p$. We write $\rho := \frac{\lambda}{\mu}$.

Proposition 4.5.1 *Let $\lambda < \mu$ and consider the SMD with parameters λ and μ. Then the invariant measure ν_∞ is not the product measure δ_ρ.*

To prove this proposition, we will show that the assumption that $\nu_\infty = \delta_\rho$ leads to a contradiction. In fact we will see that, if the initial configuration of the SMD is chosen according to δ_ρ, the probability that we see a particle at position 0 and a particle at position 1 at time t, is not constant as a function of t. We use the relation between the generator and the semigroup as given in Theorem 4.3.2. We also need a relation as in Theorem 4.3.2 for some special functions which are neither bounded nor monotone as stated in the following lemma. We use the following subset of Ω_{dec},

$$\Omega_{dec}^{\gamma} := \left\{ \eta \in \Omega_{dec} : \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} (1 - \eta(i)) = \gamma \right\}.$$

Lemma 4.5.2 *Let $\gamma > 0$ and let g_1 and g_2 be defined by*

$$g_1(\eta) := \mathbf{1}_{\{\eta(0)=0,\eta(1)=1\}}(\eta)(1 + l_\eta(0))$$

and

$$g_2(\eta) := \mathbf{1}_{\{\eta(0)=1,\eta(1)=0\}}(\eta)(1 + l_\eta(1)).$$

Then for $\eta \in \Omega_{dec}^{\gamma}$ and $t < \frac{\gamma}{4(\lambda+\mu)e}$,

$$S(t)g_1(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i g_1(\eta)}{i!} < \infty$$

and

$$S(t)g_2(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i g_2(\eta)}{i!} < \infty.$$

Proof: We prove the statement for g_1, the proof for g_2 proceeds analogously. Our strategy is as follows. We will write g_1 as a sum of monotone 1-local functions which are not bounded. Then we prove a relation as in Theorem 4.3.2 for these functions, by writing them as an increasing limit of monotone, local and bounded functions for which Theorem 4.3.2 holds.

Let $\eta \in \Omega_{dec}^{\gamma}$ and $t < \frac{\gamma}{4(\lambda+\mu)e}$. Define h_1 by

$$h_1(\eta) := \mathbf{1}_{\{\eta(1)=1\}}(1 + l_\eta(0)),$$

and h_2 by

$$h_2(\eta) := \mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(1 + l_\eta(0)).$$

We will show that

$$S(t)h_1(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i h_1(\eta)}{i!} < \infty, \tag{4.13}$$

and that

$$S(t)h_2(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i h_2(\eta)}{i!} < \infty. \tag{4.14}$$

This suffices, since if (4.13) and (4.14) hold, we have that

$$\begin{aligned} S(t)g_1(\eta) &= S(t)h_1(\eta) - S(t)h_2(\eta) \\ &= \sum_{i=0}^{\infty} \frac{t^i G^i h_1(\eta)}{i!} - \sum_{i=0}^{\infty} \frac{t^i G^i h_2(\eta)}{i!} \\ &= \sum_{i=0}^{\infty} \frac{t^i G^i (h_1(\eta) - h_2(\eta))}{i!} \\ &= \sum_{i=0}^{\infty} \frac{t^i G^i g_1(\eta)}{i!}. \end{aligned}$$

We prove (4.13). The proof of (4.14) proceeds analogously and is omitted. Define h_M ($M \geq 3$) by

$$h_M(\eta) := \min\{h_1(\eta), M\}.$$

Observe that by monotone convergence,

$$S(t)h_1(\eta) = \lim_{M\to\infty} S(t)h_M(\eta).$$

Since h_M is monotone and bounded, we find by Theorem 4.3.2 that

$$S(t)h_1(\eta) = \lim_{M\to\infty} \sum_{i=0}^{\infty} \frac{t^i G^i h_M(\eta)}{i!},$$

and we will apply the dominated convergence theorem to bring the limit into the sum.

We prove that for $\eta \in \Omega_{dec}^{\gamma}$ and all M,

$$|G^i h_M(\eta)| \leq (2(\lambda+\mu))^i (|I_{-(i+1)}(\eta)| + \cdots + |I_{i+1}(\eta)|)^{i+1}. \tag{4.15}$$

The proof proceeds by induction on i. We will only show (4.15) for $i = 1$, the rest of the proof is analogous to the proof of Lemma 4.3.4 and therefore omitted. Recall that

$$\begin{aligned} Gh_M(\eta) &= \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda l_\eta(i) \left(h_M\left(T_i(\eta)\right) - h_M(\eta)\right) \\ &\quad + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(h_M\left(T_i(\eta)\right) - h_M(\eta)\right) \\ &\quad + \sum_{i=-n}^{n} \mathbf{1}_{\{\eta(i)=0\}} \lambda \left(h_M\left(T_i(\eta)\right) - h_M(\eta)\right). \end{aligned}$$

Since h_M is a 1-local function,

$$h_M(T_i(\eta)) - h_M(\eta) = 0, \quad \text{for } i \in \mathbb{Z} \setminus \{R_{-2}(\eta), \cdots, R_1(\eta)\}.$$

Notice further that

$$h_M(\eta) \leq |I_0(\eta)| + |I_1(\eta)|$$

and that for $i \in \{R_{-2}(\eta), \cdots, R_1(\eta)\}$,

$$h_M(T_i(\eta)) \leq |I_0(\eta)| + |I_1(\eta)| + |I_2(\eta)|.$$

We conclude that

$$\begin{aligned} |Gh_M(\eta)| &\leq 2(|I_0(\eta)| + |I_1(\eta)| + |I_2(\eta)|) \\ &\quad \times \sum_{i \in \{R_{-2}(\eta), \cdots, R_1(\eta)\}} \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) + \mu \\ &\leq 2(\lambda + \mu)(|I_{-2}(\eta)| \cdots + |I_2(\eta)|)^2, \end{aligned}$$

which proves (4.15) for the case $i = 1$.

Observe that for η and t as above,

$$\sum_{i=0}^{\infty} \frac{t^i (2(\lambda+\mu))^i (|I_{-(i+1)}(\eta')| + \cdots + |I_{i+1}(\eta')|)^{i+1}}{i!} < \infty,$$

which follows from Lemma 4.3.5. Applying the dominated convergence theorem leads to the desired result. □

Let $X_i(t)$ be independent Poisson processes with parameter λ and recall the definition of the special empty points (s.e.p.) as in Lemma 4.4.3. The following lemma deals with the expectation of the first positive s.e.p. at time

$$t^* := \frac{1-\rho}{5(\lambda+\mu)e},$$

in the case that the initial configuration is chosen according to δ_ρ.

Lemma 4.5.3 *Suppose η is chosen according to δ_ρ. Let*

$$J := \min\{i \geq 1 : \ i \textit{ is a s.e.p. at time } t^* \}.$$

Then

$$E(J) < \infty.$$

Proof: We call $i \geq 1$ a *nice* point at time t^*, if for $j \geq i$:

$$\sum_{n=1}^{j} (1 - \eta(n) - X_n(t^*)) \geq 1$$

and we define

$$J^* := \min\{i : i \text{ is a nice point at time } t^* \}.$$

Observe that, in general, a nice point does not have to be a special empty point, but that the first positive nice point, J^*, is also a special empty point. So $E(J) < E(J^*)$.

We show that $E(J^*) < \infty$, which suffices. Since for all i,

$$E\left(1-\eta(i)-X_i(t^*)\right) = 1-\rho-\frac{(1-\rho)\lambda}{5(\lambda+\mu)e} > 0,$$

it follows from the large deviations result, Theorem 5.11.2 in Grimmett and Stirzaker (1992), that there exist $0 < c < 1$ and $N > 0$ such that for all $n > N$,

$$P\left(\sum_{j=1}^{n}(1-\eta(j)-X_j(t^*)) \leq 0\right) < c^n.$$

We find

$$\begin{aligned} E(J^*) &= \sum_{j=0}^{N} P(J^* > j) + \sum_{j=N+1}^{\infty} P(J^* > j) \\ &\leq N+1+\sum_{j=N+1}^{\infty}\sum_{n=j+1}^{\infty} P\left(1-\eta(n)-X_n(t^*) \leq 0\right) \\ &\leq N+1+\sum_{j=N+1}^{\infty}\sum_{n=j+1}^{\infty} c^n < \infty, \end{aligned}$$

so we are done. □

Lemma 4.5.4 *Let $0 < \rho < 1$. For $t < t^*$,*

$$\frac{d}{dt}\int_{\Omega_{dec}^{1-\rho}}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\,d\delta_\rho = \int_{\Omega_{dec}^{1-\rho}}\frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\,d\delta_\rho.$$

Proof: Let $0 < \rho < 1$ be given and denote the semigroup of the SMD with parameters λ and μ by $S_{\lambda,\mu}(t)$. According to Theorem 16.8 in Billingsley (1986), it suffices to show that there exists a δ_ρ-integrable function g such that for $t < t^*$ and $\eta \in \Omega_{dec}^{1-\rho}$,

$$\left|\frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\right| \leq g(\eta), \tag{4.16}$$

We prove (4.16). Observe that for $t \leq t^*$,

$$\left|\frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\right| =$$

$$
\begin{aligned}
&= \left|\sum_{n=1}^{\infty}\frac{nt^{n-1}G^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\right| \\
&= \left|\sum_{n=0}^{\infty}\frac{t^nG^n(G\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta))}{n!}\right| \\
&= \left|\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=0,\eta(1)=1\}}(\eta)\lambda(1+l_\eta(0))}{n!}\right. \\
&\quad \left.+\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=0\}}(\eta)\lambda(1+l_\eta(1))-2\mu\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta))}{n!}\right| \\
&= \big|S_{\lambda,\mu}(t)\mathbf{1}_{\{\eta(0)=0,\eta(1)=1\}}(\eta)\lambda(1+l_\eta(0)) + S_{\lambda,\mu}(t)\left(\mathbf{1}_{\{\eta(0)=1,\eta(1)=0\}}(\eta)\lambda(1+l_\eta(1)\right) \\
&\quad -2\mu S_{\lambda,\mu}(t)\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)\big|
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\le S_{\lambda,\mu}(t)\lambda(1+l_\eta(0))+S_{\lambda,\mu}(t)\lambda(1+l_\eta(1))+2\mu \\
&\le S_{\lambda,0}(t)\lambda(1+l_\eta(0))+S_{\lambda,0}(t)\lambda(1+l_\eta(1))+2\mu \\
&\le S_{\lambda,0}(t^*)\lambda(1+l_\eta(0))+S_{\lambda,0}(t^*)\lambda(1+l_\eta(1))+2\mu,
\end{aligned}
$$

where the last two equalities hold by Theorem 4.3.2 and Lemma 4.5.2. Now let J be as defined in Lemma 4.5.3 and observe that

$$S_{\lambda,0}(t^*)\lambda(1+l_\eta(0))+S_{\lambda,0}(t^*)\lambda(1+l_\eta(1))+2\mu$$

is a δ_ρ integrable function, since

$$
\begin{aligned}
&\int_{\Omega_{dec}^{1-\rho}}\left(S_{\lambda,0}(t^*)\lambda(1+l_\eta(0))+S_{\lambda,0}(t^*)\lambda(1+l_\eta(1))+2\mu\right)\,d\delta_\rho = \\
&= 2\lambda+2\mu+2\lambda\int_{\Omega_{dec}^{1-\rho}} l_\eta(0)\,d\delta_\rho S_{\lambda,0}(t^*) \\
&\le 2\lambda+2\mu+2\lambda E(J)<\infty,
\end{aligned}
$$

by Lemma 4.5.3. Observe that at time t^*, the block of ones to the right of 0 cannot be longer than the distance from zero to the first positive s.e.p., which explains the last inequality. ☐

Proof of Theorem 4.5.1: Let $0<\rho<1$ be given. Assume that $\nu_\infty=\delta_\rho$. Then δ_ρ is invariant, which implies that

$$\frac{d}{dt}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)\,d\delta_\rho=0, \tag{4.18}$$

for all $t\ge 0$. We shall show that this cannot be true. Since δ_ρ concentrates on $\Omega_{dec}^{1-\rho}$ and $\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}$ is a monotone, local and bounded function, we get from Theorem 4.3.2 that for $t<t^*$,

$$\frac{d}{dt}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)\,d\delta_\rho=\frac{d}{dt}\int_{\Omega_{dec}^{1-\rho}}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\,d\delta_\rho.$$

By Lemma 4.5.4, (4.17) and the invariance of δ_ρ, we may write

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_{dec}^{1-\rho}}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\,d\delta_\rho \\
&= \int_{\Omega_{dec}^{1-\rho}}\frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^nG^n\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}(\eta)}{n!}\,d\delta_\rho \\
&= \int_{\Omega_{dec}^{1-\rho}} S(t)\mathbf{1}_{\{\eta(0)=0,\eta(1)=1\}}(\eta)\lambda(1+l_\eta(0))\,d\delta_\rho \\
&\quad + \int_{\Omega_{dec}^{1-\rho}} S(t)\mathbf{1}_{\{\eta(0)=1,\eta(1)=0\}}(\eta)\lambda(1+l_\eta(1))\,d\delta_\rho \\
&\quad -2\mu\int_{\Omega_{dec}^{1-\rho}} S(t)\mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}\,d\delta_\rho \\
&= \int_{\Omega_{dec}^{1-\rho}} \mathbf{1}_{\{\eta(0)=0,\eta(1)=1\}}(\eta)\lambda(1+l_\eta(0))\,d\delta_\rho \\
&\quad + \int_{\Omega_{dec}^{1-\rho}} \mathbf{1}_{\{\eta(0)=1,\eta(1)=0\}}(\eta)\lambda(1+l_\eta(1))\,d\delta_\rho \\
&\quad -2\mu\int_{\Omega_{dec}^{1-\rho}} \mathbf{1}_{\{\eta(0)=1,\eta(1)=1\}}\,d\delta_\rho \\
&= \sum_{n=1}^{\infty}(1-\rho)^2\rho^n(n+1)\lambda + \sum_{n=1}^{\infty}(1-\rho)^2\rho^n n\lambda - 2\mu\rho^2 \\
&= \frac{\lambda^2}{\mu} - \frac{\lambda^3}{\mu^2} \neq 0,
\end{aligned}$$

which contradicts (4.18), so $\nu_\infty \neq \delta_\rho$. □

4.6 Positive correlations

In this section we discuss some further details of the invariant measure ν_∞. We assume that $\lambda < \mu$, so we are in the situation where a non-trivial stationary distribution exists. Recall that $\rho := \frac{\lambda}{\mu}$. The space $C(\Omega)$ is the Banach space of continuous functions on Ω, with norm

$$||f|| := \sup_{\eta\in\Omega}|f(\eta)|.$$

The set of increasing continuous functions on Ω is denoted by $\mathcal{M}_c$. A measure ν on Ω is said to have positive correlations if for all $f, g \in \mathcal{M}_c$,

$$\int_\Omega fg\,d\nu \geq \int_\Omega f\,d\nu \int_\Omega g\,d\nu.$$

When we say that a process is Feller, we use Definition 1.2 (p. 8) in Liggett (1985). The main results of this section are the following two theorems.

Theorem 4.6.1 *The measure ν_∞ has positive correlations.*

Theorem 4.6.2 *Suppose that the initial state of the SMD is chosen according to ν_∞. The following inequalities hold for the correlation between the initial state of the process at site zero and the state of the process at site zero at time t:*

$$0 \leq \int_\Omega (\eta(0) - \rho) S(t)(\eta(0) - \rho)\, d\nu_\infty \leq \rho(1-\rho)e^{-\mu t}.$$

Recall that $S_n(t)$ is the generator of the n-process on Ω. The idea of the proof of Theorem 4.6.1 is as follows. We prove that for continuous functions f, $S_n(t)f$ is also continuous (this is sometimes called the Feller property). Then we show that if ν has positive correlations, then also $\nu S_n(t)$ has positive correlations. Sending first n and then t to infinity will lead to the desired result. This method is indicated in Liggett (1985) p. 80. The Feller property of the n-process is not used in the proof of Theorem 4.6.1. Nevertheless we will prove this, since it is an interesting result in itself, and we use the method of the proof later on. Furthermore, we use the Feller property of the n-process in the proof of Theorem 4.6.2.

Proposition 4.6.3 *If $f \in C(\Omega)$, then also $S_n(t)f \in C(\Omega)$.*

Proof: Let $X_i(t)$, $D_i(t)$ and $\hat{\eta}_{n,t}$ be defined as in Section 4.4. We will show that for all $\eta \in \Omega$,

$$\lim_{\eta' \to \eta} S_n(t) f(\eta') = S_n(t) f(\eta).$$

Observe that for $l > n$,

$$d(\eta', \eta) = \frac{1}{l}$$

implies that

$$d(\hat{\eta}'_{n,t}, \hat{\eta}_{n,t})) \leq \frac{1}{l}$$

(here we use the same sequences X_i and D_i to construct both $\hat{\eta}'_{n,t}$ and $\hat{\eta}_{n,t}$).

Let $f \in C(\Omega)$, $\eta \in \Omega$, $t \geq 0$, and $\epsilon > 0$ be given. We show that there exists an $l > 0$ such that

$$d(\eta, \eta') < \frac{1}{l}$$

implies that

$$|S_n(t)f(\eta) - S_n(t)f(\eta')| < \epsilon,$$

which suffices. Since Ω is a compact metric space, f is uniformly continuous. Hence take a $\gamma > 0$ so that for all $\xi_1, \xi_2 \in \Omega$ with $d(\xi_1, \xi_2) < \gamma$,

$$|f(\xi_1) - f(\xi_2)| < \epsilon.$$

Now fix some $l > n$ such that $\frac{1}{l} < \gamma$. Then

$$d(\eta', \eta) < \frac{1}{l}$$

implies that

$$d(\hat{\eta}_{n,t}, \hat{\eta}'_{n,t}) < \frac{1}{l},$$

which implies that

$$|f(\hat{\eta}_{n,t}) - f(\hat{\eta}'_{n,t})| < \epsilon,$$

which yields

$$\begin{aligned} |S_n(t)f(\eta) - S_n(t)f(\eta')| &= |E(f(\hat{\eta}_{n,t})) - E(f(\hat{\eta}'_{n,t}))| \\ &\leq E|f(\hat{\eta}_{n,t}) - f(\hat{\eta}'_{n,t})| < \epsilon. \end{aligned}$$

□

Lemma 4.6.4 *Let ν be a measure on Ω with positive correlations. Then for all t, the measure $\nu S_n(t)$ has positive correlations.*

Proof: We write the semigroup of the n-process as a limit of semigroups $S_{k,n}(t)$ of (k,n)-processes (to be defined) for k tending to infinity, where the (k,n)-processes are monotone Feller processes and have bounded generators. The (k,n)-processes will have the property that if ν has positive correlations, then also $\nu S_{k,n}(t)$ has positive correlations, and taking limits will prove the statement.

Recall that $l^n_\eta(i)$ was defined to be the number of occupied sites in $[-n,n]$ to the right of site i until the nearest site that is unoccupied:

$$l^n_\eta(i) := \#\{j \in \mathbb{Z} \cap [-n,n]\colon\ j > i \text{ and for all } i < j' \leq j\colon\ \eta(j') = 1\}.$$

For $k > n$, in a (k,n)-process on Ω, particles enter the system as in the n-processes, but only particles at the sites in $[-k,k]$ can leave the system. So for $k > n$, the (k,n) process is the process associated to the generator $G_{k,n}$ which is for all $f \in C(\Omega)$ given by

$$\begin{aligned} G_{k,n}f(\eta) &= \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda l^n_\eta(i) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ &\quad + \sum_{i=-k}^{i=k} \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ &\quad + \sum_{i=-n}^{n} \mathbf{1}_{\{\eta(i)=0\}} \lambda \left(f\left(T_i(\eta)\right) - f(\eta)\right). \end{aligned}$$

Since $G_{k,n}$ is a bounded Markov pregenerator, it is also a Markov generator (Proposition 2.8, p. 15, Liggett (1985)). The (k,n)-processes are Feller, which can be proved in the same way as Proposition 4.6.3. We claim that for all $f \in C(\Omega)$,

$$\lim_{k\to\infty} ||S_{k,n}(t)f - S_n(t)f|| = 0, \tag{4.19}$$

which we prove now. Assume that $k > n$. Use the arrival and departure processes as indicated in the proof of Lemma 4.6.3 to construct a coupling of the state of the (k,n)-process and the n-process at time t in the following way. Use the same procedure as before to construct the $\hat{\eta}_{n,t}$. In this coupling, for the (k,n)-process, an arrival at a site which is already occupied, causes a flip at the nearest unoccupied site to the left and only departures at sites in $[-k, k]$ can occur. We denote the state of the (k, n)-process at time t in the coupling by $\hat{\eta}_{(k,n)}(t)$. We see that, if both processes start from the same initial state, the distance between the coupled processes at any time t cannot be larger than $\frac{1}{k}$, since the processes coincide at sites in $[-k, k]$. Now let $\epsilon > 0$ be given and take l as in the proof of Proposition 4.6.3. Then for $k \geq l$ and for all η,

$$d(\hat{\eta}_{(k,n)}(t), \hat{\eta}_{n,t}) \leq \frac{1}{l},$$

so for all η,

$$|S_{k,n}(t)f(\eta) - S_n(t)f(\eta)| < \epsilon,$$

which implies that also

$$||S_{k,n}(t)f - S_n(t)f|| < \epsilon.$$

So

$$||S_{k,n}(t)f - S_n(t)f|| \to 0,$$

for all $f \in C(\Omega)$, $(k \to \infty)$, and (4.19) is proved.

Let ν have positive correlations. Since the (k, n)-process can only jump to comparable states (that is, if the process jumps from η to ξ then either $\xi < \eta$ or $\eta < \xi$), it follows from Theorem 2.14 p. 80 Liggett (1985), that $\nu S_{n,k}(t)$ has positive correlations. So for all $f, g \in \mathcal{M}_c$,

$$\int_\Omega fg\, d\nu S_{k,n}(t) \geq \int_\Omega f\, d\nu S_{k,n}(t) \int_\Omega g\, d\nu S_{k,n}(t),$$

or equivalently,

$$\int_\Omega S_{k,n}(t) fg\, d\nu \geq \int_\Omega S_{k,n}(t) f\, d\nu \int_\Omega S_{k,n}(t) g\, d\nu.$$

Sending k to infinity, we find by (4.19) that

$$\int_\Omega S_n(t) fg\, d\nu \geq \int_\Omega S_n(t) f\, d\nu \int_\Omega S_n(t) g\, d\nu,$$

or

$$\int_\Omega fg\, d\nu S_n(t) \geq \int_\Omega f\, d\nu S_n(t) \int_\Omega g\, d\nu S_n(t),$$

so $\nu S_n(t)$ has positive correlations whenever ν has positive correlations. □

Proof of Theorem 4.6.1: Let δ_0 be the Dirac measure on $\{0\}^{\mathbb{Z}}$. This measure has positive correlations. It follows from Lemma 4.6.4 that $\delta_0 S_n(t)$ has positive correlations, so for all $f, g \in \mathcal{M}_c$,

$$\int_\Omega S_n(t) fg\, d\delta_0 \geq \int_\Omega S_n(t) f\, d\delta_0 \int_\Omega S_n(t) g\, d\delta_0.$$

Sending n to infinity, we claim that

$$\int_\Omega S(t) fg\, d\delta_0 \geq \int_\Omega S(t) f\, d\delta_0 \int_\Omega S(t) g\, d\delta_0. \tag{4.20}$$

To see this, for the right hand side, we have that $S_n(t)f(\eta) \uparrow S(t)f(\eta)$ and $S_n(t)g(\eta) \uparrow S(t)g(\eta)$ by definition, and we can apply the monotone convergence theorem. Observe that we must be a bit careful when we take the limit $n \to \infty$ in $\int_\Omega S_n(t) fg \, d\nu_{\delta_0}$, since fg does not need to be a monotone function. But since we can write fg as a sum of monotone functions by

$$\begin{aligned} fg &= (f - \min_{\xi\in\Omega} f(\xi))(g - \min_{\xi\in\Omega} g(\xi)) \\ &\quad + \min_{\xi\in\Omega} f(\xi) g + \min_{\xi\in\Omega} g(\xi) f - \min_{\xi\in\Omega} f(\xi) \min_{\xi\in\Omega} g(\xi), \end{aligned}$$

we see that

$$\lim_{n\to\infty} \int_\Omega S_n(t) fg \, d\delta_0 = \int_\Omega S(t) fg \, d\delta_0.$$

We can rewrite (4.20) as

$$\int_\Omega fg \, d\delta_0 S(t) \geq \int_\Omega f \, d\delta_0 S(t) \int_\Omega g \, d\delta_0 S(t).$$

Taking the limit $t \to \infty$ gives

$$\int_\Omega fg \, d\nu_\infty \geq \int_\Omega f \, d\nu_\infty \int_\Omega g \, d\nu_\infty,$$

so ν_∞ has positive correlations. □

Proof of Theorem 4.6.2: Observe that

$$\begin{aligned} \int_\Omega (\eta(0) - \rho) S(t)(\eta(0) - \rho) \, d\nu_\infty &= \int_\Omega \eta(0) S(t) \eta(0) \, d\nu_\infty \\ &\quad - \int_\Omega \rho S(t) \eta(0) \, d\nu_\infty - \int_\Omega \eta(0) S(t) \rho \, d\nu_\infty \\ &\quad + \int_\Omega \rho S(t) \rho \, d\nu_\infty \\ &= \int_\Omega \eta(0) S(t) \eta(0) \, d\nu_\infty - \rho^2. \end{aligned}$$

Let δ_1 be the measure which concentrates on $\{1\}^{\mathbb{Z}}$.

We start with the first inequality. Observe that by Proposition 4.6.3, $S_n(t)\eta(0) \in \mathcal{M}_c$ for all n. By Theorem 4.6.1 we get that

$$\begin{aligned} \int_\Omega \eta(0) S(t) \eta(0) \, d\nu_\infty - \rho^2 &= \lim_{n\to\infty} \int_\Omega \eta(0) S_n(t) \eta(0) \, d\nu_\infty - \rho^2 \\ &\geq \lim_{n\to\infty} \int_\Omega \eta(0) \, d\nu_\infty \int_\Omega S_n(t) \eta(0) \, d\nu_\infty - \rho^2 \\ &\geq \rho\rho - \rho^2 = 0. \end{aligned}$$

For the second inequality we get, using the computations of the proof of Theorem 4.1.1, that

$$\int_\Omega (\eta(0) - \rho) S(t)(\eta(0) - \rho) d\nu_\infty - \rho^2 = \rho \int_\Omega S(t) \eta(0) \, d\nu_\infty(\eta | \eta(0) = 1) - \rho^2$$

$$
\begin{aligned}
&\le \rho \int_\Omega S(t)\eta(0)\, d\delta_1 - \rho^2 \\
&= \rho((1-\rho)e^{-\mu t} + \rho) - \rho^2 \\
&= \rho(1-\rho)e^{-\mu t},
\end{aligned}
$$

which proves the theorem. □

Chapter 5

Construction of a bricklayer process

5.1 Introduction

In Balàzs (2001) a bricklayer model is introduced, which is a process on $\tilde{\Omega} = \mathbb{Z}^{\mathbb{Z}}$. The system is a nearest neighbour system with unbounded rates. Balàzs does not construct this system, but (under the assumption it exists) he achieves various results on the invariant measure of the process, which turns out to be a product measure.

As motivated in Section 4.1, the question whether such a process exists is interesting from a mathematical point of view. In this chapter we will give a formal construction of the bricklayer process. Configurations of the process are denoted by $\tilde{\omega} \in \tilde{\Omega}$ and the random state of the process at time t and with initial configuration $\tilde{\omega}$ is denoted by $\tilde{\omega}_t$. The construction deals with the case that the rates are (sub)linear, and the initial configuration is chosen according to a (spatially) stationary ergodic measure such that the expectation of $|\tilde{\omega}(0)|$ is finite.

We first give an informal description of the bricklayer process. Let $r : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ be an increasing function, with the property that for all $z \in \mathbb{Z}$,

$$r(z)r(-z+1) = 1.$$

The dynamics of the process can be described as

$$(\tilde{\omega}(i), \tilde{\omega}(i+1)) \to (\tilde{\omega}(i) - 1, \tilde{\omega}(i+1) + 1) \text{ at rate } r(\tilde{\omega}(i)) + r(-\tilde{\omega}(i+1)).$$

The process can be interpreted as a bricklayer process in the following sense. Think of an infinite wall, built of bricks, as in Figure 5.1. We identify the wall with a vector $\tilde{\omega} \in \tilde{\Omega}$, where $\tilde{\omega}(i)$ denotes the height difference between the column of bricks between the sites $(i-1)$ and i and the sites i and $(i+1)$. Imagine you take a walk along the top of this wall, from right to left. Then the height difference is positive if you go up, and negative if you go down. For example, in Figure 5.1, $\tilde{\omega}(0) = 1$, $\tilde{\omega}(1) = 3$, $\tilde{\omega}(3) = 2$, $\tilde{\omega}(6) = -1$ and $\tilde{\omega}(7) = 3$. At each site i, a bricklayer is present, putting a brick to his right at rate $r(\tilde{\omega}(i))$ and to his left at rate $r(-\tilde{\omega}(i))$.

Define for $i \in \mathbb{Z}$ transformations $\tilde{A}_i : \tilde{\Omega} \to \tilde{\Omega}$ by

$$\tilde{A}_i(\tilde{\omega})(x) = \begin{cases} \tilde{\omega}(x) - 1 & \text{if } x = i \\ \tilde{\omega}(x) + 1 & \text{if } x = i+1 \\ \tilde{\omega}(x) & \text{if } x \neq i, i+1. \end{cases}$$

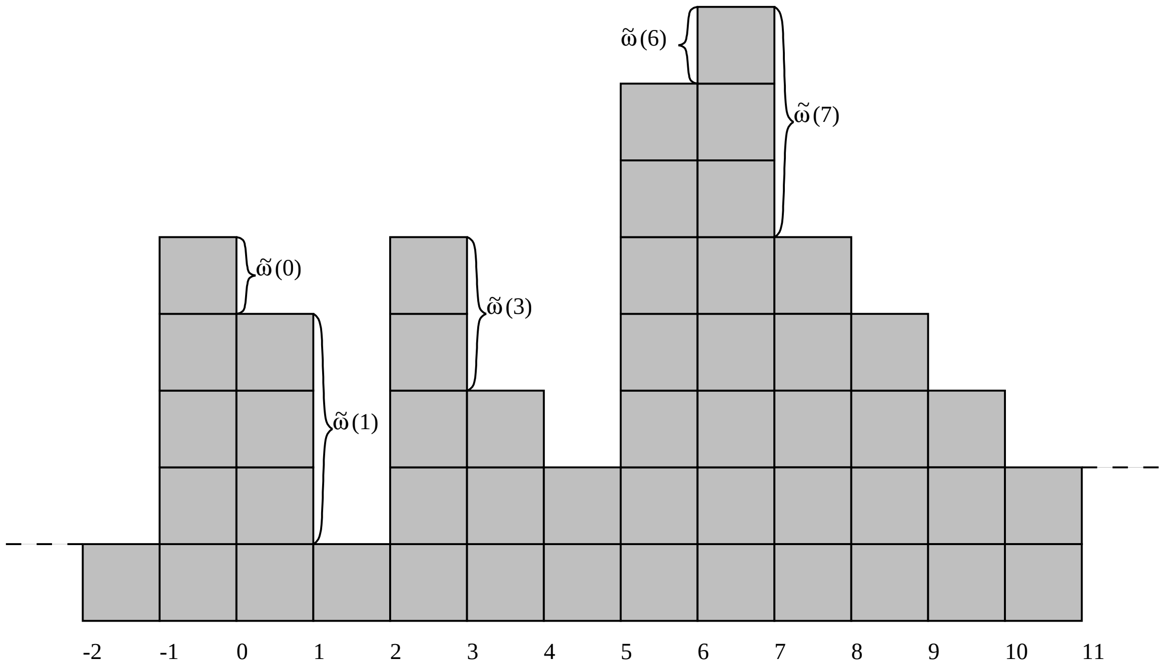

Figure 5.1: Bricklayer process

Then the formal generator $\tilde{G}$ of the bricklayer process is, for $f : \tilde{\Omega} \to \mathbb{R}$, given by

$$\tilde{G}f(\omega) = \sum_i \left(r(\tilde{\omega}(i)) + r(-\tilde{\omega}(i+1))\right) \left(f(\tilde{A}_i(\tilde{\omega})) - f(\tilde{\omega})\right).$$

We will construct the bricklayer process in the case that there exist $a, b \in \mathbb{R}_{\geq 0}$ such that for $z \geq 1$, $r(z) \leq az + b$. We first only allow bricklayers at the positions in $[-n, n]$ to lay bricks and call the corresponding process the bricklayer n-process. Then we show that, if the initial configuration is chosen according to an ergodic stationary measure for which $E(|\tilde{\omega}(0)|) < \infty$, we can almost surely define a limiting process, by making the region in which the bricks are laid larger and larger. We define this process to be the bricklayer process. To do this, we compare the bricklayer process to a particle system (called the *block process*), which we can construct relatively easy, because of its intrinsic monotonicity. This block process is constructed by considering limits of so called block n-processes. In the block process, for any time t, there is (almost surely) a positive density of sites that have not changed until time t. We shall make a coupling of the block n-process and the bricklayer n-process such that if certain sites change in the bricklayer n-process, then the corresponding sites *always* change in the block n-process. The fact that the limit of the block n-processes is well defined, will lead to a well defined limit of bricklayer n-processes.

In Section 5.2 we construct the block process, and we construct the bricklayer process in Section 5.3.

Since Balàzs (2001) uses a relation between the semigroup $\tilde{S}(t)$ of the bricklayer process and its formal generator, we will prove that such a relation indeed holds for the process we constructed.

5.2 The block process

5.2.1 Introduction

We describe and construct a particle system with state space $\Omega = \mathbb{N}^{\mathbb{Z}}$. The particles in this system are interpreted as blocks, and we will call the process the *block process*. The space Ω is equipped with the product topology and the Borel σ-algebra. Elements of Ω are typically denoted by ω, and ω_t denotes the (random) state of the process at time t if the initial configuration was ω. We interpret a configuration $\omega \in \Omega$ as an infinite wall, in which at site i, there is a column of $\omega(i)$ blocks (see Figure 5.2). For instance, in Figure 5.2, $\omega(1) = 2$ and $\omega(6) = 0$. Notice the difference with the bricklayer process, where $\tilde{\omega}(i)$ denotes a height *difference* in the wall at site i, while in the block process, $\omega(i)$ gives the *height* of the wall at site i.

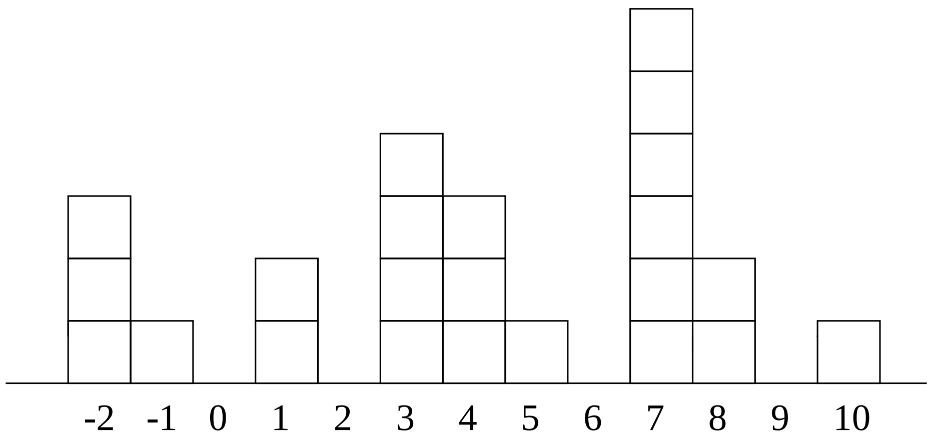

Figure 5.2: Block process

Suppose that the process starts in configuration $\omega \in \Omega$. Informally, the dynamics of the process can be described as follows. For all $i \in \mathbb{Z}$, at rate $\omega(i) + \omega(i+1)$, one block is added to site i and one block is added to site $i+1$. So

$$(\omega(i), \omega(i+1)) \to (\omega(i) + 1, \omega(i+1) + 1), \text{ at rate } \omega(i) + \omega(i+1).$$

This description does not give rise to a well defined process on Ω for all $\omega \in \Omega$. One can imagine that it is possible that for certain initial configurations, the process comes into a state in which there are sites with infinitely many blocks. We will make precise in which case the process can be constructed on the space Ω later.

We define, for $i \in \mathbb{Z}$, adding transformations $A_i : \Omega \to \Omega$ by

$$A_i(\omega)(x) = \begin{cases} \omega(x) + 1 & \text{if } x = i, i+1 \\ \omega(x) & \text{if } x \neq i, i+1. \end{cases}$$

The formal generator G of the process is then, for functions $f : \Omega \to \mathbb{R}$, given by

$$Gf(\omega) = \sum_i (\omega(i) + \omega(i+1)) \left(f(A_i(\omega)) - f(\omega) \right).$$

In Section 5.2.2 we will construct the block process on Ω, if the initial configuration ω of the block process is chosen according to an ergodic stationary measure with $E(\omega(0)) < \infty$. This construction uses the monotonicity of the process.

In Section 5.2.3 we see that, for ω chosen as above, for any time t there is a positive density of sites that have not changed until time t, almost surely. So at these sites almost surely no blocks were added. We will call these sites *special points at time t*. These sites will turn out to be useful when we construct the bricklayer process later.

Finally, Section 5.2.4 deals with the relation between the semigroup $S(t)$ of the block process and its formal generator. We include this section since the idea of the proof is also used in the proof of the relation between the semigroup and the formal generator of the bricklayer process.

5.2.2 Construction of the block process

We construct the block process as a process with initial configurations in Ω and configurations at time $t > 0$ in $(\mathbb{N} \cup \{\infty\})^{\mathbb{Z}}$. The idea of the construction is as follows. We first assume that changes from ω to $A_i(\omega)$ occur only for $i \in [-n, n]$, with rates as described in Section 5.2.1. Then we show that we can define a limiting process, for $n \to \infty$. This process will be the block process. The construction proceeds in three steps.

Step 1. We define the block n-process on Ω, in which for $i \in [-n, n]$ and $\omega \in \Omega$, ω changes to $A_i(\omega)$ at rate $\omega(i) + \omega(i+1)$. We write $\omega_{n,t}$ for the state of the block n-process at time t if the initial configuration was ω.

Let $S_n(t)$ be the semigroup of the block n-process, that is to say that for measurable and bounded functions $f : \Omega \to \mathbb{R}$,

$$S_n(t)f(\omega) := E\left(f(\omega_{n,t})\right)$$

(and we use the same notation for unbounded measurable f, if $E(f(\omega_{n,t})) < \infty$). The generator of the block n-process is then, for f as above, given by

$$\begin{aligned} G_n f(\omega) &= \lim_{t \downarrow 0} \frac{S_n(t)f(\omega) - f(\omega)}{t} \\ &= \sum_{i=-n}^{n-1} (\omega(i) + \omega(i+1)) \left(f(A_i(\omega)) - f(\omega)\right) \\ &\quad + \omega(-n) \left(f(A_{-n-1}(\omega)) - f(\omega)\right) + \omega(n) \left(f(A_n(\omega)) - f(\omega)\right). \end{aligned}$$

For notational convenience we define, for $\omega \in \Omega$, $l_n(\omega) \in (\mathbb{R}_{\geq 0})^{\mathbb{Z}}$ by

$$l_n(\omega)(i) = \begin{cases} \omega(i) + \omega(i+1) & \text{if } i \in [-n, n-1] \\ \omega(-n) & \text{if } i = -n-1 \\ \omega(n) & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$G_n f(\omega) = \sum_{i \in \mathbb{Z}} l_n(\omega)(i) \left(f(A_i(\omega)) - f(\omega)\right).$$

Step 2. Let $\mathcal{M}$ be the collection of measurable, bounded and increasing functions from $\Omega \to \mathbb{R}$. The block n-processes are monotone in the parameter n, that is to say that for $f \in \mathcal{M}$,

$$S_{n+1}(t)f(\omega) \geq S_n(t)f(\omega).$$

This can be seen when we use the following method to construct a coupling $(\omega^c_{n,t})_{n=1}^{\infty}$ of the processes $(\omega_{n,t})_{n=1}^{\infty}$. In this coupling, when blocks are added at certain sites in the block n-process, they are also added at the corresponding sites in all block k-processes with $k \geq n$. We make this precise.

We write $\mathbb{N}^+ = \{1, 2, \ldots\}$. Let for $i \in \mathbb{Z}, j \in \mathbb{N}^+$, $X_{i,j}(t)$ be independent Poisson processes with parameter 1. We use these Poisson processes for all (coupled) block n-processes to decide when and whether blocks are added as follows. In the block n-process, if the configuration at a certain moment is ω and $l_n(\omega)$ is positive, we use the Poisson processes $X_{i,1}(t), \ldots, X_{i,l_n(\omega)}(t)$ to decide whether and when ω changes to $A_i(\omega)$. In this coupling, the block n-process with initial configuration ω is denoted by $\omega^c_{n,t}$.

Observe that, if $\omega_n \leq \omega_{n+1}$ and ω_n is the configuration of blocks in the block n-process and ω_{n+1} is the configuration of blocks in the block $(n+1)$-process, then in this coupling,

$$(\omega_n, \omega_{n+1}) \to (A_i(\omega_n), A_i(\omega_{n+1})) \text{ at rate } l_n(\omega_n)(i)$$

and

$$(\omega_n, \omega_{n+1}) \to (\omega_n, A_i(\omega_{n+1})) \text{ at rate } l_{n+1}(\omega_{n+1})(i) - l_n(\omega_n)(i).$$

So in this coupling for all $n \in \mathbb{N}$,

$$\omega^c_{n,t} \leq \omega^c_{n+1,t},$$

which implies that the block n-processes are monotone in n.

Notice that the block n-process is also monotone in ω, that is, for $\omega_1 \leq \omega_2$ and $f \in \mathcal{M}$,

$$S_n(t)f(\omega_1) \leq S_n(t)f(\omega_2).$$

Finally, the block n-process is monotone in time, that is, for $t_1 \leq t_2$, $f \in \mathcal{M}$ and $\omega \in \Omega$,

$$S_n(t_1)f(\omega) \leq S_n(t_2)f(\omega).$$

Step 3. Let $\omega \in \Omega$. We use the coupling described in Step 2 to define $\omega_t \in (\mathbb{N} \cup \{\infty\})^{\mathbb{Z}}$ by

$$\omega_t := \lim_{n \to \infty} \omega^c_{n,t}.$$

This is well-defined, because of the monotonicity of the processes in the parameter n. We define this process to be the block process.

We are not too interested in those initial configurations for which $\omega_t(i) = \infty$ with positive probability, for some $i \in \mathbb{Z}$ and $t > 0$. To describe the next result, we need some more notation. Let $\mathcal{P}$ be the state space of the Poisson processes

$$\left(X_{i,j}(t)\right)_{i \in \mathbb{Z}, j \in \mathbb{N}^+},$$

and let ρ be the measure on $\mathcal{P}$ which has the property that the processes $X_{i,j}(t)$ are i.i.d. Poisson processes with parameter 1. Elements of $\mathcal{P}$ are typically denoted by χ and we can interpret ω_t and $\omega_{n,t}^c$ as a function of the initial configuration ω and the realisation of the Poisson processes χ, therefore we sometimes write $\omega_t(\omega, \chi)$. Elements of the product space $\Omega \times \mathcal{P}$ are generally denoted by (ω, χ).

Theorem 5.2.1 *Let ν be an ergodic stationary measure on Ω with*

$$E_\nu(\omega(0)) := \int_\Omega \omega(0)\, d\nu(\omega) < \infty.$$

Then

$$(\nu \times \rho)\left((\omega, \chi) : \omega_t \in \Omega, \forall t\right) = 1.$$

To prove this theorem, we use the following lemmas:

Lemma 5.2.2 *Let ν be an ergodic stationary measure on Ω with*

$$E_\nu(\omega(0)) < \infty.$$

Then for all $t \geq 0$ and all $n \in \mathbb{N}$,

$$\frac{d}{dt} \int_\Omega S_n(t)\omega(0)\, d\nu = \int_\Omega \frac{d}{dt} S_n(t)\omega(0)\, d\nu. \tag{5.1}$$

Proof: Fix some $t^* > 0$. We show that (5.1) is true for all $t < t^*$, which is sufficient, since t^* is arbitrary. To do this, it suffices to show that there exists a ν-integrable function g, such that for $t < t^*$,

$$\frac{d}{dt} S_n(t)\omega(0) \leq g(\omega)$$

(see Theorem 16.8 in Billingsley (1986)). Define the function $D : \Omega \to \mathbb{R}$ by

$$D(\omega) := \sum_{i=-n-1}^{n+1} \omega(i).$$

The function $D(\omega)$ counts the number of blocks present at the sites in $[-n-1, n+1]$. Since the block n-process is monotone in time,

$$\begin{aligned} \frac{d}{dt} S_n(t) D(\omega) &= S_n(t) G_n D(\omega) \\ &\leq 4 S_n(t) D(\omega) \\ &\leq 4 S_n(t^*) D(\omega). \end{aligned}$$

Observe that $S_n(t)D(\omega)$ is non-negative for all $t \geq 0$, so we have that for $t \leq t^*$,

$$S_n(t) D(\omega) \leq D(\omega) e^{4t}.$$

We conclude that

$$\frac{d}{dt}S_n(t)D(\omega) \le 4S_n(t^*)D(\omega) \le 4D(\omega)e^{4t^*},$$

which is ν-integrable because

$$\int_\Omega 4D(\omega)e^{4t^*}\,d\nu = 4(2n+3)e^{4t^*}\int_\Omega \omega(0)\,d\nu(\omega) < \infty.$$

This proves the lemma. □

We write ν_t for the measure of ω_t, if the initial configuration ω has measure ν on Ω. The measure ν_t is (a priori) a measure on the space $(\mathbb{Z}\cup\{\infty\})^{\mathbb{Z}}$.

Lemma 5.2.3 *Let ν be an ergodic stationary measure on Ω, then ν_t is an ergodic stationary measure on $(\mathbb{Z}\cup\{\infty\})^{\mathbb{Z}}$.*

The proof is similar to the proof of Proposition 4.4.1 and therefore omitted.

Proof of Theorem 5.2.1: Let ν be an ergodic stationary measure on Ω with $E_\nu(\omega(0)) < \infty$. Since the process ω_t is monotone in t, it suffices to show that for arbitrary t,

$$(\nu\times\rho)\left((\omega,\chi) : \omega_t \in \Omega\right) = 1. \tag{5.2}$$

Let t be arbitrary but fixed. We shall show that the expected number of blocks at site i, if the initial configuration is chosen according to ν, is finite; that is we show that

$$\int_{\Omega\times\mathcal{P}} \omega_t(i)\,d(\nu\times\rho) < \infty. \tag{5.3}$$

Having (5.3), implies that for all i, $\omega_t(i) < \infty$, $(\nu\times\rho)$-almost surely and (5.2) follows. According to Lemma 5.2.3 we only need to prove (5.3) for the case $i = 0$. Since

$$\int_{\Omega\times\mathcal{P}} \omega_t(0)\,d(\nu\times\rho) = \lim_{n\to\infty}\int_\Omega S_n(t)\omega(0)\,d\nu, \tag{5.4}$$

by monotone convergence, it suffices to show that $\int_\Omega S_n(t)\omega(0)\,d\nu$ is uniformly bounded in n. To do this, we obtain a differential inequality for the function

$$\int_\Omega S_n(t)\omega(0)\,d\nu.$$

By Lemma 5.2.2 and the observation in the proof of Lemma 5.2.3,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega S_n(t)\omega(0)\,d\nu &= \int_\Omega \frac{d}{dt}S_n(t)\omega(0)\,d\nu \\
&= \int_\Omega S_n(t)G_n\omega(0)\,d\nu \\
&= \int_\Omega S_n(t)(\omega(-1)+2\omega(0)+\omega(1))\,d\nu \\
&\le 2\int_\Omega S_n(t)\omega(0)\,d\nu + 2\int_\Omega S_{n+1}(t)\omega(0)\,d\nu. \qquad (5.5)
\end{aligned}$$

By monotonicity of the block n-processes in the parameter n and by computations as in the proof of Lemma 5.2.2,

$$\begin{aligned}
&S_{n+1}(t)\omega(0) - S_n(t)\omega(0)\\
&\leq \; S_{n+1}(t)\sum_{i=-n-2}^{n+2}\omega(i) - S_n(t)\sum_{i=-n+1}^{n-1}\omega(i)\\
&\leq \; \sum_{i=-n-2}^{n+2}\omega(i)e^{4t} - \sum_{i=-n+1}^{n-1}\omega(i)e^{4t}\\
&= \; (\omega(-n-2)+\omega(-n-1)+\omega(-n)+\omega(n)+\omega(n+1)+\omega(n+2))e^{4t}.
\end{aligned}$$

Combining this with (5.5), leads to

$$\frac{d}{dt}\int_\Omega S_n(t)\omega(0)\,d\nu \leq 4\frac{d}{dt}\int_\Omega S_n(t)\omega(0)\,d\nu + 12E_\nu(\omega(0))e^{4t},$$

which implies that for all n,

$$\int_\Omega S_n(t)\omega(0)\,d\nu \leq \left(12E_\nu(\omega(0))t + E_\nu(\omega(0))\right)e^{4t}. \tag{5.6}$$

This is finite and independent of n, so we are done. □

5.2.3 Existence of special points

Let $\omega \in \Omega$ and $\chi \in \mathcal{P}$. We call i a special point (s.p.) at time t for (ω,χ) if $\omega_t(\omega,\chi)(i) = \omega(i)$. So if i is a s.p. at time t, then until time t, no blocks were added to site i in any of the coupled block n-processes with initial configuration ω. We will prove the following theorem:

Theorem 5.2.4 *Let ν be an ergodic stationary measure with $E_\nu(\omega(0)) < \infty$. Then there exists $\epsilon_t > 0$, such that*

$$(\nu\times\rho)\left((\omega,\chi) : \omega_t(0) = \omega(0)\right) \geq \epsilon_t.$$

Proof: Let ν be an ergodic stationary measure with $E_\nu(\omega(0)) < \infty$ and write

$$h_\nu(t) = \left\lceil \left(12E_\nu(\omega(0))t + E_\nu(\omega(0))\right)e^{4t}\right\rceil.$$

Then by Lemma 5.2.3, (5.6) and the monotone convergence theorem, we find that

$$\int_{\Omega\times\mathcal{P}} (\omega_t(-1) + 2\omega_t(0) + \omega_t(1))\,d(\nu\times\rho) \leq 4h_\nu(t).$$

This implies that there exists a $\delta_t > 0$ such that

$$(\nu\times\rho)\Big((\omega,\chi) : \omega_t(-1) + 2\omega_t(0) + \omega_t(1) \leq 5h_\nu(t)\Big) \geq \delta_t,$$

which in turn implies that there exists an $\epsilon_t \geq 0$, such that

$$(\nu\times\rho)\left((\omega,\chi) : \omega_t(0) = \omega(0)\right) \geq \epsilon_t,$$

which is what we wanted to prove. □

5.2.4 The relation between $S(t)$ and G

Let $f : \Omega \to \mathbb{R}$ be a measurable function. If ω and t are such that $\omega_t \in \Omega$, ρ-almost surely, we write

$$S(t)f(\omega) := E\left(f(\omega_t)\right),$$

provided that this expectation is finite. We call $S(t)$ the semigroup of the block process. As in for the SMD discussed in Chapter 4, there is for some 'special' functions and configurations a relation between the semigroup of the block process and its formal generator.

To describe the result, we need some notation. For $N \in \mathbb{N}$, we call a measurable function $f : \Omega \to \mathbb{R}$ *N-dependent*, if for all $\omega_1, \omega_2 \in \Omega$ with $\omega_1(x) = \omega_2(x)$ for all $x \in [-N, N]$,

$$f(\omega_1) = f(\omega_2).$$

We call a configuration $\omega \in \Omega$ *decent*, if there exists $0 < a(\omega) < \infty$ such that

$$\limsup_{n\to\infty} \frac{\omega(-n) + \cdots + \omega(n)}{2n+1} \leq a(\omega).$$

The set of decent configurations is denoted by Ω_{dec}. We shall prove the following result.

Theorem 5.2.5 *Let $\omega \in \Omega_{dec}$ and let $f : \Omega \to \mathbb{R}$ be continuous, bounded and N-dependent for some $N \in \mathbb{N}$. Then $Gf(\omega)$ is well defined, for*

$$t < \frac{1}{8a(\omega)e},$$

$S(t)f(\omega)$ is well defined and

$$S(t)f(\omega) = \sum_{i=0}^{\infty} \frac{t^i G^i f(\omega)}{i!}.$$

Therefore,

$$\lim_{t\downarrow 0} \frac{S(t)f(\omega) - f(\omega)}{t}$$

exists and is equal to $Gf(\omega)$.

To prove the theorem, we need some lemmas. We leave the proofs of these lemmas to the reader, since they are very similar to the proofs in Section 4.3 (but even simpler).

Lemma 5.2.6 *Let $f : \Omega \to \mathbb{R}$ be N-dependent and bounded. Then*

$$|G^i f(\omega)| \leq ||f||_\infty \left(4 \sum_{i=-N-i}^{N+i} \omega(i)\right)^i,$$

and the same bound holds also for $|G^i_n f(\omega)|$.

Lemma 5.2.7

$$|G^i\omega(0)| \leq \left(4\sum_{i=-N-i}^{N+i}\omega(i)\right)^i,$$

and the same bound holds also for $|G_n^i\omega(0)|$.

Proof of Theorem 5.2.5: Let $\omega \in \Omega_{dec}$ and let $f : \Omega \to \mathbb{R}$ be continuous, bounded and N-dependent for some $N \in \mathbb{N}$. We write $t^* := \frac{1}{8a(\omega)e}$. Since f is N-dependent, the sum in

$$Gf(\omega) = \sum_i (\omega(i) + \omega(i+1))\,(f(A_i(\omega)) - f(\omega))$$

runs only over $i \in \{-N-1, N\}$, so $Gf(\omega)$ is well defined.

To show that $S(t)f(\omega)$ is well defined for $t < t^*$, we prove that for $t < t^*$,

$$\omega_t = \lim_{n\to\infty} \omega_{n,t}^c \in \Omega, \tag{5.7}$$

ρ-almost surely. To prove (5.7), it suffices to show that

$$E(\omega_t(i)) = \int_{\mathcal{P}} \omega_t\, d\rho < \infty,$$

for all $i \in \mathbb{Z}$. We show this for $i = 0$, the proof for $i \in \mathbb{Z} \setminus \{0\}$ proceeds analogously. Let $t < t^*$. Observe first that by Lemma 5.2.7 and Lemma 4.3.5,

$$\sum_{i=0}^{\infty} \frac{t^i G_n^i\omega(0)}{i!} < \infty$$

and therefore

$$S_n(t)\omega(0) = \sum_{i=0}^{\infty} \frac{t^i G_n^i\omega(0)}{i!}.$$

Then by the monotone convergence theorem, Lemma 5.2.7, Lemma 4.3.5 and the dominated convergence theorem,

$$\begin{aligned} E(\omega_t(0)) &= \lim_{n\to\infty} S_n(t)\omega(0) \\ &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{t^i G_n^i\omega(0)}{i!} \\ &\leq \sum_{i=0}^{\infty} \frac{t^i |G^i\omega(0)|}{i!} < \infty. \end{aligned}$$

Observe that since f is continuous and, for $t < t^*$, $\omega_t = \lim_{n\to\infty} \omega_{n,t}^c \in \Omega$, ρ-almost surely,

$$f(\omega_t) = \lim_{n\to\infty} f(\omega_{n,t}^c),$$

ρ-almost surely. By the dominated convergence theorem, for $t < t^*$,

$$\begin{aligned} S(t)f(\omega) &= \lim_{n\to\infty} S_n(t)f(\omega) \\ &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{t^i G_n^i f(\omega)}{i!}. \end{aligned}$$

By Lemma 5.2.6, Lemma 4.3.5 and the dominated convergence theorem we can bring the limit into the sum and conclude that for $t < t^*$,

$$S(t)f(\omega) = \sum_{i=0}^{\infty} \frac{t^i G^i f(\omega)}{i!}.$$

This proves the theorem. □

5.3 The bricklayer process

5.3.1 Construction of the bricklayer process

In this section we construct the bricklayer process. We will deal with the case in which $r(z) = z$, for $z \geq 1$ and (by definition) $r(z) = \frac{1}{r(-z+1)}$, for $z \leq 0$. The construction can easily be generalised to bricklayer processes where, for $z \geq 1$, $r(z) \leq az + b$, for some $a, b \in \mathbb{R}_{\geq 0}$, but we leave this to the reader.

We first define the bricklayer n-process, in which only bricklayers in $[-n, n]$ are allowed to lay bricks, as mentioned in Section 5.1. We denote the state of the bricklayer n-process with initial configuration $\tilde{\omega}$ by $\tilde{\omega}_{n,t}$. Let $\tilde{\Omega}$ be equipped with the product topology and the Borel σ-algebra. For bounded measurable functions $f : \tilde{\Omega} \to \mathbb{R}$ we define the semigroup $\tilde{S}_n(t)$ of the bricklayer n-process by

$$\tilde{S}_n(t)f(\tilde{\omega}) := E\left(f(\tilde{\omega}_{n,t})\right)$$

(and again the same notation is used for measurable unbounded f, if $E\left(f(\tilde{\omega}_{n,t})\right) < \infty$). The generator $\tilde{G}_n$ of the bricklayer n-process is, for f as above, given by

$$\begin{aligned} \tilde{G}_n f(\tilde{\omega}) &:= \lim_{t\downarrow 0} \frac{\tilde{S}_n(t)f(\tilde{\omega}) - f(\tilde{\omega})}{t} \\ &= \sum_{i=-n}^{n-1} \left(r(\tilde{\omega}(i)) + r(-\tilde{\omega}(i+1))\right) \left(f(\tilde{A}_i(\tilde{\omega})) - f(\tilde{\omega})\right) \\ &\quad + r(-\tilde{\omega}(-n)) \left(f(\tilde{A}_{-n-1}(\tilde{\omega})) - f(\tilde{\omega})\right) \\ &\quad + r(\tilde{\omega}(n)) \left(f(\tilde{A}_n(\tilde{\omega})) - f(\tilde{\omega})\right). \end{aligned}$$

We define a coupling

$$(\tilde{\omega}^c_{n,t}; \omega^c_{n,t})_{n=1}^{\infty}$$

of the bricklayer n-processes and the block n-processes. Let $X_{i,j}(t)$, ρ and $\omega^c_{n,t}$ be as defined in Section 5.2.2 and use the processes $X_{i,j}(t)$ in the same way as for the block

n-processes to decide whether and when bricks are added in the (coupled) bricklayer n-processes. The resulting bricklayer n-processes are denoted by $\tilde{\omega}^c_{n,t}$. We will write

$$\tilde{\omega}_t := \lim_{n\to\infty} \tilde{\omega}^c_{n,t}$$

if the limit is well defined. We shall prove the following theorem.

Theorem 5.3.1 *Let $\tilde{\nu}$ be an ergodic stationary measure on $\tilde{\Omega}$ with*

$$\int_{\tilde{\Omega}} |\tilde{\omega}(0)|\, d\tilde{\nu} < \infty,$$

and let $r : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ be defined by $r(z) = z$ for $z \geq 1$ and $r(z) = \frac{1}{1-z}$ for $z \leq 0$. Then

$$(\tilde{\nu} \times \rho)\Big((\omega, \chi) : \tilde{\omega}_t \text{ exists and } \tilde{\omega}_t \in \tilde{\Omega}, \forall t\Big) = 1.$$

We introduce some more notation. For $\tilde{\omega} \in \tilde{\Omega}$, we define $\tilde{l}_n(\omega) \in (\mathbb{R}_{\geq 0})^{\mathbb{Z}}$ by

$$\tilde{l}_n(\tilde{\omega})(i) = \begin{cases} r(\tilde{\omega}(i)) + r(-\tilde{\omega}(i+1)) & \text{if } i \in [-n, n-1] \\ r(-\tilde{\omega}(-n)) & \text{if } i = -n-1 \\ r(\tilde{\omega}(n)) & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Recall that in the block n-process, for $\omega \in \Omega$, $l_n(\omega)(i)$ is the rate at which ω flips to $A_i(\omega)$. Similarly, in the bricklayer n-process, for $\tilde{\omega} \in \tilde{\Omega}$, $\tilde{l}_n(\tilde{\omega})(i)$ is the rate at which $\tilde{\omega}$ flips to $\tilde{A}_i(\tilde{\omega})$.

Lemma 5.3.2 *Let $\tilde{\omega} \in \tilde{\Omega}$ and define $\omega \in \Omega$ by $\omega(i) = |\tilde{\omega}(i)| + 1$. Then for all $t \geq 0$,*

$$\tilde{l}_n(\tilde{\omega}^c_{n,t}) \leq l_n(\omega^c_{n,t}), \ \rho\text{-a.s.} \tag{5.8}$$

So the above lemma states that in the coupling, for (related) initial values $\tilde{\omega}$ and ω as in the lemma, every time that a transition occurs involving the sites i and $i+1$ in the bricklayer n-process, a transition also occurs involving the sites i and $i+1$ in the block n-process.

Proof of Lemma 5.3.2: Let $\tilde{\omega} \in \tilde{\Omega}$ and let $\omega \in \Omega$ as in the lemma. If $\tilde{\omega}^c_{n,t}(i) = 0$, then $\omega^c_{n,t}(i) \geq 1$, so

$$r(\tilde{\omega}^c_{n,t}(i)) \leq \omega^c_{n,t}(i) \tag{5.9}$$

and

$$r(-\tilde{\omega}^c_{n,t}(i)) \leq \omega^c_{n,t}(i). \tag{5.10}$$

Observe that it suffices to show that for $t \geq 0$ and $i \in \mathbb{Z}$,

$$|\tilde{\omega}^c_{n,t}(i)| \leq \omega^c_{n,t}(i), \tag{5.11}$$

ρ-a.s., since if (5.11) holds, we have that also for $i \in \mathbb{Z}$ with $\tilde{\omega}^c_{n,t}(i) \neq 0$,

$$r(\tilde{\omega}^c_{n,t}(i)) \leq |\tilde{\omega}^c_{n,t}(i)| \leq \omega^c_{n,t}(i) \tag{5.12}$$

and

$$r(-\tilde{\omega}^c_{n,t}(i)) \leq |\tilde{\omega}^c_{n,t}(i)| \leq \omega^c_{n,t}(i). \tag{5.13}$$

From (5.9), (5.10), (5.12) and (5.13) we conclude (5.8).

We prove (5.11). In the coupling, let $L(i)$ be the time of the i^{th} change in the block n-process. The proof proceeds in two steps.

1. We prove that (5.11) holds for $t \leq L(1)$.

2. We prove that if (5.11) is true for $t \leq L(r)$, then (5.11) is also true for $t \leq L(r+1)$.

This suffices, since the sequence $L(1), L(2), \ldots$ increases to ∞, ρ-almost surely.

Step 1. At time 0, $|\tilde{\omega}(i)| \leq \omega(i)$, so $\tilde{l}_n(\tilde{\omega}) \leq l_n(\omega)$. As long as $t \leq L(1)$, nothing happens, so for those t clearly

$$|\tilde{\omega}^c_{n,t}(i)| \leq \omega^c_{n,t}(i).$$

At time $L(1)$ something changes in the block n-process, let us assume this change involves the sites k and $k+1$. Then

$$\omega^c_{n,L(1)}(i) = \begin{cases} \omega(i) & \text{if } i \neq k, k+1 \\ \omega(i) + 1 & \text{if } i = k, k+1. \end{cases}$$

Now there are two possibilities. Either there was also a change in the bricklayer n-process, or there was not. If there was no change, (5.11) is true for $t \leq L(1)$. If there was a change,

$$\tilde{\omega}^c_{n,L(1)}(i) = \begin{cases} \tilde{\omega}(i) & \text{if } i \neq k, k+1 \\ \tilde{\omega}(i) - 1 & \text{if } i = k \\ \tilde{\omega}(i) + 1 & \text{if } i = k+1, \end{cases}$$

and in this case (5.11) also holds for $t \leq L(1)$.

Step 2. Suppose that (5.11) is true for $t \leq L(r)$. Then, for $t \leq L(r)$, $\tilde{l}_n(\tilde{\omega}^c_{n,t}) \leq l_n(\omega^c_{n,t})$. Using a similar argument as in Step 1, it follows that (5.11) then also holds for $t \leq L(r+1)$. □

Proof of Theorem 5.3.1: Let $\tilde{\nu}$ be given as in the theorem. It suffices to show that for arbitrary t,

$$(\tilde{\nu} \times \rho)\Big((\tilde{\omega}, \chi) : \tilde{\omega}_s \text{ exists and } \tilde{\omega}_s \in \tilde{\Omega}, \forall s \leq t\Big) = 1. \tag{5.14}$$

If $\tilde{\omega} \in \tilde{\Omega}$ is chosen according to $\tilde{\nu}$, then we define $\omega \in \Omega$, given by $\omega(i) = |\tilde{\omega}(i)| + 1$, to have distribution ν on Ω. The measure ν is a factor of $\tilde{\nu}$ and therefore an ergodic stationary measure, and

$$\int_{\Omega} \omega(0)\, d\nu < \infty.$$

We use the special points defined in Section 5.2.3. Observe that, if T is the suitable shift operator on $\Omega \times \mathcal{P}$ (and $\tilde{\Omega} \times \mathcal{P}$), then for all $k \in \mathbb{Z}$, k is a s.p. at time t for (ω^*, χ)

if and only if 0 is a s.p. at time t for $T^k(\omega^*, \chi)$. By Theorem 5.2.4 and the ergodic theorem, there exists $\epsilon_t > 0$ such that

$$(\nu \times \rho) \left((\omega^*, \chi) : \begin{array}{l} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega^*, \chi) \geq \epsilon_t \text{ and} \\ \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega^*, \chi) \geq \epsilon_t \end{array} \right) = 1.$$

For related $\tilde{\omega}$ and ω as above,

$$(\tilde{\nu} \times \rho) \left((\tilde{\omega}, \chi) : \begin{array}{l} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega, \chi) \geq \epsilon_t \text{ and} \\ \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega, \chi) \geq \epsilon_t \end{array} \right) = 1. \tag{5.15}$$

We observe that by Lemma 5.3.2, for related $\tilde{\omega}$ and ω as above and $\chi \in \mathcal{P}$, if

$$\omega_t(i) = \omega(i),$$

then also

$$\tilde{\omega}_s(i) = \tilde{\omega}(i), \ \forall s \leq t.$$

Further, if $\tilde{\omega}$ and χ have the properties that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega, \chi) \geq \epsilon_t$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} \mathbf{1}_{\{i \text{ is s.p.}\}}(\omega, \chi) \geq \epsilon_t,$$

then both $\tilde{\omega}_s$ and ω_s exist for $s \leq t$ and $\tilde{\omega}_t \in \tilde{\Omega}$ (ρ-almost surely). Combining this with (5.15) proves (5.14). □

5.3.2 The relation between $\tilde{S}(t)$ and $\tilde{G}$

Suppose that $f : \tilde{\Omega} \to \mathbb{R}$ is a measurable function, and that $\tilde{\omega}$ and t are such that $\tilde{\omega}_s \in \tilde{\Omega}$ for all $s \leq t$, ρ-almost surely. Provided that

$$E\left(f(\tilde{\omega}_t)\right) = \int_{\mathcal{P}} f(\tilde{\omega}_t)\, d\rho < \infty,$$

we write

$$\tilde{S}(t)f(\tilde{\omega}) := E\left(f(\tilde{\omega}_t)\right),$$

and we call $\tilde{S}(t)$ the semigroup of the bricklayer process. The following result deals with the relation between $\tilde{S}(t)$ and $\tilde{G}$.

We first need some notation. A measurable function $f : \tilde{\Omega} \to \mathbb{R}$ is called N-*dependent*, if for all $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\Omega}$ with $\tilde{\omega}_1(x) = \tilde{\omega}_2(x)$ for all $x \in [-N, N]$,

$$f(\tilde{\omega}_1) = f(\tilde{\omega}_2).$$

A configuration $\tilde{\omega} \in \tilde{\Omega}$ is called *decent*, if there exists $0 < \tilde{a}(\tilde{\omega}) < \infty$ such that

$$\limsup_{n\to\infty} \frac{|\tilde{\omega}(-n)| + \cdots + |\tilde{\omega}(n)|}{2n+1} \leq \tilde{a}(\tilde{\omega}).$$

The set of decent configurations is denoted by $\tilde{\Omega}_{dec}$. We write

$$t(\tilde{\nu}) := \frac{1}{8(E_{\tilde{\nu}}\left(|\tilde{\omega}(0)|\right) + 1)e}.$$

Theorem 5.3.3 *Let $\tilde{\nu}$ be a stationary ergodic measure, with $E_{\tilde{\nu}}\left(|\tilde{\omega}(0)|\right) < \infty$. Let $f : \tilde{\Omega} \to \mathbb{R}$ be a continuous, bounded and N-dependent function for some $N \in \mathbb{N}$. Then*

$$\tilde{\nu}\left(\tilde{\omega} : \textit{for } t < t(\tilde{\nu}), \tilde{S}(t)f(\tilde{\omega}) = \sum_{i=0}^{\infty} \frac{t^i \tilde{G}^i f(\tilde{\omega})}{i!}\right) = 1.$$

Proof: Let f and $\tilde{\nu}$ be as above. Observe first that if $\tilde{\omega} \in \tilde{\Omega}_{dec}$ is such that

$$\limsup_{n\to\infty} \frac{|\tilde{\omega}(-n)| + \cdots + |\tilde{\omega}(n)|}{2n+1} \leq E_{\nu}\left(|\tilde{\omega}(0)|\right),$$

and for $t < t(\tilde{\nu})$, $\tilde{\omega}_t$ exists and $\tilde{\omega}_t \in \tilde{\Omega}$, ρ-almost surely, then, analogously to the proof of Theorem 5.2.5, for $t < t(\tilde{\nu})$,

$$\tilde{S}(t)f(\tilde{\omega}) = \sum_{i=0}^{\infty} \frac{t^i \tilde{G}^i f(\tilde{\omega})}{i!}.$$

To finish the proof, we observe that $\tilde{\nu}$ concentrates on configurations with

$$\limsup_{n\to\infty} \frac{|\tilde{\omega}(-n)| + \cdots + |\tilde{\omega}(n)|}{2n+1} \leq E_{\tilde{\nu}}\left(|\tilde{\omega}(0)|\right),$$

and that according to Theorem 5.3.1,

$$\begin{aligned}\tilde{\nu}\Big(\tilde{\omega} : \tilde{\omega}_t \text{ exists and } \tilde{\omega}_t \in \tilde{\Omega},\ \text{for } t < t(\tilde{\nu}),\ \rho\text{-a.s.}\Big) &= \\ (\tilde{\nu} \times \rho)\Big((\tilde{\omega}, \chi) : \tilde{\omega}_t \text{ exists and } \tilde{\omega}_t \in \tilde{\Omega} \text{ for } t < t(\tilde{\nu})\Big) &= 1.\end{aligned}$$

$\square$

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Samenvatting

We bekijken een aantal wachtrijsystemen op cirkels. In dit soort systemen kiezen klanten een willekeurige positie op een cirkel, terwijl de bediende volgens een bepaalde strategie over de cirkel reist om de klanten te bedienen.

Eén van de meest eenvoudige systemen die je kunt verzinnen, is het systeem waarbij klanten volgens een Poisson proces met parameter λ op de cirkel aankomen en vervolgens een uniforme positie kiezen. De bediende reist met een constante snelheid en in een vaste richting over de cirkel en stopt als hij een klant tegenkomt. Hij bedient deze klant en reist daarna weer verder, in dezelfde richting als eerst. Bedieningstijden zijn onafhankelijk van elkaar, hebben dezelfde verdeling en een verwachte lengte van $\frac{1}{\mu}$. We noemen dit systeem *het gewone systeem*. *Het systeem met de gretige bediende* is een variant op dit systeem. Dit systeem is bijna hetzelfde als het gewone systeem, alleen reist de bediende hier altijd in de richting van de dichtstbijzijnde klant.

De eerste vraag die rijst bij het bestuderen zulke systemen, is of het systeem stabiel is. Voor verschillende systemen is het antwoord op deze vraag bekend. Er zijn echter ogenschijnlijk eenvoudige wachtrijsystemen waarvan je verwacht dat ze stabiel zijn, maar waarvoor dit nog niet bewezen is, zoals bijvoorbeeld het wachtrijsysteem met de gretige bediende. In het tweede hoofdstuk van dit proefschrift geven we een nieuw bewijs voor de stabiliteit van twee specifieke wachtrijsystemen op een cirkel. We hopen dat deze bewijzen bijdragen tot het krijgen van een beter begrip voor de stabiliteit van dit soort wachtrijsystemen en uiteindelijk ook leiden tot een bewijs van de stabiliteit van het wachtrijsysteem met de gretige bediende.

Het gewone systeem is stabiel als $\lambda < \mu$, onafhankelijk van de snelheid van de bediende. Ons bewijs is gebaseerd op de gemiddelde reistijd van de bediende tussen twee klanten. Het idee van het bewijs is ook toepasbaar op een variant op het systeem met de gretige bediende, waarin de klanten alleen op een eindig aantal vaste stations mogen wachten. Dit systeem is ook stabiel voor $\lambda < \mu$.

In hoofdstuk 3 behandelen we zogenaamde *k-systemen*, waarbij k een positief geheel getal is. Deze systemen gedragen zich als het gewone systeem, met dit verschil dat als er een klant aankomt terwijl er al k klanten op de cirkel zijn, deze klant wordt weggestuurd en niet meer terugkeert. We bewijzen dat de stationaire verdelingen van de k-systemen zwak convergeren naar de stationaire verdeling van het gewone systeem. Hoewel dit resultaat intuïtief niet zo vreemd is, lijkt het toch verrassend lastig te bewijzen.

In de hoofdstukken 4 en 5 bekijken we twee specifieke deeltjessystemen met interactie. In deze systemen is er een aantal deeltjes op iedere positie in $\mathbb{Z}$ en per positie komen er met exponentieel verdeelde tussenpozen deeltjes bij of verdwijnen er deeltjes. De parameter van deze tussenpozen varieert steeds en hangt af van de toestand van het systeem. We noemen deze parameter de *snelheid* van de veranderingen.

Dit soort deeltjessystemen is de laatste twintig jaar uitgebreid bestudeerd door zowel wis- als natuurkundigen, die deze systemen verschillend benaderen. De eerste vraag die wiskundigen zichzelf stellen is of deze systemen wel echt bestaan, dat wil zeggen of je een systeem kunt construeren waarin veranderingen inderdaad met de beschreven snelheden optreden. In het geval waarin de snelheid van de veranderingen per positie begrensd is en alleen afhangt van het aantal deeltjes op naburige posities, is dit probleem opgelost. Als de snelheid echter onbegrensd is en af kan hangen van de toestand van het hele systeem, is het op voorhand niet zo duidelijk of het systeem echt te construeren is. Een volgende vraag is of het systeem stationaire verdelingen heeft en welke eigenschappen die verdelingen hebben.

In hoofdstuk 4 construeren we een een systeem waarin de snelheid van de veranderingen onbegrensd is en niet alleen afhangt van de toestand op de naburige posities. De toestandsruimte van dit systeem is $\{0,1\}^{\mathbb{Z}}$ en enen veranderen in nullen met snelheid μ en nullen veranderen in enen met een snelheid van λ keer één plus het aantal enen rechts van de nul tot we de volgende één zien. We bewijzen dat dit systeem een unieke stationaire verdeling heeft en bekijken enkele eigenschappen van deze stationaire verdeling.

Hoofdstuk 5 gaat over zogenaamde *metselaars-systemen*. Deze systemen zijn geïntroduceerd door Balázs. We kunnen deze systemen construeren als we bepaalde restricties leggen op de snelheid van de veranderingen.

Het idee van de constructies is ongeveer als volgt. We definiëren eerst systemen waarin alleen veranderingen optreden op de middelste $2n + 1$ posities. Dan laten we zien dat we deze systemen kunnen koppelen, zodanig dat de limiet van de systemen bestaat als we het gebied waarin de veranderingen plaatsvinden steeds groter maken. Deze limiet wordt ons 'echte' proces.

Curriculum Vitae

Corrie Quant werd geboren op 27 juni 1974 te Amersfoort. Van 1986 tot 1992 bezocht zij het Menso Alting College in Hoogeveen, waar zij in juli 1992 haar VWO-diploma behaalde. Aansluitend studeerde zij wiskunde aan de Universiteit Utrecht en haalde in juni 1997 haar doctoraaldiploma. In september 1997 begon zij onder begeleiding van Ronald Meester aan haar promotie-onderzoek, eerst in dienst van de Universiteit Utrecht en later van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

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