Star points on cubic surfaces

(met een samenvatting het Nederlands)

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THESIS

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by

Tu Chanh Nguyen

born on 23 January 1965 in Da Nang.

To My Mother, Wife and Son

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Contents

Chapter 0

INTRODUCTION

0.1   Some historical remarks

Cubic surfaces have been studied since the last century, initially by G. Salmon [Sa] and A. Cayley [C1] in 1848. Since then, the theory of twenty-seven lines on a non-singular cubic surface fascinated many mathematicians. There were a lot of papers and books devoted to this beautiful object. Besides the results by G. Salmon and A. Caley, the following publications were influential: J. Steiner [St], L. Cremona [Cr], R. Sturm with a purely geometric theory of cubic surfaces; L. Schläfli [Sch1, Sch2] with results on the classification of cubic surfaces and the structure of the configuration of twenty-seven lines; C. Wiener, Clebsch, F. Klein [K], Rodenberg and W. H. Blythe [Bl1, Bl2] with the construction of models of cubic surfaces; C. F. Geiser [Gei], H. G. Zeuthen [Ze], H. F. Baker [Ba] studied the relation between twenty-seven lines of a cubic surface and twenty-eight bitangents of a plane quartic curve; C. Jordan, Maschke, Burkhardt, Witting, Dickson, Kühnen, Weber, Kasner with results on the twenty-seven lines from a group theoretic point of view and the Galois group of the equation of the twenty-seven lines. In his book published in 1911, "The Twenty-Seven Lines upon the Cubic Surface" [He], A. Henderson gave a clear historical summary on the research of the subject in the last century and a lot of references. We quote the first sentences in the historical summary of the book:

> " While it is doubtless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year by year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of the discovery of the twenty-seven lines upon the cubic surface. The literature of the subject is very extensive. In a bibliography on curves and surfaces compiled by J. E. Hill, of Columbia University, New York, the section on cubic surfaces contained two hundred and five titles. The Royal Society of London Catalogue of Scientific Papers, 1800-1900, volume for Pure Mathematics (1908), contains very many more. "

These results turned out to be useful in research of many mathematicians. Modern algebraic geometry and other related branches of mathematics introduced powerful tools which were used to study cubic surfaces. The theory of cubic surfaces with a specific configuration of lines is still an fascinating topic. There are more papers and

books dealing with cubic surfaces and related topics. Among them, we mention: B. Serge [S1,S2], Yu. I. Manin [Ma1, Ma2, Ma3, Ma4], F. Bardelli and A. Del Centina [Bar], [B-D], J. Sekiguchi [Se1, Se2], I. Naruki [Na], [Na-Se], J. W. Bruce and C. T. C. Wall [B-W], A. Geramita [Ge], M. Brundu and A. Logar [B-L], B. Hassett, B. Hunt [Hu1, Hu2].

0.2   Non-singular cubic surfaces with star points

A cubic surface in $\mathbb{P}^3$ is given by a non-zero cubic homogeneous polynomial in 4 variables. Fixing an ordering of monomials of degree 3 in the polynomial ring $k[x_0, x_1, x_2, x_3]$, each cubic surface defines a point in $\mathbb{P}^{19}$. A non-singular cubic surface X contains twenty-seven lines. There exist at most 3 lines among these twenty-seven lines through a given point of X. A *star point* (also called *Eckardt point*) on a non-singular cubic surface is the intersection point of three lines on the surface. Not every non-singular cubic surface has a star point. In fact, the subset of $\mathbb{P}^{19}$ corresponding to non-singular cubic surfaces with at least one star point is a locally closed subvariety of codimension 1. A non-singular cubic surface does not have more than 18 star points. This was proved by Serge [S1] in 1946. We give another proof in Chapter 2. In 1876, F. E. Eckardt considered non-singular cubic surfaces possessing star points [Ec] (see also [S1], p. 147). In his book [S1], B. Serge classified non-singular cubic surfaces possessing star points. He proved a criterion stating that a non-singular cubic surface has as many Eckardt points as it has harmonic homologies into itself. A *harmonic homology* of the central point P and the fundamental plane H is a projective transformation of $\mathbb{P}^3$ which maps a point Q into its harmonic conjugate $Q' \in \overline{PQ}$ with respect to P and the intersection point $\overline{PQ} \cap H$ ([S1], §98). Using this criterion, he determined all classes of non-singular cubic surfaces with respect to possible numbers of star points ([S1], p. 154) by considering polynomials defining surfaces and determining the number of harmonic homologies.

In this Ph.D. thesis, we denote by H_k the subset of $\mathbb{P}^{19}$ consisting of points corresponding to non-singular cubic surfaces with at least k star points. We study these H_k as subvarieties of $\mathbb{P}^{19}$ and their images in a compactification of the moduli space M of non-singular cubic surfaces. In the next section, we describe a construction of one compactification $\overline{M}$ of the moduli space of non-singular cubic surfaces. We obtain a morphism $\phi : \mathbb{P}^{19} - \Delta \longrightarrow M$, where Δ is the locus of singular cubic surfaces. We consider the irreducibility, the local closedness and the dimension of H_k inside $\mathbb{P}^{19}$. Especially, we are interested in studying boundaries of $\phi(H_k)$ in the compactification $\overline{M}$.

0.3   Moduli space of non-singular cubic surfaces and a compactification

We construct the coarse moduli space M of non-singular cubic surfaces and its compactification as well, in which we describe the subspaces $\phi(H_k)$ and we study their boundaries. Moduli theory is an area of algebraic geometry. The goal of moduli theory is to classify certain objects by constructing spaces which parametrize *isomorphism classes* of such geometric objects. This if often achieved by invariant theory, in con-

structing a quotient of a parameter space with respect to an equivalence relation (given by declaring isomorphic objects to be represented by equivalent points). The fine moduli space of marked non-singular cubic surfaces has been constructed by I. Naruki [N] and its closure is known as the "Cross ratio variety". He also constructed a space which is isomorphic to the Mumford compactification of the coarse moduli space of non-singular cubic surfaces. Another approach to construct coarse moduli spaces is via geometric invariant theory. We describe this approach in order to construct M and its compactification.

We have a natural action of the group variety $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. This action is good enough in the sense that the quotient spaces of *semi-stable points* and of *stable points* do exist (Chapter 3, Section 2). Moreover, the quotient space $\overline{M} := \mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{ss}$ of semi-stable points is projective; the image of the subset of $\mathbb{P}^{19}$ corresponding to all non-singular cubic surfaces is affine and it is a coarse moduli space. We have $\mathrm{PGL}(3)\backslash(\mathbb{P}^{19} - \Delta) = M \subset \overline{M}$. Hence, the space $\overline{M}$ is a compactification of the coarse moduli space M.

0.4   Singular cubic surfaces

The locus $\Delta \subset \mathbb{P}^{19}$ of singular cubic surfaces is a closed subset of codimension 1. Some classifications of non-singular cubic surfaces can be found in [B-W] or [B-L]. We are interested in singular cubic surfaces which correspond to semi-stable and stable points under the action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$ in the sense of geometric invariant theory. A cubic surface with only isolated singularities has at most 4 singular points.

In this thesis, we consider singular cubic surfaces with only isolated singularities. A point P on a cubic surface X with only isolated singularities is called a *singular point of type A_1* if the tangent cone at P is an irreducible quadric surface. The subset of stable points consists of the points corresponding to non-singular cubic surfaces and singular cubic surfaces with only A_1 singularities. The subset of $\mathbb{P}^{19}$ corresponding to singular cubic surfaces with only A_1 singularities is locally closed of codimension 1. A point P on a cubic surface X with only isolated singularities is called a *singular point of type A_2* if the tangent cone at P consists of two distinct planes whose intersection line does not lie on X. The subset of semi-stable points consists of the points corresponding to non-singular cubic surfaces and singular cubic surfaces with A_1 or A_2 singularities. We denote by $i\mathcal{A}_1 j\mathcal{A}_2$ the subset of Δ consisting of the points corresponding to the cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2; we have $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$. We use $j\mathcal{A}_2$ and $i\mathcal{A}_1$ instead of $0\mathcal{A}_1 j\mathcal{A}_2$ and $i\mathcal{A}_1 0\mathcal{A}_2$ respectively. In the quotient space $\overline{M} := \mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{ss}$, the image of all semi-stable but non-stable points is just one point. We will give a proof for this fact in Chapter 3, Section 4.

0.5   Blowing-up of $\mathbb{P}^2$ at 6 points

One of the main methods used to study cubic surfaces (at least for the case of semi-stable ones) in this thesis is the blowing-up of $\mathbb{P}^2$ at 6 points. We say that 6 distinct points $P_1, \ldots, P_6$ in $\mathbb{P}^2$ are *in general position* if no three are collinear and all six points

do not lie on a conic. A set of 6 points in general position determines an element (called a *6-point scheme*) in the Hilbert scheme of zero-dimensional closed subschemes of length 6 in $\mathbb{P}^2$. Let $\mathcal{P}$ be a 6-point scheme consisting of 6 points $P_1 \ldots, P_6$ in general position. Let Y be the blowing-up of $\mathbb{P}^2$ at $P_1, \ldots, P_6$. We see that the linear space $\mathcal{L}_\mathcal{P}$ of cubic forms in four variables which are zero at $P_1, \ldots, P_6$ has linear dimension 4. Let $\{f_1, f_2, f_3, f_4\}$ be a basis of $\mathcal{L}_\mathcal{P}$. Consider the rational map

$$
\begin{array}{ccc}
\mathbb{P}^2 & \overset{\psi}{\dashrightarrow} & \mathbb{P}^3 \\
P & \longmapsto & \big(f_1(P) : f_2(P) : f_3(P) : f_4(P)\big).
\end{array}
$$

Let X be the closure of the image of ψ. Then X is a non-singular cubic surface. Moreover, we have an isomorphism $Y \xrightarrow{\sim} X$. The twenty-seven lines of X are the following:

- six exceptional curves $\tilde{P}_i$ corresponding to P_i for $1 \le i \le 6$,

- six strict transforms $\tilde{C}_i$ of the conics C_i through $\{P_1, \ldots, P_6\} - \{P_i\}$ for $1 \le i \le 6$,

- fifteen strict transforms $\tilde{l}_{ij}$ of the lines $l_{ij} = \overline{P_i P_j}$ for $1 \le i < j \le 6$.

Conversely, let X be any non-singular cubic surface. For any choice of 6 mutually skew lines $L_1, \ldots, L_6 \subset X$, there exist $P_1, \ldots, P_6 \in \mathbb{P}^2$ in general position such that the blowing-up of $\mathbb{P}^2$ at these points gives a surface isomorphism with X and $\tilde{P}_i = L_i$ for $1 \le i \le 6$. There are 51840 of such choices. Star points on a non-singular cubic surface can be recognized by the configuration of these 6 points.

Example 1. Let $P_1, \cdots, P_6$ be 6 points of $\mathbb{P}^2$ in general position such that $\overline{P_1 P_2} \cap \overline{P_3 P_4} \cap \overline{P_5 P_6} = \{O\}$, see Figure 0.1, (a). Then the blowing-up of $\mathbb{P}^2$ at $P_1, \ldots, P_6$ has at least one star point, which is the image of the point O.

(a)   (b)
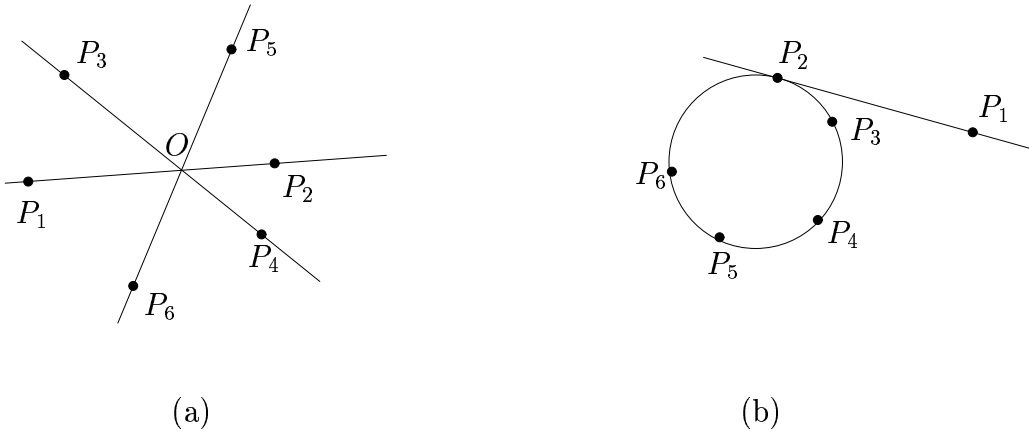
Figure 0.1: Configurations show the only ways of getting a star point

Example 2. Another configuration of 6 points which gives a star point on the corresponding cubic surface is as follows. Let $P_1, \cdots, P_6$ be 6 points of $\mathbb{P}^2$ in general position such that the line $\overline{P_1 P_2}$ is tangent to the conic C_1 containing the 5 points $P_2, \ldots, P_5$ at P_2. Then the star point is the intersection point of the lines $\tilde{P}_2, \tilde{C}_1$ and $\tilde{l}_{12}$, see Figure 0.1, (b).

In fact, any star point on a given non-singular cubic surface is recognized by the above two configurations when viewing the cubic surface by the blowing-up process.

We can determine all configurations of 6 points in $\mathbb{P}^2$ corresponding to the types of non-singular cubic surfaces with a given number of star points.

A question arises naturally: is there a similar correspondence between a singular, semi-stable cubic surface and a 6-point scheme in some relevant configuration of its points? In this thesis, we show such a correspondence. Namely, let X be a semi-stable cubic surface. Then there exists a 6-point scheme $\mathcal{P}$ such that the linear system $\mathcal{L}_\mathcal{P}$ of cubic forms in four variables through $\mathcal{P}$ has dimension 4; furthermore, for any basis of $\mathcal{L}_\mathcal{P}$, the closure of the image of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the basis is isomorphic to X. In this case, we have a morphism $Y \longrightarrow X$, where Y is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$. In general, this is a blowing-down which is not an isomorphism. A close study of such 6-point schemes enable us to determine the number of lines, the number of singularities of X and their configuration as well.

For example, let $\mathcal{P}_1$ be a 6-point scheme consisting of 6 mutually different points $P_1, \ldots, P_6$ lying on an irreducible conic. Then the corresponding cubic surface is singular with exactly one A_1 singularity. The image (via any rational map defined by a choice of basis of $\mathcal{L}_{\mathcal{P}_1}$) of the irreducible conic is the singular point. Let $\mathcal{P}_2$ be another 6-point scheme consisting of 6 mutually different points $P_1, \ldots, P_6$ such that no four points lie on a line, three points P_1, P_2, P_3 as well as three points P_4, P_5, P_6 are collinear. Then the corresponding cubic surface X is singular with exactly one A_2 singularity. The image of $\overline{P_1 P_2} \cup \overline{P_4 P_5}$ is the singular point of X .

0.6 Specialization

We see that the correspondence between semi-stable cubic surfaces and certain 6-point schemes gives us a way to study the subsets H_k as well as $i\mathcal{A}_1 j\mathcal{A}_2$ for $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$. In this way, we can find all irreducible components of the subsets H_k and we can determine the inclusion relationship between these irreducible components. Moreover, this gives a way to study specializations in families of non-singular cubic surfaces, especially to study boundaries of the irreducible components of H_k inside $(\mathbb{P}^{19})^{ss}$ or at least of their images inside the compactification $\overline{M} = \mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{ss}$.

We give some examples as illustrations.

Example 3. Let $P_1, \ldots, P_6$ be 6 points in general position. Let l be the line $\overline{P_1 P_6}$. Let P_6^t be a moving point on l, keeping $P_1, \ldots, P_5$ fixed. We see that, except for a finite number of positions, the 6 points $P_1, \ldots, P_5, P_6^t$ are in general position. Therefore, we have a family $\mathcal{X} \longrightarrow T$ of cubic surfaces, non-singular above a dense open subset of $T = \overline{P_1 P_6}$. If $P_6^{t_0}$ lies on the irreducible conic through $P_1, \ldots, P_5$, we get a specialization position, where the corresponding cubic surface is a singular cubic surface X_0 with exactly one A_1 singularity. We can assume that the twenty-seven lines on fibers of the family $\mathcal{X} \longrightarrow T$ are given by twenty-seven sections in $\mathcal{G}(1, 3) \times T \longrightarrow T$. A question arises: what are the specializations of these sections of lines on the surface X_0? For each $1 \leq i \leq 6$, the sections corresponding to the lines $\tilde{P}_i$ and $\tilde{C}_i$ specialize to the same line among the 21 lines of X_0. The other 15 lines of X_0 are contained in the 15 sections corresponding to the lines $\tilde{l}_{ij}$ for $1 \leq i < j \leq 6$.

Example 4. Let $P_1, \ldots, P_6$ be 6 points in general position satisfying the condition $\overline{P_1 P_6} \cap \overline{P_2 P_3} \cap \overline{P_4 P_5} = \{O\}$, see Figure 0.2, (a).

Consider the family determined as in the previous example (the moving point P_6^t moves on the line $l = \overline{P_1 O}$). We see that the family is contained in H_1. If P_6^t lies on the irreducible conic through $P_1, \ldots, P_5$, we get a specialization position, where the corresponding cubic surface X_0 is singular with an A_1 singularity. The point determined by X_0 is contained in $\overline{H_1} \cap \overline{\mathcal{A}_1} \cap (\mathbb{P}^{19})^{ss}$; this intersection is an irreducible component of the boundary $\Delta H_1 := \overline{H_1} \cap \Delta \cap (\mathbb{P}^{19})^{ss}$ of H_1. If P_6^t moves to O, the 6 points $P_1, \ldots, P_5$ and O define a specialization in the family and the corresponding cubic surface is singular with two A_1 singularities. These two singularities are the images of lines $\overline{P_2 P_3}$ and $\overline{P_4 P_5}$. Moreover, we show in this thesis that the closure of the subset $2\mathcal{A}_1$ in $(\mathbb{P}^{19})^{ss}$ is another irreducible component of the boundary ΔH_1.

(a)   (b)
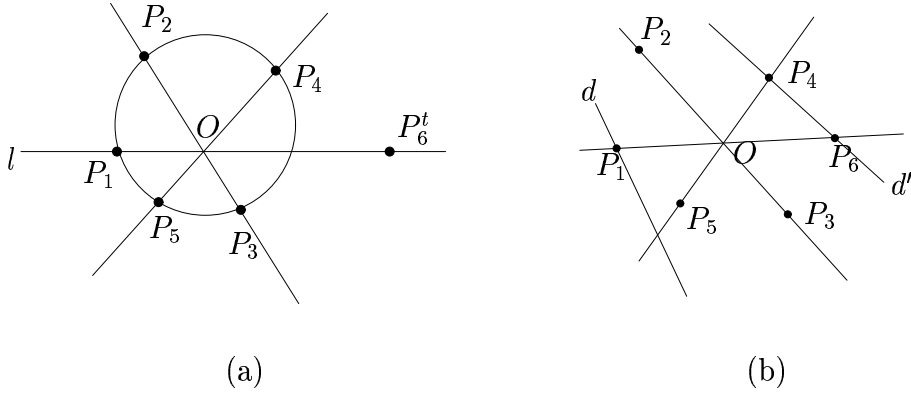
Figure 0.2: Configurations corresponding to points in the boundary of H_1

Example 5. Suppose that $P_1, \ldots, P_6, O$ satisfy the conditions above. Fix the 4 points P_2, P_3, P_4, P_5. Fix a line d through P_1 not containing any point of $\{O, P_2, P_3, P_4, P_5\}$. Let d' be the line $\overline{P_4 P_6}$. Let P_1^t move on d and let P_6^t be the point on d' such that the three points P_1^t, P_6^t and O are collinear, see Figure 0.2, (b).

Except for a finite number of values of t, the six points $P_2, \ldots, P_5, P_1^t$ and P_6^t are in general position and the blowing-up of $\mathbb{P}^2$ at those six points is isomorphic to a non-singular cubic surface with at least one star point. Therefore, we have another family in H_1. There exists t_0 such that $P_6^{t_0} = P_4$ and $P_1^{t_0} = d \cap \overline{P_4 P_5}$. The four points $P_1^{t_0}, P_2, P_3, P_5$ and the double point P_4 define a specialization in the family and the corresponding cubic surface X_0 is singular with one A_2 singularity. The singular point of X_0 is the image of the line $\overline{P_4 P_5}$.

In this thesis, we prove that the closure of the subset $\mathcal{A}_2 := 0\mathcal{A}_1 1\mathcal{A}_2$ in $(\mathbb{P}^{19})^{ss}$ is one irreducible component of ΔH_1. We show that the boundary ΔH_1 consists of the three irreducible components as described above.

0.7 Results in this thesis

The main results of this thesis are in Chapter 2 and 3. Chapter 1 deals with some basis facts on cubic surfaces and actions of group varieties. We study non-singular

cubic surfaces with star points in Chapter 2. We describe the specific configurations of six points in general position corresponding to non-singular cubic surfaces with a given number of star points. We consider the irreducibility, the local closedness and the dimension of H_k. Moreover, we determine the inclusion relationship between the irreducible components of these H_k.

In Chapter 3, we study the boundaries of the subsets H_k inside $\mathbb{P}^{19}$ and the boundaries of their images in the compactification $\overline{M}$. To do so, we describe in Section 3.1 a classification of singular cubic surfaces and compute the number of singular points, the number of lines on each singular cubic surface with their configuration. This classification can be found in [B-W]. Moreover, we compute the codimension of these classes and determine the relationship between their closures. In Section 3.2, we prove the basic fact that semi-stable cubic surfaces are those containing at most A_2 singularities and stable cubic surfaces are those containing at most A_1 singularities. This fact were mentioned in [GIT], p. 80 as well as in [Mu3], p. 51, but we do not know a reference for a proof. In Section 3.3, we study semi-stable cubic surfaces. We prove that for any semi-stable cubic surface X, there exists a 6-point scheme $\mathcal{P}$ such that the linear system $\mathcal{L}_{\mathcal{P}}$ of cubic forms through $\mathcal{P}$ has dimension 4; furthermore, for any basis of $\mathcal{L}_{\mathcal{P}}$, the closure of the image of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the basis is isomorphic to X. We also define and compute the multiplicities of lines and triple intersections on semi-stable cubic surfaces. Section 3.4 contains several results on the boundaries of H_k and the boundaries of the images of H_k in $\overline{M}$. We prove that the image of all semi-stable but non-stable points in the quotient space $\overline{M}$ is just one point.

Chapter 1

Preliminaries

Throughout the thesis, we work on schemes and varieties over an algebraic closed field with characteristic 0.

1.1 General position

Definition: Let P_1, P_2, P_3, P_4, P_5 and P_6 be points of $\mathbb{P}^2$. They are called *in general position* if no 3 of them are collinear and not all of them lie on a conic.

Denote:

$$\Phi = \{\mathcal{P} = (P_1, P_2, P_3, P_4, P_5, P_6) \mid P_1, P_2, P_3, P_4, P_5, P_6 \text{ are in general position}\}.$$

Proposition 1.1.1. *As a subset of* $(\mathbb{P}^2)^6$, *the set* Φ *is an open subvariety of dimension 12.*

Proof. Clear. $\square$

1.2 The space of lines in $\mathbb{P}^3$

Definition: Let l be a line in $\mathbb{P}^3$. Let $a, b \in l$ be two points such that $a \neq b$, given by $a = (a_0 : a_1 : a_2 : a_3)$ and $b = (b_0 : b_1 : b_2 : b_3)$. Denote $(P_{ij})_{0 \leq i < j \leq 3}$ where $P_{ij} = a_i b_j - a_j b_i$, as a point of $\mathbb{P}^5$. This point does not depend on the choice of a, b on l. The P_{ij} are called the Plücker coordinates of l in $\mathbb{P}^5$.

Lemma 1.2.1. *The set of lines in* $\mathbb{P}^3$ *in Plücker coordinates is a closed subvariety of* $\mathbb{P}^5$ *given by* $\mathcal{G} = V(v_{01}v_{23} - v_{02}v_{13} + v_{03}v_{12}) \subset \mathbb{P}^5$.

Proof. [Mu1], p. 173 $\square$

1.3   The space of cubic surfaces in $\mathbb{P}^3$

Definition: Fix an ordering in the set $\{x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \mid \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 3\}$ of monomials of degree 3 in $k[x_0, x_1, x_2, x_3]$. In $\mathbb{P}^3$, each cubic surface $X \subset \mathbb{P}^3$ is given by a non-zero homogeneous polynomial:

$$X = V\Big(\sum_{\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 3} c_\alpha x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \Big),$$

(i.e not all c_α equal to zero). It corresponds with a point $c = (c_\alpha) \in \mathbb{P}^{19}$.

Conversely, for each point $c = (c_\alpha) \in \mathbb{P}^{19}$, let H_c be the cubic surface $V(\sum c_\alpha x^\alpha)$ in $\mathbb{P}^3$, let

$$\mathcal{H} = \bigcup_{c \in \mathbb{P}^{19}} H_c \subset \mathbb{P}^3 \times \mathbb{P}^{19}.$$

Then $\mathcal{H} = V(\sum c_\alpha x^\alpha)$. Therefore $\mathcal{H}$ is a closed subvariety of $\mathbb{P}^3 \times \mathbb{P}^{19}$ of dimension 21.

Let $\mathcal{H} \supset S = \{(x, c) \mid H_c \text{ is singular}\}$, let $\Delta = p_2(S)$, where $p_2 : \mathbb{P}^3 \times \mathbb{P}^{19} \longrightarrow \mathbb{P}^{19}$ is the projection. Then $\mathbb{P}^{19} - \Delta$ parameterizes the set of non-singular cubic surfaces. In fact, there exists a polynomial $D \in k[T_0, \dots, T_{19}]$ such that $\Delta = V(D)$, see [GIT] p. 79. In particular, this implies that:

Proposition 1.3.1. $\mathbb{P}^{19} - \Delta$ *is an open subset of* $\mathbb{P}^{19}$.

Proposition 1.3.2. *The blowing-up $\tilde{X}$ of $\mathbb{P}^2$ at six points in general position is isomorphic to a non-singular cubic surface in $\mathbb{P}^3$.*

Proof. [H], V.4.7.   □

Proposition 1.3.3. *The linear space of plane cubic forms passing through 6 points in general position has dimension 4. A choice of a basis for this linear space determines a rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$. The closure of the image of this rational map is a non-singular cubic surface. This cubic surface contains exactly 27 lines. They are:*

   (i) the exceptional curves $\tilde{P}_i$, for $i = 1, \dots, 6$ (six of these),

  (ii) the strict transform $\tilde{l}_{ij}$ of the line l_{ij} in $\mathbb{P}^2$ containing P_i and P_j, $1 \leq i < j \leq 6$ (fifteen of these), and

 (iii) the strict transform $\tilde{C}_j$ of the conic C_j in $\mathbb{P}^2$ containing the five P_i for $i \neq j, j = 1, \dots, 6$ (six of these).

Proof. [H], V.4.9.   □

From now on, we use notations $\tilde{P}_i, \tilde{l}_{ij}, \tilde{C}_i, C_i$ for the lines and curves as in Proposition 1.3.3.

Proposition 1.3.4. *Let X be a non-singular cubic surface in $\mathbb{P}^3$, and let $E_1, \dots, E_6$ be any set of six mutually skew lines chosen among the 27 lines on X. Then there exists a morphism $\pi : X \longrightarrow \mathbb{P}^2$, making X isomorphic to the blowing-up of that $\mathbb{P}^2$ with six points $P_1, \dots, P_6$ (no 3 collinear and not all 6 on a conic) such that $E_1, \dots, E_6$ are the exceptional curves for π.*

Proof. [H], V.4.10 and [Mu1], 8.22, 8.23. $\qquad\square$

Proposition 1.3.5. *Let* $P_1, ..., P_5$ *in* $\mathbb{P}^2$ *and no 3 of them are collinear. Then there exists a unique, irreducible conic containing* $P_1, ..., P_5$.

Proof. [H], V.4.2. $\qquad\square$

1.4 Algebraic groups and actions of algebraic groups on varieties

In this section, we recall some basic facts about algebraic groups and actions of them on varieties. These definitions and notations can be found in [N], Chapter 3, §1 or [GIT], Chapter 0, §1.

Definition: An *algebraic group* is a group G together with a structure of algebraic variety on G such that the maps:

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g, g') & \mapsto & gg' \end{array}$$

and

$$\begin{array}{ccc} G & \longrightarrow & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of algebraic varieties.

A *homomorphism of algebraic groups* is a map which is simultaneously a homomorphism of groups and a morphism of algebraic varieties.

Definition: An *action* of an algebraic group on a variety X is a morphism

$$\varphi : G \times X \longrightarrow X$$

such that for any $g, g' \in G$ and $x \in X$ we have

$$\varphi(g, \varphi(g', x)) = \varphi(gg', x) \text{ and } \varphi(e, x) = x$$

where e is the identity element of G.

For convenience, one usually write gx for $\varphi(g, x)$. So that the above conditions become

$$g(g'x) = (gg')x \text{ and } ex = x.$$

Definition: Let G be an algebraic group acting on a variety X; let $x \in X$. The *stabilizer G_x* of x is the closed subgroup $G_x = \{g \in G \mid gx = x\}$ of G.

The *orbit $\mathcal{O}(x)$* of x is the subset $\mathcal{O}(x) = \{gx \mid g \in G\}$ of X.

A point x (a subset W) of X is said to be *invariant* under G if $gx = x$ ($gW = W$) for all $g \in G$.

Definition: Let G be an algebraic group acting on varieties X and Y. A morphism $\phi : X \longrightarrow Y$ is a *G-morphism* if $\phi(gx) = g\phi(x)$ for all $x \in X$ and $g \in G$.

If G acts trivially on Y (i.e. $gy = y$ for all $g \in G$ and $y \in Y$) then a $G-$morphism is called $G-invariant$.

Example 6. The projective general linear group $\mathrm{PGL}(n)$ is an algebraic group. Let

$$F = \sum_{\alpha_0 + \cdots + \alpha_3 = 3} c_\alpha x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

be a non-zero cubic form in $k[x_0, \ldots, x_3]$. Let $A \in \mathrm{PGL}(3)$. Denote X for the column matrix of x_0, x_1, x_2, x_3. Then $F(AX)$ is a non-zero cubic form in $k[x_0, x_1, x_2, x_3]$. This induces a natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. In Chapter 3, we shall study this action.

Chapter 2

Non-singular cubic surfaces with star points

In this chapter, we study non-singular cubic surfaces. A star point (also called an Eckardt point) on a non-singular cubic surface is the point of intersection of 3 lines on the surface. We denote by H_k the subset of points corresponding to non-singular cubic surfaces in $\mathbb{P}^3$ with at least k star points. For every k, we determine the dimension and the number of irreducible components of H_k. We also determine the subset of H_k which consists of points corresponding to surfaces with exactly a given number of star points.

A non-singular cubic surface can be viewed as the blowing-up of $\mathbb{P}^2$ at 6 points in general position. A close study of the configuration of 6 points in $\mathbb{P}^2$ enables us to describe the sets of points in $\mathbb{P}^{19} - \Delta$ corresponding to non-singular cubic surfaces with a given number of star points.

In his book "The Non-Singular Cubic Surfaces" [S], B. Segre gave a classification of non-singular cubic surfaces in $\mathbb{P}^3$ with a given number of Eckardt points. Our methods give another proof of the possible number of star points on a non-singular cubic surface.

2.1   Star points. Star triples. Star-Steiner sets

Each non-singular cubic surface X has 45 tritangent planes, i.e the planes containing 3 lines of X. Moreover, there exist at most 3 lines among 27 lines of X through any point $P \in X$. In this case they form one of 45 tritangent planes of X ([R], pp. 102, 103).

Definition: A *star point* of a non-singular cubic surface is the intersection point of three lines on the surface.

Definition: Let $H_k \subset \mathbb{P}^{19} - \Delta$ denote *the set of points corresponding to cubic surfaces with at least k star points.*

In blowing-up of $\mathbb{P}^2$ at 6 points in general position $P_1, \ldots, P_6$, we see that each tritangent plane is defined uniquely by a triple of lines in form $(\tilde{P}_i \tilde{C}_j \tilde{l}_{ij})$ or $(\tilde{l}_{ij} \tilde{l}_{mn} \tilde{l}_{kh})$. So, sometimes, we use these triples of lines to denote the tritangent planes.

Definition: Let $\mathcal{T}$ denote the set of 45 triples of lines on a given non-singular cubic surface X, which span the tritangent planes. If a triple in $\mathcal{T}$ forms a star point then it

is called a *star triple*.

Remark 2.1.1. Let T_1 and T_2 be 2 triples in $\mathcal{T}$ having no line in common. Each line of T_1 meets exactly one line of T_2. There exists uniquely another triple $T_3 \in \mathcal{T}$ such that each line in T_3 forms one tritangent plane with one line of T_1 and one of T_2. A such set of 3 triples in $\mathcal{T}$ is called a *Steiner set*.

Example 7. The following are Steiner sets:

(i) $\left\{ (\tilde{P}_1\tilde{C}_2\tilde{l}_{12}), (\tilde{P}_2\tilde{C}_3\tilde{l}_{23}), (\tilde{P}_3\tilde{C}_1\tilde{l}_{13}) \right\}$;

(ii) $\left\{ (\tilde{P}_1\tilde{C}_2\tilde{l}_{12}), (\tilde{P}_3\tilde{C}_4\tilde{l}_{34}), (\tilde{l}_{14}\tilde{l}_{23}\tilde{l}_{56}) \right\}$;

(iii) $\left\{ (\tilde{l}_{14}\tilde{l}_{23}\tilde{l}_{56}), (\tilde{l}_{35}\tilde{l}_{16}\tilde{l}_{24}), (\tilde{l}_{26}\tilde{l}_{45}\tilde{l}_{13}) \right\}$.

Remark 2.1.2.

(i) By blowing-up, we see that any Steiner set of a given non-singular cubic surface X has one of three following forms:

   (a) $S_1 = \left\{ (\tilde{P}_i\tilde{C}_j\tilde{l}_{ij}), (\tilde{P}_j\tilde{C}_k\tilde{l}_{jk}), (\tilde{P}_k\tilde{C}_i\tilde{l}_{ik}) \right\}$,

   (b) $S_2 = \left\{ (\tilde{P}_i\tilde{C}_j\tilde{l}_{ij}), (\tilde{P}_k\tilde{C}_h\tilde{l}_{kh}), (\tilde{l}_{ih}\tilde{l}_{jk}\tilde{l}_{mn}) \right\}$,

   (c) $S_3 = \left\{ (\tilde{l}_{mn}\tilde{l}_{kh}\tilde{l}_{ij}), (\tilde{l}_{ki}\tilde{l}_{mj}\tilde{l}_{nh}), (\tilde{l}_{jh}\tilde{l}_{in}\tilde{l}_{mk}) \right\}$,

where the indices belong to $\{1, \ldots, 6\}$ and the different indices denote different numbers.

(ii) The 9 lines of a Steiner set of a given non-singular cubic surface X can form a Steiner set in exactly two ways. For example $\left\{ (\tilde{P}_1\tilde{C}_2\tilde{l}_{12}), (\tilde{P}_2\tilde{C}_3\tilde{l}_{23}), (\tilde{P}_3\tilde{C}_1\tilde{l}_{13}) \right\}$ and $\left\{ (\tilde{P}_1\tilde{C}_3\tilde{l}_{13}), (\tilde{P}_3\tilde{C}_2\tilde{l}_{23}), (\tilde{P}_2\tilde{C}_1\tilde{l}_{12}) \right\}$. In the other words, any Steiner set $S = \left\{ (l_1 l_2 l_3), (d_1 d_2 d_3), (m_1 m_2 m_3) \right\}$ can be reordered and written in matrix form:

$$
\begin{matrix}
l_1 & l_2 & l_3 \\
d_1 & d_2 & d_3 \\
m_1 & m_2 & m_3
\end{matrix}
$$

such that each row or each column of the above matrix expresses a triple in $\mathcal{T}$ and 3 triples from the 3 rows as well as from the 3 columns form a Steiner set.

We recall the concept and properties of the quadratic transformation φ_{123} (see[H], V 4.2.3) corresponding to 3 given points $P_1 = (1:0:0), P_2 = (0:1:0), P_3 = (0:0:1)$,

$$
\begin{array}{ccc}
\mathbb{P}^2 & \overset{\varphi_{123}}{\dashrightarrow} & \mathbb{P}^2 \\
(x_0 : x_1 : x_2) & \longmapsto & (x_1 x_2 : x_0 x_2 : x_0 x_1).
\end{array}
$$

Its inverse is $(y_0 : y_1 : y_2) \longmapsto (y_1 y_2 : y_0 y_2 : y_0 y_1)$. So the quadratic transformation φ_{123} gives an isomorphism:

$$
\mathbb{P}^2 - V(x_0 x_1 x_2) \overset{\cong}{\longrightarrow} \mathbb{P}^2 - V(y_0 y_1 y_2).
$$

Let V be the blowing-up of $\mathbb{P}^2$ at P_1, P_2, P_3. We have the following diagram:

$$V$$

$$\pi \qquad \psi$$

$$\mathbb{P}^2_X \quad - - - - \overset{\varphi_{123}}{-} - - - \to \mathbb{P}^2_Y,$$

where ψ is the blowing-up of $\mathbb{P}^2_Y$ at $Q_1 = (1:0:0), Q_2 = (0:1:0), Q_3 = (0:0:1)$ such that:

(i) The exceptional curves with respect to Q_1, Q_2, Q_3 are the strict transforms of $\overline{P_2 P_3}, \overline{P_1 P_3}$ and $\overline{P_1 P_2}$, respectively.

(ii) The strict transforms of $\overline{Q_1 Q_2}$, $\overline{Q_2 Q_3}$ and $\overline{Q_3 Q_1}$ are the exceptional curves $\tilde{P}_3, \tilde{P}_1$ and $\tilde{P}_2$ of P_3, P_1 and P_2, respectively.

(iii) $\varphi_{123} = \psi \circ \pi^{-1}$.

Then through the quadratic transformation:

a) The points P_1, P_2, P_3 correspond to $l_{23} = \overline{Q_2 Q_3}$, $l_{13} = \overline{Q_1 Q_3}$, $l_{12} = \overline{Q_1 Q_2}$, respectively.

b) An irreducible conic passing through P_1, P_2, P_3 corresponds to a line not containing any Q_1, Q_2, Q_3; conversely, a line not containing any P_1, P_2, P_3 corresponds to an irreducible conic passing through Q_1, Q_2, Q_3.

c) An irreducible conic containing 2 of P_1, P_2, P_3, say P_1 and P_2, corresponds to an irreducible conic containing Q_1 and Q_2.

d) A line containing only one point of P_1, P_2, P_3, say P_1, corresponds to a line containing only Q_1 of Q_1, Q_2, Q_3.

Proposition 2.1.3. *Let* $\mathcal{P} = (P_1, \dots, P_6) \in \Phi$. *Then the blowing up of* $\mathbb{P}^2_X$ *at* $\mathcal{P} = (P_1, \dots, P_6)$ *is isomorphic to the blowing up of* $\mathbb{P}^2_Y$ *at* $\mathcal{Q} = \Big(Q_1, Q_2, Q_3,$ $\varphi(P_4), \varphi(P_5), \varphi(P_6)\Big) \in \Phi$, *where* P_i, Q_i *for* $1 \le i \le 3$ *have coordinates as above and* φ *is the quadratic transformation with respect to* P_1, P_2, P_3.

(Note that, we say "blowing up at $\mathcal{P} = (P_1, \dots, P_6)$" in stead of "blowing up at $P_1, \dots, P_6$").

Proof. Let X and Y be the blowing-ups of $\mathbb{P}^2_X$ at $\mathcal{P} = (P_1, \dots, P_6)$ and $\mathbb{P}^2_Y$ at $(Q_1, Q_2, Q_3, \varphi(P_4), \varphi(P_5), \varphi(P_6))$, respectively, where P_i, Q_i have coordinates as above and φ is the quadratic transformation w.r.t P_i's, for $i \in \{1, 2, 3\}$. Consider the diagram
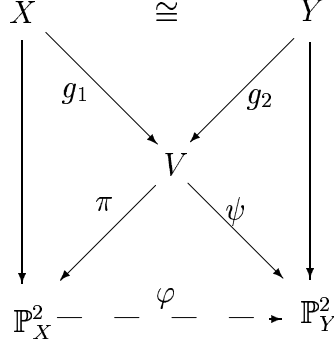
where $V \xrightarrow{\pi} \mathbb{P}_X^2$ and $V \xrightarrow{\psi} \mathbb{P}_Y^2$ are blowing-ups of $\mathbb{P}_X^2$ at P_1, P_2, P_3 and of $\mathbb{P}_Y^2$ at Q_1, Q_2, Q_3, respectively; where $X \xrightarrow{g_1} V$ and $Y \xrightarrow{g_2} V$ are the blowing-ups of V at $\pi^{-1}(P_i)$ and V at $\psi^{-1}(\varphi(P_i))$ respectively, for $i \in \{4,5,6\}$.

Since $\psi \circ \pi^{-1} = \varphi$, so $\pi^{-1}(P_i) = \psi^{-1}(\varphi(P_i))$ for $i = 4, 5, 6$. This implies that the blowing-ups of $\mathbb{P}_X^2$ at $\mathcal{P}$ and $\mathbb{P}_Y^2$ at $\mathcal{Q}$ are exactly the blowing up of V at $\pi^{-1}(P_i)$ for $i = 4, 5, 6$.

Therefore $X \cong Y$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 2.1.4. Let $\mathcal{P}$ and $\mathcal{P}'$ be two elements of Φ. If $\mathcal{P}'$ can be obtained from $\mathcal{P}$ by a quadratic transformation then the blowing-ups of $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic.

From now on we denote (ijk) for the quadratic transformation with respect to P_i, P_j, P_k for $i, j, k \in \{1, \ldots, 6\}$.

Lemma 2.1.5. *For any $i, j \in \{1, 2, 3\}$, each Steiner set of the form S_i can be transformed to an element of the form S_j by quadratic transformations.(Here S_i is used as in (2.1.2)).*

Proof. Let $s \in S_2$, we can assume (up to permutations of the permutation group of 6 letters) that:
$$s = \left\{ (\tilde{P}_1 \tilde{C}_2 \tilde{l}_{12}), (\tilde{P}_3 \tilde{C}_4 \tilde{l}_{34}), (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56}) \right\} \text{ then}$$

(124) exchanges s with $\left\{ (\tilde{P}_4 \tilde{C}_2 \tilde{l}_{24}), (\tilde{P}_3 \tilde{C}_4 \tilde{l}_{34}), (\tilde{P}_2 \tilde{C}_3 \tilde{l}_{23}) \right\} \in S_1$;

(135) exchanges s with $\left\{ (\tilde{l}_{35} \tilde{l}_{46} \tilde{l}_{12}), (\tilde{l}_{15} \tilde{l}_{26} \tilde{l}_{34}), (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56}) \right\} \in S_3$.

This means that one member of S_1 and one of S_3 can be exchanged by a sequence of two quadratic transformations. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 2.1.6. *Let $S = \{(l_1 l_2 l_3), (d_1 d_2 d_3), (t_1 t_2 t_3)\}$ be a Steiner set of a given non-singular cubic surface X and $m_1 = (l_1 l_2 l_3) \cap (d_1 d_2 d_3)$, $m_2 = (l_1 l_2 l_3) \cap (t_1 t_2 t_3)$, $m_3 = (d_1 d_2 d_3) \cap (t_1, t_2, t_3)$. Then m_1, m_2 and m_3 have a common point.*

Proof. Since m_1, m_2 are contained in the hyperplane H spanned by $(l_1l_2l_3)$, the line m_1 meets m_2. Similarly, the line m_1 meets m_3, the line m_2 meets m_3. If $m_1 = m_2$ then $m_3 = m_1$. Otherwise, since $m_3 \not\subset H = \mathrm{span}(m_1, m_2)$, this implies that m_1, m_2 and m_3 have a common point. $\square$

Corollary 2.1.7.

(i) *Assume the hypothesis as in (2.1.6) and suppose that $(l_1d_1t_1)$ forms a star point A. Then $m_1 \cap m_2 \cap m_3 \supset \{A\}$.*

(ii) *Let $S = \{(l_1l_2l_3), (d_1d_2d_3), (t_1t_2t_3)\}$ be a Steiner set of a given non-singular cubic surface X. If $(l_1l_2l_3)$ and $(d_1d_2d_3)$ form 2 star points A_1 and A_2, respectively, then $(t_1t_2t_3)$ forms another star point A_3 and A_1, A_2, A_3 lie on a line.*

Proof.

(i) Follows directly from the proposition.

(ii) By renumbering the triples of S, we can assume that $\big\{(l_1d_1t_1), (l_2d_2t_2), (l_3d_3t_3)\big\}$ is another Steiner set formed by 9 lines of S (2.1.2). Then apply the proposition and (i) for the Steiner set $\big\{(l_1d_1t_1), (l_2d_2t_2), (l_3d_3t_3)\big\}$. See also [S], p. 147.

$\square$

Definition: A Steiner set such that every of the 3 members gives a star point is called a *star-Steiner set*.

Proposition 2.1.8. *A non-singular cubic surface does not have three star triples which have a line in common.*

Proof. Up to quadratic transformations, we can assume that the common line is the strict transform of some conic C_i. The three star points come from tangent points of 3 tangent lines from P_i. This is impossible! $\square$

2.2   A study of H_1

For each $x \in H_1$, and view the corresponding cubic surface X_x as the blowing-up of $\mathbb{P}^2$ at 6 points in general position, then each star triple of X_x is of the form $(\tilde{P}_i\tilde{C}_j\tilde{l}_{ij})$ or $(\tilde{l}_{ij}\tilde{l}_{mn}\tilde{l}_{kh})$. As we have seen in (2.1.4) and (2.1.5), we can consider X_x as the blowing-up of $\mathbb{P}^2$ at $\mathcal{P} = (P_1, \ldots, P_6) \in \Phi$ where $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$. For this, if there is another one, say $(l_{ij}l_{kh}l_{mn})$, then the permutation $(i1)(j2)(k3)(m5)(n6)(h4)$ of S_6 makes it correspond to the previous one.

Let

$$\mathcal{L} = \Big\{ \big([X_3^2], L_1, \cdots, L_6\big) \mid X_3^2 \text{ is a non-singular cubic surface in } \mathbb{P}^3;$$

$$L_i \text{ is a line on } X_3^2;\ L_i \cap L_j = \emptyset \text{ for } 1 \le i \neq j \le 6 \Big\} \subset \mathbb{P}^{19} \times \mathcal{G}^6.$$

Consider:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{p} & \mathbb{P}^{19} - \Delta \\ \left([X_3^2], L_1, \cdots, L_6\right) & \longmapsto & [X_3^2]. \end{array}$$

We know that for any $x \in \mathbb{P}^{19} - \Delta$, the number of elements of $p^{-1}(x)$ is 51840. This implies that, $\dim \mathcal{L} = 19$, see [H], V.4.10.1 or [Mu1], p. 180.

Let

$$\mathcal{D} = \left\{ \left(\mathcal{P}, X, [X_3^2], L_1, \cdots, L_6\right) \mid \mathcal{P} \in \Phi; X \text{ is the blowing-up of } \mathbb{P}^2 \text{ at } \mathcal{P}; \right.$$

$$\left. L_i \text{ is the exceptional curve w.r.t. } P_i \in \mathcal{P}; \ X_3^2 \text{ is an embedding of } X \text{ in } \mathbb{P}^3 \right\}.$$

Consider

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Theta} & \Phi \\ \left(\mathcal{P}, X, [X_3^2], L_1, \cdots, L_6\right) & \longmapsto & \mathcal{P}. \end{array}$$

Note that for any $\mathcal{P} \in \Phi$, we have $\Theta^{-1}(\mathcal{P}) \cong \mathrm{Aut}\mathbb{P}^3 \cong \mathrm{PGL}(3, k) \cong \mathrm{GL}(4, k)/k^*$, see [H], II.7.1.1. Consequently, $\dim\Theta^{-1}(\mathcal{P}) = 15$.

Consider

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Gamma} & \mathcal{L} \\ \left(\mathcal{P}, X, [X_3^2], L_1, \cdots, P_6\right) & \mapsto & \left([X_3^2], L_1, \cdots, L_6\right). \end{array}$$

Note that for any $\alpha \in \mathcal{L}$, we have $\Gamma^{-1}(\alpha) \cong \mathrm{Aut}\mathbb{P}^2 \cong \mathrm{PGL}(2, k) \cong \mathrm{GL}(3, k)/k^*$. Consequently, $\dim\Gamma^{-1}(\alpha) = 8$, see [H], II.7.1.1.

Let $K_1 = \left\{ (P_1, \ldots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} \neq \emptyset \right\}$, $D_1 = \Theta^{-1}(K_1)$ and $L_1 = \Gamma(D_1)$. Then we have $H_1 = p(L_1)$.

Theorem 2.2.1. *The set H_1 is a closed, irreducible subvariety of $\mathbb{P}^{19} - \Delta$ of dimension 18.*

Proof. We first prove that K_1 is irreducible of dimension 11. Let

$$\mathcal{F}_1 = \left\{ (Q, l_1, l_2, l_3) \mid l_i \in \mathcal{G}; Q \in l_i; l_i \neq l_j \text{ for } 1 \leq i < j \leq 3 \right\} \subset \mathbb{P}^2 \times \mathcal{G}^3,$$

$$\mathcal{F}_2 = \left\{ (Q, l_1, l_2, l_3, P_1, \ldots, P_6) \mid (Q, l_1, l_2, l_3) \in \mathcal{F}_1; P_i \neq P_j \text{ for } 1 \leq i < j \leq 3; \right.$$

$$\left. P_1, P_2 \in l_1; P_3, P_4 \in l_2; P_5, P_6 \in l_3; \ (P_1, \ldots, P_6) \in \Phi \right\} \subset \mathcal{F}_1 \times (\mathbb{P}^2)^6.$$

Consider $\mathcal{F}_2 \xrightarrow{p_2} \mathcal{F}_1 \xrightarrow{p_1} \mathbb{P}^2$ where p_1, p_2 are the projections. Since p_1 is surjective and every fiber is irreducible and has dimension 3, this follows that $\mathcal{F}_1$ is also irreducible and $\mathcal{F}_1$ has dimension 5. Moreover, the map p_2 is surjective and every fiber is irreducible and has dimension 6, so $\mathcal{F}_2$ is irreducible and has dimension 5+6=11. Finally the projection $\mathcal{F}_2 \longrightarrow K_1$ is an isomorphism. This implies K_1 is irreducible and has dimension 11.

Consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\Gamma} & \mathcal{L} \\
\Theta \downarrow & & p \downarrow \\
\Phi & & \mathbb{P}^{19} - \Delta.
\end{array}
$$

This induces

$$
\begin{array}{ccc}
D_1 & \xrightarrow{\Gamma} & L_1 \\
\Theta \downarrow & & p \downarrow \\
K_1 & & H_1.
\end{array}
$$

Our task is now to prove that D_1 is irreducible and then this implies that $H_1 = p \circ \Gamma(D_1)$ is irreducible. Let $\mathcal{L}_{\mathcal{P}}$ denote the set of cubic forms passing through the 6 points $P_1, \ldots, P_6$ of a given $\mathcal{P}$ in Φ. This set is a linear vector space of dimension 4. Let

$$
Y = \Big\{ (\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \mid \mathcal{P} = (P_1, \ldots, P_6) \in K_1;
$$
$$
\varphi_1, \varphi_2, \varphi_3, \varphi_4 \text{ are non-zero cubic forms passing through } P_1, \ldots, P_6 \Big\} \subset K_1 \times (\mathbb{P}^9)^4
$$

and
$$
U = \Big\{ (\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \in Y \mid \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \text{ forms a basis of } \mathcal{L}_{\mathcal{P}} \Big\}.
$$

Then we have the following commutative diagram:

$$
\begin{array}{ccccc}
U & \overset{\text{open}}{\hookrightarrow} & Y & \overset{\text{closed}}{\hookrightarrow} & K_1 \times (\mathbb{P}^9)^4 \\
& \searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle p} & \\
& & K_1 & &
\end{array}
$$

where the map $p : K_1 \times (\mathbb{P}^9)^4 \longrightarrow K_1$ is the projection. The map $Y \xrightarrow{g} K_1$ in the above diagram is surjective and every fiber is isomorphic to $(\mathbb{P}^3)^4$, therefore Y is irreducible. Consequently U is irreducible. For any basis $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ of $\mathcal{L}_{\mathcal{P}}$, there exist embeddings in $\mathbb{P}^3$ of the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$; namely embeddings come from:

$$
\begin{array}{ccc}
\mathbb{P}^2 & - - - \to & \mathbb{P}^3 \\
(x_0 : x_1 : x_2) & \longmapsto & (\lambda_1 \varphi_1 : \lambda_2 \varphi_2 : \lambda_3 \varphi_3 : \lambda_4 \varphi_4),
\end{array}
$$

for $\lambda_i \in k^*$ such that $\{\lambda_1 \varphi_1, \lambda_2 \varphi_2, \lambda_3 \varphi_3, \lambda_4 \varphi_4\}$ is a basis of $\mathcal{L}_{\mathcal{P}}$. Therefore, the set D_1 is isomorphic to an open subset of $U \times \mathbb{P}^3$. So that D_1 is irreducible, too. Consequently, the set D_1 has dimension 11+12+3=26. Therefore, $\dim(H_1)$=26-8=18. For the closedness of H_1, see the proof of the Theorem 2.3.1. $\qquad \square$

Remark 2.2.2. We can prove the irreducibility of H_1 in another way as follows.

Consider the subset V_1 of $\mathbb{P}^{19}$ consisting of points whose corresponding cubic surfaces have 3 lines meeting at one point. Given $x \in V_1$, we can choose a system of coordinates of $\mathbb{P}^3$ such that X_x is determined by a non-zero cubic form

$$Qz + axy(x - y) \tag{2.1}$$

where Q is a quadratic form in x, y, z, t and $a \in k$. Each such non-zero cubic form corresponds to a point of $\mathbb{P}^{10}$. Let $S \subset \mathbb{P}^{19}$ denote the set of all points in H_1 determined by non-zero cubic forms (2.1). In particular, the subset S is irreducible.

Consider the surjective map $\mathrm{PGL}(3) \times S \longrightarrow H_1$ deduced by the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. This implies that H_1 is irreducible.

However, we do not see how we can use this method for the case H_k for $k > 1$.

2.3   A study of H_2 and H_3

Definition: Let

$$H_2^{(2)} = \{x \in H_2 \mid \text{ the surface } X_x \text{ has a pair of star triples having one line in common}\},$$

$$H_2^{(3)} = \{x \in H_2 \mid \text{ the surface } X_x \text{ has 2 star triples having no line in common}\}.$$

For each $x \in H_2^{(2)}$, view X_x as blowing-up of $\mathbb{P}^2$ at 6 points in general position. Moreover, by Remark 2.1.4 and Proposition 2.1.5, we can assume that X_x has a pair of star triples of one of the following forms:

(i) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$, $(\tilde{l}_{12}\tilde{l}_{35}\tilde{l}_{46})$;

(ii) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$, $(\tilde{l}_{12}\tilde{l}_{36}\tilde{l}_{45})$;

(iii) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$, $(\tilde{l}_{12}\tilde{C}_1\tilde{P}_2)$;

(iv) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$, $(\tilde{l}_{12}\tilde{C}_2\tilde{P}_1)$.

The first and the second just differ by the permutation (56). The third and the fourth differ by the permutation (12). Moreover, the quadratic transformation (135) exchanges the first and the third.

Let

$$K_2^{(2)} = \left\{ (P_1, \ldots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{Q_1\}, l_{12} \cap l_{35} \cap l_{46} = \{Q_2\} \right\},$$

see Figure 2.1.

Let $D_2^{(2)} = \Theta^{-1}(K_2^{(2)})$, $L_2^{(2)} = \Gamma(D_2^{(2)})$. Then by the argument above, we have $H_2^{(2)} = p(L_2^{(2)})$.

Similarly, up to permutations and quadratic transformations, we can assume that each cubic surface corresponding to an element of the subsets $H_2^{(3)}$, has a pair of star triples of one of the following kinds:

Figure 2.1: The configuration for members of $K_2^{(2)}$

(i) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56}), (\tilde{P}_i\tilde{C}_j\tilde{l}_{ij})$;

(ii) $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56}), (\tilde{l}_{45}\tilde{l}_{26}\tilde{l}_{13})$.

But it is easy to see that the quadratic transformation(imn) for $j \notin \{m, n\}$ and $\{1, 2\} \cap \{i, m, n\} \neq \emptyset$, $\{3, 4\} \cap \{i, m, n\} \neq \emptyset$, $\{5, 6\} \cap \{i, m, n\} \neq \emptyset$ exchanges (ii) to (i). Moreover, if there exists another pair, say $(\tilde{l}_{ij}\tilde{l}_{kh}\tilde{l}_{mn}), (\tilde{l}_{hm}\tilde{l}_{jn}\tilde{l}_{ik})$, then the permutation $(i1)(j2)(k3)(h4)(m5)(n6)$ exchanges it to $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56}), (\tilde{l}_{45}\tilde{l}_{26}\tilde{l}_{13})$.
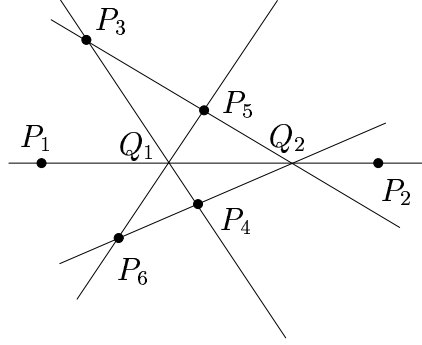 Let

$$K_2^{(3)} = \left\{ (P_1, \dots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{Q_1\}, l_{13} \cap l_{45} \cap l_{26} = \{Q_2\} \right\},$$

see Figure 2.2. Let $D_2^{(3)} = \Theta^{-1}(K_2^{(3)})$, $L_2^{(3)} = \Gamma(D_2^{(3)})$, and then by the above argument,
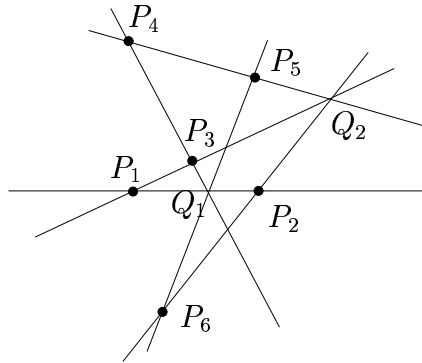
Figure 2.2: The configuration for members of $K_2^{(3)}$

we have $H_2^{(3)} = p(L_2^{(3)})$.

Theorem 2.3.1. *The set H_2 is closed in $\mathbb{P}^{19} - \Delta$ and has two irreducible components $H_2^{(2)}$ and $H_2^{(3)}$ of dimension 17.*

Proof. We have the following diagram:

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\;\Gamma\;} & \mathcal{L} \\
{\scriptstyle\Theta}\big\downarrow & & {\scriptstyle p}\big\downarrow \\
\Phi & & \mathbb{P}^{19} - \Delta.
\end{array}
$$

This induces

$$
\begin{array}{ccc}
D_2^{(2)} & \xrightarrow{\;\Gamma\;} & L_2^{(2)} \\
{\scriptstyle\Theta}\big\downarrow & & {\scriptstyle p}\big\downarrow \\
K_2^{(2)} & & H_2^{(2)}.
\end{array}
$$

Suppose that $K_2^{(2)}$ is irreducible, we prove that $D_2^{(2)}$ is irreducible then this implies that $H_2^{(2)} = p \circ \Gamma(D_2^{(2)})$ is irreducible.

Let

$$
Y = \Big\{ (\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \mid \mathcal{P} = (P_1, \dots, P_6) \in K_2^{(2)};
$$

$$
\varphi_1, \varphi_2, \varphi_3, \varphi_4 \text{ are non-zero cubic forms passing through } P_1, \dots, P_6 \Big\} \subset K_2^{(2)} \times (\mathbb{P}^9)^4
$$

and

$$
U = \Big\{ (\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \in Y \mid \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \text{ forms a basis of } \mathcal{L}_{\mathcal{P}} \Big\},
$$

where $\mathcal{L}_{\mathcal{P}}$ is used as on page 19.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
U & \overset{\text{open}}{\hookrightarrow} & Y & \overset{\text{closed}}{\hookrightarrow} & K_2^{(2)} \times (\mathbb{P}^9)^4 \\
 & \searrow & \big\downarrow{\scriptstyle g} & \swarrow{\scriptstyle p} & \\
 & & K_2^{(2)} & &
\end{array}
$$

where the map $p : K_2^{(2)} \times (\mathbb{P}^9)^4 \longrightarrow K_2^{(2)}$ is the projection. The map $Y \xrightarrow{\;g\;} K_2^{(2)}$ is surjective and every fiber is isomorphic to $(\mathbb{P}^3)^4$. Therefore Y is irreducible. Consequently, the set U is irreducible. As in the proof of (2.2.1), we see that $D_2^{(2)}$ is isomorphic to an open subset of $U \times \mathbb{P}^3$. In particular $D_2^{(2)}$ is irreducible.

Next, we prove that $K_2^{(2)}$ is irreducible of dimension 10. Let

$$
\mathcal{F}_1 = \{ (Q_1, Q_2) \mid Q_i \in \mathbb{P}^2, Q_1 \neq Q_2 \} \overset{\text{open}}{\hookrightarrow} (\mathbb{P}^2)^2,
$$

$$
\mathcal{F}_2 = \Big\{ (Q_1, Q_2, l_1, l_2, l_3, l_4) \mid (Q_1, Q_2) \in \mathcal{F}_1; l_i \in \mathcal{G}; \{Q_1\} = l_1 \cap l_2, \{Q_2\} = l_3 \cap l_4 \Big\}
$$

$$
\subset \mathcal{F}_1 \times \mathcal{G}^4
$$

and

$$\mathcal{F}_3 = \left\{ (Q_1, Q_2, l_1, l_2, l_3, l_4, P_1, P_2) \mid (Q_1, Q_2, l_1, l_2, l_3, l_4) \in \mathcal{F}_2; \; P_1, P_2 \in \overline{Q_1 Q_2} \right\}$$
$$\subset \mathcal{F}_2 \times (\mathbb{P}^1)^2.$$

Consider $\mathcal{F}_3 \xrightarrow{p_1} \mathcal{F}_2 \xrightarrow{p_2} \mathcal{F}_1$ where the maps p_1, p_2 are projections. Since $\mathcal{F}_1$ is irreducible of dimension 4, the map p_2 is surjective and every fiber is irreducible of dimension 4, so $\mathcal{F}_2$ is irreducible of dimension 8. Also, the map p_1 is surjective and every fiber is irreducible of dimension 2, so $\mathcal{F}_3$ is irreducible of dimension 8+2=10. Moreover $K_2^{(2)}$ is an open subset of $\mathcal{F}_3$ if we let $\{P_3\} = l_1 \cap l_3, \{P_4\} = l_1 \cap l_4, \{P_5\} = l_2 \cap l_3, \{P_6\} = l_2 \cap l_6$. Therefore $K_2^{(2)}$ is irreducible of dimension 10. This implies that $\dim(H_2^{(2)}) = 10+15-8=17$.

Similarly, we prove that $K_2^{(3)}$ is irreducible of dimension 10 and then $H_2^{(3)}$ is irreducible of dimension 17.

Finally, we prove that $H_2^{(2)}$, $H_2^{(3)}$ and H_1 are closed. Let

$$\mathcal{M} = \left\{ (x, l_1, \ldots, l_{27}) \mid x \in \mathbb{P}^{19} - \Delta; \; l_i \subset X_x \right\} \subset (\mathbb{P}^{19} - \Delta) \times \mathcal{G}^{27}.$$

Then the projection $p : \mathcal{M} \longrightarrow \mathbb{P}^{19} - \Delta$ is finite.

Let $H_\alpha = \left\{ (x, l_1, \ldots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset \right\}$. Consider the maps:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f_{ijk}} & \mathcal{G}^3 \\ (x, l_1, \ldots, l_{27}) & \mapsto & (l_i, l_j, l_k) \end{array} \quad \text{for } 1 \le i < j < k \le 27.$$

Let $L = \left\{ (l_1, l_2, l_3) \mid l_1 \cap l_2 \cap l_3 \neq \emptyset \right\} \subset \mathcal{G}^3$. It is easy to see that the set $K = \left\{ (l_1, l_2) \in \mathcal{G}^2 \mid l_1 \cap l_2 \neq \emptyset \right\}$ is closed in $\mathcal{G}^2$. Consider the maps:

$$\begin{array}{ccc} \mathcal{G}^3 & \xrightarrow{g_{ij}} & \mathcal{G}^2 \\ (l_1, l_2, l_3) & \mapsto & (l_i, l_j) \end{array} \quad \text{for } 1 \le i < j \le 3.$$

Then $L = g_{12}^{-1}(K) \cap g_{13}^{-1}(K) \cap g_{23}^{-1}(K)$ is closed in $\mathcal{G}^3$. Therefore $H_\alpha = f_{123}^{-1}(L)$ is closed in $\mathcal{M}$. Since H_1 is the image of H_α via p, it is a closed subset of $\mathbb{P}^{19} - \Delta$.

Let $H_\beta = \left\{ (x, l_1, \ldots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset; \; l_1 \cap l_4 \cap l_5 \neq \emptyset \right\}$. Then $H_\beta = f_{123}^{-1}(L) \cap f_{145}^{-1}(L)$ is a closed in $\mathcal{M}$ and $H_2^{(2)} = p(H_\beta)$ is closed in $\mathbb{P}^{19} - \Delta$.

Similarly, let $H_\gamma = \left\{ (x, l_1, \ldots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset; \; l_4 \cap l_5 \cap l_6 \neq \emptyset \right\}$. Then $H_\gamma = f_{123}^{-1}(L) \cap f_{456}^{-1}(L)$ is closed in $\mathcal{M}$ and $H_2^{(3)} = p(H_\gamma)$ is closed in $\mathbb{P}^{19} - \Delta$. $\square$

Remark 2.3.2. The argument used to prove the closedness of $H_2^{(2)}$, $H_2^{(3)}$ and H_1 above will be used several times in the rest of the chapter.

Corollary 2.3.3. $H_3 = H_2^{(3)}$. *Consequently H_3 is closed in $\mathbb{P}^{19} - \Delta$, irreducible of dimension 17.*

Proof. By (2.1.7 (ii)), for any $x \in H_2^{(3)}$, we have $X_x \in H_3$. Conversely, if $x \in H_3$ then the corresponding surface X_x has at least 3 star triples and there exist 2 of them having no line in common. For this, suppose X has 2 star triples, say T_1 and T_2 having one

line in common. Let T_3 be another star triple which has one line in common with T_2. Then T_1 and T_3 have no line in common, otherwise this situation contradicts against the fact that the five tritangent planes containing a given line of any non-singular cubic surface in $\mathbb{P}^3$ are different. So $x \in H_2^{(3)}$. The rest of the corollary follows from the theorem. $\qquad \square$

Corollary 2.3.4. *The set H_1 generically consists of points corresponding to cubic surfaces with exactly 1 star point.*

2.4 A study of H_4

Recall that H_4 is the set of points corresponding to non-singular cubic surfaces with at least 4 star points. Since $H_4 \subset H_3 = H_2^{(2)}$, this implies that for each $x \in H_4$, the surface X_x has a star-Steiner set. Moreover, if X_x has a star-Steiner $S = \{T_1, T_2, T_3\}$ and another star triple T having 2 lines in common with S, then T has all lines in common with S. This follows from (2.1.2). Therefore, the set H_4 consists of elements in one of the 3 following subsets:

$$H_4^{(4)} = \big\{ [X] \in H_4 \mid X \text{ has one star-Steiner } S \text{ and another}$$
$$\text{star triple } T \text{ having 3 lines in common with } S \big\};$$

$$H_4^{(6)} = \big\{ [X] \in H_4 \mid X \text{ has one star-Steiner S and another}$$
$$\text{star triple } T \text{ having 1 line in common with } S \big\};$$

$$H_4^{(9)} = \big\{ [X] \in H_4 \mid X \text{ has one star-Steiner } S \text{ and another}$$
$$\text{star triple } T \text{ having no line in common with } S \big\}.$$

A. A study of $H_4^{(4)}$

By (2.1.5) and up to permutations, we can assume that for each $x \in H_4^{(4)}$, the corresponding cubic surface X_x has the star-Steiner set $S = \{ (\tilde{l}_{13}\tilde{l}_{24}\tilde{l}_{56}), (\tilde{P}_3\tilde{C}_4\tilde{l}_{34}), (\tilde{P}_2\tilde{C}_1\tilde{l}_{12}) \}$. Therefore, the subsets $H_4^{(4)}$, up to permutations, consists of points corresponding to surfaces which possesses S and another star triple in one of the following kinds:

(i) S and $(\tilde{P}_3\tilde{C}_1\tilde{l}_{13})$,

(ii) S and $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$.

We see that the quadratic transformation(234) exchanges S and $(\tilde{P}_3\tilde{C}_1\tilde{l}_{13})$ of kind (i) to $\{(\tilde{P}_3\tilde{C}_1\tilde{l}_{13}), (\tilde{P}_2\tilde{C}_4\tilde{l}_{24}), (\tilde{l}_{56}\tilde{l}_{34}\tilde{l}_{12})\}$ and $(\tilde{l}_{13}\tilde{l}_{24}\tilde{l}_{56})$. Moreover, the permutation (23) makes $\{(\tilde{P}_3\tilde{C}_1\tilde{l}_{13}), (\tilde{P}_2\tilde{C}_4\tilde{l}_{24}), (\tilde{l}_{56}\tilde{l}_{34}\tilde{l}_{12})\}$ and $(\tilde{l}_{13}\tilde{l}_{24}\tilde{l}_{56})$ correspond to S and $(\tilde{l}_{12}\tilde{l}_{34}\tilde{l}_{56})$, respectively.

Definition:

$$K_4^{(4)} = \left\{ (P_1, \ldots, P_6) \in \Phi \mid l_{13} \cap l_{24} \cap l_{56} = \{Q\}; \, l_{12}, l_{13} \text{ are tangent to } C_1 \right\},$$
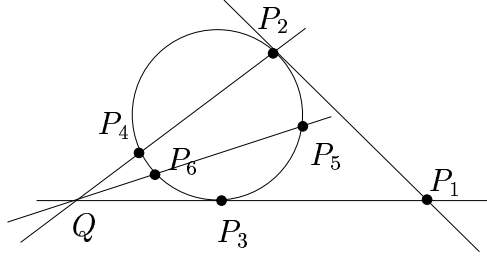
see Figure 2.3.

Figure 2.3: The configuration for members of $K_4^{(4)}$

Let $D_4^{(4)} = \Theta^{-1}(K_4^{(4)})$, $(L_4)^{(4)} = \Gamma(D_4^{(4)})$. By the argument above, we have $H_4^{(4)} = p(L_4^{(4)})$.

Definition: Let $\mathcal{C} \subset \mathbb{P}^5$ denote the space parameterizing non-singular conics in $\mathbb{P}^2$. This is an open subvariety of $\mathbb{P}^5$.

Theorem 2.4.1. *The subset $H_4^{(4)} \subset H_4$ is closed in $\mathbb{P}^{19} - \Delta$, irreducible of dimension 16.*

Proof. We first prove that $K_4^{(4)}$ is irreducible of dimension 9.
Let

$$\mathcal{F}_1 = \Big\{ \big([C], P_1, P_2, P_3\big) \mid C \text{ is a non-singular conic in } \mathbb{P}^2; \, P_1 \notin C;$$
$$P_2, P_3 \in C; \, P_2 \neq P_3; \, l_{12}, l_{13} \text{ are tangent to } C \Big\} \subset \mathcal{C} \times (\mathbb{P}^2)^3,$$

$$\mathcal{F}_2 = \Big\{ \big(Q, [C], P_1, P_2, P_3\big) \mid \big([C], P_1, P_2, P_3\big) \in \mathcal{F}_1; \, Q \in l_{13}; \, Q \notin \{P_1, P_3\} \Big\} \subset \mathcal{F}_1 \times \mathbb{P}^2,$$

$$\mathcal{F}_3 = \Big\{ \big(Q, [C], P_1, P_2, P_3, P_4, P_5, P_6\big) \mid \big(Q, [C], P_1, P_2, P_3\big) \in \mathcal{F}_2; \, \{P_4\} = \overline{P_2 Q} \cap C;$$
$$P_5, P_6 \in C; \, l_{56} \cap l_{13} = \{Q\}; \, (P_1, \ldots, P_6) \in \Phi \Big\} \subset \mathcal{F}_2 \times (\mathbb{P}^2)^3.$$

Consider

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{C} \\ \big([C], P_1, P_2, P_3\big) & \mapsto & [C]. \end{array}$$

The map p_1 is surjective and every fiber is irreducible of dimension 2. This implies that $\mathcal{F}_2$ is irreducible of dimension $5 + 2 = 7$. Similarly, the map:

$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{p_2} & \mathcal{F}_1 \\ \big(Q, [C], P_1, P_2, P_3\big) & \mapsto & \big([C], P_1, P_2, P_3\big) \end{array}$$

is surjective and every fiber is irreducible of dimension 1. Therefore $\mathcal{F}_2$ is irreducible of dimension 8. Finally, consider:

$$\begin{array}{ccc} \mathcal{F}_3 & \xrightarrow{p_3} & \mathcal{F}_2 \\ \big(Q, [C], P_1, \ldots, P_6\big) & \mapsto & \big(Q, [C], P_1, P_2, P_3\big). \end{array}$$

The map p_3 is surjective and each fiber is naturally isomorphic to $A = \big\{(P_5, P_6) \in C \times C \mid Q \in l_{56};\ P_5 \neq P_6\big\}$. Consider:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ (P_5, P_6) & \mapsto & P_5. \end{array}$$

The map h is surjective and every fiber consists of one point. Therefore A is irreducible of dimension 1. This implies $\mathcal{F}_3$ is irreducible of dimension 8+1=9. So is $K_4^{(4)}$, since the projection $\mathcal{F}_3 \longrightarrow K_4^{(4)}$ is an isomorphism.

Next, we prove that the $H_4^{(4)}$ is irreducible of dimension 16. We have the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Gamma} & \mathcal{L} \\ \Theta \downarrow & & p \downarrow \\ \Phi & & \mathbb{P}^{19} - \Delta. \end{array}$$

This induces

$$\begin{array}{ccc} D_4^{(4)} & \xrightarrow{\Gamma} & L_4^{(4)} \\ \Theta \downarrow & & p \downarrow \\ K_4^{(4)} & & H_4^{(4)}. \end{array}$$

We shall prove that $D_4^{(4)}$ is irreducible. This implies that $H_4^{(4)} = p \circ \Gamma(D_4^{(4)})$ is irreducible. Hence $H_4^{(4)}$ has dimension 9-8+15=16.

Let

$$Y = \Big\{(\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \mid \mathcal{P} = (P_1, \ldots, P_6) \in K_4^{(4)};$$

$\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are non-zero cubic forms passing through $P_1, \ldots, P_6\Big\} \subset K_4^{(4)} \times (\mathbb{P}^9)^4$

and

$$U = \Big\{(\mathcal{P}, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \in Y \mid \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \text{ forms a basis of } \mathcal{L}_\mathcal{P}\Big\},$$

where $\mathcal{L}_\mathcal{P}$ is used as on page 19.

Consider the following commutative diagram:

$$U \overset{\text{open}}{\hookrightarrow} Y \overset{\text{closed}}{\hookrightarrow} K_4^{(4)} \times (\mathbb{P}^9)^4$$

(diagram with maps g, p into $K_4^{(4)}$)

where the map $p : K_4^{(4)} \times (\mathbb{P}^9)^4 \longrightarrow K_4^{(4)}$ is the projection. The map $Y \overset{g}{\longrightarrow} K_4^{(4)}$ is surjective and every fiber is isomorphic to $(\mathbb{P}^3)^4$, therefore Y is irreducible. Consequently U is irreducible.

As in the proof of (2.2.1), we see that $D_4^{(4)}$ is isomorphic to an open subset of $U \times \mathbb{P}^3$. So $D_4^{(4)}$ is irreducible.

For the closedness of $H_4^{(4)}$, see the proof of (2.4.5). $\qquad\square$

B. A study of $H_4^{(6)}$

For convenience, from now on, for any two star triples T_1, T_2 of a non-singular cubic surface X such that T_1 and T_2 do not have any line in common, we denote $T_1 T_2$ for the third star triple of the star-Steiner set determined by T_1 and T_2. By this notation, we have, $T_1 T_2 = T_2 T_1$; $T_1(T_1 T_2) = T_2$; $T_2(T_1 T_2) = T_1$,...

For any $x \in H_4^{(6)}$, the surface X_x possesses a pair (S, U) where $S = \{T_1, T_2, T_1 T_2\}$ is a star-Steiner set and U is another star triple which has only one line in common with S and we can assume with T_1. We can assume that $T_1 = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12})$; $U = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13})$. Then the star triple T_2, up to permutations, has one of following kinds:

(i) $T_2 = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34})$ and then $T_1 T_2 = (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56})$;

(ii) $T_2 = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14})$ and then $T_1 T_2 = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24})$;

(iii) $T_2 = (\tilde{C}_5 \tilde{P}_4 \tilde{l}_{45})$ and then $T_1 T_2 = (\tilde{l}_{14} \tilde{l}_{25} \tilde{l}_{36})$.

Note that if T_2 has form $(\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24})$ or $(\tilde{l}_{ij} \tilde{l}_{mn} \tilde{l}_{kh})$ then $T_1 T_2$ has the form (i) or (iii).

Lemma 2.4.2. *Any element of each kind in the above list can be transformed to any other kind in the list by permutations and quadratic transformations.*

Proof.
If $T_2 = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14})$ (kind (ii)) then $T_1 T_2 = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24})$ and $U(T_1 T_2) = (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56})$. It is easy to see that $S' = \{U, T_1 T_2, U(T_1 T_2)\}$ together T_1 form one of kind (i).

If $T_2 = (\tilde{C}_5 \tilde{P}_4 \tilde{l}_{45})$ (kind (iii)) then $T_1 T_2 = (\tilde{l}_{14} \tilde{l}_{25} \tilde{l}_{36})$, $T_2 U = (\tilde{l}_{14} \tilde{l}_{35} \tilde{l}_{26})$. Under the "action" of the quadratic transformation(356), the star triples $T_2 U$, T_2 and $T_1 T_2$ are exchanged with $T_1' = (\tilde{C}_2 \tilde{P}_6 \tilde{l}_{26})$, $T_2' = (\tilde{C}_5 \tilde{P}_4 \tilde{l}_{45})$ and $U' = (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25})$, respectively. It is easy to see that $S' = \{T_1', T_2', T_1' T_2'\}$ together U' form one of kind (i). $\qquad\square$

Moreover, if any one of kind (i) possesses a pair $S = \{(\tilde{C}_i \tilde{P}_j \tilde{l}_{ij}), (\tilde{C}_h \tilde{P}_k \tilde{l}_{hk}), (\tilde{l}_{ik} \tilde{l}_{jh} \tilde{l}_{mn}\}$ and $U = (\tilde{C}_i \tilde{P}_h \tilde{l}_{ih})$ then the action of the permutation $(i1)(j2)(h3)(k4)(m5)(n6)$ let it correspond to the pair $\{(\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56})\}$ and $(\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13})$.

Definition:

$$K_4^{(6)} = \left\{ (P_1, \ldots, P_6) \in \Phi \mid l_{14} \cap l_{23} \cap l_{56} = \{S\}; \; l_{12}, l_{13} \text{ are tangent to } C_1 \right\},$$
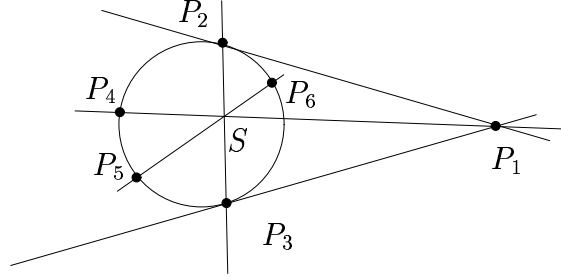
see Figure 2.4.

Figure 2.4: The configuration for members of $K_4^{(6)}$

Let $D_4^{(6)} = \Theta^{-1}(K_4^{(6)})$ and $L_4^{(6)} = \Gamma(D_4^{(6)})$. By the above argument and by (2.1.4), we have $H_4^{(6)} = p(L_4^{(6)})$.

Theorem 2.4.3. *The subset $H_4^{(6)} \subset H_4$ is closed in $\mathbb{P}^{19} - \Delta$, irreducible of dimension 16.*

Proof. We prove first that $K_4^{(6)}$ is irreducible of dimension 9. Let

$$\mathcal{F}_1 = \Big\{ ([C], P_1, P_2, P_3) \mid C \text{ is a non-singular conic in } \mathbb{P}^2; \, P_1 \notin C;$$

$$P_2, P_3 \in C; \; P_2 \neq P_3; \; l_{12}, l_{13} \text{ are tangent to } C \Big\} \subset \mathcal{C} \times (\mathbb{P}^2)^3,$$

$$\mathcal{F}_2 = \Big\{ ([C], P_1, P_2, P_3, P_4) \mid ([C], P_1, P_2, P_3) \in \mathcal{F}_1; \; P_4 \in C; \; P_4 \notin \{P_2, P_3\} \Big\} \subset \mathcal{F}_1 \times \mathbb{P}^2,$$

$$\mathcal{F}_3 = \Big\{ ([C], P_1, P_2, P_3, P_4, P_5, P_6) \mid ([C], P_1, P_2, P_3, P_4) \in \mathcal{F}_2;$$

$$P_5, P_6 \in C; \; l_{56} \cap l_{23} \cap l_{14} = \{S\}; \; (P_1, \ldots, P_6) \in \Phi \Big\} \subset \mathcal{F}_2 \times (\mathbb{P}^2)^2.$$

Consider

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{C} \\ ([C], P_1, P_2, P_3) & \mapsto & [C]. \end{array}$$

The map p_1 is surjective and every fiber is irreducible of dimension 2. This implies that $\mathcal{F}_2$ is irreducible of dimension $5 + 2 = 7$. Similarly, the map:

$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{p_2} & \mathcal{F}_1 \\ ([C], P_1, P_2, P_3, P_4) & \mapsto & ([C], P_1, P_2, P_3) \end{array}$$

is surjective and every fiber is irreducible of dimension 1. Therefore $\mathcal{F}_2$ is irreducible of dimension 8. Finally, consider:

$$
\begin{array}{ccc}
\mathcal{F}_3 & \xrightarrow{p_3} & \mathcal{F}_2 \\
\big([C], P_1, \ldots, P_6\big) & \mapsto & \big([C], P_1, P_2, P_3, P_4\big).
\end{array}
$$

Then the map p_3 is surjective and each fiber is naturally isomorphic to $A = \big\{ (P_5, P_6) \in C \times C \mid S \in l_{56}; \ P_5 \neq P_6 \big\}$. Consider:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
(P_5, P_6) & \mapsto & P_5.
\end{array}
$$

Then the map h is surjective and every fiber consists of one point. Therefore A is irreducible of dimension 1. This implies that $\mathcal{F}_3$ is irreducible of dimension 8+1=9. So is $K_4^{(4)}$ since the projection $\mathcal{F}_3 \longrightarrow K_4^{(4)}$ is an isomorphism.

Next, we prove that $H_4^{(6)}$ is irreducible of dimension 16. The proof is analogous to that of the proof of (2.4.1) when $K_4^{(4)}$ is substituted by $K_4^{(6)}$. Also, for closedness of $H_4^{(6)}$, see the proof of (2.4.5). $\qquad\square$

Proposition 2.4.4. *Each $X \in H_4^{(6)}$ has at least 6 star triples, which form 4 star-Steiner sets and each star triple has exactly one line in common with another star triple among the 6 ones above. Hence $H_4^{(6)} \subset H_6$.*

Proof. By (2.4.2), we can assume that X possesses a pair $\{S, U\}$, where $S = \big\{ T_1 = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), T_2 = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), \ T_1 T_2 = (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56}) \big\}$ and $U = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13})$.

It is easy to see that X has 2 more star triples, namely: $T_2 U = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14})$ and $U(T_1 T_2) = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24})$. Moreover, they form 4 star-Steiner sets, namely: $\{T_1, T_2, T_1 T_2\}$, $\{T_2, U, U T_2\}$, $\{U, T_1 T_2, U(T_1 T_2)\}$ and $\{T_1, U T_2, U(T_1 T_2)\}$. $\qquad\square$

C. A study of $H_4^{(9)}$

Recall that $H_4^{(9)}$ is the set of points corresponding to non-singular cubic surfaces such that each surface has a pair $\{S, U\}$, where S is a star-Steiner set and U is another star triple having no line in common with S. We can assume that each cubic surface X corresponding to an element of $H_4^{(9)}$ has the star-Steiner set $S = \{(\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56}), (\tilde{l}_{15} \tilde{l}_{24} \tilde{l}_{36}), (\tilde{l}_{45} \tilde{l}_{26} \tilde{l}_{13})\}$. This implies that U has the form $(\tilde{P}_i \tilde{C}_j \tilde{l}_{ij})$. Furthermore, we can assume that $U = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14})$ by using a suitable permutation. For this,

- $(41)(23)$ if $i = 4, j = 1$;

- (164) if $i = 4, j = 6$ or $i = 6, j = 1$;

- $(kh)(j4)(i1)$ for $k, h \in \{1, \ldots, 6\} - \{i, j, 1, 4\}$ in others.

Definition:

$$
K_4^{(9)} = \Big\{ (P_1, \ldots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{S_1\}; \ l_{15} \cap l_{24} \cap l_{36} = \{S_2\};
$$

$$
l_{14} \text{ is tangent to } C_1 \text{ at } P_4 \Big\}.
$$

Let $D_4^{(9)} = \Theta^{-1}(K_4^{(9)})$ and $(L_4)^{(9)} = \Gamma(D_4^{(9)})$. By the above argument, we have $H_4^{(9)} = p(L_4^{(9)})$.

Theorem 2.4.5. *The subset $H_4^{(9)} \subset H_4$ is closed in $\mathbb{P}^{19} - \Delta$ and has two irreducible components of dimension 16.*

Proof. We consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ \Gamma\ } & \mathcal{L} \\
\Theta \downarrow & & p \downarrow \\
\Phi & & \mathbb{P}^{19} - \Delta.
\end{array}
$$

This induces

$$
\begin{array}{ccc}
D_4^{(9)} & \xrightarrow{\ \Gamma\ } & L_4^{(9)} \\
\Theta \downarrow & & p \downarrow \\
K_4^{(9)} & & H_4^{(9)}.
\end{array}
$$

We shall prove that $K_4^{(9)}$ has two irreducible components of dimension 9. Then this implies that the $H_4^{(9)}$ has two irreducible components of dimension 16 (see, for example, the proof of (2.4.1)).

Let

$$
\mathcal{F}_1 = \left\{ (P_1, P_2, P_3, P_4) \in (\mathbb{P}^2)^4 \mid P_i \neq P_j \ \forall i \neq j; \ P_i \notin l_{jk} \text{ for } 1 \leq i, j, k \leq 4 \right\}.
$$

$$
\mathcal{F}_2 = \left\{ (P_1, P_2, P_3, P_4, S) \mid (P_1, P_2, P_3, P_4) \in \mathcal{F}_1; \ S \in l_{24}; \ S \notin l_{23} \cup l_{14} \cup l_{13} \right\} \subset \mathcal{F}_1 \times \mathbb{P}^2.
$$

$$
\mathcal{F}_3 = \Big\{ (P_1, P_2, P_3, P_4, P_5, P_6, S) \mid (P_1, P_2, P_3, P_4, S) \in \mathcal{F}_2; \ (P_1, \ldots, P_6) \in \Phi;
$$

$$
l_{36} \cap l_{15} \cap l_{24} = \{S\}; \ l_{14} \text{ is tangent to } C_1 \text{ at } P_4 \Big\} \subset \mathcal{F}_2 \times (\mathbb{P}^2)^2.
$$

First of all, we see that the set $\mathcal{F}_1$ is an open subset of $(\mathbb{P}^2)^4$. Therefore it irreducible of dimension 8. Consider

$$
\begin{array}{ccc}
\mathcal{F}_2 & \xrightarrow{\ p_1\ } & \mathcal{F}_1 \\
(P_1, P_2, P_3, P_4, S) & \mapsto & (P_1, P_2, P_3, P_4).
\end{array}
$$

The map p_1 is surjective and every fiber is irreducible of dimension 1. This implies that $\mathcal{F}_1$ is irreducible of dimension $8 + 1 = 9$. Now, consider

$$
\begin{array}{ccc}
\mathcal{F}_3 & \xrightarrow{\ p_2\ } & \mathcal{F}_2 \\
(P_1, \ldots, P_6, S) & \mapsto & (P_1, P_2, P_3, P_4, S).
\end{array}
$$

We prove that this map is $2 : 1$ and the set $\mathcal{F}_3$ consists of two components, each of them is isomorphic to an open subset of $\mathcal{F}_2$. Choose a system of coordinates of $\mathbb{P}^2$ such that $P_1 = (-1 : 0 : 1), P_2 = (1 : 1 : 0), P_3 = (-1 : 1 : 0), P_4 = (0 : 0 : 1)$ and $S = (b : b : 1)$ for $b \in k \cup \{\infty\}$. By a straight forward computation we easily prove that

except for a finite number of values of b, the fiber of p_2 at a given point consists of two distinct points. Moreover, the set $\mathcal{F}_3$ is isomorphic to the set $K_4^{(9)}$.

Finally, we have to show that the subsets $H_4^{(4)}$, $H_4^{(6)}$ and $H_4^{(9)}$ are closed in $\mathbb{P}^{19} - \Delta$. Repeat the argument and notations used in the proof of the closedness of H_1 and H_2 in (2.3.1). Consider the following sets:

$$G_4^{(4)} = \big\{ (x, l_1, \dots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset; \ l_1 \cap l_4 \cap l_5 \neq \emptyset; \ l_2 \cap l_6 \cap l_7 \neq \emptyset \big\},$$

$$G_4^{(6)} = \big\{ (x, l_1, \dots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset; \ l_1 \cap l_4 \cap l_5 \neq \emptyset; \ l_6 \cap l_7 \cap l_8 \neq \emptyset \big\},$$

and

$$G_4^{(9)} = \big\{ (x, l_1, \dots, l_{27}) \in \mathcal{M} \mid l_1 \cap l_2 \cap l_3 \neq \emptyset; \ l_4 \cap l_5 \cap l_6 \neq \emptyset; \ l_7 \cap l_8 \cap l_9 \neq \emptyset \big\}.$$

We have

$$G_4^{(4)} = f_{123}^{-1}(L) \cap f_{145}^{-1}(L) \cap f_{267}^{-1}(L) \underset{\text{closed}}{\hookrightarrow} \mathcal{M},$$

$$G_4^{(6)} = f_{123}^{-1}(L) \cap f_{145}^{-1}(L) \cap f_{678}^{-1}(L) \underset{\text{closed}}{\hookrightarrow} \mathcal{M},$$

and

$$G_4^{(9)} = f_{123}^{-1}(L) \cap f_{456}^{-1}(L) \cap f_{789}^{-1}(L) \underset{\text{closed}}{\hookrightarrow} \mathcal{M}.$$

In particular the subsets $H_4^{(4)} = p(G_4^{(4)})$, $H_4^{(6)} = p(G_4^{(6)})$ and $H_4^{(9)} = p(G_4^{(9)})$ are closed in $\mathbb{P}^{19} - \Delta$. $\qquad\qquad\square$

Proposition 2.4.6. *Each cubic surface corresponding to an element of $H_4^{(9)}$ contains at least 9 star points and at least 12 star-Steiner sets.*

Proof. We can assume that X is isomorphic to the blowing up of $\mathbb{P}^2$ at $\mathcal{P} = (P_1, \dots, P_6) \in K_4^{(9)}$. We compute all star triples "generated" by the given star triples $(\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14})$, $(\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$ and $(\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36})$ as follows:

(1)	$(\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14})$,	**(6)**	$(\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25})$,
(2)	$(\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$,	**(7)**	$(\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16})$,
(3)	$(\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36})$,	**(8)**	$(\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46})$,
(4)	$(\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23})$,	**(9)**	$(\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35})$
(5)	$(\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45})$,		

and they form 12 following star-Steiner sets:

(1) $\big\{ (\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14}), \ (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56}), \ (\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23}) \big\}$; **(7)** $\big\{ (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56}), \ (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46}), \ (\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35}) \big\}$;

(2) $\big\{ (\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14}), \ (\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36}), \ (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25}) \big\}$; **(8)** $\big\{ (\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36}), \ (\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35}), \ (\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16}) \big\}$;

(3) $\big\{ (\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14}), \ (\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35}), \ (\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45}) \big\}$; **(9)** $\big\{ (\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36}), \ (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46}), \ (\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23}) \big\}$;

(4) $\big\{ (\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14}), \ (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46}), \ (\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16}) \big\}$; **(10)** $\big\{ (\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45}), \ (\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23}), \ (\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16}) \big\}$;

(5) $\big\{ (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56}), \ (\tilde{l}_{24} \tilde{l}_{15} \tilde{l}_{36}), \ (\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45}) \big\}$; **(11)** $\big\{ (\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45}), \ (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25}), \ (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46}) \big\}$;

(6) $\big\{ (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56}), \ (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25}), \ (\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16}) \big\}$; **(12)** $\big\{ (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25}), \ (\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23}), \ (\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35}) \big\}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.4.7. The above nine star triples contain all the 27 lines of the surface. Each star triple occurs in 4 of the above 12 star-Steiner sets.

We have $\dim H_2 = \dim H_2^{(2)} = \dim H_2^{(3)} = 17$ and $\dim H_4 = 16$. We conclude:

Corollary 2.4.8. *The subset $H_2^{(2)}$ respectively $H_2^{(3)}$ generically consists of point corresponding to cubic surfaces with exactly 2, respectively 3, star points.*

2.5   A study of H_5 and H_6

Theorem 2.5.1. $H_5 = H_6 = H_4^{(6)} \cup H_4^{(9)}$.

Proof. The inclusion $H_4^{(6)} \cup H_4^{(9)} \subset H_6 \subset H_5$ is trivial by (2.4.4) and (2.4.6). The proof is done if we show that $H_5 \subset H_4^{(6)} \cup H_4^{(9)}$. Let $x \in H_5$. Since $H_5 \subset H_4 = H_4^{(4)} \cup H_4^{(6)} \cup H_4^{(9)}$, we need only consider the case $x \in H_4^{(4)}$. This means that the corresponding surface X_x has a pair (S, T) where S is a star-Steiner set and T is another star triple which has all lines in common with S. Let U be the fifth star triple of X_x. If U has no line or only one line in common with S then $x \in H_4^{(6)} \cup H_4^{(9)}$. Consider the case U has all lines in common with S. We can assume that

$$S = \left\{ (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}),\ (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}),\ (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}) \right\},$$

$T = (\tilde{C}_1 \tilde{P}_4 \tilde{l}_{14})$ and $U = (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$. Note that at the same time the triple $(\tilde{C}_3 \tilde{P}_2 \tilde{l}_{23})$ forms another star point. Our task is now to prove that the three lines $\tilde{l}_{24}$, $\tilde{l}_{35}$ and $\tilde{l}_{16}$ form a star triple, hence x belongs to $H_4^{(9)}$. Choose a system of coordinates for $\mathbb{P}^2$ such that $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 := (1 : 1 : 1)$ and $P_4 = (0 : 0 : 1)$. So the conic containing P_2, P_3, P_4 and being tangent to l_{12} and l_{14} at P_2 and P_4 respectively is given by $x_0^2 - x_1 x_2 = 0$. Moreover, we have $S_0 = (1 : 1 : 0) \in l_{12} \cap l_{34}$, $S_1 = (1 : 0 : 1) \in l_{14} \cap l_{23}$. It is easy to see that $l_{24} = V(x_0)$ and $\overline{S_0 S_1} = V(x_1 + x_2 - x_0)$. Therefore either

$$P_5 = (1 : \epsilon : -\epsilon^2) \text{ and } P_6 = (1 : -\epsilon^2 : \epsilon)$$

or

$$P_6 = (1 : \epsilon : -\epsilon^2) \text{ and } P_5 = (1 : -\epsilon^2 : \epsilon),$$

where ϵ is a primitive cube root of unit. A computation shows that the three lines l_{24}, l_{35} and l_{16} have one point in common. $\qquad\square$

Remark 2.5.2. Later, we will see in (2.7.2) that, in this case, the surface has exactly 18 star points.

2.6   A study of H_7, H_8 and H_9

Recall that H_7, H_8 and H_9 are the subsets corresponding to non-singular cubic surfaces with at least 7, 8 and 9 star points, respectively.

Lemma 2.6.1. *Let $x \in H_4^{(6)}$ and let $T_1, \ldots, T_6$ be the six star triples of X_x determined by the given pair (S, U) as in the proof of (2.4.4). Let V be another star triple of X_x.*

(i) *If V has all line in common with one of 4 star-Steiner sets determined by $\{T_1, \ldots, T_6\}$ then X_x has at least 10 star points and at least 10 star-Steiner sets.*

(ii) *Otherwise, the surface X_x has at least 18 star points and at least 42 star-Steiner sets.*

Proof.

(i) We can assume that:

$$S = \left\{ (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), \ (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), \ (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}) \right\}$$

and $U = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13})$. Then V is one of the triples $(\tilde{P}_2 \tilde{C}_3 \tilde{l}_{23})$ and $(\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$, up to permutations. It is easy to find all star triples determined by S, U and V.

(a) If $V = (\tilde{P}_2 \tilde{C}_3 \tilde{l}_{23})$ then:

$$
\begin{array}{llll}
(1)\, T_1 & = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), & (2)\ T_2 & = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), \\
(3)\, T_1 T_2 & = (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}), & (4)\ U & = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \\
(5)\, V & = (\tilde{P}_2, \tilde{C}_3 \tilde{l}_{23}), & (6)\ T_2 U & = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14}), \\
(7)\, U(T_1 T_2) & = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), & (8)\ V(T_2 U) & = (\tilde{l}_{13} \tilde{l}_{24} \tilde{l}_{56}), \\
(9)\, V(U(T_1 T_2)) & = (\tilde{C}_4 \tilde{P}_3 \tilde{l}_{34}), & (10)\ UV & = (\tilde{C}_2 \tilde{P}_1 \tilde{l}_{12}).
\end{array}
$$

(b) If $V = (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$ then:

$$
\begin{array}{llll}
(1)\, T_1 & = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), & (2)\ T_2 & = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), \\
(3)\, T_1 T_2 & = (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}), & (4)\ U & = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \\
(5)\, V & = (\tilde{P}_2, \tilde{C}_3 \tilde{l}_{23}), & (6)\ T_2 U & = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14}), \\
(7)\, U(T_1 T_2) & = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), & (8)\ V(T_2 U) & = (\tilde{C}_2 \tilde{P}_3 \tilde{l}_{23}), \\
(9)\, V(U(T_1 T_2)) & = (\tilde{C}_3 \tilde{P}_1 \tilde{l}_{13}), & (10)\ UV & = (\tilde{C}_4 \tilde{P}_2 \tilde{l}_{24}).
\end{array}
$$

Moreover, we can see that, in fact they are of the same kind. For this, in the case (a), we choose $S' = \left\{ (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \ (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), \ (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}) \right\}$, $U' = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12})$ and $V' = (\tilde{l}_{13} \tilde{l}_{24} \tilde{l}_{56})$. Then $\left\{ S', U', V' \right\}$ shows that X has the form of (b). Similarly, in the case (b), we choose $S' = \left\{ (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \ (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), \ (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}) \right\}$, $U' = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12})$ and $V' = (\tilde{C}_2 \tilde{P}_3 \tilde{l}_{23})$. Then $\left\{ S', U', V' \right\}$ shows that X has the form as in (a).

The 10 star triples of X as in (a) form 10 star-Steiner sets. They are as follows:

$$
\begin{array}{lll}
1.\ \left\{ (1),\, (2),\, (3) \right\} & 2.\ \left\{ (7),\, (9),\, (5) \right\} & 3.\ \left\{ (1),\, (7),\, (6) \right\} \\
4.\ \left\{ (4),\, (7),\, (3) \right\} & 5.\ \left\{ (4),\, (2),\, (6) \right\} & 6.\ \left\{ (1),\, (9),\, (8) \right\} \\
7.\ \left\{ (6),\, (8),\, (5) \right\} & 8.\ \left\{ (2),\, (8),\, (10) \right\} & 9.\ \left\{ (4),\, (5),\, (10) \right\} \\
10.\ \left\{ (18),\, (9),\, (10) \right\}
\end{array}
$$

where (n) denotes the star triple numbered (n) in the list (a) of star triples of X_x.

(ii) Recall that, the six star triples of X_x determined by $\{S, U\}$ form 4 star-Steiner sets:

- $\{T_1, T_2, T_1 T_2\} = S$;
- $\{U, T_2, U T_2\}$;
- $\{T_1, U T_2, T_1 (U T_2)\}$;
- $\{U, T_1 T_2, T_1 (U T_2)\}$.

Therefore, if V does not satisfy the hypothesis of (i) then V has no line in common with one of the above star-Steiner sets. We can assume (S, U) as in (i) and V has no line in common with S. Furthermore, we can assume that V has no line in common with U, otherwise, we can choose $S' = S$, $U' = T_2 U$ and $V' = V$, then V' has no line in common with U'. Therefore, the star triple V is one of the forms $(\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25})$, $(\tilde{C}_4 \tilde{P}_5 \tilde{l}_{45})$, $(\tilde{C}_5 \tilde{P}_1 \tilde{l}_{15})$ and $(\tilde{l}_{a5} \tilde{l}_{24} \tilde{l}_{36})$. (Note that there are four other cases for V when P_5 is substituted by P_6 in the above list, but it is easy to see that they just differ from each other by the permutation (56), which does not change S or U).

It is not hard to find all star triples determined by $\{S, U, V\}$ for each case of V. Namely,

(a) When $V = (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25})$ or $V = (\tilde{C}_5 \tilde{P}_1 \tilde{l}_{15})$, then the 18 star triples generated by $\{S, U, V\}$ are:

$$
\begin{array}{llll}
(1) & = (\tilde{C}_6 \tilde{P}_3 \tilde{l}_{36}), & (2) & = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), \\
(3) & = (\tilde{l}_{15} \tilde{l}_{23} \tilde{l}_{46}), & (4) & = (\tilde{l}_{25} \tilde{l}_{16} \tilde{l}_{34}), \\
(5) & = (\tilde{C}_3 \tilde{P}_5 \tilde{l}_{35}), & (6) & = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14}), \\
(7) & = (\tilde{l}_{13} \tilde{l}_{26} \tilde{l}_{45}), & (8) & = (\tilde{C}_5 \tilde{P}_1 \tilde{l}_{15}), \\
(9) & = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), & (10) & = (\tilde{l}_{12} \tilde{l}_{36} \tilde{l}_{45}), \\
(11) & = (\tilde{C}_2 \tilde{P}_5 \tilde{l}_{25}), & (12) & = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \\
(13) & = (\tilde{C}_6 \tilde{P}_2 \tilde{l}_{26}), & (14) & = (\tilde{C}_5 \tilde{P}_6 \tilde{l}_{56}), \\
(15) & = (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46}), & (16) & = (\tilde{l}_{16} \tilde{l}_{24} \tilde{l}_{35}), \\
(17) & = (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56}), & (18) & = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}).
\end{array}
$$

(b) When $V = (\tilde{C}_4 \tilde{P}_5 \tilde{l}_{45})$, or $V = (\tilde{l}_{15} \tilde{l}_{24} \tilde{l}_{36})$, then the 18 star triples generated by $\{S, U, V\}$ are:

$$
\begin{array}{llll}
(1) & = (\tilde{C}_3 \tilde{P}_6 \tilde{l}_{36}), & (2) & = (\tilde{l}_{12} \tilde{l}_{35} \tilde{l}_{46}), \\
(3) & = (\tilde{C}_4 \tilde{P}_5 \tilde{l}_{45}), & (4) & = (\tilde{l}_{15} \tilde{l}_{24} \tilde{l}_{36}), \\
(5) & = (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}), & (6) & = (\tilde{l}_{13} \tilde{l}_{25} \tilde{l}_{46}), \\
(7) & = (\tilde{C}_2 \tilde{P}_4 \tilde{l}_{24}), & (8) & = (\tilde{C}_6 \tilde{P}_5 \tilde{l}_{56}), \\
(9) & = (\tilde{C}_5 \tilde{P}_2 \tilde{l}_{25}), & (10) & = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13}), \\
(11) & = (\tilde{C}_5 \tilde{P}_3 \tilde{l}_{35}), & (12) & = (\tilde{l}_{15} \tilde{l}_{26} \tilde{l}_{34}), \\
(13) & = (\tilde{C}_4 \tilde{P}_1 \tilde{l}_{14}), & (14) & = (\tilde{l}_{14} \tilde{l}_{23} \tilde{l}_{56}), \\
(15) & = (\tilde{C}_2 \tilde{P}_6 \tilde{l}_{26}), & (16) & = (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}), \\
(17) & = (\tilde{C}_6 \tilde{P}_1 \tilde{l}_{16}), & (18) & = (\tilde{l}_{16} \tilde{l}_{23} \tilde{l}_{45}).
\end{array}
$$

And we can see, in fact, they are of the same kind. For this, in the case of (a), we can choose $S' = S$, $U' = U$ and $V' = (\tilde{C}_4 \tilde{P}_6 \tilde{l}_{46})$. Then $\{S', U', V'\}$ shows that X_x has the form (b). Similarly, in the case (b), we choose $S' = S$, $U' = U$ and $V' = (\tilde{C}_2 \tilde{P}_6 \tilde{l}_{26})$. Then $\{S', U', V'\}$ shows that X_x has the form of (a). The surface X_x in case (a) has at least (but in fact exactly!) 42 star-Steiner sets. There are 3 star-Steiner sets consisting of all the 27 lines of X_x, namely:

$$
\begin{array}{llll}
& (12) & (13) & (10) \\
(18): & \tilde{C}_1 & \tilde{P}_2 & \tilde{l}_{12} \\
(1): & \tilde{P}_3 & \tilde{C}_6 & \tilde{l}_{36} \\
(7): & \tilde{l}_{13} & \tilde{l}_{26} & \tilde{l}_{45}
\end{array}
\qquad
\begin{array}{llll}
& (2) & (5) & (4) \\
(11): & \tilde{C}_2 & \tilde{P}_5 & \tilde{l}_{25} \\
(9): & \tilde{P}_4 & \tilde{C}_3 & \tilde{l}_{34} \\
(16): & \tilde{l}_{24} & \tilde{l}_{35} & \tilde{l}_{16}
\end{array}
$$

$$
\begin{array}{llll}
& (6) & (14) & (3) \\
(15): & \tilde{C}_4 & \tilde{P}_6 & \tilde{l}_{46} \\
(8): & \tilde{P}_1 & \tilde{C}_5 & \tilde{l}_{15} \\
(17): & \tilde{l}_{14} & \tilde{l}_{56} & \tilde{l}_{23}.
\end{array}
$$

There are 6 star-Steiner sets from the above list, see 2.1.2 (iii). Others are:

1. $\{(2),\ (15),\ (13)\}$ 2. $\{(17),\ (10),\ (4)\}$
3. $\{(11),\ (7),\ (15)\}$ 4. $\{(18),\ (6),\ (2)\}$
5. $\{(6),\ (16),\ (13)\}$ 6. $\{(3),\ (7),\ (4)\}$
7. $\{(18),\ (14),\ (4)\}$ 8. $\{(18),\ (3),\ (5)\}$
9. $\{(9),\ (3),\ (13)\}$ 10. $\{(18),\ (16),\ (15)\}$
11. $\{(5),\ (10),\ (15)\}$ 12. $\{(12),\ (16),\ (14)\}$
13. $\{(8),\ (16),\ (1)\}$ 14. $\{(8),\ (13),\ (4)\}$
15. $\{(17),\ (5),\ (13)\}$ 16. $\{(3),\ (2),\ (1)\}$
17. $\{(6),\ (7),\ (5)\}$ 18. $\{(18),\ (9),\ (17)\}$
19. $\{(8),\ (2),\ (10)\}$ 20. $\{(11),\ (17),\ (1)\}$
21. $\{(9),\ (12),\ (6)\}$ 22. $\{(12),\ (11),\ (3)\}$
23. $\{(17),\ (16),\ (7)\}$ 24. $\{(5),\ (1),\ (14)\}$
25. $\{(9),\ (10),\ (14)\}$ 26. $\{(18),\ (11),\ (8)\}$
27. $\{(12),\ (17),\ (2)\}$ 28. $\{(2),\ (7),\ (14)\}$
29. $\{(16),\ (3),\ (10)\}$ 30. $\{(9),\ (1),\ (15)\}$
31. $\{(12),\ (15),\ (4)\}$ 32. $\{(11),\ (14),\ (13)\}$
33. $\{(12),\ (8),\ (5)\}$ 34. $\{(11),\ (6),\ (10)\}$
35. $\{(9),\ (8),\ (7)\}$ 36. $\{(6),\ (1),\ (4)\}$

$\square$

Definition: Let $H_{10}^{(10)}$ and $H_{10}^{(18)}$ denote the subsets of $H_4^{(6)}$ consisting of all points as in the cases (i) and (ii) respectively of Lemma 2.6.1.

Remark 2.6.2.

(i) The 10 star triples of each cubic surface X corresponding to an element of $H_{10}^{(10)}$ determined by $\{S,\ U,\ V\}$ as in the lemma consist of 15 lines during 27 lines of X. Each star triple occurs in two star-Steiner sets of the 10 star-Steiner sets.

(ii) The 18 star triples of each cubic surface X corresponding to an element of $H_{10}^{(18)}$ determined by $\{S,\ U,\ V\}$ as in the lemma consist of 27 lines of X. Each star triple occurs in seven star-Steiner sets of the 42 star-Steiner sets.

Corollary 2.6.3. $H_7 = H_8 = H_9 = H_{10}^{(10)} \cup H_4^{(9)}$.

Proof. First of all, we have $H_9 \subset H_8 \subset H_7 \subset H_6 = H_4^{(6)} \cup H_4^{(9)}$, see (2.5.1). Therefore $H_7 = (H_7 \cap H_4^{(6)}) \bigcup (H_7 \cap H_4^{(9)}) = (H_7 \cap H_4^{(6)}) \bigcup H_4^{(9)}$. On the other hand, it follows from (2.6.1) that $(H_7 \cap H_4^{(6)}) = H_{10}^{(10)} \cup H_{10}^{(18)}$. Moreover, $H_{10}^{(18)} \subset H_4^{(9)}$. The inclusion $H_{10}^{(10)} \cup H_4^{(9)} \subset H_9$ is clear. $\qquad\square$

Theorem 2.6.4. *The set $H_{10}^{(10)}$ is closed in $\mathbb{P}^{19} - \Delta$, irreducible of dimension 15.*

Proof. Let $x \in H_{10}^{(10)}$. By (2.6.1), we can assume that X_x has a set $\{S, U, V\}$, where $S = \big\{ (\tilde{C}_1 \tilde{P}_2 \tilde{l}_{12}),\ (\tilde{C}_3 \tilde{P}_4 \tilde{l}_{34}),\ (\tilde{l}_{23} \tilde{l}_{14} \tilde{l}_{56}) \big\}$ is a star-Steiner set, $U = (\tilde{C}_1 \tilde{P}_3 \tilde{l}_{13})$ and $V = (\tilde{l}_{12} \tilde{l}_{34} \tilde{l}_{56})$ are star triples.

Let

$$K_{10}^{(10)} = \Big\{ (P_1, \dots, P_6) \in \Phi \mid l_{12} \cap l_{34} \cap l_{56} = \{S_2\};\ l_{14} \cap l_{23} \cap l_{56} = \{S_1\};$$

$$l_{12}\ \text{and}\ l_{13}\ \text{is tangent to}\ C_1 \Big\},\ \text{see Figure 2.5.}$$
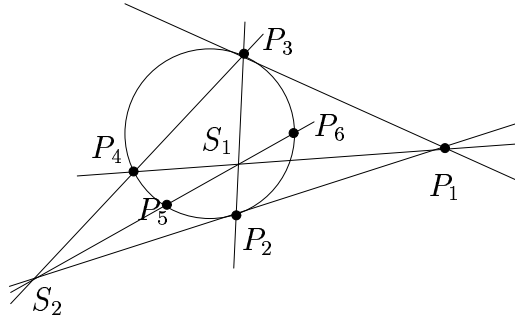

Figure 2.5: The configuration for members of $K_{10}^{(10)}$

Let $D_{10}^{(10)} = \Theta^{-1}(K_{10}^{(10)})$ and $L_{10}^{(10)} = \Gamma(D_{10}^{(10)})$. Therefore $H_{10}^{(10)} = p(L_{10}^{(10)})$. Consider the diagram:

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ \Gamma\ } & \mathcal{L} \\
{\scriptstyle \Theta}\big\downarrow & & {\scriptstyle p}\big\downarrow \\
\Phi & & \mathbb{P}^{19} - \Delta.
\end{array}
$$

This induces

$$
\begin{array}{ccc}
D_{10}^{(10)} & \xrightarrow{\ \Gamma\ } & L_{10}^{(10)} \\[4pt]
{\scriptstyle \Theta}\downarrow & & {\scriptstyle p}\downarrow \\[4pt]
K_{10}^{(10)} & & H_{10}^{(10)}.
\end{array}
$$

Consider

$$
\mathcal{F}_1 = \Big\{ \big([C], P_1, P_2, P_3\big) \mid C \text{ is a non-singular conic in } \mathbb{P}^2;\ P_1 \notin C;
$$

$$
P_2, P_3 \in C;\ P_2 \neq P_3;\ l_{12}, l_{13} \text{ are tangent to } C \Big\} \subset \mathcal{C} \times (\mathbb{P}^2)^3,
$$

$$
\mathcal{F}_2 = \Big\{ \big(S_1, [C], P_1, P_2, P_3\big) \mid \big([C], P_1, P_2, P_3\big) \in \mathcal{F}_1;\ S_1 \in l_{23};\ S_1 \notin \{P_2, P_3\} \Big\} \subset \mathcal{F}_1 \times \mathbb{P}^2,
$$

$$
\mathcal{F}_3 = \Big\{ \big(S_1, [C], P_1, P_2, P_3, P_4, P_5, P_6\big) \mid \big(S_1, [C], P_1, P_2, P_3\big) \in \mathcal{F}_2;\ \{P_4\} = \overline{P_1 S_1} \cap C;
$$

$$
P_5, P_6 \in C;\ l_{56} \cap l_{23} = \{S_1\};\ l_{12} \cap l_{34} \cap l_{56} \neq \emptyset;\ (P_1, \dots, P_6) \in \Phi \Big\} \subset \mathcal{F}_2 \times (\mathbb{P}^2)^3.
$$

Consider

$$
\begin{array}{ccc}
\mathcal{F}_1 & \xrightarrow{\ p_1\ } & \mathcal{C} \\
\big([C], P_1, P_2, P_3\big) & \mapsto & [C].
\end{array}
$$

The map p_1 is surjective and every fiber is irreducible of dimension 2. This implies that $\mathcal{F}_1$ is irreducible of dimension $5 + 2 = 7$. Similarly, the map:

$$
\begin{array}{ccc}
\mathcal{F}_2 & \xrightarrow{\ p_2\ } & \mathcal{F}_1 \\
\big(S_1, [C], P_1, P_2, P_3\big) & \mapsto & \big([C], P_1, P_2, P_3\big)
\end{array}
$$

is surjective and every fiber is irreducible of dimension 1. Therefore $\mathcal{F}_2$ is irreducible of dimension 8. Finally, consider:

$$
\begin{array}{ccc}
\mathcal{F}_3 & \xrightarrow{\ p_3\ } & \mathcal{F}_2 \\
\big(S_1, [C], P_1, \dots, P_6\big) & \mapsto & \big(S_1, [C], P_1, P_2, P_3\big).
\end{array}
$$

It is easy to see that the fiber of $\big(S_1, [C], P_1, P_2, P_3\big)$ consists of one pair determined by:

$$
\{P_4\} = \overline{P_1 S_1} \cap C,
$$

$$
\{S_2\} = l_{34} \cap l_{12},
$$

$$
\{P_5, P_6\} = \overline{S_1 S_2} \cap C,
$$

with a remark that the 2 points of the pair $\{P_5, P_6\}$ can be interchanged. Consequently, the set $K_{10}^{(10)}$ has dimension 8. Use the same argument as in (2.4.1), we see that $H_{10}^{(10)}$ is irreducible of dimension 15. By the same argument used in (2.3.1), we see that $H_{10}^{(10)}$ is closed in $\mathbb{P}^{19} - \Delta$. $\qquad\square$

Similarly, we can assume that for each $x \in H_{10}^{(18)}$, the surface X_x is isomorphic to the blowing-up of $\mathbb{P}^2$ at one element of the following set

$$K_{10}^{(18)} = \Big\{ (P_1, \dots, P_6) \in \Phi \mid l_{15} \cap l_{24} \cap l_{36} = \{S_1\}; \ l_{14} \cap l_{23} \cap l_{56} = \{S_2\};$$

$$l_{12} \text{ and } l_{13} \text{ is tangent to } C_1 \Big\}.$$

Let $D_{10}^{(18)} = \Theta^{-1}(K_{10}^{(18)})$ and $L_{10}^{(18)} = \Gamma(D_{10}^{(18)})$. Therefore $H_{10}^{(18)} = p(L_{10}^{(18)})$.

Theorem 2.6.5. *The set $H_{10}^{(18)}$ is closed in $\mathbb{P}^{19} - \Delta$ and has two irreducible components of dimension 15.*

Proof. It is sufficient to prove that $K_{10}^{(18)}$ has two irreducible components of dimension 8. Consider

$$\mathcal{F}_1 = \Big\{ ([C], P_1, P_2, P_3) \mid C \text{ is a non-singular conic in } \mathbb{P}^2; P_1 \notin C;$$

$$P_2, P_3 \in C; \ P_2 \neq P_3; \ l_{12}, l_{13} \text{ are tangent to } C \Big\} \subset \mathcal{C} \times (\mathbb{P}^2)^3,$$

$$\mathcal{F}_2 = \Big\{ (S, [C], P_1, P_2, P_3) \mid ([C], P_1, P_2, P_3) \in \mathcal{F}_1; \ S \in l_{23}; \ S \notin \{P_2, P_3\} \Big\} \subset \mathcal{F}_1 \times \mathbb{P}^2$$

and

$$\mathcal{F}_3 = \Big\{ (S, [C], P_1, P_2, P_3, P_4, P_5, P_6) \mid (S, [C], P_1, P_2, P_3) \in \mathcal{F}_2; \ \{P_4\} = \overline{P_1 S} \cap C;$$

$$P_5, P_6 \in C; \ l_{56} \cap l_{23} = \{S\}; \ l_{15} \cap l_{24} \cap l_{36} \neq \emptyset; \ (P_1, \dots, P_6) \in \Phi \Big\} \subset \mathcal{F}_2 \times (\mathbb{P}^2)^3.$$

Consider

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{C} \\ ([C], P_1, P_2, P_3) & \mapsto & [C]. \end{array}$$

The map p_1 is surjective and every fiber is irreducible of dimension 2. This implies that $\mathcal{F}_2$ is irreducible of dimension $5 + 2 = 7$. Similarly, the map

$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{p_2} & \mathcal{F}_1 \\ (S, [C], P_1, P_2, P_3) & \mapsto & ([C], P_1, P_2, P_3) \end{array}$$

is surjective and every fiber is irreducible of dimension 1. Therefore $\mathcal{F}_2$ is irreducible of dimension 8. Finally, consider

$$\begin{array}{ccc} \mathcal{F}_3 & \xrightarrow{p_3} & \mathcal{F}_2 \\ (S, [C], P_1, \dots, P_6) & \mapsto & (S, [C], P_1, P_2, P_3). \end{array}$$

To see that any fiber of $(S, [C], P_1, P_2, P_3)$ consists of two points, we choose coordinates of $\mathbb{P}^2$ such that $P_1 = (0 : 1 : 0)$, $P_2 = (-1 : 0 : 1)$, $P_3 = (1 : 0 : 1)$ and $P_4 = (0 : 1 : 1)$. Therefore the conic containing P_2, P_3 and P_4, being tangent to l_{12} and l_{13} at P_2 and P_3 respectively, is determined by $X^2 + Y^2 - Z^2 = 0$. Then either $P_5 = (1 - \epsilon^2 : 2\epsilon : 1 + \epsilon^2)$, $P_6 = (\epsilon^2 - 1 : -2\epsilon : 1 + \epsilon^2)$ or $P_5 = (1 + \epsilon : -2\epsilon^2 : 1 - \epsilon)$, $P_6 = (-\epsilon - 1 : 2\epsilon^2 : 1 - \epsilon)$, where ϵ is a primitive cube root of -1.

By the same argument used in (2.3.1), we see that $H_{10}^{(18)}$ is closed in $\mathbb{P}^{19} - \Delta$. $\square$

Proposition 2.6.6. $H_{10}^{(10)} \cap H_{10}^{(18)} = \emptyset$.

Proof. Let $x \in H_{10}^{(18)}$ and assume that X_x has at least 18 star triples as in page 35:

$$
\begin{array}{cccccccccc}
& (12) & (13) & (10) & & (2) & (5) & (4) & & (6) & (14) & (3) \\
(18): & \tilde{C}_1 & \tilde{P}_2 & \tilde{l}_{12} & (11): & \tilde{C}_2 & \tilde{P}_5 & \tilde{l}_{25} & (15): & \tilde{C}_4 & \tilde{P}_6 & \tilde{l}_{46} \\
(1): & \tilde{P}_3 & \tilde{C}_6 & \tilde{l}_{36} & (9): & \tilde{P}_4 & \tilde{C}_3 & \tilde{l}_{34} & (8): & \tilde{P}_1 & \tilde{C}_5 & \tilde{l}_{15} \\
(7): & \tilde{l}_{13} & \tilde{l}_{26} & \tilde{l}_{45} & (16): & \tilde{l}_{24} & \tilde{l}_{35} & \tilde{l}_{16} & (17): & \tilde{l}_{14} & \tilde{l}_{56} & \tilde{l}_{23}
\end{array}
$$

If $x \in H_{10}^{(10)}$ then X_x has a set $\{S, U, V\}$, where S is a star-Steiner set, U is a star triple with all lines in common with S and V is a star triple which has only one line l in common with S.

Note that, the above 18 star triples consist of all 27 lines of X_x and each line of X_x does not occur in more than two star triples (2.1.7). Therefore, the star triples in $\{S, U, V\}$ belong to the above 18 star triples. In particular, the pair $\{S, U\}$ forms one of the three above matrices. (This means that if three star triples of S are three rows of a matrix then U is one of the columns of that matrix. Similarly, if three star triples of S are three columns of a matrix then U is one of the rows of that matrix).

Since V has one line in common with S, the star triple V is one of the rows or the columns of the matrix formed by S and U. But V then has all lines in common with S. This is a contradiction. $\qquad\square$

Corollary 2.6.7. *Each cubic surface corresponding to an element of $H_{10}^{(10)}$ has exactly 10 star points.*

Proof. Let $x \in H_{10}^{(10)}$. Suppose that X_x has another star triple W which does not belong to the set L of ten star triples determined by a set $\{S, U, V\}$ as (2.6.2.(i)). The ten star triples of L consist of 15 lines, each of them occurs in exactly two star triples. Therefore, the three lines of W do not appear in the 15 lines of L. The set $\{S, U, W\}$ shows that x belongs to $H_{10}^{(18)}$. But by the previous proposition, we have $H_{10}^{(10)} \cap H_{10}^{(18)} = \emptyset$. $\qquad\square$

2.7   A study of H_k with $k \geq 10$

Theorem 2.7.1. $H_{10} = H_{10}^{(10)} \cup H_{10}^{(18)} = H_4^{(4)} \cap H_4^{(6)}$.

Proof.

(i) We prove that $H_{10} \subset H_{10}^{(10)} \cup H_{10}^{(18)}$.

Since $H_{10} \subset H_9 = H_{10}^{(10)} \cup H_4^{(9)}$ then $H_{10} = (H_{10} \cap H_{10}^{(10)}) \bigcup (H_{10} \cap H_4^{(9)}) = H_{10}^{(10)} \bigcup (H_{10} \cap H_4^{(9)})$. Consider $x \in (H_{10} \cap H_4^{(9)})$. Since $x \in H_4^{(9)}$, the surface X_x has at least 9 star triples, which contain all 27 lines of X_x, see (2.4.6). Therefore the tenth star triple W has all lines in common with three ones , say T_1, T_2 and T_3 of the above 9 star triples. There exists another star triple $T_4 \notin \{T_1, T_2, T_3\}$ such that $S = \{T_1, T_4, T_1 T_4\}$ and W show that x belongs to $H_4^{(6)}$. Then $\{S, W\}$ and T_2 show that x belongs to $H_{10}^{(18)}$.

(ii) The inclusion $(H_{10}^{(10)} \cup H_{10}^{(18)}) \subset (H_4^{(4)} \cap H_4^{(6)})$ is clear.

(iii) We prove that $(H_4^{(4)} \cap H_4^{(6)}) \subset H_{10}$. Recall that each cubic surface X corresponding to an element $x \in H_4^{(6)}$ has at least 6 star triples $\{T_1, \ldots, T_6\}$ and each star triple has exactly one line in common with another star triple in $\{T_1, \ldots, T_6\}$, see (2.4.4). Therefore, if $x \in H_4^{(4)}$ then the surface X has another star triple U, which does not belong to the set $\{T_1, \ldots, T_6\}$. The conclusion follows from (2.6.1).

$\square$

Corollary 2.7.2. *A non-singular cubic surface does not have more than* 18 *star points. Consequently* $H_k = \emptyset$ *for* $k > 18$.

Proof. Let X be a cubic surface corresponding to an element $x \in H_{18}$. Then $x \in H_{10} = H_{10}^{(10)} \cup H_{10}^{(18)}$. Since $H_{10}^{(10)}$ consists of points corresponding to surfaces with exactly 10 star points (2.6.7), this implies that $x \in H_{10}^{(18)}$. We can assume that X has the set L consisting of 18 star points generated by a set $\{S, U, V\}$ as in (2.6.1). But as we have seen in the proof of (2.6.6), the set L consists of all 27 lines of X, each line occurs in two star triples. Therefore any star triple of X belongs to L. This means that X_x does not have more than 18 star points. $\square$

Corollary 2.7.3. $H_k = H_{10}^{(18)}$ *for* $10 < k \le 18$.

Proof. For any $10 < k \le 18$, $H_k \subset H_{10} = H_{10}^{(10)} \cup H_{10}^{(18)}$. Hence $H_k = (H_k \cap H_{10}^{(10)}) \bigcup (H_k \cap H_{10}^{(18)}) = (H_k \cap H_{10}^{(10)}) \bigcup H_{10}^{(18)}$. But $H_k \cap H_{10}^{(10)} = \emptyset$ by (2.6.7). $\square$

In the rest of this section, we want to study some other properties of $H_{18} = H_{10}^{(18)}$.

Proposition 2.7.4. $H_{18} = H_4^{(4)} \cap H_4^{(9)} = H_4^{(6)} \cap H_4^{(9)}$.

Proof.

(i) $H_{18} = H_4^{(4)} \cap H_4^{(9)}$.
The inclusion $H_{18} \subset H_4^{(4)} \cap H_4^{(9)}$ is clear. Suppose $x \in H_4^{(4)} \cap H_4^{(9)}$ then the surface X_x has a pair $\{S', U'\}$, where S' is a star-Steiner set and U' is another star triple with no line in common with S'. Moreover, the surface X_x has a pair $\{S, U\}$, where S is a star-Steiner set and U is another star triple which has all line in common with S. Let L be the set of 9 star triples generated by $\{S', U'\}$. Let $S = \{T_1, T_2, T_1 T_2\}$. Note that any two star triples in L have no line in common. So there exists one star triple V in L which does not have any line in common with S. Then the pair of star-Steiner set $S_1 = \{T_1, V, V T_1\}$ and S shows that x belongs to $H_4^{(6)}$. So the set $\{S_1, U, T_2\}$ or $\{S_1, U, T_2\}$ shows that x belongs to $H_{10}^{(18)} = H_{18}$.

(ii) $H_{18} = H_4^{(6)} \cap H_4^{(9)}$.
The inclusion $H_{18} \subset H_4^{(6)} \cap H_4^{(9)}$ is clear. Suppose that $x \in H_4^{(6)} \cap H_4^{(9)}$. Let L denote the same set as in the proof of (i). Moreover, since $x \in H_4^{(6)}$, the corresponding cubic surface X_x has a pair of star triples T_1 and T_2 such that they have one line in common. Both T_1 and T_2 do not belong to L by the configuration of L, see (2.4.6). This implies that T_1 has all lines in common with L. Therefore $x \in H_4^{(4)}$ and the result follows from (i).

□

From (2.5.1), (2.6.3), (2.6.4), (2.6.5), (2.7.1) and (2.7.4) we have the following results:

Corollary 2.7.5. *The sets $H_4^{(4)}$, $H_4^{(6)}$ and $H_4^{(9)}$ generically consist of points corresponding to cubic surfaces with exactly 4, 6 and respectively 9 star points.*
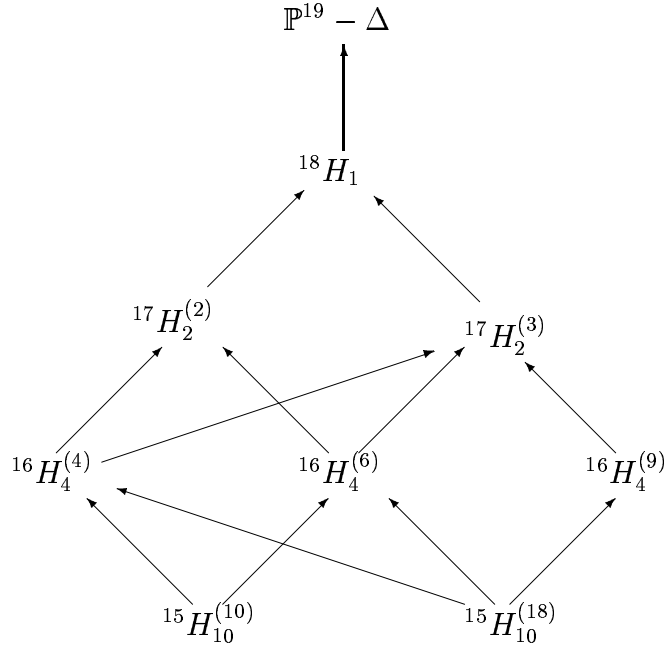
Figure 2.6: A diagram explaining properties of $H_k^{(m)}$.

We give a survey of the results obtained in Figure 2.6. In the diagram of the figure:

(i) the number n in the left top of the symbol $^n H_k^{(m)}$ denotes the dimension of $H_k^{(m)}$;

(ii) the vectors mean the inclusion relations;

(iii) the symbol (m) indicates that generically in the set $H_k^{(m)}$ the corresponding surface has exactly m star points, see (2.3.4), (2.4.8), (2.6.7), (2.6.1), (2.7.2) and (2.7.5).

Other main results are:

- H_1 is irreducible and $\dim H_1 = 18$;

- $H_2 = H_2^{(2)} \cup H_2^{(3)}$ and $\dim H_2 = 17$;

- $H_3 = H_2^{(3)}$ and $\dim H_3 = 17$ and H_3 is irreducible;

- $H_4 = H_4^{(4)} \cup H_4^{(6)} \cup H_4^{(9)}$ and $\dim H_4 = 16$;

- $H_5 = H_6 = H_4^{(6)} \cup H_4^{(9)}$ and $\dim H_k = 16$, for $k \in \{5, 6\}$;

- $H_7 = H_8 = H_9 = H_4^{(9)} \sqcup H_{10}^{(10)}$ and $\dim H_k = 16$, for $k \in \{7, 8, 9\}$ and the union is disjoint;

- $H_{10} = H_{10}^{(10)} \sqcup H_{10}^{(18)}$ and $\dim H_{10} = 15$ and the union is disjoint;

- $H_k = H_{10}^{(18)}$ and $\dim H_k = 15$, for $11 \leq k \leq 18$;

- $H_k = \emptyset$ for $k > 18$.

Chapter 3

On the moduli spaces of non-singular cubic surfaces with star points and compactifications

3.1 Cubic surfaces with only isolated singularities

In this section, we classify all cubic surfaces with only isolated singularities. This gives various classes and we shall see that these classes are locally closed subsets in $\mathbb{P}^{19}$. We shall determine their codimensions, the number of lines and singular points on the surface corresponding to each element of these classes. Finally, we show the inclusion relationship between the closures of these classes.

Lemma 3.1.1. *A line in $\mathbb{P}^3$ which contains two singular points of a given cubic surface lies on the cubic surface.*

Proof. This is clear from the fact that a line passing though 2 singular points of a cubic surface has the intersection multiplicity at least 4. $\square$

Lemma 3.1.2. *A line on a cubic surface with only isolated singularities does not contain 3 singular points.*

Proof. Let X be a cubic surface with only isolated singularities. Suppose that there exist 3 singular points lying on a line. Choose a system of coordinates such that 3 singular points are: $P_1 = (0 : 0 : 0 : 1), P_2 = (0 : 0 : 1 : 0), P_3 = (0 : 0 : 1 : 1)$. The surface X is given by an equation:

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i.

We have $\partial F/\partial x_i = x_3 \partial f_2/\partial x_i + \partial f_3/\partial x_i$ for $i = 0, 1, 2$ and $\partial F/\partial x_3 = f_2$. Then $(\partial F/\partial x_i)(P_2) = (\partial f_3/\partial x_i)(P_2) = (\partial f_3/\partial x_i)(P_3) = 0$ for $i = 0, 1, 2$ and $f_2(P_2) = f_2(P_3) = 0$. Moreover, since $(\partial F/\partial x_i)(P_3) = (\partial f_2/\partial x_i)(P_3) + (\partial f_3/\partial x_i)(P_3) = 0$, then $(\partial f_2/\partial x_i)(P_3) = 0$ for $i = 0, 1, 2$. This implies that $(\partial F/\partial x_i)(P) = 0$ for any $0 \le i \le 3$ and $P = (0 : 0 : a : b)$ on the line $l = V(x_0, x_1)$. This contradicts the fact that X has only isolated singularities. $\square$

Lemma 3.1.3. *Let X be a cubic surface with only isolated singularities. Let $P_0 \in X$ be a singular point. Suppose that the tangent cone at P_0 is a quadric surface. Then there exist at most 6 lines on X through P_0. Consequently, there exist at most 7 singular points on the surface X.*

Proof. Choose coordinates such that X is given as the zeros of

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and $P_0 = (0 : 0 : 0 : 1)$. Let $d = V(a_0 x_0 + a_1 x_1 + a_2 x_2, b_0 x_0 + b_1 x_1 + b_2 x_2)$ be a line on X through P_0. Let $P_1 = (c_0 : c_1 : c_2 : 0) = V(x_3) \cap d$. We have $F(P_1) = f_3(P_1) = 0$. It is clear that $F(c_0 : c_1 : c_2 : 1) = f_2(P_1) = 0$. So $P_1 \in V(f_2, f_3, x_3)$. Since X has only isolated singularities and $f_2 \neq 0$, we see that $V(f_2, f_3, x_3)$ is a 0-dimensional variety which contains at most 6 points. $\qquad\square$

Remark 3.1.4. In fact, the lines through the singular point P_0 are completely determined by the intersection points of the curves $V_{\mathbb{P}^2}(f_2)$ and $V_{\mathbb{P}^2}(f_3)$ on the projective plane $H = V(x_3)$. Namely, let $Q = (c_0 : c_1 : c_2 : 0) \in V(f_2, f_3, x_3)$. Then the line $l = \overline{PQ}$ is given by $V(c_0 x_1 - c_1 x_0, c_0 x_2 - c_2 x_0)$. So $l \subset X$. Moreover, since $V(f_2)$ is the tangent cone TC_P of X at P, we see that the lines through P lie on the tangent cone TC_P.

We use the definition of types of isolated singularities as in the sense of V. I. Arnol'd to classify all cubic surfaces with only isolated singularities. About the types $A_i, D_j, E_k, \hat{E}_6$ singularities, we refer to papers of Arnol'd [Ar1] and [Ar2] or the paper of J. W. Bruce and C. T. C. Wall [B-W] for general definitions and properties. In the case of cubic surfaces, the above types of isolated singularities can be characterized as follows.

A singular point P on a cubic surface with only isolated singularities is:

- A_1 if the tangent cone TC_P is an irreducible quadric surface;

- A_2 if the tangent cone TC_P factors into two different planes such that the intersection line does not lie on the surface;

- A_i for $i = 3, 4, 5$ if the tangent cone TC_P factors into two different planes such that the intersection line lies on the surface and there exist exactly $8 - i$ distinct lines through P. Moreover, in the case of A_5, the 3 lines through P lie on one component of TC_P.

- D_4 (D_5, E_6) if the tangent cone TC_P is a double plane and there exist exactly 3 (2, 1 respectively) distinct lines through P.

- $\hat{E}_6$ if the tangent cone is an irreducible cubic surface.

Remark 3.1.5. Let a cubic surface with only isolated singularities be given by $F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2)$ as before. We see that the above types of isolated singularities can be characterized by $\mathrm{rank}(f_2)$ and the configuration of $V_{\mathbb{P}^2}(f_2, f_3)$. For instance, the point P is A_1 singular if $\mathrm{rank}(f_2) = 3$, is A_2 singular if $\mathrm{rank}(f_2) = 2$ and the singular point of $V_{\mathbb{P}^2}(f_2)$ does not lie on $V_{\mathbb{P}^2}(f_3)$, is A_3 if $\mathrm{rank}(f_2) = 2$ and the singular point of $V_{\mathbb{P}^2}(f_2)$ is a unique double point of $V_{\mathbb{P}^2}(f_2, f_3)$, and so on.

Definition: Let I be a homogeneous ideal of $S = k[x_0, \ldots, x_r]$ for $r = 2, 3$ such that $\mathrm{Proj}(S/I)$ consists of one point. This point is called a multiple point if the length of S/I is greater than 1. In particular, this point is called a double (triple) point if the length is 2 (respectively 3).

Proposition 3.1.6. *Let X be a cubic surface with only isolated singularities given by $F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2)$ as before with $f_2 \neq 0$. Let $P = (0 : 0 : 0 : 1) \in X$. If X has another singular point Q then the line $l = \overline{PQ}$ intersects the hyperplane $V(x_3)$ at a point which is a multiple point of $V_{\mathbb{P}^2}(f_2, f_3)$. Consequently, the surface X has at most 4 isolated singularities.*

Proof. We can choose coordinates such that the singular point $Q = (0 : 0 : 1 : 0)$. We have $\partial F / \partial x_i = x_3 \partial f_2 / \partial x_i + \partial f_3 / \partial x_i$ for $i = 0, 1, 2$ and $\partial F / \partial x_3 = f_2$. It is clear that $f_2(Q) = 0$ and $(\partial f_3 / \partial x_i)(Q) = 0$ for $i = 0, 1, 2$. This implies that the point $(0 : 0 : 1)$ is a singular point of $V_{\mathbb{P}^2}(f_3)$ and therefore is a multiple point of $V_{\mathbb{P}^2}(f_2, f_3)$. $\square$

Conversely, we have:

Proposition 3.1.7. *Let X be a cubic surface given as in the previous proposition. Suppose that $V_{\mathbb{P}^2}(f_2, f_3)$ contains a k-multiple point which is not a singular point of $V_{\mathbb{P}^2}(f_2)$. Then the surface X has another singular point Q which is in type A_{k-1} and lies on the line connecting $P = (0 : 0 : 0 : 1)$ and the multiple point. Moreover, the point Q is contained in $V(x_3)$ if and only if the multiple point is a singular point of $V_{\mathbb{P}^2}(f_3)$.*

Proof. Let $R = (a_0 : a_1 : a_2)$ be a multiple point of $V_{\mathbb{P}^2}(f_2, f_3)$. If $V_{\mathbb{P}^2}(f_3)$ is singular at R then X is singular at $Q = (a_0 : a_1 : a_2 : 0) \in V(x_3)$. Consider the case when $V_{\mathbb{P}^2}(f_3)$ is not singular at R. Since R is multiple point, we have $((\partial f_2 / \partial x_0)(R) : (\partial f_2 / \partial x_1)(R) : (\partial f_2 / \partial x_3)(R)) = ((\partial f_3 / \partial x_0)(R) : (\partial f_3 / \partial x_1)(R) : (\partial f_3 / \partial x_3)(R))$. Then there exists $b \in k^*$ such that $(\partial f_2 / \partial x_i)(R) = b(\partial f_3 / \partial x_i)(R)$ for all $i = 0, 1, 2$. This implies that the point $(a_0 : a_1 : a_2 : b)$ is a singular point of X, and this point is not contained in $V(x_3)$.

For the type of singularity at Q, we refer to [B-W], Lemmas 2 and 3. $\square$

Lemma 3.1.8. *Let X be a cubic surface with only isolated singularities. If X has a singular point in type either D_4 or D_5 or E_6 or $\hat{E}_6$, then X has only one singular point.*

Proof. Except for the case of $\hat{E}_6$ singularity, we can choose coordinates such that X is given by $F = x_3 x_0^2 + f_3(x_0, x_1, x_2)$ where f_3 is a homogeneous polynomial of degree 3. We have $\partial F / \partial x_0 = 2x_0 x_3 + \partial f_3 / \partial x_0$, $\partial F / \partial x_i = \partial f_3 / \partial x_i$ for $i = 1, 2$ and $\partial F / \partial x_3 = x_0^2$. Suppose that X has another singular point $Q = (0 : a_1 : a_2 : a_3)$. This implies that any point $(0 : a_1 : a_2 : x_3)$ on the line $V(x_0, a_1 x_2 - a_2 x_1)$ is singular. This is a contradiction. For $\hat{E}_6$, the result is clear from the definition. $\square$

By (3.1.7) and (3.1.8), we conclude that the list of all cases of isolated singularities of cubic surfaces is as follows.

- $A_1, \quad 2A_1, \quad 3A_1, \quad 4A_1,$

- $A_2, \quad 2A_2, \quad 3A_2,$

- $A_1 A_2, \quad 2A_1 A_2, \quad A_1 2A_2, \quad A_1 A_3, \quad A_1 A_4, \quad A_1 A_5,$

- $D_4, \quad D_5, \quad E_6,$

- $\hat{E}_6.$

Remark 3.1.9. This list of singularities can be found in [B-W] where several deep properties, especially for $\hat{E}_6$, were achieved.

Definition: We denote by $i\mathcal{A}_1 j\mathcal{A}_2$ the subset in $\mathbb{P}^{19}$ corresponding to all cubic surfaces with exactly i singular points of A_1 type and j singular points of A_2 type.

Remark 3.1.10. We are interested in $i\mathcal{A}_1 j\mathcal{A}_2$ since we shall prove later that their points correspond to all semi-stable, singular cubic surfaces (see the next section for the definition of semi-stable). At the moment, we want to know the codimensions of these classes in $\mathbb{P}^{19}$ as well as the relationship between them and the number of lines on each member. First of all, we need a lemma.

Lemma 3.1.11. *Let $\mathcal{X}_n$ be the projective space of homogeneous polynomials of degree n in two variables. Any subset of $\mathcal{X}_n$ consisting of polynomials with solutions corresponding to a given partition of n is irreducible and its dimension equals to the number of distinct solutions. Furthermore, let A and B be two such subsets corresponding to given partitions P_A and P_B respectively. If P_A and P_B can be written $P_A = (a_1, \ldots, a_r)$ and $P_B = (b_{11}, \ldots, b_{1t_1}, \ldots, b_{r1}, \ldots, b_{rt_r})$ such that $\sum_{i=1}^{r} a_i = n$ and $\sum_{k=1}^{t_j} b_{jk} = a_j$ for $1 \leq j \leq r$, then $\overline{A} \subset \overline{B}$.*

Proof. A homogeneous polynomial of degree n in two variables, as a point of $\mathcal{X}_n$, is completely determined by its zeros $P_1, \ldots, P_n$, which are considered as points in $\mathbb{P}^1$. This defines a morphism $\gamma : (\mathbb{P}^1)^n \longrightarrow \mathcal{X}_n$ which is finite and surjective.

Let A be the subset of $\mathcal{X}_n$ corresponding to the given partition $P_A = (a_1, \ldots, a_r)$, where $\sum_{i=1}^{r} a_i = n$. Consider the morphism:

$$
\begin{array}{ccc}
(\mathbb{P}^1)^r & \longrightarrow & (\mathbb{P}^1)^n \\
(P_1, \ldots, P_r) & \mapsto & (\underbrace{P_1, \ldots, P_1}_{a_1 \text{ times}}, \ldots, \underbrace{P_r, \ldots, P_r}_{a_r \text{ times}}).
\end{array}
$$

This is an inclusion. The image $\gamma((\mathbb{P}^1)^r - \Delta)$ is the subset A. So the set A is irreducible of dimension r.

Let B be the subset corresponding to the partition $P_B = (b_{11}, \ldots, b_{1t_1}, \ldots, b_{r1}, \ldots, b_{rt_r})$ such that $\sum_{k=1}^{t_j} b_{jk} = a_j$ for $1 \leq j \leq r$. The inclusion $(\mathbb{P}^1)^r \longrightarrow (\mathbb{P}^1)^n$ factors through $(\mathbb{P}^1)^r \longrightarrow (\mathbb{P}^1)^{\sum_{i=1}^{r} t_i} \longrightarrow (\mathbb{P}^1)^n$, where

$$
\begin{array}{ccc}
(\mathbb{P}^1)^r & \longrightarrow & (\mathbb{P}^1)^{\sum_{i=1}^{r} t_i} \\
(P_1, \ldots, P_r) & \mapsto & (\underbrace{P_1, \ldots, P_1}_{t_1 \text{ times}}, \ldots, \underbrace{P_r, \ldots, P_r}_{t_r \text{ times}}).
\end{array}
$$

The fact $\overline{A} \subset \overline{B}$ follows immediately. $\qquad\square$

Proposition 3.1.12. *Let $X_1 = i_1\mathcal{A}_1 j_1\mathcal{A}_2$ and $X_2 = i_2\mathcal{A}_1 j_2\mathcal{A}_2$ be two classes such that $i_1 > i_2$.*

(i) If $j_1 = j_2$ then $\overline{X_1} \subset \overline{X_2}$.

(ii) If $i_1 + j_1 = i_2 + j_2$ then $\overline{X_1} \supset \overline{X_2}$.

(iii) The subset $i\mathcal{A}_1 j\mathcal{A}_2$ has codimension $i + 2j$.

Proof. Any singular cubic surface with only isolated singularities can be given by a polynomial of the form:

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i. The subset $i\mathcal{A}_1 j\mathcal{A}_2$ consists of elements characterized by $\operatorname{rank}(f_2)$ and a specific partition of 6. Namely, the set $i\mathcal{A}_1 j\mathcal{A}_2$ is characterized either by $\operatorname{rank}(f_2) = 3$ and the partition $(2^{i-1}, j^3, 1^{8-2i-3j})$ of 6 if $i > 0$ or by $\operatorname{rank}(f_2) = 2$, and the partition $(2^i, 3^{j-1}, i^{9-2i-3j})$ of 6 if $j > 0$ plus the requirement that no point of $V_{\mathbb{P}^2}(f_2, f_3)$ is the singular point of $V_{\mathbb{P}^2}(f_2)$. This implies that the codimensions of $\mathcal{A}_1$ and $\mathcal{A}_2$ are 1 and 2 respectively. Moreover, we note that a double point (triple point) of $V_{\mathbb{P}^2}(f_2, f_3)$ makes the number of distinct points drop 1 (2 respectively). Applying the lemma, we see that the dimension of the set $i\mathcal{A}_1 j\mathcal{A}_2$ for $i > 0$, drops $(i - 1) + 2j$. Since the codimension of $\mathcal{A}_1$ is 1, the codimension of $i\mathcal{A}_1 j\mathcal{A}_2$ is $1 + (i - 1) + 2j = i + 2j$. Similarly, the dimension of the set $j\mathcal{A}_2$ drops $2(j - 1)$. Since the codimension of $\mathcal{A}_2$ is 2, the codimension of $j\mathcal{A}_2$ is $2 + 2(j - 1) = 2j$. This proves (iii).

The results in (i) and (ii) follow from the second part of the lemma. Namely:

(i) If $i_1 > i_2$ and $j_1 = j_2$.

    (a) If $j_1 = j_2 = 0$. We can assume that $\operatorname{rank}(f_2) = 3$ for any element in X_1 and X_2. This means that the partitions corresponding to X_1 and X_2 are $(2^{i_1-1}, 1^{8-2i_1})$ and $(2^{i_2-1}, 1^{8-2i_2})$ respectively. Since $i_1 > i_2$, the second part of the lemma can be applied.

    (b) If $j_1 = j_2 > 0$. We can assume that X_1 and X_2 are characterized by $\operatorname{rank}(f_2) = 2$ and partitions $(2^{i_1}, 3^{j_1-1}, 1^{9-2i_1-3j_1})$ and $(2^{i_2}, 3^{j_2-1}, 1^{9-2i_2-3j_2})$ respectively. Since $i_1 > i_2$, the second part of the lemma can be applied.

(ii) If $i_1 + j_1 = i_2 + j_2$ and $i_1 > i_2$.

    (a) If $j_1 = 0$ then X_1 can be characterized by $\operatorname{rank}(f_2) = 3$ and the partition $(2^{i_1-1}, 1^{8-2i_1})$. The set X_2 can be characterized by $\operatorname{rank}(f_2) = 2$ and the partition $(2^{i_2}, 3^{j_2-1}, 1^{9-2i_2-3j_2})$. With condition $i_1 = i_2 + j_2$ in mind, we can check that $i_1 \neq 4$ and the second part of the lemma can be applied for this case.

    (b) If $j_1 \neq 0$. Since $j_2 > j_1 > 0$, we can assume that X_1 and X_2 are characterized by $\operatorname{rank}(f_2) = 2$ and the partitions $(2^{i_1}, 3^{j_1-1}, 1^{9-2i_1-3j_1})$ and $(2^{i_2}, 3^{j_2-1}, 1^{9-2i_2-3j_2})$ respectively. If $j_2 = 3$ then $i_2 = 0$. This implies that either $(i_1, j_1) = (1, 2)$ or $(i_1, j_1) = (2, 1)$. The result follows from the lemma. If $j_2 = 2$ then $j_1 = 1$. The partition corresponding to X_1 is $(2^{i_1}, 1^{6-2i_1})$ with $i_1 < 3$. The partition corresponding to X_2 is $(2^{i_2-1}, 3, 1^{5-2i_2})$. Then the lemma can be applied to this case.

$\square$

Remark 3.1.13. We write $X_1 \longrightarrow X_2$ for the fact $\overline{X_1} \subset \overline{X_2}$; the proposition implies the following diagram:

$$
\begin{array}{ccccc}
\mathcal{A}_1 & \longleftarrow & 2\mathcal{A}_1 & \longleftarrow & 3\mathcal{A}_1 & \longleftarrow & 4\mathcal{A}_1 \\
\end{array}
$$
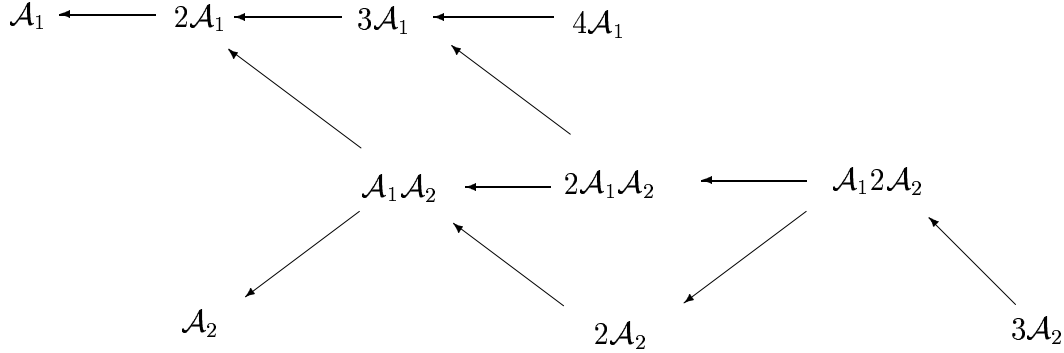
where classes are in the same column if and only if they have the same codimension.

For the rest of the section, we determine the number of lines on each cubic surface in these classes. We need a lemma.

Lemma 3.1.14. *Let X be a cubic surface with only isolated singularities. Suppose that X is given by $F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2)$ as before with $f_2 \neq 0$. Let d be a line on X passing through $P = (0 : 0 : 0 : 1)$. Then d contains another singular point of X if and only if there exists a hyperplane H such that $H \cap X = 2d \cup l$ where l is a line on X. Furthermore if $P \in l$ then P is not an A_1 singularity.*

Proof. Suppose that d contains another singular point Q; we change coordinates such that $Q := (0 : 0 : 1 : 0)$ and $d = V(x_0, x_1)$. Then we have $f_2 = a_1 x_0^2 + a_2 x_0 x_1 + a_3 x_0 x_2 + a_4 x_1^2 + a_5 x_1 x_2$. It is easy to check that $(0 : 0 : 1)$ is a singular point of $V_{\mathbb{P}^2}(f_3)$, see (3.1.7). So $f_3 = x_2 g_2(x_0, x_1) + g_3(x_0, x_1)$, where g_i for $i = 2, 3$ is a homogeneous polynomial of degree i.

Let $H = V(ax_0 - x_1)$ where $a \in (k \cup \{\infty\})$, is a hyperplane containing d. Then $H \cap X$ is determined by $x_3 x_0 [(a_1 + aa_2 + a_4 a^2) x_0 + (a_5 a + a_3) x_2] + b x_0^2 h_1(x_0, x_2)$ in H. Then there exists a such that $a_5 a + a_3 = 0$ and we have the desired result.

Conversely, suppose that there exists a hyperplane H such that $H \cap X = 2d \cup l$. We can assume that $H = V(x_1)$ and $d = V(x_0, x_1)$. This implies that $x_3 f_2(x_0, 0, x_2) + f_3(x_0, 0, x_2) = x_0^2 t_1(x_0, x_2)$ where $t_1 \in k[x_0, x_2]$. So $f_2 = a_1 x_0^2 + x_1 h_1(x_0, x_1, x_2)$ and $f_3 = x_0^2 g_1(x_0, x_2) + x_1 g_2(x_0, x_1, x_2)$. If f_3, as a polynomial in $k[x_0, x_1, x_2]$, is not singular at $Q = (0 : 0 : 1)$ then it is easy to check that $\big((\partial f_3/\partial x_0)(Q) : (\partial f_3/\partial x_1)(Q) : (\partial f_3/\partial x_2)(Q)\big) = \big((\partial f_2/\partial x_0)(Q) : (\partial f_2/\partial x_1)(Q) : (\partial f_2/\partial x_2)(Q)\big) = (0 : 1 : 0)$. This means that Q is a double point of $V_{\mathbb{P}^2}(f_2, f_3)$.

Finally, suppose that $H \cap X = 2d \cup l$ and $P \in l$. We can assume that $l = V(x_1, ax_0 + bx_2)$ and $f_2 = a_1 x_0^2 + a_2 x_0 x_1 + a_3 x_0 x_2 + a_4 x_1^2 + a_5 x_1 x_2$. Since $H \cap X = 2d \cup l$, we have $a_1 x_3 x_0^2 + x_3 a_3 x_0 x_2 + f_3(x_0, 0, x_2) = c x_0^2 (ax_0 + bx_2)$ for $c \in k^*$ and this implies that $a_1 = a_3 = 0$. Then we have $\text{rank}(f_2) \leq 2$. $\square$

Determining the number of lines on a cubic surface corresponding to an element in $i\mathcal{A}_1 j\mathcal{A}_2$ and their configuration

$\mathcal{A}_1$. Let X be the cubic surface corresponding to an element of $\mathcal{A}_1$. We know that there exist exactly 6 lines on X through the singular point P of X. We note that any line which does not contain P intersects with at least one line through P. By the lemma, this line intersects exactly 2 lines through P. There are 15 of them. Since the cone containing these 6 lines is irreducible, all of the 15 lines are mutually different. Therefore there exist exactly 21 lines on X.

$\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $\mathcal{A}_2$. We know that there exist exactly 6 lines on X through the singular point P of X. These 6 lines lie on two different components H_1 and H_2 of the tangent cone at P. By the same argument as in the case of $\mathcal{A}_1$, we see that d is another line on X if and only if d intersects one of the three lines in H_1 and one of the three lines in H_2. So, there exist exactly 15 lines on X.

$2\mathcal{A}_1$. Let X be the cubic surface corresponding to an element of $2\mathcal{A}_1$. Let d be the line connecting the two singular points P_1 and P_2 of X. There are 4 other lines through each of P_1 and P_2. Let t_1 be a line containing only one singular point. Then there exists a line t_2 through another singular point such that $t_1 \cap t_2 \neq \emptyset$. By the lemma, there is a line l intersecting d such that $2d + l$ is a triple tangent of X. This line does not contain any singular point. By the same argument as before, we see that if m is another line on X which does not intersect d, then m intersects exactly one pair of the lines through P_1 and one pair of the lines through P_2. There are 6 such lines. Therefore, there exist exactly $1 + 2.4 + 1 + 6 = 16$ lines on X.

$3\mathcal{A}_1$. Let X be the cubic surface corresponding to an element of $3\mathcal{A}_1$. Denote by P_i for $i = 1, 2, 3$ the 3 singular points of X. Let d_{ij} for $1 \leq i < j \leq 3$ be the line connecting P_i and P_j. There exist exactly 2 other lines through each of P_i for $i = 1, 2, 3$. We denote them by a_i, b_i for $i = 1, 2, 3$. As in the case of $2\mathcal{A}_1$, there exists another line intersecting d_{ij}, denoted by l_{ij}, such that $2d_{ij} \cup l_{ij}$ is a hyperplane intersection of X. The line l_{ij} does not contain any of the singular points and intersects the two lines a_k, b_k for $k \notin \{i, j\}$. Therefore, there exist exactly $3 + 2.3 + 3 = 12$ lines on X.

$4\mathcal{A}_1$. Let X be the cubic surface corresponding to an element of $4\mathcal{A}_1$. Denote by P_i for $1 \leq i \leq 4$ the 4 singular points of X. Let d_{ij} for $1 \leq i < j \leq 4$ be the line connecting two points P_i and P_j. So the 4 singular points and 6 lines d_{ij} form a tetrahedron in $\mathbb{P}^3$. With the same argument as above, we see that there exists another line l_{ij} intersecting d_{ij} such that $2d_{ij} \cup l_{ij}$ is a hyperplane intersection of X. The line l_{ij} does not contain any singular point and intersects the line opposite to d_{ij}. So, there exist exactly $6 + 3 = 9$ lines on X.

$\mathcal{A}_1\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $\mathcal{A}_1\mathcal{A}_2$. Let d be the line connecting the two singular points P_1 and P_2 of X where P_1 is the A_1 singular point. There exist 3 other lines, denoted by m_1, m_2, m_3, through P_1 and there exist 4 other lines through P_2. We know that the tangent cone of X at P_2

factors into 2 hyperplanes H_1 and H_2. We can assume that $H_1 \cap X = 2d \cup l$, where l is one of the 4 other lines through P_2; the plane H_2 contains the other 3 lines, which are denoted by n_1, n_2, n_3. The line l does not intersect any line of m_i for $i = 1, 2, 3$. Each of the lines m_1, m_2, m_3 through P_1 intersects one of the lines n_1, n_2, n_3 through P_2. Finally, there exists another line l_{ij} intersecting m_i and m_j for $1 \leq i < j \leq 3$. The line l_{ij} intersects l and one of the lines n_1, n_2, n_3. So, there exist exactly $1 + 3 + 4 + 3 = 11$ lines on X.

$2\mathcal{A}_1\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $2\mathcal{A}_1\mathcal{A}_2$. Let P_1, P_2 be the two A_1 singularities and Q be the A_2 singularity of X. There are 3 lines connecting each pair of the 3 singular points. There exists exactly one other line through each of P_1, P_2 and there exist exactly 2 other lines through Q. There exists another line intersecting the two lines through Q and meeting the line connecting P_1, P_2. So, there exist exactly $3 + 1 + 1 + 2 + 1 = 8$ lines on X.

$2\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $2\mathcal{A}_2$. Let d be the line intersecting the two singular points P_1 and P_2 of X. One component of the tangent cone at P_i for $i = 1, 2$ intersects X at $3d$. The second component contains the 3 other lines on X through P_i. Each line through P_1 different from d intersects a line through P_2 and different from d. There are no other lines. So, there exist exactly $1 + 3 + 3 = 7$ lines on X.

$\mathcal{A}_1 2\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $\mathcal{A}_1 2\mathcal{A}_2$. Let Q_1, Q_2 be two A_2 singularities of X. There are 3 lines connecting the 3 singular points. There exists exactly one other line l_i through Q_i for $i = 1, 2$ and there are no other lines through the A_1 singular point. The two lines l_1 and l_2 intersect. There are no other lines. So there exist exactly $3 + 1 + 1 = 5$ lines on X.

$3\mathcal{A}_2$. Let X be the cubic surface corresponding to an element of $3\mathcal{A}_2$. There exist 3 lines connecting three singular points of X. Let d be one of these 3 lines. Then one component of the tangent cone at a singular point on d intersects X at $3d$. It is clear that there are no other lines on X. So there exist exactly 3 lines on X.

3.2   Stable and semi-stable cubic surfaces

In this section, we study the natural action of the projective general linear group $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$, the parameterizing space of cubic surfaces in $\mathbb{P}^3$. We determine stable, semi-stable cubic surfaces in the sense of the geometric invariant theory and study the correspondent quotient spaces.

A. Linear actions of reductive groups on projective varieties

We describe briefly basic fundamentals of the geometric invariant theory. In this subsection, as well as through this section, we use mainly definitions and properties from the book by P. E. Newstead [N] and the book by D. Mumford [GIT].

Definition: A linear action of an algebraic group G on the affine space $\mathbb{A}^{n+1}$ is an action induced by a morphism of algebraic groups $G \longrightarrow \mathrm{GL}(n + 1)$ and the natural

action of $\mathrm{GL}(n+1)$ on $\mathbb{A}^{n+1}$. If the group morphism $G \longrightarrow \mathrm{GL}(n+1)$ is an injective then G is called a linear algebraic group.

Example 8. The groups $\mathrm{SL}(n)$, $\mathrm{PGL}(n)$ are linear algebraic groups. The fact that $\mathrm{SL}(n) \hookrightarrow \mathrm{GL}(n)$ is clear. We show that $\mathrm{PGL}(n) \hookrightarrow \mathrm{GL}(n^2)$. Consider the action of $\mathrm{GL}(n)$ on the linear space of $n \times n$ matrices $\mathbf{M}_n$ defined by

$$
\begin{array}{ccc}
\mathrm{GL}(n) \times \mathbf{M}_n & \longrightarrow & \mathbf{M}_n \\
(A, M) & \mapsto & AMA^{-1}.
\end{array}
$$

This defines a morphism $\phi : \mathrm{GL}(n) \to \mathrm{GL}(\mathbf{M}_n)$ where $\mathrm{GL}(\mathbf{M}_n)$ is the algebraic group of linear automorphisms of the vector space $\mathbf{M}_n$.

We see that $\mathrm{Ker}(\phi) \cong \mathcal{G}_m^1$, where $\mathcal{G}_m^1$ is the 1-dimensional torus. It is clear that

$$
\mathrm{GL}(n)/\mathrm{Ker}(\phi) = \mathrm{GL}(n)/\mathcal{G}_m^1 \cong \mathrm{PGL}(n).
$$

Hence we get an injective homomorphism: $\mathrm{PGL}(n) \hookrightarrow \mathrm{GL}(\mathbf{M}_n) \cong \mathrm{GL}(n^2)$.

Definition: A linearization of an action of an algebraic group G on a projective variety X in $\mathbb{P}^n$ is a linear action of G on $\mathbb{A}^{n+1}$ which induces the given action of G on X.

Definition: A linear action of an algebraic group G on a projective variety X in $\mathbb{P}^n$ is an action of G together a linearization of this action.

Definition: The maximal connected normal solvable subgroup of a linear algebraic group G is called the radical of G. A linear algebraic group is called reductive if its radical is a torus (i.e. isomorphic to $(\mathcal{G}_m^r)(k) = (k^*)^r$ for some integer r).

Remark 3.2.1.

(i) In fact, the radical of a connected linear algebraic group G is the identity component of the intersection of all Borel subgroups of G (see [B], 11.12). Recall that a subgroup of a connected linear algebraic group is said to be Borel if it is a maximal connected solvable subgroup. In $\mathrm{GL}(n+1)$, any Borel subgroup is isomorphic to the subgroup of upper triangular matrices ([Sp], 7.2.11) and therefore is isomorphic to the subgroup of lower triangular matrices. This implies that the radical of $\mathrm{GL}(n+1)$ is the diagonal group $\mathbb{D}_{n+1}$. So $\mathrm{GL}(n+1)$ is reductive. Since $\mathrm{SL}(n+1)$ ($\mathrm{PGL}(n)$ respectively) is a normal connected subgroup (quotient, respectively) of $\mathrm{GL}(n+1)$, the group $\mathrm{SL}(n+1)$ ($\mathrm{PGL}(n)$, respectively) is a reductive group (see [B], 11.14).

(ii) There are the concepts of linear reductive group and geometric reductive group. In the case of characteristic 0, they coincide with the concept of reductive group. We refer the reader, who is interested in these topics, to [N], chapter 3, §1, or [N-M].

Definition: Let X be a projective variety in $\mathbb{P}^n$. For a given linear action of a reductive group G on X, a point $x \in X$ is called:

- *semi-stable* if there exists an invariant homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of positive degree such that $f(x) \neq 0$;

- *stable* if it is semi-stable and $\dim \mathcal{O}(x) = \dim(G)$ and the induced action of G on X_f is closed, where $\mathcal{O}(x)$ is the orbit of x.

Let X^{ss}, X^s denote the set of semi-stable (stable) points of X with respect to the action of G.

Remark 3.2.2. The definition of "stable" corresponds to Mumford's "properly stable" (see [GIT], Definition 1.8), where he used the notation $X^s_{(0)}$ for this set.

Lemma 3.2.3. *For a linear action of an reductive group on a projective variety X, the subsets X^{ss} and X^s are open in X.*

Proof. See [N], 3.13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Definition: (Categorical quotient, good quotient and geometric quotient).

(i) Let G be an algebraic group acting on a variety X. A categorical quotient of X by G is a pair (Y, ϕ) consisting of a variety Y and a morphism $\phi : X \longrightarrow Y$ satisfying:

   (a) ϕ is G-invariant,

   (b) for any variety Z and a G-invariant morphism $\psi : X \longrightarrow Z$, there exists uniquely a morphism $\chi : Y \longrightarrow Z$ such that $\chi \cdot \phi = \psi$.

(ii) Let G be an algebraic group acting on a variety X. A good quotient of X by G is a pair (Y, ϕ) consisting of a variety Y and an affine morphism $\phi : X \longrightarrow Y$ satisfying:

   (a) ϕ is $G-$invariant,

   (b) ϕ is surjective,

   (c) if U is open in Y then $\phi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\phi^{-1}(U))$ is an isomorphism of $\mathcal{O}_Y(U)$ onto $(\mathcal{O}_X(\phi^{-1}(U)))^G$,

   (d) if W is a closed invariant subset of X, then $\phi(W)$ is closed,

   (e) if W_1, W_2 are disjoint closed invariant subsets of X then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

(iii) The pair (Y, ϕ) is called a geometric quotient if it is a good quotient and for any $y \in Y$ the set $\phi^{-1}(y)$ consists of a single orbit.

Remark 3.2.4. A good quotient is a categorical quotient (see [N], 3.11). In fact, the conditions (a), (c), (d) and (e) imply that (Y, ϕ) is a categorical quotient (see [GIT], Chapter 0, §2.(6)).

Theorem 3.2.5. *Let X be a projective variety in $\mathbb{P}^n$. For any linear action of a reductive group G on X:*

(i) *there exists a good quotient (Y, ϕ) of X^{ss} by G and Y is projective;*

(ii) *there exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and (Y^s, ϕ) is a geometric quotient of X^s;*

(iii) for $x_1, x_2 \in X^{ss}$,

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{\mathcal{O}(x_1)} \cap \overline{\mathcal{O}(x_2)} \cap X^{ss} \neq \emptyset;$$

(iv) for $x \in X^{ss}$,

$$x \text{ is stable } \Leftrightarrow \dim \mathcal{O}(x) = \dim G \text{ and } \mathcal{O}(x) \text{ is closed in } X^{ss}.$$

Proof. See [N], 3.14. $\qquad\square$

Proposition 3.2.6. *The concept of good quotient and geometric quotient are local with respect to Y, i.e:*

(i) if (Y, ϕ) is a good (geometric) quotient of X by G and U is open in Y then (U, ϕ) is a good (geometric) quotient of $\phi^{-1}(U)$ by G.

(ii) if $\phi : X \longrightarrow Y$ is a morphism and $\{U_i\}$ is an open covering of Y such that (U_i, ϕ) is a good (geometric) quotient of $\phi^{-1}(U_i)$ by G for all i, then (Y, ϕ) is a good (geometric) quotient of X by G.

Proof. See [N], 3.10. $\qquad\square$

Proposition 3.2.7. *Suppose that for a given moduli problem, there exists a family U parameterized by X with local universal property. Suppose further that an algebraic group G acts on X in such a way that $U_{x_1} \sim U_{x_2}$ if and only if x_1 and x_2 lie in the same orbit of this action. Then*

(i) any coarse moduli space is a categorical quotient of X by G;

(ii) a categorical quotient (Y, ϕ) of X by G is a coarse moduli space if and only if for any $y \in Y$, the set $\phi^{-1}(y)$ consists of one single orbit.

Proof. See [N], 2.13. $\qquad\square$

B. The action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$

We consider the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$, the parameterizing space of cubic surfaces in $\mathbb{P}^3$. Although this action is not linear, we see that the definitions of semi-stability and stability are well-defined. For this, we lift the action of $\mathrm{PGL}(3)$ to the natural action of $\mathrm{SL}(4)$ on $\mathbb{P}^{19}$ induced by the action of $\mathrm{PGL}(3)$. It is clear that the action of $\mathrm{SL}(4)$ on $\mathbb{P}^{19}$ is linear. Since the natural morphism $\mathrm{SL}(4) \longrightarrow \mathrm{PGL}(3)$ is surjective with a finite kernel and the set of invariant homogeneous polynomials as well as the orbit of any point in $\mathbb{P}^{19}$ with respect to these actions are the same, we can view semi-stable (stable) points w.r.t. the action of $\mathrm{SL}(4)$ are semi-stable (stable) points w.r.t. the action of $\mathrm{PGL}(3)$. Moreover, we have:

Proposition 3.2.8. *The action of* $\mathrm{PGL}(3)$ *on* $\mathbb{P}^{19}$ *has all properties* $(i) - (iv)$ *as in Theorem 3.2.5.*

Proof. It is clear that the action of $\mathrm{SL}(4)$ on $\mathbb{P}^{19}$ satisfies the hypothesis of Theorem 3.2.5. The result follows when we look at the definitions of good quotient and geometric quotient and keep in mind that the set of invariant homogeneous polynomials, the orbit of any point in $\mathbb{P}^{19}$, the set of semi-stable, stable points w.r.t the actions of $\mathrm{SL}(4)$ and $\mathrm{PGL}(3)$ are the same.

In particular, their quotient spaces are the same. $\qquad\square$

Corollary 3.2.9. *The quotient space* $\mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{ss}$ *is a projective variety. The varieties* $\mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{s}$ *and* $M := \mathrm{PGL}(3)\backslash(\mathbb{P}^{19} - \Delta)$ *are coarse moduli spaces. Moreover, the latter is affine but not a fine moduli space.*

Proof. The first two conclusions follow immediately from the previous proposition and (3.2.7). Moreover $\mathrm{PGL}(3)\backslash(\mathbb{P}^{19} - \Delta)$ is affine since $\mathbb{P}^{19} - \Delta$ is affine and the discriminant defining Δ is an invariant homogeneous polynomial. We prove that $M = \mathrm{PGL}(3)\backslash(\mathbb{P}^{19} - \Delta)$ is not a fine moduli space.

Consider 2 cubic surfaces in $\mathbb{P}^3$ given by $C_1 = V(x_0^3 + x_1^3 + x_2^3 + x_3^3)$ and $C_2 = (tx_0^3 + x_1^3 + x_2^3 + x_3^3)$ where $t \in \mathbb{Q}(\omega)$ for ω is the primitive cube root of unity. We can choose t in such a way that $t \notin (\mathbb{Q}(\omega))^3$. The 27 lines of C_1 are given by (see [Mu1], p. 177):

$$x_0 + \omega^i x_1 = x_2 + \omega^j x_3 = 0$$
$$x_0 + \omega^i x_2 = x_1 + \omega^j x_3 = 0$$
$$x_0 + \omega^i x_3 = x_1 + \omega^j x_2 = 0$$

for $0 \le i, j \le 2$.

Similarly, the 27 lines of C_2 are given by:

$$\sqrt[3]{t}\, x_0 + \omega^i x_1 = x_2 + \omega^j x_3 = 0$$
$$\sqrt[3]{t}\, x_0 + \omega^i x_2 = x_1 + \omega^j x_3 = 0$$
$$\sqrt[3]{t}\, x_0 + \omega^i x_3 = x_1 + \omega^j x_2 = 0$$

for $0 \le i, j \le 2$.

If C_1 and C_2 are isomorphic over $\mathbb{Q}(\omega)$, each line of C_1 is isomorphic to some line of C_2 over $\mathbb{Q}(\omega)$. Since the lines of C_1 are defined over $\mathbb{Q}(\omega)$ but the lines of C_2 are not, the surfaces C_1 and C_2 are not isomorphic over $\mathbb{Q}(\omega)$.

But it is easy to see that they are isomorphic over $\mathbb{Q}(\omega, \sqrt[3]{t})$. (For example, take the change of coordinates $\xi_0 = \sqrt[3]{t}\, x_0, \xi_i = x_i$, for $i = 1, 2, 3$, then C_2 is defined by the same equation as C_1).

Now the situation follows from the fact:

Lemma 3.2.10. *Suppose that* M *is a moduli space and* X_1, X_2 *are two objects defined over a field* K. *If there exists a field extension* $K \subset L$ *such that* $X_1 \not\cong_K X_2$ *but* $X_1 \cong_L X_2$, *then* M *is not a fine moduli space.*

(Proof of the lemma). Suppose that M is a fine moduli space. The family $[X_i]$ for $i = 1, 2$ over $\mathrm{Spec}(K)$ defines a morphism $\phi_i : \mathrm{Spec}(K) \longrightarrow M$. The families of X_1 and X_2 are the same over $\mathrm{Spec}(L)$, so they define the same morphism $\phi' : \mathrm{Spec}(L) \longrightarrow M$ such that the diagram

$$\text{Spec}(K) \; \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \; M$$

$$\phi'$$

$$\text{Spec}(L)$$

commutes. Let $y \in M$ be the image of these morphisms in M. The above diagram corresponds to the following commutative diagram:

$$\mathcal{O}_{M,y} \; \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \; K$$

$$\varphi'$$

$$L$$

where φ_1, φ_2 and φ' are homomorphisms determined by ϕ_1, ϕ_2 and ϕ' respectively. Note that the map $K \longrightarrow L$ is the inclusion. This implies that $\varphi_1 = \varphi_2$ and $X_1 \cong X_2$ over K. Contradiction!

$\square$

C. Semi-stable and stable cubic surfaces

In this subsection, we give explicitly the sets of semi-stable and stable points of $\mathbb{P}^{19}$ under the action of PGL(3). To do so, we first state a powerful criterion of stability originally considered by D. Hilbert and improved by D. Mumford and C. S. Seshadri. A complete description concerning to this criterion can be found in [GIT], Chapter 2, §1 or in [N], Chapter 4, §2. Secondly, we apply this criterion to find out all semi-stable and stable points in $\mathbb{P}^{19}$.

C.1. A criterion of stability

Definition: An 1-parameter subgroup (1-PS) of an algebraic group G is a non-trivial homomorphism of algebraic groups $\lambda : k^* \longrightarrow G$.

Let G be a reductive group acting linearly on a projective variety in $\mathbb{P}^n$. An 1-PS of G induces a linear action of k^* on k^{n+1}. A linear action of k^* on k^{n+1} can be diagonalized ([B], 4.6). In other words, there exists a basis $\{e_0, \cdots, e_n\}$ of k^{n+1} and there exist integers r_i for $0 \le i \le n$ such that $\lambda(t) e_i = t^{r_i} e_i$.

Let $x \in X$. Let $\hat{x} \in \mathbb{A}^{k+1}$ be a point over x. Then $\hat{x} = \sum_{i=0}^n \hat{x}_i e_i$. Let

$$\mu(x, \lambda) = \max\{-r^i \mid \hat{x}_i \neq 0\}.$$

Theorem 3.2.11. *Let G be a reductive group acting linearly on a projective variety in $\mathbb{P}^n$. Then a point $x \in X$ is:*

$$\textit{semi-stable} \quad \Leftrightarrow \mu(x, \lambda) \ge 0 \textit{ for every 1-PS } \lambda \textit{ of } G,$$
$$\textit{stable} \quad \Leftrightarrow \mu(x, \lambda) > 0 \textit{ for every 1-PS } \lambda \textit{ of } G.$$

Proof. See [GIT], Theorem 2.1 or [N], Theorem 4.9. □

In our case, we have the action of SL(4) on $\mathbb{P}^{19}$. Any 1-PS λ of SL(4) conjugates to one of the form:

$$\lambda(t) = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^c & 0 \\ 0 & 0 & 0 & t^d \end{pmatrix} \tag{3.1}$$

where $a + b + c + d = 0$ and $(a, b, c, d) \in (\mathbb{Z}^4)^*$. The theorem above can be rephrased:

Theorem 3.2.12. *Consider the action of* PGL(3) *on* $\mathbb{P}^{19}$. *A point* $x \in \mathbb{P}^{19}$ *is semi-stable (respectively stable) if* $\mu(gx, \lambda) \geq 0$ *(respectively* > 0) *for every* $g \in$ PGL(3) *and for every 1-PS of the form (3.1).*

Note that for a point $x \in \mathbb{P}^{19}$ given by a cubic form:

$$\sum_{i+j+k+t=3} a_{ijkt} x_0^i x_1^j x_2^k x_3^t,$$

we have

$$\mu(x, \lambda) = \max\{ai + bj + ck + dt \mid a_{ijkt} \neq 0\}, \tag{3.2}$$

where λ is given in form (3.1).

C.2. Semi-stable and stable cubic surfaces

Lemma 3.2.13. *The set* $\mathbb{P}^{19} - \Delta$ *is contained in* $(\mathbb{P}^{19})^s$.

Proof. In fact, this is just a particular case of [GIT], Proposition 4.2. The result follows from the fact that the discriminant D defining Δ is an invariant homogeneous polynomial and the stabilizer of any $x \in \mathbb{P}^{19} - \Delta$ is finite. □

We consider the case of singular cubic surfaces. The main result of this subsection is the following theorem.

Theorem 3.2.14. *On the action of* PGL(3) *on* $\mathbb{P}^{19}$, *we have:*

(i) *The subset of stable points consists of points in* $\mathbb{P}^{19} - \Delta$ *and those of types* $i\mathcal{A}_1$ *for* $1 \leq i \leq 4$.

(ii) *The subset of semi-stable points consists of points in* $\mathbb{P}^{19} - \Delta$ *and all those of types* $i\mathcal{A}_1 j\mathcal{A}_2$ *for* $2i + 3j \leq 9$, $i \leq 4$ *and* $(i, j) \neq (3, 1)$.

(See the definition of $i\mathcal{A}_1 j\mathcal{A}_2$ in the previous section).

Proof.
a) Suppose that there exists an element $x \in i\mathcal{A}_1$ for some $0 \leq i \leq 4$, which is not stable. This means that there exists an 1-PS λ of SL(4) in the form (3.1) such that

$\mu(x, \lambda) \leq 0$. As we know, the corresponding cubic surface X_x can be given by a homogeneous polynomial:

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and $\text{rank}(f_2) = 3$. We can choose coordinates such that $f_2 = x_1^2 - x_0 x_2$ and $f_3 = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^3$ (see [B-W], p. 248). So we can write:

$$F = x_3(x_1^2 - x_0 x_2) + a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^3.$$

Moreover, we can assume that $Q = (0 : 0 : 1 : 0) \in V(f_2, f_3, x_3)$ and in the case the surface has more than one singular point, the point Q is a multiple point. Since $x \in i\mathcal{A}_1$, we have $a_4 \neq 0$. From the formula (3.2) for computing $\mu(x, \lambda)$ we have values $2b + d$, $a + c + d$ and $2b + c$ corresponding to monomials $x_3 x_1^2$, $x_0 x_2 x_3$ and $x_1^2 x_2$, respectively, where $(a, b, c, d) \in (\mathbb{Z}^4)^*$ and $a + b + c + d = 0$. Since $\mu(x, \lambda) \leq 0$, all values $2b + d$, $a + c + d$ and $2b + c$ are not positive. If $a_0 \neq 0$, then we have $a \leq 0$ also. Since $a + c + d = -b \leq 0$, then $b \geq 0$. We have $2b + d + 2b + c = 4b + c + d = 4b - a + a + c + d = 3b - a \leq 0$. This implies that $a = b = 0$. But this forces $c = d = 0$ also. Contradiction!

Suppose that $a_0 = 0$. If $a_1 \neq 0$, then $2a + b \leq 0$. Then we have $2b + d + 2b + c + 2a + b = 5b + a + (a + b + c) = 4b + a \leq 0$. This implies that $8b + a + d + c = 7b \leq 0$. So $b = 0$ and this implies that $a = c = d = 0$. Contradiction!

If $a_0 = a_1 = 0$, then $a_2 \neq 0$. Otherwise, the point $(1 : 0 : 0)$ is a triple point of $V_{\mathbb{P}^2}(f_2, f_3)$. Since $a_2 \neq 0$, we have $2b + a \leq 0$. Then we have $2b + d + 2b + c + 2b + a = 6b + (a + b + c) = 5b \leq 0$. This implies $b = 0$ and similarly we have a contradiction. Therefore, every element of $i\mathcal{A}_1$ for $1 \leq i \leq 4$ is stable.

b) Next, we prove that if $x \in i\mathcal{A}_1 j\mathcal{A}_2$ for $j > 0$, then x is semi-stable but not stable. We know that x can be given by a homogeneous polynomial:

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and $\text{rank}(f_2) = 2$ and the singular point of $V_{\mathbb{P}^2}(f_2)$ does not lie on $V_{\mathbb{P}^2}(f_3)$. We can choose coordinates such that (see [B-W], p. 249) $f_2 = x_0 x_1$ and

$$f_3 = x_0(a_0 x_0^2 + a_1 x_0 x_2 + a_2 x_2^2) + x_1(a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2) + a_6 x_2^3,$$

where $a_6 \neq 0$. So we can write:

$$F = x_3 x_0 x_1 + x_0(a_0 x_0^2 + a_1 x_0 x_2 + a_2 x_2^2) + x_1(a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2) + a_6 x_2^3.$$

Suppose that x is not semi-stable. Then there exists an 1-PS λ of $SL(4)$ such that $\mu(x, \lambda) < 0$. As in the above case, by the formula (3.2), this means that there exists a 4-tuple $(a, b, c, d) \in (\mathbb{Z}^4)^*$ such that $a + b + c + d = 0$ and $a + b + d \leq 0$ and $3c \leq 0$. But it is easy to see that there is no such 4-tuple. So x is semi-stable.

We prove that x is not stable. It is enough to find a 4-tuple $(a, b, c, d) \in (\mathbb{Z}^4)^*$ such that $a + b + c + d = 0$ and all values $a + b + d$, $3a$, $2a + c$, $a + 2c$, $3b$, $2b + c$, $b + 2c$ and $3c$ are not positive. We can choose $b = c = 0$, $a = -1$ and $d = 1$.

c) Finally, let x be a point corresponding to a singular cubic surface. Suppose that x does not belong to any $i\mathcal{A}_1 j\mathcal{A}_2$ for $2i + 3j \leq 9$, $i \leq 4$ and $(i,j) \neq (3,1)$. We prove that x is not semi-stable. It is enough to find an 1-PS λ of SL(4) such that $\mu(x, \lambda) < 0$.

(c_1) The case that the corresponding cubic surface X_x is a reducible cubic surface. We can assume that X_x is given by a cubic form $F = x_3 f_2(x_0, x_1, x_2, x_3)$ where f_2 is a homogeneous polynomial of degree 2. The monomials of F have forms $x_0^{t_1} x_1^{t_2} x_2^{t_3} x_3^{3-t_1-t_2-t_3}$ where $0 \leq t_1 + t_2 + t_3 \leq 2$. If we choose the 1-PS λ of the form (3.1) with $a = b = c = 1$, $d = -3$ then $at_1 + bt_2 + ct_3 + d(3 - t_1 - t_2 - t_3) = 4(t_1 + t_2 + t_3) - 9 \leq 8 - 9 = -1$. So $\mu(x, \lambda) < 0$.

(c_2) The case that the corresponding cubic surface X_x is an irreducible, reduced surface with non-isolated singularities. This implies that a general hyperplane section is an irreducible cubic curve and therefore has only one singular point. This means that the singular locus is a line. We can assume that the line of singular locus is given by $V(x_0, x_1)$. Then the surface can be given by a polynomial $F = f_3(x_0, x_1) + x_2 f_2(x_0, x_1) + x_3 g_2(x_0, x_1)$ where g_2 and f_i for $i = 2, 3$ are homogeneous polynomials of degree 2 and i respectively (see [B-W], p. 252). Every monomial of F has one of the following forms

$$x_0^r x_1^{3-r} \text{ for } 0 \leq r \leq 3, \quad x_0^t x_1^{2-t} x_2, \quad x_0^u x_1^{2-u} x_3 \text{ for } 0 \leq t, u \leq 2. \qquad (3.3)$$

We choose an 1-PS λ in form (3.1) with $a = b = -1$, $c = d = 1$ then from (3.3), we have:

$$r(a - b) + 3b = -3; \quad (a - b)t + 2b + c = -1; \quad (a - b)u + 2b + d = -1.$$

So we have $\mu(x, \lambda) < 0$.

(c_3) The case that x is of the type D_4 or D_5 or E_6. We can choose coordinates such that x is given by a polynomial $F = x_3 x_0^2 + f_3(x_0, x_1, x_2)$ where f_3 is a homogeneous polynomial of degree 3. Moreover, we can assume that $Q = (0 : 0 : 1 : 0) \in V(f_3)$. This means that the coefficient of the monomial x_2^3 in f_3 is zero. The monomials of F are $x_3 x_0^2$ and $x_0^{3-i-j} x_1^i x_2^j$ for $0 \leq i + j \leq 3$ and $j < 3$. We choose an 1-PS λ in form (3.1) with $a = b = -3$, $c = 1$ and $d = 5$. Then we have $2a + d = -1$ and $(3 - i - j)a + ib + jc = -3(3 - i - j) - 3i + j = -9 + 4j < 0$. So $\mu(x, \lambda) < 0$.

(c_4) The case that x is of the type $\hat{E}_6$. We may assume that x is given by an irreducible homogeneous polynomial $f_3(x_0, x_1, x_2)$. The monomials of f_3 are $x_0^{3-i-j} x_1^i x_2^j$ for $0 \leq i + j \leq 3$. Then we can choose an 1-PS λ in form (3.1) with $a = b = c = -1$ and $d = 3$. Then we have $(3 - i - j)a + ib + jc = -3$. So $\mu(x, \lambda) < 0$.

(c_5) The case that x belongs to one of $\mathcal{A}_1\mathcal{A}_3$, $\mathcal{A}_1\mathcal{A}_4$, $\mathcal{A}_1\mathcal{A}_5$, $2\mathcal{A}_1\mathcal{A}_3$. Assume that $P = (0 : 0 : 0 : 1)$ is an A_1 singularity of the corresponding cubic surface X_x. Then X_x is given by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i. Moreover, as in the case a), we can choose coordinates such that $f_2 = x_1^2 - x_0 x_2$ and

$$f_3 = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^3,$$

see [B-W], p. 248. So we can write:

$$F = x_3(x_1^2 - x_0x_2) + a_0x_0^3 + a_1x_0^2x_1 + a_2x_0x_1^2 + a_3x_1^3 + a_4x_1^2x_2 + a_5x_1x_2^2 + a_6x_2^3.$$

Moreover, we can assume that $Q = (0 : 0 : 1 : 0) \in V(f_2, f_3, x_3)$ is a multiple point with multiplicity at least 4. This implies that $a_i = 0$ for all $3 \le i \le 6$. The monomials of F are $x_3x_1^2$, $x_3x_0x_2$, x_0^3, $x_0^2x_1$ and $x_0x_1^2$. We choose an 1-PS λ in form (3.1) with $a = d = -3, b = 1, c = 5$. Then $2b + d = -1$, $a + c + d = -1$, $3a = -9$, $2a + b = -5$ and $a + 2b = -1$. So $\mu(x, \lambda) < 0$.

(c_6) Finally, we consider the case that x belongs to one of $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$. As in the case b), we can assume that x is given by

$$F = x_3x_0x_1 + x_0(a_0x_0^2 + a_1x_0x_2 + a_2x_2^2) + x_1(a_3x_1^2 + a_4x_1x_2 + a_5x_2^2).$$

The monomials of F are $x_3x_0x_1$, x_0^3, $x_0^2x_2$, $x_0x_2^2$, x_1^3, $x_1^2x_2$, $x_1x_2^2$. We choose an 1-PS λ in form (3.1) with $a = b = -3$, $c = 1$, $d = 5$. Then $a + b + d = -1, 3a = 3b = -6, 2a + c = 2b + c = -5, a + 2c = b + 2c = -1$. So $\mu(x, \lambda) < 0$.

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

3.3 The csurfaces of 6-point schemes in almost general position

In this section, we make a further study of semi-stable cubic surfaces; we generalize a useful method which was used in Chapter 2 to study non-singular cubic surfaces. We know that the blowing-up of $\mathbb{P}^2$ at 6 points in general position is isomorphic to a non-singular cubic surface. Conversely, each non-singular cubic surface is isomorphic to the blowing-up of $\mathbb{P}^2$ at 6 points in general position. We will define the concepts of 6-point scheme, 6-point scheme in almost general position and the csurface of a 6-point scheme in almost general position. The last two ones can be considered as the generalization of the concepts of general position and the blowing-up of $\mathbb{P}^2$ at 6 points in general position. We will prove that any semi-stable cubic surface is isomorphic to the csurface of a 6-point scheme in almost general position with a specific configuration. By this way, we can study properties of semi-stable cubic surfaces by checking corresponding 6-point schemes in $\mathbb{P}^2$; we can compute the multiplicities of lines and triple intersections on semi-stable cubic surfaces. Especially, in this way, we can study boundaries of subvarieties parameterizing non-singular cubic surfaces with a given number of star points as well as the boundaries of their moduli spaces in $\mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{ss}$.

A. 6-point schemes and 6-point schemes in almost general position

Definition: A *6-point scheme* is a closed subscheme in $\mathbb{P}^2$ of dimension zero and of length 6. Any 6-point scheme $\mathcal{P}$ defines a formal cycle $c(\mathcal{P}) = \sum n_iP_i$ for $\sum n_i = 6$; the set of the points P_i is called *the support* of $\mathcal{P}$ and denoted by $\mathrm{Supp}(\mathcal{P})$. If the linear system of all cubic forms passing through a 6-point scheme $\mathcal{P}$ has (linear) dimension 4, then $\mathcal{P}$ is called a 6-point scheme *in almost general position*.

Remark 3.3.1.

(i) Any subscheme of $\mathbb{P}^2$ consisting of 6 distinct points in general position is a 6-point scheme in almost general position. Let $\mathcal{H}^{gp}$ denote the subset of all 6-point schemes in general position.

(ii) Let Hilb_n denote the Hilbert scheme of zero-dimensional closed subschemes of length n in $\mathbb{P}^2$. For the definition, construction and the existence of Hilbert schemes, we refer to [G], Exp. 221, Section 3. It is well-known that Hilb_n is non-singular, projective and irreducible of dimension $2n$ (see [F], 2.4 and [E-H], p. 136). We denote by $\mathcal{H}^a$ the subscheme of Hilb_6 consisting of all 6-point schemes in almost general position.

(iii) Let $\mathcal{P} \in \mathrm{Hilb}_6$. Let $I \subset k[x_0, x_1, x_2]$ is the homogeneous ideal defining $\mathcal{P}$. "A cubic form $f_3 \in k[x_0, x_1, x_2]$ passing through $\mathcal{P}$" means that f is an element of I.

(iv) Let $\mathcal{P} \in \mathrm{Hilb}_6$. Let $h_{\mathcal{P}}$ be the Hilbert function corresponding to $\mathcal{P}$. Then $\mathcal{P}$ is a 6-point scheme in almost general position if and only if $h_{\mathcal{P}}(3) = 6$.

Lemma 3.3.2. *Let $\mathcal{P} \in \mathcal{H}^a$ and let l be any line in $\mathbb{P}^2$ such that $l \cap \mathcal{P} \neq \emptyset$. Then the length of $l \cap \mathcal{P}$ is not greater than 4.*

Proof. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms passing through $\mathcal{P}$. Let f_3 be an element of $\mathcal{L}_{\mathcal{P}}$. If the length of $l \cap \mathcal{P}$ is at least 4 then f_3 factors into the linear form defining l and a quadratic form passing $\mathcal{P} - \{l \cap \mathcal{P}\}$. Suppose that the length of $l \cap \mathcal{P}$ is greater than 4. This implies that the linear space of quadratic forms passing through $\mathcal{P} - \{l \cap \mathcal{P}\}$ has dimension greater than 4. This means that the dimension of $\mathcal{L}_{\mathcal{P}}$ is greater than 4. A contradiction! $\qquad\square$

Definition: Let $\mathcal{P} \in \mathcal{H}^a$. We say that $\mathcal{P}$ is a 6-point scheme with *no 4 points on a line* if there does not exist any line l in $\mathbb{P}^2$ satisfying that the length of $l \cap \mathcal{P}$ equals to 4. Denote $\mathcal{H}^o$ for the subset of 6-point schemes with no 4 points on a line.

Lemma 3.3.3. *The subscheme $\mathcal{H}^a$ in Hilb_6 has dimension 12 . The same holds for $\mathcal{H}^o$.*

Proof. Consider the morphism $c : \mathcal{H}_6 \longrightarrow S_6\backslash(\mathbb{P}^2)^6$ which maps each $\mathcal{P} \in \mathrm{Hilb}_6$ to the formal cycle of $\mathcal{P}$. Let $U = S_6\backslash((\mathbb{P}^2)^6 - \Delta)$, where Δ is the diagonal closed subset of $(\mathbb{P}^2)^6$. Let U^a (respectively U^o) be the subset of U consisting of elements with 6 distinct points such that no 5 (respectively 4) points collinear. It is clear that U^a and U^o are open subsets of U. Note that $c^{-1}(U_a) \subset \mathcal{H}^a$ and $c^{-1}(U^o) \subset \mathcal{H}^o$. The morphism c induces isomorphisms from $c^{-1}(U^a)$ to U^a and from $c^{-1}(U^o)$ to U^o. $\qquad\square$

B. Csurfaces of 6-point schemes in almost general position

Lemma 3.3.4. *Let $\mathcal{P} \in \mathcal{H}^o$. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms passing through $\mathcal{P}$.*

(i) *The base locus of $\mathcal{L}_{\mathcal{P}}$ is the support of $\mathcal{P}$.*

(ii) *Let $\{f_1, \ldots, f_4\}$ be a basis of $\mathcal{L}_{\mathcal{P}}$. Consider the morphism*

$$\psi: \quad \begin{array}{ccc} \mathbb{P}^2 - Supp(\mathcal{P}) & \longrightarrow & \mathbb{P}^3 \\ P & \longmapsto & (f_1(P) : f_2(P) : f_3(P) : f_4(P)). \end{array}$$

Let X be the closure of the image of ψ. Then X is a cubic surface.

(iii) *If $\{g_1, \ldots, g_4\}$ is another basis of $\mathcal{L}_{\mathcal{P}}$ and X' is the cubic surface obtained as in (ii), then X and X' are isomorphic.*

Proof.

(i) Let $P \in \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$. Since $\mathcal{P}$ does not have 4 points on a line, there exists a cubic form in $\mathcal{L}_{\mathcal{P}}$ which does not contain P. This implies that the base locus of $\mathcal{L}_{\mathcal{P}}$ is the support of $\mathcal{P}$.

(ii) Let Q_1, Q_2 be two general points in $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$. The linear subspaces consisting of cubic forms through $\mathcal{P} \cup \{Q_1\}$ and $\mathcal{P} \cup \{Q_1, Q_2\}$ respectively have dimension 3 and 2. This implies that there exists a cubic form in $\mathcal{L}_{\mathcal{P}}$ which contains Q_1 but does not contain Q_2 and conversely. This means that ψ is injective over an open subset of $\mathbb{P}^2$. Moreover, any two general cubic forms in $\mathcal{L}_{\mathcal{P}}$ have 3 other points in common which do not belong to $\mathcal{P}$. This implies that X is a cubic surface.

(iii) Let $A = (a_{ij})_{4 \times 4}$ be the matrix of changing bases from $\{f_1, \ldots, f_4\}$ to $\{g_1, \ldots, g_4\}$. Then A defines a projective transformation which transforms X to X'.

$\square$

Definition: A *csurface* is an algebraic variety Y such that there exists a cubic surface $X \subset \mathbb{P}^3$ such that $X \cong Y$.

From the lemma, we see that each $\mathcal{P} \in \mathcal{H}^o$ determines uniquely (up to isomorphisms) a csurface, which is called the csurface of $\mathcal{P}$. If $\mathcal{P} \in \mathcal{H}^{gp}$, then the csurface of $\mathcal{P}$ is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$.

Consider the case that $\mathcal{P}_0$ contains 4 points on a line. There exists a line l such that $l \cap \mathcal{P}_0$ has length 4. In this case the base locus of $\mathcal{L}_{\mathcal{P}_0}$ contains all points on l. Each cubic form in $\mathcal{L}_{\mathcal{P}_0}$ factors into the linear form defining l and a quadratic form passing through $\mathcal{P}_0 - \{\mathcal{P}_0 \cap l\}$, which is contained in a unique line d. In other words, we can write $\mathcal{L}_{\mathcal{P}_0} = l.\mathcal{Q}$, where $\mathcal{Q}$ is the linear space of quadratic forms passing through $\mathcal{P}_0 - \{\mathcal{P}_0 \cap l\}$. So each choice of a basis of $\mathcal{L}_{\mathcal{P}_0}$ reduces to a basis of $\mathcal{Q}$. The closure of the map defined by this basis of $\mathcal{Q}$ is a quadric surface Q in $\mathbb{P}^3$. Let H be the hyperplane in $\mathbb{P}^3$ corresponding to the quadratic form in $\mathcal{Q}$ defining $C = l \cup d$. Set $C_{\mathcal{P}_0} := Q \cup H$. The surface $C_{\mathcal{P}_0}$ is called the csurface of $\mathcal{P}_0$.

Definition: Let $P_0 = (1 : 0 : 0)$, $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$. Let $\varphi : \mathbb{P}^2 - - \to \mathbb{P}^2$ be the quadratic transformation with respect to P_0, P_1 and P_2 (see [H], V.4.2.3). Let C be the cubic curve given by

$$F = \sum a_{ijk} x_0^i x_1^j x_2^k \text{ for } i + j + k = 3 \text{ and } 0 \le i, j, k \le 2. \tag{3.4}$$

The cubic curve defined by $F_{\varphi} := \sum a_{ijk} y_0^{2-i} y_1^{2-j} y_2^{2-k}$ in $\mathbb{P}^2$ is called the image of C by φ and is denoted by C_{φ}.

Remark 3.3.5.

(i) A cubic curve given by a polynomial as in (3.4) contains the 3 points P_0, P_1 and P_2 and its image contains the 3 points $Q_0 = (1 : 0 : 0)$, $Q_1 = (0 : 1 : 0)$ and $Q_2 = (0 : 0 : 1)$ in the second $\mathbb{P}^2$.

(ii) Let φ^{-1} be the inverse of φ. We know that $\varphi^{-1} = \varphi$. It is clear that $(F_\varphi)_{\varphi^{-1}} = F$ where F is a polynomial as in (3.4).

From now on, when we consider a quadratic transformation with respect to $P_i = (1 : 0 : 0), P_j = (0 : 1 : 0), P_k = (0 : 0 : 1)$ from $\mathbb{P}^2$ to $\mathbb{P}^2$, we use notation Q_i, Q_j, Q_k for the points which are the images of lines $\overline{P_j P_k}, \overline{P_i P_k}$ and $\overline{P_i P_j}$, respectively. This means that $Q_i = (1 : 0 : 0), Q_j = (0 : 1 : 0)$ and $Q_k = (0 : 0 : 1)$ in the second $\mathbb{P}^2$.

Lemma 3.3.6. *Let C be a cubic curve given by a polynomial as in (3.4).*

(i) *If C is non-singular at any P_i then C_φ equals to the closure of $\varphi(C - \{P_0, P_1, P_2\})$.*

(ii) *If C is singular at only one P_i then C_φ factors into $d_{jk} := \overline{Q_j Q_k}$ for $\{i, j, k\} = \{0, 1, 2\}$ and the closure of $\varphi(C - \{P_0, P_1, P_2\})$.*

(iii) *If C factors into $l_{jk} := \overline{P_j P_k}$ and a conic then C_φ is singular at Q_i for $\{i\} = \{0, 1, 2\} - \{j, k\}$.*

(iv) *Suppose that C is singular at P_0, P_1 and P_2. This means that $C = l_{ij} \cup l_{ik} \cup l_{jk}$. Then $C_\varphi = d_{ij} \cup d_{ik} \cup d_{jk}$.*

(v) *Suppose that C is singular at P_i, P_j and non-singular at P_k.*

    (a) *If $C = 2l_{ij} \cup l$ then C_φ factors into d_{ik}, d_{jk} and the closure of $\varphi(l - \{P_0, P_1, P_2\})$.*

    (b) *If C factors into l_{ij} and an irreducible conic C_0 containing P_0, P_1, P_2 then C_φ factors into d_{ik}, d_{jk} and the closure of $\varphi(C_0 - \{P_0, P_1, P_2\})$.*

(vi) *If C is irreducible and C contains a given point P on the line l_{ij} for $i, j \in \{0, 1, 2\}$, then C_φ is tangent at P_k to a direction which is determined uniquely by P, where $\{k\} = \{0, 1, 2\} - \{i, j\}$. Conversely, if C is non-singular at P_k and tangent to a given direction at P_k for $k \in \{0, 1, 2\}$, then C_φ contains a specific point $P \in \overline{Q_i Q_j}$ defined by the direction, where $\{i, j\} = \{0, 1, 2\} - \{k\}$.*

Proof.

(i) Let $P \in C - \{P_0, P_1, P_2\}$ and $P = (x_0 : x_1 : x_2)$. We have $\varphi(P) = (x_1 x_2 : x_0 x_2 : x_0 x_1)$. Since $F_\varphi(\varphi(P)) = 0$, we have $\varphi(P) \in C_\varphi$. This implies that the closure of $\varphi(C - \{P_0, P_1, P_2\})$ is contained in C_φ. Moreover, since C is non-singular at any P_i, a general cubic curve containing P_0, P_1, P_2 intersects C at 3 other distinct points. This implies that the closure of $\varphi(C - \{P_0, P_1, P_2\})$ is a reduced cubic curve. So F_φ is a homogeneous polynomial defining this closure.

(ii) Suppose that C is singular at $P_0 = (1 : 0 : 0)$. Then C is given by

$$F = a_1 x_0 x_1^2 + a_2 x_0 x_2^2 + a_3 x_1^2 x_2 + a_4 x_1 x_2^2 + a_5 x_0 x_1 x_2.$$

We see that

$$\begin{aligned} F_\varphi &= a_1 y_0 y_2^2 + a_2 y_0 y_1^2 + a_3 y_0^2 y_2 + a_4 y_0^2 y_1 + a_5 y_0 y_1 y_2 \\ &= y_0 (a_1 y_2^2 + a_2 y_1^2 + a_3 y_0 y_2 + a_4 y_0 y_1 + a_5 y_1 y_2). \end{aligned}$$

This implies that the line $d_{12} = V(y_0) \subset C_\varphi$. Moreover, for any $P \in C - \{P_0, P_1, P_2\}$, we have $\varphi(P) \in C_\varphi$. This implies that the closure of $\varphi(C - \{P_0, P_1, P_2\})$ is contained in C_φ.

Conversely, let T be the closure of $\varphi(C - \{P_0, P_1, P_2\})$. Then T is a reduced conic curve. This means that $d \cup T$ is a reduced cubic curve. This implies that $C_\varphi = d \cup T$.

(iii) Suppose that C factors into $l_{12} := \overline{P_1 P_2}$ and a conic C_0 containing $P_0 = (1 : 0 : 0)$. Then C is defined by $F = x_0(a_1 x_2^2 + a_2 x_1^2 + a_3 x_0 x_2 + a_4 x_0 x_1 + a_5 x_1 x_2)$. We see that $F_\varphi = a_1 y_0 y_1^2 + a_2 y_0 y_2^2 + a_3 y_1^2 y_2 + a_4 y_1 y_2^2 + a_5 y_0 y_1 y_2$. Therefore C_φ is singular at $Q_0 = (1 : 0 : 0)$.

(iv) It is clear from the definition of C_φ.

(v) and (vi). Use the same argument as above.

$\square$

Lemma 3.3.7. *Let $\mathcal{P} \in \mathcal{H}^o$. Suppose that $\mathcal{P}$ contains 3 distinct points P_1, P_2 and P_3. Suppose further that there exists a cubic form in $\mathcal{L}_\mathcal{P}$ which is non-singular at any P_i for $i = 1, 2, 3$. Let φ be the quadratic transformation with respect to P_1, P_2 and P_3. Then the set $\varphi(\mathcal{L}_\mathcal{P}) := \{F\varphi \mid F \in \mathcal{L}_\mathcal{P}\}$ is a 4-dimensional linear space whose base locus is of dimension 0.*

Proof. Choose coordinates such that $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. Suppose that the base locus of $\varphi(\mathcal{L}_\mathcal{P})$ contains an irreducible component Y of positive dimension. Since φ is one-to-one in $\mathbb{P}^2 - V(x_0 x_1 x_2)$, the variety Y is contained in $V(y_0 y_1 y_2)$. Assume that Y contains the line $d_{12} = V(y_0)$. This means that for any $F \in \mathcal{L}_\mathcal{P}$, we have $F_\varphi = y_0 g_2(y_0, y_1, y_2)$ where g_2 is a homogeneous polynomial of degree 2 and vanishes at $Q_3 = (0 : 0 : 1)$. Then $F = (F_\varphi)_{\varphi^{-1}}$ is singular at $P_1 = (1 : 0 : 0)$, see (3.3.6). A contradiction! $\square$

Definition: Let $\mathcal{P} \in \mathcal{H}^o$ satisfy the conditions as in the previous lemma. Let I be the ideal generated by all cubic forms in $\varphi(\mathcal{L}_\mathcal{P})$. The scheme defined by this ideal is called the image of $\mathcal{P}$ and denoted by $\varphi(\mathcal{P})$.

Proposition 3.3.8. *Every semi-stable cubic surface is isomorphic to the csurface of some 6-point scheme in almost general position with no 4 points on a line.*

Proof. Let X be a semi-stable cubic surface. If X is a non-singular cubic surface then X is isomorphic to the blowing-up of a 6-point scheme in general position. We consider the case that X is singular.

Suppose that X does not have any A_2 singularity. By choosing coordinates, we may assume X to be defined by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i. The scheme $\mathcal{P} = V_{\mathbb{P}^2}(f_2, f_3)$ defines an element in $\mathcal{H}^o$. The 6-point scheme $\mathcal{P}$ is contained in an irreducible conic curve defined by f_2 and the cycle $c(\mathcal{P})$ corresponds to a partition $(2^{i-1}1^k)$ of 6. Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Since $\mathcal{P}$ does not contain any triple point, we see that the cubic forms $x_0 f_2, x_1 f_2$ and $x_2 f_2$ are elements of $\mathcal{L}_\mathcal{P}$. Moreover, we have $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ is a basis of $\mathcal{L}_\mathcal{P}$.

Consider the morphism $\psi : \mathbb{P}^2 - \operatorname{Supp}(\mathcal{P}) \longrightarrow \mathbb{P}^3$ determined by this basis. Then we see that $F(x_0 f_2, x_1 f_2, x_2 f_2, -f_3) = -f_3 f_2^3 + f_3 f_2^3 = 0$. This means that X is isomorphic to the csurface of $\mathcal{P}$.

Consider the case that X contains at least one A_2 singularity. By choosing coordinates, we may assume X to be defined by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and f_2 is reducible. The scheme $\mathcal{P} = V_{\mathbb{P}^2}(f_2, f_3)$ defines an element in H^o which corresponds to a partition $(3^{j-1}2^i 1^k)$ of 6, where $j \geq 1$. Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Note that, if $\mathcal{P}$ has a multiple point then the direction at the multiple point is contained in the reducible conic defined by f_2. This implies that the cubic forms $x_0 f_2, x_1 f_2$ and $x_2 f_2$ are elements of $\mathcal{L}_\mathcal{P}$. Moreover, we have $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ is a basis of $\mathcal{L}_\mathcal{P}$. As above, we see that X is isomorphic to the csurface of $\mathcal{P}$. $\qquad\square$

Remark 3.3.9. Let $\mathcal{P} \in \mathcal{H}^o$ such that the csurface of $\mathcal{P}$ is isomorphic to a semi-stable cubic surface and the support of $\mathcal{P}$ contains at least 3 distinct points. Let P_1, P_2, P_3 be some 3 distinct points contained in $\mathcal{P}$. Choose coordinates such that $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. Let φ be the quadratic transformation with respect to P_1, P_2 and P_3. As in the proof of the previous lemma, there exist a basis of $\mathcal{L}_\mathcal{P}$ of the form $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ where $f_2, f_3 \in k[x_0, x_1, x_2]$ are homogeneous polynomials such that the csurface of $\mathcal{P}$ is isomorphic to the surface $X = V(x_3 f_2 + f_3)$.

On the other hand, we see that $\{(x_0 f_2)_\varphi, (x_1 f_2)_\varphi, (x_2 f_2)_\varphi, -(f_3)_\varphi\}$ is a basis of the linear space $\varphi(\mathcal{L}_\mathcal{P})$. Consider the morphism:

$$\begin{aligned} \mathbb{P}^2 - \operatorname{Supp}(\varphi(\mathcal{P})) &\longrightarrow \mathbb{P}^3 \\ (y_0 : y_1 : y_2) &\longmapsto ((x_0 f_2)_\varphi : (x_1 f_2)_\varphi : (x_2 f_2)_\varphi : (-f_3)_\varphi) \end{aligned}$$

defined by this basis. The closure of the image of this morphism is a surface Y. We will see that the surface Y is isomorphic to X. For this, let $f_2 = a_1 x_0 x_1 + a_2 x_0 x_2 + a_3 x_1 x_2$. Then f_2 defines a conic curve containing P_1, P_2, P_3. We have:

$$\begin{aligned} (x_0 f_2)_\varphi &= y_1 y_2 (a_1 y_2 + a_2 y_1 + a_3 y_0), \\ (x_1 f_2)_\varphi &= y_0 y_2 (a_1 y_2 + a_2 y_1 + a_3 y_0), \\ (x_2 f_2)_\varphi &= y_0 y_1 (a_1 y_2 + a_2 y_1 + a_3 y_0). \end{aligned}$$

Let $h_1 = a_1 y_2 + a_2 y_1 + a_3 y_0$ and $F = x_3 f_2 + f_3$. We have

$$F\big((x_0 f_2)_\varphi, (x_1 f_2)_\varphi, (x_2 f_2)_\varphi, (-f_3)_\varphi\big)$$
$$= (-f_3)_\varphi f_2(y_1 y_2 h_1, y_0 y_2 h_1, y_0 y_1 h_1) + f_3(y_1 y_2 h_1, y_0 y_2 h_1, y_0 y_1 h_1)$$
$$= (-f_3)_\varphi h_1^2 f_2(y_1 y_2, y_0 y_2, y_0 y_1) + h_1^3 f_3(y_1 y_2, y_0 y_2, y_0 y_1).$$

Note that

$$f_2(y_1 y_2, y_0 y_2, y_0 y_1) = a_1 y_0 y_1 y_2^2 + a_2 y_0 y_1^2 y_2 + a_3 y_0^2 y_1 y_2$$
$$= y_0 y_1 y_2 (a_1 y_2 + a_2 y_1 + a_3 y_0) = y_0 y_1 y_2 h_1,$$

and $f_3(y_1 y_2, y_0 y_2, y_0 y_1) = y_0 y_1 y_2 (f_3)_\varphi$. So we have

$$F\big((x_0 f_2)_\varphi, (x_1 f_2)_\varphi, (x_2 f_2)_\varphi, (-f_3)_\varphi\big) = 0.$$

Since F is irreducible, the surface Y is defined by the polynomial F. This implies that $\varphi(\mathcal{P})$ is a 6-point scheme in almost general position. Therefore, we have proved the following proposition.

Proposition 3.3.10. *Let $\mathcal{P} \in \mathcal{H}^o$. Suppose that the csurface of $\mathcal{P}$ is isomorphic to a semi-stable cubic surface and the support of $\mathcal{P}$ contains at least 3 distinct points. Let φ be the quadratic transformation with respect to some 3 distinct points of $\mathcal{P}$. Then the subscheme $\varphi(\mathcal{P})$ is a 6-point scheme in almost general position and the csurface of $\varphi(\mathcal{P})$ is isomorphic to the csurface of $\mathcal{P}$.*

C. Multiplicity of a line on a semi-stable cubic surface. Triple intersection and multiplicity

We recall how the 27 lines of a non-singular cubic surface are obtained by blowing-up $\mathbb{P}^2$ at a 6-point scheme in general position.

Let $\mathcal{P}$ be a 6-point scheme in general position. Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Let $\{f_1, \ldots, f_4\}$ be a basis of $\mathcal{L}_\mathcal{P}$. Consider the morphism:

$$\psi: \quad \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P}) \quad \longrightarrow \quad \mathbb{P}^3$$
$$P \quad \longmapsto \quad \big(f_1(P) : f_2(P) : f_3(P) : f_4(P)\big).$$

The closure of the image of ψ is a non-singular cubic surface X with exactly 27 lines obtained as follows.

(i) Consider the two-dimensional linear subspace S_{P_i} for $1 \le i \le 6$ consisting of cubic forms which are singular at P_i. This subspace determines uniquely a line on X which we denote by $\tilde{P}_i$. The surface X has 6 lines of this type.

(ii) Consider the two-dimensional linear subspace S_{ij}, for $1 \le i < j \le 6$ consisting of all cubic forms which factor into the linear form defining $l_{ij} = \overline{P_i P_j}$ and quadratic forms passing through $\mathcal{P} - \{P_i, P_j\}$. This subspace determines uniquely a line on X which is denoted by $\tilde{l}_{ij}$. The line $\tilde{l}_{ij}$ is the closure of the image of $l_{ij} - \{P_i, P_j\}$. The surface X has 15 lines of this type.

(iii) Consider the two-dimensional linear subspace S_{C_i} for $1 \leq i \leq 6$ consisting of all cubic forms which factor into the quadratic form defining the conic C_i through $\{P_1, \ldots, P_6\} - \{P_i\}$ and linear forms vanishing at P_i. This subspace determines uniquely a line on X, which is denoted by $\tilde{C}_i$. The line $\tilde{C}_i$ is the closure of the image of $C_i - \{P_1, P_2, P_3\}$. So the surface has 6 lines of this type.

Let $\mathcal{L}$ be the scheme consisting of pairs $(x, l) \in \mathbb{P}^{19} \times \mathcal{G}$ where l corresponds to a line on the cubic surface X_x defined by x. Let $\mathcal{L} \longrightarrow \mathbb{P}^{19}$ be the projection. Let $\mathcal{H}$ be the set of 6-point schemes such that the csurface of each element in $\mathcal{H}$ is isomorphic to a semi-stable cubic surface. There exists a morphism from an open covering of $\mathcal{H}$ to $\mathbb{P}^{19}$ mapping each $\mathcal{P} \in \mathcal{H}$ into a point corresponding to an embedding of the csurface of $\mathcal{P}$. The pullback of $\mathcal{L} \longrightarrow \mathbb{P}^{19}$ via this morphism gives a morphism $p : \mathcal{L}^0 \longrightarrow \mathcal{H}$, which is proper.

Let $\mathcal{H}^{gp}$ be the subset of $\mathcal{H}$ consisting of 6-point schemes in general position. Let $(\mathbb{P}^2)_0^6 \subset (\mathbb{P}^2)^6$ be the subset of 6-tuples $(P_1, \ldots, P_6)$ such that the 6 points $P_1, \ldots, P_6$ are in general position. Let $f : (\mathbb{P}^2)_0^6 \longrightarrow H^{gp}$ be the natural map, which is finite and surjective. We have $M := k(\mathcal{H}) = k(\mathcal{H}^{gp}) \subset k((\mathbb{P}^2)_0^6) =: N$. Define $\widetilde{\mathcal{H}}$ as the normalization of $\mathcal{H}$ in $M \subset N$; we have $g : \widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ is finite, proper, surjective and it extends f.

Let $\widetilde{\mathcal{L}}$ be the fibered product of $g : \widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ and $p : \mathcal{L}^0 \longrightarrow \mathcal{H}$. We have sections $\mathcal{P}_i^0, \mathcal{P}_{ij}^0, \mathcal{C}_i^0 : (\mathbb{P}^2)_0^6 \longrightarrow \widetilde{\mathcal{L}}$ corresponding to 27 lines on non-singular cubic surfaces.

Proposition 3.3.11. *The sections $\mathcal{P}_i^0, \mathcal{C}_i^0$ for $1 \leq i \leq 6$ and $\mathcal{P}_{ij}^0$ for $1 \leq i < j \leq 6$ extend uniquely to morphisms $\mathcal{P}_i, \mathcal{C}_i$ and $\mathcal{P}_{ij}$ respectively from $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{L}}$.*

Proof. Note that the morphism $\mathcal{P}_i^0 : (\mathbb{P}^2)_0^6 \longrightarrow \widetilde{\mathcal{L}}$ is injective. Denote by L_i^0 the image of $\mathcal{P}_i^0$. Then we have an isomorphism $\mathcal{P}_i^0 : (\mathbb{P}^2)_0^6 \longrightarrow L_i^0$. Let $\overline{L_i^0}$ be the closure of L_i^0 inside $\widetilde{\mathcal{L}}$. Since the morphism $\overline{L_i^0} \longrightarrow \widetilde{\mathcal{H}}$ is quasi-finite (see (3.3.8) and Section 3.1, p. 49) and proper, it is finite. By Zariski's Main Theorem ([Mu2], p. 288), the morphism $\mathcal{P}_i^0$ extends uniquely to an isomorphism $\mathcal{P}_i : \widetilde{\mathcal{H}} \longrightarrow \overline{L_i^0}$. Similarly, we have isomorphisms $\mathcal{P}_{ij}$ and $\mathcal{C}_i$. $\qquad\square$

By composing with $\widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$, we can consider the morphisms $\mathcal{P}_i, \mathcal{C}_i$ for $1 \leq i \leq 6$ and $\mathcal{P}_{ij}$ for $1 \leq i < j \leq 6$ above as morphisms from $\widetilde{\mathcal{H}}$ to $\mathcal{L}$.

Corollary 3.3.12. *Let $(x, l) \in \mathcal{L}$, where the corresponding cubic surface X_x is semi-stable and isomorphic to the csurface of a given 6-point scheme in $\mathcal{H}$. Then (x, l) is contained in one of the images of the morphisms $\mathcal{P}_i, \mathcal{C}_i$ for $1 \leq i \leq 6$ and $\mathcal{P}_{ij}$ for $1 \leq i < j \leq 6$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{L}} & \overset{\pi}{\longrightarrow} & \mathcal{L}^0 \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{H}} & \overset{g}{\longrightarrow} & \mathcal{H}.
\end{array}
\tag{3.5}
$$

Suppose that surface X_x is isomorphic to the csurface of an element $y \in \mathcal{H}$. Since g is surjective, there exists $z \in \widetilde{\mathcal{H}}$ such that $g(z) = y$. If $x \in \mathbb{P}^{19} - \Delta$, then the result is clear

from the fact of obtaining the lines on a non-singular cubic surface. Consider the case that X_x is singular. We see in page 71 when studying $i\mathcal{A}_1 j\mathcal{A}_2$ for $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$ that the line l corresponds to a two-dimension linear subspace of the forms P_i, C_i for $1 \leq i \leq 6$ or S_{ij} for $1 \leq i < j \leq 6$. This implies that (x, l) is contained in the image of one of the morphisms $\mathcal{P}_i$, $\mathcal{C}_i$ for $1 \leq i \leq 6$ and $\mathcal{P}_{ij}$ for $1 \leq i < j \leq 6$. $\quad\square$

Definition: Let $(x, l) \in \mathcal{L}$, where the correponding cubic surface X_x is isomorphic to the csurface of a given 6-point scheme $\mathcal{P} \in \mathcal{H}$. The number of the morphisms $\mathcal{P}_i$, $\mathcal{C}_i$ for $1 \leq i \leq 6$ and $\mathcal{P}_{ij}$ for $1 \leq i < j \leq 6$ whose images contains (x, l) is called *the multiplicity of* l. Consequently the multiplicity of any line on a given semi-stable cubic surface is at least one.

Remark 3.3.13. Let l be a line on a semi-stable cubic surface X isomorphic to the csurface of a 6-point scheme $\mathcal{P}$. Consider all linear subspaces of $\mathcal{L}_\mathcal{P}$ of the forms S_{P_i}, S_{C_i} for $1 \leq i \leq 6$ and S_{ij} for $1 \leq i < j \leq 6$ which determine l. Then the number of these linear subspaces is the multiplicity of l (see pp. 71-81).

The multiplicity of l can be computed with the help of specializations. Namely, let Γ_0 be an one-dimensional family of non-singular cubic surfaces such that X is the fiber of $x_0 \in \overline{\Gamma_0}$. Suppose further that there exist 27 sections corresponding to the 27 lines over Γ_0; the line l is contained in one of these sections. The multiplicity of l is the number of sections containing l. The multiplicity of l is independent on the family chosen.

Proposition 3.3.14. *Let X be a semi-stable cubic surface and l be a line on X.*

(i) *Suppose that l contains exactly one singular point.*

    (a) *If the singular point is A_1, then l is of multiplicity 2.*

    (b) *If the singular point is A_2, then l is of multiplicity 3.*

(ii) *Suppose that l contains 2 singular points.*

    (a) *If both of singularities are A_1, then l is of multiplicity 4.*

    (b) *If both of singularities are A_2, then l is of multiplicity 9.*

    (c) *If two singularities are of different types, then l is of multiplicity 6.*

(iii) *If l does not contain any singular point, then l is of multiplicity 1.*

Proof.
(1) Consider the case that X contains only A_1 singularities. By choosing coordinates, we assume that X is given by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 1, 2$ is a homogeneous polynomial of degree i. Let $\mathcal{P}$ be the 6-point scheme $V_{\mathbb{P}^2}(f_2, f_3)$. Let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the points P_i for $1 \leq i \leq 6$ are unnecessarily different. We know that X is the closure of the image of the morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P}) \longrightarrow \mathbb{P}^3$ determined by the basis $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ of $\mathcal{L}_\mathcal{P}$. Let C be the conic curve in $\mathbb{P}^2 = V(x_3)$ defined by f_2. It is clear that the image of any

point on $C - \mathrm{Supp}(\mathcal{P})$ is the point $S = (0 : 0 : 0 : 1)$, which is an A_1 singularity. Let P_i is a point in the support of $\mathcal{P}$. Each cubic form in S_{P_i} factors into f_2 and a linear form vanishing at P_i. This implies that the line $\tilde{P}_i$ contains the singular point S. Moreover, we prove that $\tilde{P}_i$ is the line containing S and P_i. For this, suppose that $P_i = (1 : 0 : 0 : 0)$. Any line d containing P_i is given by $V_{\mathbb{P}^2}(a_1 x_1 + a_2 x_2)$. We see that $d \cup \tilde{P}_i = V(F, a_1 x_1 + a_2 x_2)$. The line connecting S and P_i is given by $x_1 = x_2 = 0$. This implies that $\tilde{P}_i$ is the line containing S and P_i.

Let l be a line on X containing at least one A_1 singularity; we may assume l to be one of $\tilde{P}_i$. If l contains exactly one A_1 singularity, then the corresponding point P_i is a single point of $V_{\mathbb{P}^2}(f_2, f_3)$. It is easy to check that the linear subspaces S_{P_i} and S_{C_i} are the same. Moreover, they are different from other linear subspaces of the forms S_{P_i} and S_{ij}. Therefore, the multiplicity of l is 2. If l contains two A_1 singularities, then the corresponding point P_i is a double point of $V_{\mathbb{P}^2}(f_2, f_3)$. So we may assume that in the cycle $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$, the points P_1 coincides with P_2. This implies that the linear subspaces $S_{P_1}, S_{P_2}, S_{C_1}$ and S_{C_2} are the same; in fact, the line l is of multiplicity 4.

Consider the case that l does not contain any singular point. If X has exactly one A_1 singularity, then there exist exactly 6 lines of multiplicity 2. Note that X has exactly 21 lines. This implies that the other 15 lines are of multiplicity 1. So l is of multiplicity 1. If X has exactly two A_1 singularities, then there exist exactly 8 lines with multiplicity 2; there exists one line with multiplicity 4. Note that X has exactly 16 lines. This implies that the other 7 lines of X are of multiplicity 1. So l is of multiplicity 1 in this case. If X has exactly three A_1 singularities, then there exist exactly 6 lines with multiplicity 2, there exist exactly 3 lines with multiplicity 4. In this case, the surface X has exactly 12 lines. This implies that the other 3 lines are of multiplicity 1. This means that l is of multiplicity 1. Finally, if X has exactly four A_1 singularities, then there exist exactly 6 lines with multiplicity 4. Since X has exactly 9 lines, the other 3 lines are of multiplicity 1. So l is of multiplicity 1.

(2) Consider the case that X contains at least one A_2 singularity and l is one line on X through one A_2 singularity. By choosing coordinates, we may assume X to be given by a polynomial of the form

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i; the quadratic form f_2 defines two different lines d_1, d_2 in $\mathbb{P}^2$ where the intersection point of d_1 and d_2 is not contained in $V_{\mathbb{P}^2}(f_3)$. Let $\mathcal{P}$ be the 6-point scheme defined by $V_{\mathbb{P}^2}(f_2, f_3)$. Then l is one of $\overline{SP_i}$, where P_i for $1 \le i \le 6$, are the points (not necessarily different) of $V_{\mathbb{P}^2}(f_2, f_3)$. Suppose that $l = \overline{SP_1}$. We know that the surface X is the closure of the image of the morphism ψ from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ defined by the basis $\{x_0 f_2, x_1 f_2, x_2 f_2, -f_3\}$ of $\mathcal{L}_{\mathcal{P}}$. As in the previous case, we see that the A_2 singular point $S = (0 : 0 : 0 : 1)$ is the image of $(d_1 \cup d_2) - \mathrm{Supp}(\mathcal{P})$; the line $\overline{SP_i}$ is $\tilde{P}_i$ for $1 \le i \le 6$.

If l contains only one singular point, then the corresponding point P_1 is a single point of $V_{\mathbb{P}^2}(f_2, f_3)$. We can assume that $P_1 \in d_1$. We see that $d_1 \cap \mathcal{P}$ defines a zero-dimensional subscheme of length 3, which corresponds to a formal cycle $P_1 + P_2 + P_3$ where P_2 and P_3 are not necessarily different. Note that if P_2 and P_3 coincide, then d_1 is the direction at the double point. With the same argument, we see that $d_2 \cap \mathcal{P}$ defines a zero-dimensional subscheme which corresponds to a formal cycle $P_4 + P_5 + P_6$

where the point P_4, P_5, P_6 are not necessarily different and d_2 is the direction if $d_2 \cap \mathcal{P}$ has multiple point. We see that the linear subspaces S_{P_1}, S_{C_1} and S_{23} are the same. Moreover, the subspace S_{P_1} is different from other subspaces of the forms S_{P_i} and S_{ij}. This means that the line $l(= \tilde{P}_1)$ is of multiplicity 3.

Suppose that l contains another singular point which is of A_1 type. Then the corresponding point P_1 is a double point of $V_{\mathbb{P}^2}(f_2, f_3)$. This means that in the formal cycle $P_1 + P_2 + P_3$, the point P_1 coincides with P_2. We see that the linear subspaces $S_{P_1}, S_{P_2}, S_{C_1}, S_{C_2}, S_{13}$ and S_{23} are the same and in fact, the multiplicity of the line l is 6. Similarly, if l contains two A_2 singularities, then the corresponding point P_1 is a triple point of $V_{\mathbb{P}^2}(f_2, f_3)$. This means that the subscheme $d_1 \cap \mathcal{P}$ defines a cycle $3P_1$. And we can check that the line $l = \tilde{P}_1$ is of multiplicity 9.

Consider the case that l does not contain any singularity. With the same argument used in (1), by checking case by case, we see that l is of multiplicity 1. $\qquad\square$

Definition: Let X be a semi-stable cubic surface. A *triple intersection* of X is a hyperplane intersection which factors into 3 lines. A point $P \in X$ is called a *star point* if it is contained in the intersection of all lines of some triple intersection of X. In that case, the triple intersection is also called a star triple.

Example 9. If X is a non-singular cubic surface, then a triple intersection of X is the hyperplane intersection of a tritangent plane. So there exist exactly 45 different triple intersections on X.

Definition: Let $x \in (\mathbb{P}^{19})^{ss}$. Suppose that x is a specialization of a given one-dimensional family of non-singular cubic surfaces, which locally possesses a section of star points. The specialization position of the section of star point on the corresponding surface X_x is called *a proper star point* with respect to the family. It is clear that a proper star point is a star point.

Example 10. If X is a non-singular cubic surface, then the concepts of star point and proper star point are the same as the concept of star point which we defined in the previous chapter.

Remark 3.3.15. Let X be a semi-stable cubic surface. View X as the csurface of a 6-point scheme $\mathcal{P}$. Let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the points P_i for $1 \leq i \leq 6$ are not necessarily different. Then we see that 45 triple intersections of X corresponding to $(\tilde{P}_i, \tilde{C}_i, \tilde{l}_{ij})$ for $1 \leq i \leq 6$ and $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{hk})$ for $\{i, j, m, n, h, k\} = \{1, \ldots, 6\}$ are not necessarily different. But as in the case of non-singular cubic surfaces, we show in the following proposition that any triple intersection of X is one of these.

Proposition 3.3.16. *Let X be a semi-stable cubic surface. Let T be a triple intersection of X. Then there exists a 6-point scheme $\mathcal{P}$ such that its csurface is isomorphic to X. Furthermore, let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$, where the points P_i for $1 \leq i \leq 6$ are not necessarily different. Then T corresponds to one of triple intersections of the forms $(\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$ for $1 \leq i \leq 6$ or $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$ where $\{i, j, m, n, h, k\} = \{1, \ldots, 6\}$.*

Proof.

(1) Consider the case that X contains only A_1 singularities. There exists a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ contained in an irreducible conic C such that X is the closure of the image of a morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ defined by some basis of $\mathcal{L}_{\mathcal{P}}$. The image

of $C - \mathrm{Supp}(\mathcal{P})$ is an A_1 singularity, which is denoted by S. Let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where P_i for $1 \leq i \leq 6$ are not necessarily different. Let $T = (d_1, d_2, d_3)$ be a triple intersection of X.

Suppose that there exist exactly two of the lines d_1, d_2, d_3 coinciding and containing S. We can assume that $d_1 = d_2 = \tilde{P}_1 = \tilde{P}_2$. By (3.1.13), we see that this happens only when $\tilde{P}_1$ contains (exactly) two A_1 singularities. This means that the corresponding point P_1 in $\mathcal{P}$ is a double point and l_{12} is the direction at P_1. Note that the line $\tilde{l}_{12}$ is a unique line meeting $\tilde{P}_1$ but not containing any singular point since its multiplicity is 1. This implies that $d_3 = \tilde{l}_{12}$. So, in this case, we have $T = (\tilde{P}_1, \tilde{C}_2, \tilde{l}_{12})$.

Consider the case that $d_1 = \tilde{P}_i$, $d_2 = \tilde{P}_j$ where $\tilde{P}_i \neq \tilde{P}_j$. Suppose that $d_3 = \tilde{P}_k$ for $\tilde{P}_k \notin \{\tilde{P}_i, \tilde{P}_j\}$. This means that there exists a cubic curve, which factors into the conic C and a line containing P_i, P_j, P_k. This is impossible since three points P_i, P_j, P_k are not collinear. Suppose that $d_3 = \tilde{l}_{mn}$. The corresponding subspace S_{mn} contains the cubic form $C \cup l_{ij}$ if $S_{mn} = S_{ij}$. This means that $d_3 = \tilde{l}_{ij}$ and $T = (\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$.

Consider the case that $d_1 = \tilde{l}_{ij}, d_2 = \tilde{l}_{mn}$ where $\tilde{l}_{ij} \neq \tilde{l}_{mn}$. The triple intersection T corresponds to the cubic curve which contains $l_{ij} \cup l_{mn}$. This happens only when $\{i, j\} \cup \{m, n\} = \emptyset$. Moreover, we see that $d_3 = \tilde{l}_{kh}$ where $\{i, j, m, n, k, h\} = \{1, \ldots, 6\}$. In this case, we have $T = (\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$.

(2) Consider the case that X contains at least one A_2 singularity. There exists a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ contained in two different lines l_1 and l_2, such that $l_i \cap \mathcal{P}$ for $1 \leq i \leq 2$ has length 3 and X is the closure of the image of a morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by some basis of $\mathcal{L}_{\mathcal{P}}$. Furthermore, we can assume that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the points P_i are not necessarily different and three points P_1, P_2, P_3 are contained in l_1, three points P_4, P_5, P_6 are contained in l_2. As in the case above, we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is a point S, which is an A_2 singularity; the tangent cone at S is two hyperplanes which are $H_1 = \mathrm{span}(S, l_1)$ and $H_2 = \mathrm{span}(S, l_2)$. The lines through S are $\tilde{P}_i$ for $1 \leq i \leq 6$ and they are not necessarily different.

Suppose that T is one of $H_1 \cap X$ and $H_2 \cap X$. Then $T = (\tilde{P}_1, \tilde{C}_2, \tilde{l}_{12})$ or $T = (\tilde{P}_4, \tilde{C}_5, \tilde{l}_{45})$. Consider the case that T is not one of these two hyperplane intersections. Suppose that T contains a line $\tilde{P}_i \subset H_1$ and a line $\tilde{P}_j \subset H_2$, then we see that $T = (\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$. If T is formed by three lines, all of them do not contain the singular point S, then $T = (\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$ where $\{i, m, k\} = \{1, 2, 3\}$ and $\{j, n, h\} = \{4, 5, 6\}$. $\qquad\square$

Remark 3.3.17. Let X be a semi-stable cubic surface isomorphic to the csurface of some 6-point scheme $\mathcal{P}$. We know that a line l on X is determined by a two-dimensional linear subspace of $\mathcal{L}_{\mathcal{P}}$ of the forms S_{P_i}, S_{C_i} or S_{ij} (3.3.12). The number of the linear subspaces determining the line l is the multiplicity of l. Let T be a given triple intersection of X. By the above proposition, we see that T is determined by a triple of linear subspaces of the forms either $(S_{P_i}, S_{C_j}, S_{ij})$ for $1 \leq i < j \leq 6$ or (S_{ij}, S_{mn}, S_{hk}) where $\{i, j, m, n, h, k\} = \{1, \ldots, 6\}$. Considering all the linear subspaces corresponding to three lines of T enables us to determine how many triples of the forms $(\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$ for $1 \leq i \leq 6$ and $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{hk})$ for $\{i, j, m, n, h, k\} = \{1, \ldots, 6\}$ coinciding with T. This fact allows us to talk about the *multiplicity* of T.

As in the case of the multiplicity of a line on X, the multiplicity of T can be computed with the help of specializations. Namely, let Γ_0 be an one-dimensional family of non-singular cubic surfaces such that X is the fiber over $x_0 \in \overline{\Gamma_0}$. Suppose further that there exist 45 sections corresponding to the 45 triple intersections over Γ_0; the

triple T is contained in one of these sections. The multiplicity of T is the number of sections containing T. The multiplicity of T is independent on the family chosen.

The rest of this section is used to determine the multiplicities of lines and triple intersections, the number of star points on semi-stable cubic surfaces. Moreover, we describe how to recognize the singular points as well as the lines on semi-stable cubic surfaces from the configuration of corresponding 6-point schemes. From now on, unless stating differently, when we write the formal cycle $c(\mathcal{P})$ of a given 6-point scheme $\mathcal{P}$, we always mean that the points in the cycle are mutually distinct.

$\mathcal{A}_1$ Let $x \in \mathcal{A}_1$. We know that the corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where the 6 mutually distinct points lie on an irreducible conic curve C (see Figure 3.1).
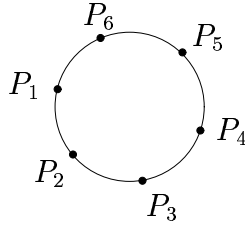
Figure 3.1: 6-point schemes corresponding to points in $\mathcal{A}_1$

By (3.3.14), we see that the image of $C - \mathrm{Supp}(\mathcal{P})$ (via any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$) is the singular point; the lines $\tilde{P}_i = \tilde{C}_i$ for $1 \le i \le 6$ are the 6 lines through the singular point. Other lines of X_x are $\tilde{l}_{ij}$ for $1 \le i < j \le 6$. The 21 lines of X_x with multiplicities correspond to the partition $(2^6, 1^{15})$ of 27. Moreover, we see that triple intersections $(\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$, $(\tilde{P}_j, \tilde{C}_i, \tilde{l}_{ij})$ and $(\tilde{P}_i, \tilde{P}_j, \tilde{l}_{ij})$ for $1 \le i < j \le 6$ are the same. This means that every triple intersection $(\tilde{P}_i, \tilde{P}_j, \tilde{l}_{ij})$ for $1 \le i < j \le 6$ is of multiplicity 2. The surface X_x has 30 distinct triple intersections which correspond to the partition $(2^{15}, 1^{15})$ of 45.

$2\mathcal{A}_1$ Let $x \in 2\mathcal{A}_1$. The corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ contained in an irreducible conic curve C and $c(\mathcal{P}) = 2P_1 + \sum_{i=2}^{5} P_i$ (see Figure 3.2,(a)).

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the double point $2P_1$ is contained in the two sections corresponding the points P_1 and P_6. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. By (3.3.14), we see that the image of $C - \mathrm{Supp}(\mathcal{P})$ is a point S_1, which is an A_1 singularity; the lines $\tilde{P}_i$ for $2 \le i \le 5$ are of multiplicity 2; the line $\tilde{P}_1$ is of multiplicity 4 and it is the line containing two singular points of X_x. The singular point S_1 is contained in the lines $\tilde{P}_i$ for $1 \le i \le 5$. Another singular point is contained in the lines $\tilde{P}_1$ and $\tilde{l}_{1i}$ for $2 \le i \le 5$, since the lines $\tilde{l}_{1i}$ for $2 \le i \le 5$ are of multiplicity 2. The 16 lines of X_x with multiplicities correspond to the partition $(4^1, 2^8, 1^7)$ of 27.
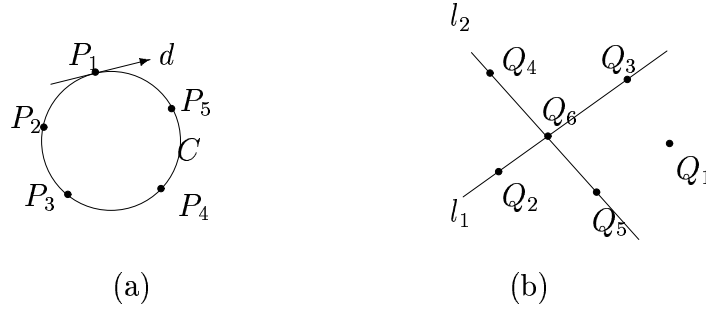
Figure 3.2: 6-point schemes giving points in $2\mathcal{A}_1$

The triple intersection $(2\tilde{P}_1, \tilde{l}_{16})$ has multiplicity 2 since it is determined by $(\tilde{P}_1, \tilde{C}_6, \tilde{l}_{16})$ and $(\tilde{C}_1, \tilde{P}_6, \tilde{l}_{16})$. Note that the line $\tilde{l}_{16}$ is the closure of the image of the direction d at the double point $2P_1$. Every triple intersection $(\tilde{P}_1, \tilde{P}_i, \tilde{l}_{1j})$ for $2 \leq i \leq 5$ has multiplicity 4 since it coincides with $(\tilde{P}_1, \tilde{C}_i, \tilde{l}_{1i})$, $(\tilde{C}_1, \tilde{P}_i, \tilde{l}_{1i})$, $(\tilde{P}_6, \tilde{C}_i, \tilde{l}_{6i})$ and $(\tilde{C}_1, \tilde{P}_i, \tilde{l}_{6i})$. Every triple intersection $(\tilde{P}_i, \tilde{P}_j, \tilde{l}_{ij})$ for $2 \leq i < j \leq 5$ has multiplicity 2 since it coincides with $(\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$ and $(\tilde{C}_i, \tilde{P}_j, \tilde{l}_{ij})$. Similarly, every triple intersection $(\tilde{l}_{1i}, \tilde{l}_{1j}, \tilde{l}_{mn})$ for $\{i, j, m, n\} = \{2, 3, 4, 5\}$ has multiplicity 2. Every triple intersection $(\tilde{l}_{16}, \tilde{l}_{ij}, \tilde{l}_{mn})$ for $\{i, j, m, n\} = \{2, 3, 4, 5\}$ has multiplicity 1. These triples do not contain any singularities. So X has 20 distinct triple intersections which correspond to the partition $(4^4, 2^{13}, 1^3)$ of 45.

We see that the intersection point of $\tilde{P}_1$ and $\tilde{l}_{16}$ is a star point of X. In the next section we will prove that this point is also a proper star point.

Remark 3.3.18. When considering X_x as the csurface of a 6-point scheme as above, it is not clear how the second singular point is obtained. Consider the quadratic transformation with respect to P_1, P_2, P_3. Then the image of $\mathcal{P}$ is a 6-point scheme $\mathcal{Q}$ consisting of 6 distinct points $\{Q_1, \ldots, Q_6\}$ where 3 points Q_2, Q_3, Q_6 lie on a line l_1; three points Q_4, Q_5, Q_6 lie on another line l_2, see Figure 3.2, (b).

By (3.3.10), the surface X_x is isomorphic to the csurface of $\mathcal{Q}$. Then 2 singular points of X are the images of 2 lines l_1, l_2 via any morphism from $\mathbb{P}^2 - \text{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{Q}$. The line corresponds to S_{Q_6} is the line containing the 2 singular points of X_x.

$\mathcal{A}_2$ Let $x \in \mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where 3 points P_1, P_2, P_3 lie on a line l_1; three points P_4, P_5, P_6 lie on another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 3.3).

Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Consider any morphism from $\mathbb{P}^2 - \text{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{P}$. By (3.3.14), the image of $(l_1 \cup l_2) - \text{Supp}(\mathcal{P})$ is the singular point. The 6 lines $\tilde{P}_i$ for $1 \leq i \leq 6$ contain the singularity and they are of multiplicity 3. The other 9 lines of X_x are $\tilde{l}_{ij}$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. These lines are of multiplicity 1. The 15 lines of X_x with multiplicities correspond to the partition $(3^6, 1^9)$ of 27.
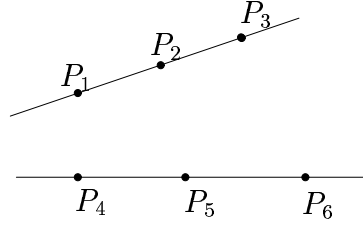
Figure 3.3: 6-point schemes corresponding to points in $\mathcal{A}_2$

Note that the linear subspaces S_{P_i}, S_{C_i} and S_{jk} for $\{i, j, k\} = \{1, 2, 3\}$ or $\{i, j, k\} = \{4, 5, 6\}$ are the same. This implies that the triple intersection $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$ has multiplicity 6 since it coincides with $(\tilde{P}_1, \tilde{C}_2, \tilde{l}_{12})$, $(\tilde{C}_1, \tilde{P}_2, \tilde{l}_{12})$, $(\tilde{P}_1, \tilde{l}_{13}, \tilde{C}_3)$, $(\tilde{C}_1, \tilde{l}_{13}, \tilde{P}_3)$, $(\tilde{l}_{23}, \tilde{P}_2, \tilde{C}_3)$ and $(\tilde{l}_{23}, \tilde{C}_2, \tilde{P}_3)$. Similarly, the triple $(\tilde{P}_4, \tilde{P}_5, \tilde{P}_6)$ has multiplicity 6. Every triple intersection $(\tilde{P}_i, \tilde{P}_j, \tilde{l}_{ij})$ for $1 \leq i \leq 3$ and $4 \leq j \leq 6$ has multiplicity 3 since it coincides with $(\tilde{P}_i, \tilde{C}_j, \tilde{l}_{ij})$, $(\tilde{C}_i, \tilde{P}_j, \tilde{l}_{ij})$ and $(\tilde{l}_{mn}, \tilde{l}_{kh}, \tilde{l}_{ij})$ where $\{m, n\} = \{1, 2, 3\} - \{i\}$, $\{k, h\} = \{4, 5, 6\} - \{j\}$. Every triple intersection $(\tilde{l}_{ij}, \tilde{l}_{mk}, \tilde{l}_{nh})$ for $\{i, m, n\} = \{1, 2, 3\}$, $\{j, k, h\} = \{4, 5, 6\}$ has multiplicity 1. So X_x has 17 distinct triple intersections. The 17 triple intersections with their multiplicities correspond to the partition $(6^2, 3^9, 1^6)$ of 45.

Moreover, we see that the singular point is a star point of X, since it is contained in all lines of the triple intersections $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$. We will prove in the next section that it is a proper star point.

$\mathcal{A}_1\mathcal{A}_2$ Let $x \in \mathcal{A}_1\mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ where $c(\mathcal{P}) = P_1 + P_2 + P_3 + P_4 + 2P_5$ such that P_4 and $2P_5$ are contained in a line l_1; three points P_1, P_2, P_3 are contained in another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 3.4, (a)).

    (a)   (b)
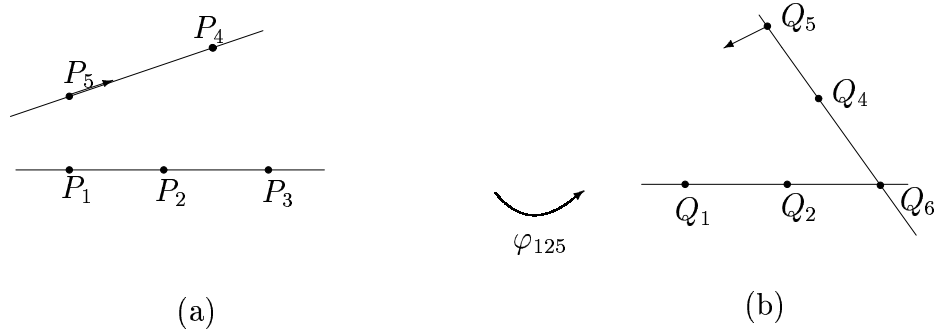
Figure 3.4: 6-point schemes corresponding to elements in $\mathcal{A}_1\mathcal{A}_2$

View X as a point in the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization

position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the double point $2P_5$ is contained in the two sections corresponding to the points P_5 and P_6. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. By (3.3.14), we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is the A_2 singularity; the line $\tilde{P}_5$ is of multiplicity 6 and is the line containing 2 singularities; the lines $\tilde{P}_i$ for $1 \leq i \leq 4$ contain the A_2 singularity and they are of multiplicity 3. Moreover, we see that the lines $\tilde{l}_{i5}$ for $1 \leq i \leq 3$ are of multiplicity 2. So they contain the A_1 singularity. The other lines of X_x are $\tilde{l}_{4i}$ for $1 \leq i \leq 3$ which are of multiplicity 1. The 11 lines of X_x with their multiplicities correspond to the partition $(6^1, 3^4, 2^3, 1^3)$ of 27.

As in the case of $\mathcal{A}_2$, we see that the triple intersections $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$ and $(2\tilde{P}_5, \tilde{P}_4)$ are of multiplicity 6; every triple intersection $(\tilde{P}_4, \tilde{P}_i, \tilde{l}_{4i})$ for $1 \leq i \leq 3$ has multiplicity 3. Every triple intersection $(\tilde{P}_5, \tilde{P}_i, \tilde{l}_{5i})$ for $1 \leq i \leq 3$ has multiplicity 6 since it coincides with $(\tilde{P}_5, \tilde{C}_i, \tilde{l}_{5i})$, $(\tilde{C}_5, \tilde{P}_i, \tilde{l}_{5i})$, $(\tilde{l}_{46}, \tilde{l}_{kh}, \tilde{l}_{5i})$, $(\tilde{P}_6, \tilde{C}_i, \tilde{l}_{6i})$, $(\tilde{C}_6, \tilde{P}_i, \tilde{l}_{6i})$ and $(\tilde{l}_{45}, \tilde{l}_{kh}, \tilde{l}_{6i})$ for $\{k, h\} = \{1, 2, 3\} - \{i\}$. Finally, every triple $(\tilde{l}_{i5}, \tilde{l}_{j5}, \tilde{l}_{k4})$ for $\{i, j, k\} = \{1, 2, 3\}$ has multiplicity 2 since it coincides with $(\tilde{l}_{i5}, \tilde{l}_{j6}, \tilde{l}_{k4})$ and $(\tilde{l}_{i6}, \tilde{l}_{j5}, \tilde{l}_{k4})$. So X has 11 distinct triple intersections. With multiplicities, the triple intersections of X_x correspond to the partition $(6^5, 3^3, 2^3)$ of 45.

The A_2 singularity is a star point of X_x, since it is the intersection of all lines of the triple intersection $(2\tilde{P}_5, \tilde{P}_4)$.

Remark 3.3.19. If we consider the above 6-point scheme, it is not clear how to obtain the A_1 singularity. Consider the quadratic transformation φ_{125} with respect to P_1, P_2, P_5. Let $\mathcal{Q} = \varphi(\mathcal{P})$ be the image of $\mathcal{P}$. We see that $c(\mathcal{Q}) = 2Q_5 + Q_1 + Q_2 + Q_4 + Q_6$, where Q_1, Q_2, Q_6 lie on the line d_1; three points Q_4, Q_5, Q_6 lie on another line d_2 (see Figure 3.4, (b)). The csurface of $\mathcal{Q}$ is isomorphic to X_x. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{Q}}$. In this case, the image of $l_1 - \{Q_1, Q_2, Q_6\}$ is the A_1 singularity; the image of $l_2 - \{Q_4, Q_5, Q_6\}$ is the A_2 singularity; the line $\tilde{Q}_6$ is the line containing two singularities.

$3\mathcal{A}_1$ Let $x \in 3\mathcal{A}_1$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ where $\mathcal{P} = 2P_1 + 2P_2 + P_3 + P_4$ and $\mathcal{P}$ is contained in an irreducible conic curve C (see Figure 3.5, (a)).

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the double point $2P_1$ is contained in the two sections corresponding to the points P_1 and P_5, the double point $2P_2$ is contained in the two sections corresponding to the points P_2 and P_6. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. As in the case of $2\mathcal{A}_1$, we see that the image of $C - \mathrm{Supp}(\mathcal{P})$ is a point S_1, which is an A_1 singularity; the lines $\tilde{P}_1$ and $\tilde{P}_2$ are of multiplicity 4; the lines $\tilde{P}_i$ for $1 \leq i \leq 4$ contain the singular point S_1 (3.3.14).

Moreover, the line $\tilde{l}_{12}$ also has multiplicity 4 since the linear subspaces S_{12}, S_{16}, S_{25} and S_{56} are the same. There exist 3 star points on X_x determined by triple
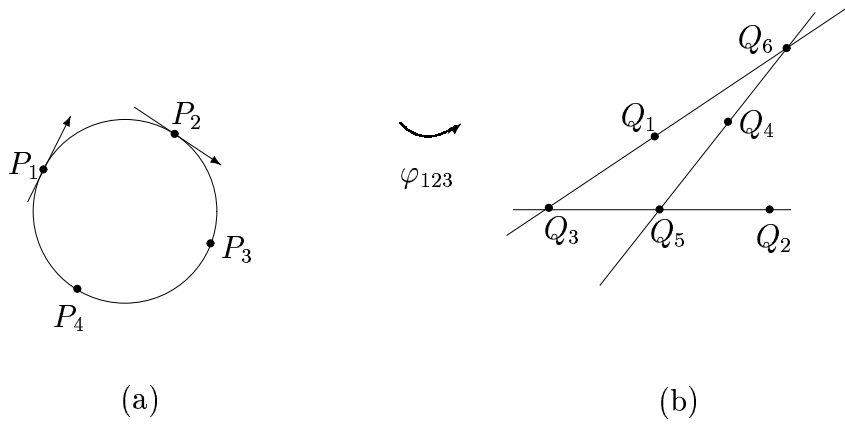
Figure 3.5: 6-point schemes corresponding to elements in $3\mathcal{A}_1$

intersections $(2\tilde{P}_1, \tilde{l}_{15}), (2\tilde{P}_2, \tilde{l}_{26})$ and $(2\tilde{l}_{12}, \tilde{l}_{34})$. We see that $\tilde{P}_1, \tilde{P}_2$ and $\tilde{l}_{12}$ are three lines connecting 3 singular points of X_x. The six lines $\tilde{P}_3, \tilde{P}_4, \tilde{l}_{13}, \tilde{l}_{14}, \tilde{l}_{23}$ and $\tilde{l}_{24}$ have multiplicity 2 and each of them contains exactly one singular point. Namely, the lines $\tilde{P}_3$ and $\tilde{P}_4$ contain $S_1 = \tilde{P}_1 \cap \tilde{P}_2$; the lines $\tilde{l}_{13}$ and $\tilde{l}_{14}$ contain $S_2 := \tilde{P}_1 \cap \tilde{l}_{12}$; the lines $\tilde{l}_{23}$ and $\tilde{l}_{24}$ contain $S_3 := \tilde{P}_2 \cap \tilde{l}_{12}$. The 12 lines of X with multiplicities correspond to the partition $(4^3, 2^6, 1^3)$ of 27.

As in the case of $2\mathcal{A}_1$, the triple intersections $(2\tilde{P}_1, \tilde{l}_{15}), (2\tilde{P}_2, \tilde{l}_{26}), (2\tilde{l}_{12}, \tilde{l}_{34})$ have multiplicity 2. The triple intersection $(\tilde{P}_1, \tilde{l}_{13}, \tilde{P}_3)$ has multiplicity 4 since it coincides with $(\tilde{P}_1, \tilde{l}_{13}, \tilde{C}_3), (\tilde{P}_5, \tilde{l}_{35}, \tilde{C}_3), (\tilde{C}_1, \tilde{l}_{13}, \tilde{P}_3)$ and $(\tilde{C}_5, \tilde{l}_{35}, \tilde{P}_3)$. Similarly, the triple intersections $(\tilde{P}_1, \tilde{l}_{14}, \tilde{P}_4), (\tilde{P}_2, \tilde{l}_{23}, \tilde{P}_3), (\tilde{P}_2, \tilde{l}_{24}, \tilde{P}_4), (\tilde{l}_{12}, \tilde{l}_{13}, \tilde{l}_{24})$ and $(\tilde{l}_{12}, \tilde{l}_{23}, \tilde{l}_{14})$ have multiplicity 4. The triple intersection $(\tilde{l}_{15}, \tilde{l}_{23}, \tilde{l}_{24})$ has multiplicity 2 since it coincides with $(\tilde{l}_{15}, \tilde{l}_{23}, \tilde{l}_{46})$ and $(\tilde{l}_{15}, \tilde{l}_{36}, \tilde{l}_{24})$. Similarly, the triple intersections $(\tilde{l}_{26}, \tilde{l}_{13}, \tilde{l}_{14})$ and $(\tilde{l}_{34}, \tilde{P}_3, \tilde{P}_4)$ have multiplicity 2. The triple intersection $(\tilde{P}_1, \tilde{P}_2, \tilde{l}_{12})$ has multiplicity 8 since it coincides with

$$(\tilde{P}_1, \tilde{C}_2, \tilde{l}_{12}), (\tilde{P}_1, \tilde{C}_6, \tilde{l}_{16}), (\tilde{C}_1, \tilde{P}_2, \tilde{l}_{12}), (\tilde{C}_1, \tilde{P}_6, \tilde{l}_{16}),$$
$$(\tilde{P}_5, \tilde{C}_2, \tilde{l}_{25}), (\tilde{P}_5, \tilde{C}_6, \tilde{l}_{56}), (\tilde{C}_5, \tilde{P}_2, \tilde{l}_{25}) \text{ and } (\tilde{C}_5, \tilde{P}_6, \tilde{l}_{56}).$$

The multiplicity of the triple$(\tilde{l}_{15}, \tilde{l}_{26}, \tilde{l}_{34})$ is 1. So the surface X has 14 distinct triple intersections, which correspond to the partition $(8^1, 4^6, 2^6, 1^1)$.

Remark 3.3.20.

(i) The surface X_x above has three star points, which correspond to the triple intersections$(2\tilde{P}_1, \tilde{l}_{15}), (2\tilde{P}_2, \tilde{l}_{26})$ and $(2\tilde{l}_{12}, \tilde{l}_{34})$. In the next section, we prove that they are proper star points.

(ii) As in the case of $2\mathcal{A}_1$, when considering X_x as the csurface of a 6-point scheme $\mathcal{P}$ as above, it is not clear how to obtain the singular points S_2 and S_3. Consider the quadratic transformation φ_{123} with respect to P_1, P_2, P_3. Let $\mathcal{Q} = \varphi(\mathcal{P})$ is the image of $\mathcal{P}$. Then we see that $c(\mathcal{Q}) = \sum_{i=1}^6 Q_i$ where 3 points Q_2, Q_3, Q_5 lie on a line d_1; three points Q_1, Q_3, Q_6 lie on

another line d_2; three points Q_4, Q_5, Q_6 lie on the third line d_3 (see Figure 3.5, (b)). We know that the csurface of $\mathcal{Q}$ is isomorphic to X_x. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{Q}}$. The three singular points of the csurface of $\mathcal{Q}$ are the images of $d_1 - \{\mathcal{Q}_2, Q_3, Q_5\}, d_2 - \{Q_1, Q_3, Q_6\}$ and $d_3 - \{Q_4, Q_5, Q_6\}$.

$4\mathcal{A}_1$ Let $x \in 4\mathcal{A}_1$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^0$ where $c(\mathcal{P}) = 2P_1 + 2P_2 + 2P_3$ and $\mathcal{P}$ is contained in an irreducible conic C (see Figure 3.6, (a)).
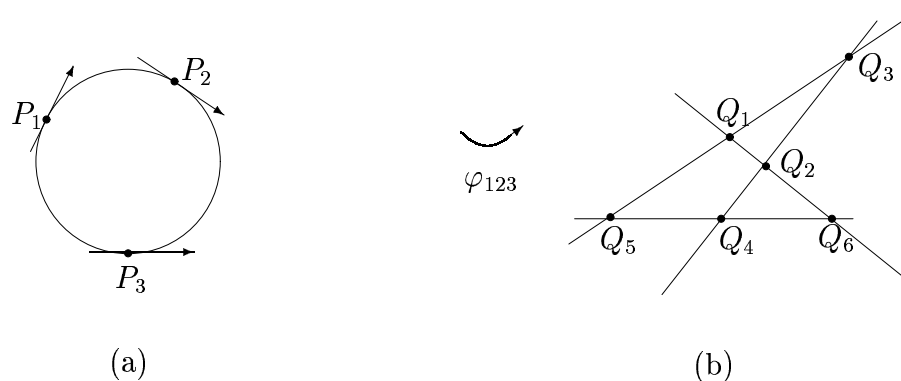

(a)   (b)

Figure 3.6: 6-point schemes corresponding to elements in $4\mathcal{A}_1$

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the double point $2P_1$ is contained in the two sections corresponding to the points P_1 and P_4, the double point $2P_2$ is contained in the two sections corresponding to the points P_2 and P_5 and the double point $2P_3$ is contained in the two sections corresponding to the points P_3 and P_6.

Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. As in the case of $3\mathcal{A}_1$, by (3.3.14), the image of $C - \mathrm{Supp}(\mathcal{P})$ is a point S_1, which is an A_1 singularity; the lines $\tilde{P}_1, \tilde{P}_2$ and $\tilde{P}_3$ have multiplicity 4 and contain S_1. We see that the linear subspaces S_{12}, S_{15}, S_{24} and S_{45} are the same. This implies that the multiplicity of $\tilde{l}_{12}$ is 4. Similarly, the lines $\tilde{l}_{13}, \tilde{l}_{23}$ are of multiplicity 4. The lines $\tilde{P}_i$ for $1 \leq i \leq 3$ and $\tilde{l}_{ij}$ for $1 \leq i < j \leq 3$ form a tetrahedron whose vertices are 4 singular points of X_x. Other lines of X_x are $\tilde{l}_{14}, \tilde{l}_{25}$ and $\tilde{l}_{36}$ which are of multiplicity 1. The 9 lines of X_x with multiplicities correspond to the partition $(4^6, 1^3)$ of 27.

As in the case of $3\mathcal{A}_1$, every triple intersection formed by three A_1 singularities is of multiplicity 8. There exist 6 triple intersections which are of the forms $(2d, l)$ where d is one of the 6 lines of multiplicity 4 and l is one of three lines $\{\tilde{l}_{14}, \tilde{l}_{25}, \tilde{l}_{36}\}$. Each of these 6 triple intersections has multiplicity 2. Finally, the triple intersection $(\tilde{l}_{14}, \tilde{l}_{25}, \tilde{l}_{36})$ has multiplicity 1. So the surface X has 11 distinct triple intersections corresponding to the partition $(8^4, 2^6, 1^1)$ of 45.

Remark 3.3.21.

(i) We see that X_x has 6 triple intersections of the form $(2d, l)$, so X_x has at least 6 star points. We see in the next section that these points are proper star points.

(ii) To see how 4 singular points of X_x are obtained, we consider the quadratic transformation φ_{123} with respect to P_1, P_2, P_3. The image $\varphi_{123}(\mathcal{P})$ consists of 6 distinct points which form a quadrilateral in $\mathbb{P}^2$ (see Figure 3.6, (b)). The csurface of $\varphi_{123}(\mathcal{P})$ is isomorphic to X_x. The 4 edges of the quadrilateral correspond to 4 singular points.

$2\mathcal{A}_1\mathcal{A}_2$ Let $x \in 2\mathcal{A}_1\mathcal{A}_2$. The corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^0$ where $c(\mathcal{P}) = 2P_1 + 2P_2 + P_3 + P_4$ such that $2P_1, P_3$ are contained in a line l_1; the points $2P_2, P_4$ are contained in another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 3.7 (a)).

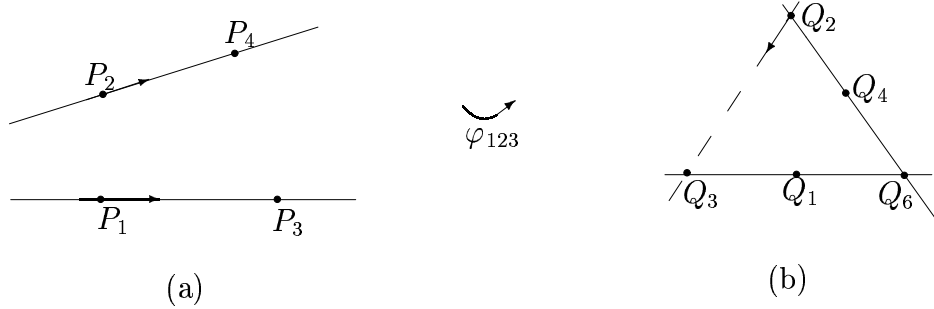(a)   (b)

Figure 3.7: 6-point schemes corresponding to elements in $2\mathcal{A}_1\mathcal{A}_2$

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points . We may assume that the double point $2P_1$ is contained in the two sections corresponding to the points P_1 and P_5, and the double point $2P_2$ is contained in the two sections corresponding to the points P_2 and P_6.

Consider any rational morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. As in the case of $\mathcal{A}_1\mathcal{A}_2$, by (3.3.14), we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is the A_2 singularity; each of the lines $\tilde{P}_1$ and $\tilde{P}_2$ has multiplicity 6 and contains the A_2 singularity and one of A_1 singularities; the lines $\tilde{P}_3$ and $\tilde{P}_4$ are of multiplicity 3; the lines $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ and $\tilde{P}_4$ contain the A_2 singularity. The line $\tilde{l}_{12}$ has multiplicity 4 since the linear subspaces S_{12}, S_{16}, S_{25} and S_{56} are the same. This implies that the line $\tilde{l}_{12}$ contains two A_1 singularities. The lines $\tilde{l}_{14}, \tilde{l}_{23}$ have multiplicity 2. The lines $\tilde{P}_1, \tilde{l}_{12}$ and $\tilde{l}_{14}$ contain one A_1 singularity. The lines $\tilde{P}_2, \tilde{l}_{12}$ and $\tilde{l}_{23}$ contain another A_1 singularity. The line $\tilde{l}_{34}$ has multiplicity 1. The 8 lines of X_x with multiplicities correspond to the partition $(6^2, 4^1, 3^2, 2^2, 1^1)$ of 27.

The triple intersection $(\tilde{P}_1, \tilde{P}_2, \tilde{l}_{12})$ is of multiplicity 12 since it coincides with

$(\tilde{P}_1, \tilde{C}_2, \tilde{l}_{12}), (\tilde{C}_1, \tilde{P}_2, \tilde{l}_{12}), (\tilde{l}_{35}, \tilde{l}_{46}, \tilde{l}_{12}), (\tilde{P}_5, \tilde{C}_2, \tilde{l}_{25}), (\tilde{C}_5, \tilde{P}_2, \tilde{l}_{25}), (\tilde{l}_{13}, \tilde{l}_{46}, \tilde{l}_{25}),$
$(\tilde{P}_1, \tilde{C}_6, \tilde{l}_{16}), (\tilde{C}_1, \tilde{P}_6, \tilde{l}_{16}), (\tilde{l}_{35}, \tilde{l}_{24}, \tilde{l}_{16}), (\tilde{P}_5, \tilde{C}_6, \tilde{l}_{56}), (\tilde{C}_5, \tilde{P}_6, \tilde{l}_{56}), (\tilde{l}_{13}, \tilde{l}_{24}, \tilde{l}_{56}).$

As in the case of $2\mathcal{A}_1$, the triple intersection $(\tilde{l}_{12}, \tilde{l}_{14}, \tilde{l}_{23})$ has multiplicity 4 since it coincides with $(\tilde{l}_{12}, \tilde{l}_{45}, \tilde{l}_{36})$, $(\tilde{l}_{16}, \tilde{l}_{45}, \tilde{l}_{23})$, $(\tilde{l}_{25}, \tilde{l}_{14}, \tilde{l}_{36})$, $(\tilde{l}_{56}, \tilde{l}_{14}, \tilde{l}_{23})$; the triple intersection $(2\tilde{l}_{12}, \tilde{l}_{34})$ has multiplicity 2.

As in the case of $\mathcal{A}_1\mathcal{A}_2$, the triple intersections $(2\tilde{P}_1, \tilde{P}_3)$, $(2\tilde{P}_2, \tilde{P}_4)$, $(\tilde{P}_1, \tilde{P}_4, \tilde{l}_{14})$ and $(\tilde{P}_2, \tilde{P}_3, \tilde{l}_{23})$ are of multiplicity 6; the triple intersection $(\tilde{P}_3, \tilde{P}_4, \tilde{l}_{34})$ has multiplicity 3. So X has 8 distinct triple intersections, which correspond to the partition $(12^1, 6^4, 4^1, 3^1, 2^1)$ of 45.

Remark 3.3.22.

(i) The surface X_x above has at least 2 star points. One of them is the A_2 singularity. Another is determined by the triple intersection $(2\tilde{l}_{12}, \tilde{l}_{34})$. In the next section, we prove that they are proper star points.

(ii) Consider the quadratic transformation φ_{123} with respect to P_1, P_2, P_3 as in the case of $\mathcal{A}_1\mathcal{A}_2$. Let $\mathcal{Q} = \varphi_{123}(\mathcal{P})$. Then we see that $c(\mathcal{Q}) = 2Q_2 + Q_1 + Q_3 + Q_4 + Q_6$, where the direction d at the double point $2Q_2$ contains Q_3; the three points Q_1, Q_3 and Q_6 are contained in a line l_1; the three points Q_2, Q_4 and Q_6 are contained in another line l_2. The csurface of $\mathcal{Q}$ is isomorphic to X_x. Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{Q}$. The images of $l_1 - \{Q_1, Q_3, Q_6\}$ and $d - \{Q_2, Q_3\}$ are two A_1 singularities; the image of $l_2 - \{Q_2, Q_4, Q_6\}$ is the A_2 singularity (see Figure 3.7 (b)).

$2\mathcal{A}_2$ Let $x \in 2\mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^0$ where $c(\mathcal{P}) = P_1 + P_2 + P_3 + 3P_4$ such that 3 point P_1, P_2, P_3 lie on a line l_1 and $3P_4$ is contained in another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 3.8, (a)).
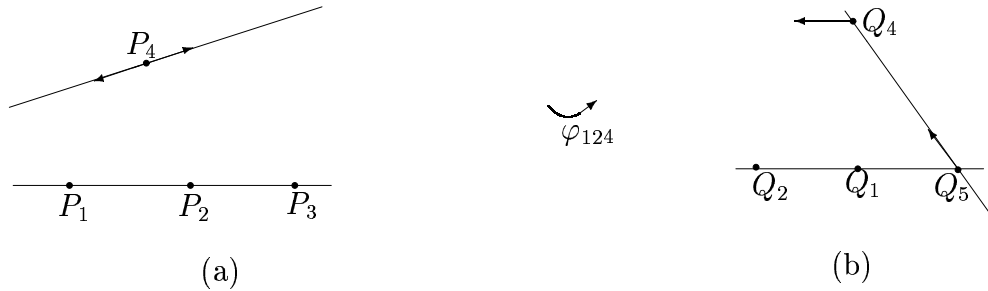

Figure 3.8: 6-point schemes corresponding to elements in $2\mathcal{A}_2$

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the triple point $3P_4$ is contained in the three sections corresponding to the points P_4, P_5 and P_6.

Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{P}$. As in the case of $\mathcal{A}_1\mathcal{A}_2$, by (3.3.14), we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$

is a point S_1, which is an A_2 singularity; the line $\tilde{P}_4$ is of multiplicity 9 and contains two singularities; the lines $\tilde{P}_i$ for $1 \leq i \leq 3$ are of multiplicity 3 and contain S_1. We see that the lines $\tilde{l}_{i4}$ for $1 \leq i \leq 3$ are of multiplicity 3 since the linear subspaces S_{i4}, S_{i5} and S_{i6} are the same. These lines contain another A_2 singularity. The 7 lines of X_x with multiplicities correspond to the partition $(9^1, 3^6)$ of 27.

As in the case of $\mathcal{A}_1\mathcal{A}_2$, the triples $(3\tilde{P}_4)$ and $(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$ are of multiplicity 6. Similarly, the triple $(\tilde{l}_{14}, \tilde{l}_{24}, \tilde{l}_{34})$ has multiplicity 6. The triple intersection $(\tilde{P}_1, \tilde{P}_4, \tilde{l}_{14})$ is of multiplicity 9 since it coincides with

$$(\tilde{P}_1\tilde{C}_4\tilde{l}_{14}), (\tilde{C}_1\tilde{P}_4\tilde{l}_{14}), (\tilde{P}_1\tilde{C}_5\tilde{l}_{15}), (\tilde{C}_1\tilde{P}_5\tilde{l}_{15}), (\tilde{P}_1\tilde{C}_6\tilde{l}_{16}),$$
$$(\tilde{C}_1\tilde{P}_6\tilde{l}_{16}), (\tilde{l}_{23}\tilde{l}_{56}\tilde{l}_{14}), (\tilde{l}_{23}\tilde{l}_{46}\tilde{l}_{15}) \text{ and } (\tilde{l}_{23}\tilde{l}_{45}\tilde{l}_{16}).$$

Similarly, the triple intersections $(\tilde{P}_2, \tilde{P}_4, \tilde{l}_{24})$ and $(\tilde{P}_3, \tilde{P}_4, \tilde{l}_{34})$ are of multiplicity 9. So, the surface X_x has 6 distinct triple intersections, which correspond to the partition $(9^3, 6^3)$ of 45.

Remark 3.3.23.

(i) The surface X_x above has the triple intersection $(3\tilde{P}_4)$. This implies that X_x has an infinite number of star points. In the next section, we will prove that indeed they are proper star points.

(ii) To see how the second singular point of X_x is obtained, we consider the quadratic transformation φ_{124} with respect to P_1, P_2, P_4. Let $\mathcal{Q} = \varphi_{124}(\mathcal{P})$. We see that $c(\mathcal{Q}) = 2Q_4 + 2Q_5 + Q_1 + Q_2$ such that the direction d at the double point $2Q_5$ contains the point Q_4 and three points Q_1, Q_2, Q_5 are contained in a line l (see Figure 3.8, (b)). The csurface of $\mathcal{Q}$ is isomorphic to X_x. The two singular points are the images of $l - \{Q_1, Q_2, Q_5\}$ and $d - \{Q_5, Q_4\}$ by any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{Q}$.

$\mathcal{A}_1 2\mathcal{A}_2$ Let $x \in \mathcal{A}_1 2\mathcal{A}_2$. The corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^0$, where $c(\mathcal{P}) = 2P_1 + P_2 + 3P_4$, such that $2P_1$ and P_2 are contained in a line l_1; the triple point $3P_4$ is contained in another line l_2; the intersection point of l_1 and l_2 does not belong to $\mathcal{P}$ (see Figure 3.9, (a)).

The point x belongs to the closure of $\mathbb{P}^{19} - \Delta$. Consider $\mathcal{P}$ as a specialization position of some family of 6-point schemes in general position. Suppose further that the family has 6 sections of points. We may assume that the triple point $3P_4$ is contained in the three sections corresponding to the points P_4, P_5 and P_6; the double point $2P_1$ is contained in the two sections corresponding to the points P_1 and P_3.

Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_\mathcal{P}$. By (3.3.14), we see that the image of $(l_1 \cup l_2) - \mathrm{Supp}(\mathcal{P})$ is a point S_1, which is an A_2 singularity; the line $\tilde{P}_4$ is of multiplicity 9 and contains two A_2 singularities; the line $\tilde{P}_1$ is of multiplicity 6 and it contains S_1 and the A_1 singularity; the line $\tilde{P}_2$ is of multiplicity 3 and it contains S_1. As in the case of $2\mathcal{A}_2$, we see that the
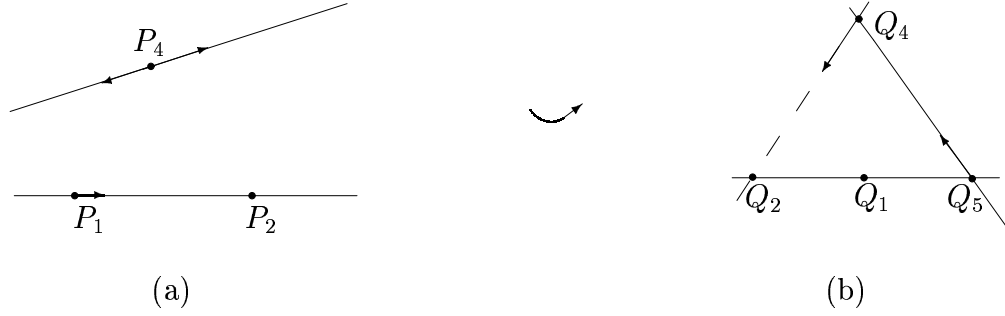
Figure 3.9: 6-point schemes corresponding to elements in $\mathcal{A}_1 2\mathcal{A}_2$

line $\tilde{l}_{24}$ is of multiplicity 3 and contains the another A_2 singularity. The surface X_x has exactly 5 lines with the partition $(9^1, 6^2, 3^2)$ of 27.

As in the case of $2\mathcal{A}_2$, the triple intersections $(3\tilde{P}_4), (2\tilde{P}_1, \tilde{P}_2)$ and $(2\tilde{l}_{14}, \tilde{l}_{24})$ are of multiplicity 6; the triple intersection $(\tilde{P}_2, \tilde{P}_4, \tilde{l}_{24})$ is of multiplicity 9. The triple intersection $(\tilde{P}_1, \tilde{P}_4, \tilde{l}_{14})$ is of multiplicity 18 since the following triple intersections are the same:

$$(\tilde{P}_1, \tilde{C}_4, \tilde{l}_{14}), (\tilde{C}_1, \tilde{P}_4, \tilde{l}_{14}), (\tilde{P}_1, \tilde{C}_5, \tilde{l}_{15}), (\tilde{C}_1, \tilde{P}_5, \tilde{l}_{15}), (\tilde{P}_1, \tilde{C}_6, \tilde{l}_{16}), (\tilde{C}_1, \tilde{P}_6, \tilde{l}_{16}),$$
$$(\tilde{l}_{23}, \tilde{l}_{56}, \tilde{l}_{14}), (\tilde{l}_{23}, \tilde{l}_{46}, \tilde{l}_{15}), (\tilde{l}_{23}, \tilde{l}_{45}, \tilde{l}_{16}), (\tilde{P}_3, \tilde{C}_4, \tilde{l}_{34}), (\tilde{C}_3, \tilde{P}_4, \tilde{l}_{34}), (\tilde{P}_3 \tilde{C}_5, \tilde{l}_{35}),$$
$$(\tilde{C}_3, \tilde{P}_5, \tilde{l}_{35}), (\tilde{P}_3, \tilde{C}_6, \tilde{l}_{36}), (\tilde{C}_3, \tilde{P}_6, \tilde{l}_{36}), (\tilde{l}_{12}, \tilde{l}_{56}, \tilde{l}_{34}), (\tilde{l}_{12}, \tilde{l}_{46}, \tilde{l}_{35}), (\tilde{l}_{12}, \tilde{l}_{45}, \tilde{l}_{36}).$$

So X has 5 distinct triple intersections corresponding to the partition $(18^1, 9^1, 6^3)$ of 45.

Remark 3.3.24.

(i) As in the case of $2\mathcal{A}_2$, since the surface X_x above has the triple $(3\tilde{P}_4)$, all points on the line $\tilde{P}_4$ are star points. Moreover, the triple intersections $(2\tilde{P}_1, \tilde{P}_2)$ and $(2\tilde{l}_{14}, \tilde{l}_{24})$ determine other star points on X_x. In the next section, we will prove that they are proper star points.

(ii) It is not clear how other singularities on X_x are obtained by considering the 6-point scheme $\mathcal{P}$ as above. Consider the quadratic transformation φ with respect to P_1, P_2, P_4. Let $\mathcal{Q} = \varphi(\mathcal{P})$. Then we see that $c(\mathcal{Q}) = 2Q_4 + 2Q_5 + Q_1 + Q_2$, where the direction d_1 at the double point $2Q_5$ contains Q_4; the direction d_2 at the double point Q_4 contains Q_2; the three points Q_1, Q_2, Q_5 are contained in a line l, (see Figure 3.9, (b)). The surface X_x is isomorphic to the csurface of $\mathcal{Q}$. Consider any morphism from $\mathbb{P}^2 - \text{Supp}(\mathcal{Q})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{Q}}$. It is clear that the image of $d_2 - \{Q_2, Q_4\}$ is the A_1 singularity; the images of $l - \{Q_1, Q_2, Q_5\}$ and $d_1 - \{Q_4, Q_5\}$ are the A_2 singularities.

$3\mathcal{A}_2$ Let $x \in 3\mathcal{A}_2$. We know that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P} \in \mathcal{H}^o$ where $c(\mathcal{P}) = 3P_1 + 3P_2$ such that

$3P_1$ and $3P_2$ are contained in 2 different lines d_1, d_2 respectively; the intersection point of d_1 and d_2 does not belong to $\mathcal{P}$ (see Figure 3.10, (a)).

(a)   (b)

Figure 3.10: 6-point schemes corresponding to points in $3\mathcal{A}_2$

Consider any morphism from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ determined by a basis of $\mathcal{L}_{\mathcal{P}}$. By (3.3.14), we see that the image of $(l_1 \cup l_2) - \mathrm{Sup}(\mathcal{P})$ is a point S_1, which is an A_2 singularity; the lines $\tilde{P}_1$ and $\tilde{P}_2$ are of multiplicity 9 and contain S_1. Moreover, as in the case of $\mathcal{A}_1 2\mathcal{A}_2$, the line $\tilde{l}_{12}$ is of multiplicity 9. The 3 lines of X together with multiplicities correspond to the partition (9^3) of 27. These 3 lines form a triangle where 3 vertices are three A_2 singularities. The surface X has 4 distinct triple intersections, namely $(3\tilde{P}_1), (3\tilde{P}_2), (3\tilde{l}_{12})$ and $(\tilde{P}_1, \tilde{P}_2, \tilde{l}_{12})$. The latter is of multiplicity 27. Each of the first three triples is of multiplicity 6. So the 4 triple intersections of X with multiplicities correspond to the partition $(27^1, 6^3)$ of 45.

Remark 3.3.25.

(i) All points on lines $\tilde{P}_1, \tilde{P}_2$ and $\tilde{l}_{12}$ are star points of X_x. Moreover, we will prove in the next section that they are proper star points.

(ii) The same question as before arises: how to see other singular points of X_x from morphisms from $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$ to $\mathbb{P}^3$ defined by bases of $\mathcal{L}_{\mathcal{P}}$? We can not apply any quadratic transformation as before since $\mathcal{P}$ does not contain 3 distinct points. Going back to the cases of $2\mathcal{A}_2$ and $\mathcal{A}_1 2\mathcal{A}_2$, a "conjecture" appears naturally : is the csurface of 6-point scheme $\mathcal{Q}$ with a configuration as in Figure 3.10, (b) isomorphic to the surface X_x? The 6-point scheme Q defines a cycle $c(\mathcal{Q}) = 2Q_1 + 2Q_2 + 2Q_3$, where the direction at $2Q_1$ contains Q_2, the direction at Q_2 contains Q_3 and the direction at Q_3 contains Q_1.

We prove that the conjecture is true. Choose coordinates such that $Q_1 = (1 : 0 : 0), Q_2 = (0 : 1 : 0)$ and $Q_3 = (0 : 0 : 1)$. The linear space $\mathcal{L}_{\mathcal{Q}}$ parameterizes all cubic forms

$$f = a_1 x_0^2 x_1 + a_2 x_1^2 x_2 + a_3 x_2^2 x_0 + a_4 x_0 x_1 x_2 \in k[x_0, x_1, x_2].$$

Choose the basis $\{x_0^2 x_1, x_1^2 x_2, x_0 x_1 x_2, x_2^2 x_0\}$ of $\mathcal{L}_{\mathcal{P}}$. Consider the morphism:

$$\begin{array}{ccc} \mathbb{P}^2 - \mathrm{Supp}(\mathcal{Q}) & \longrightarrow & \mathbb{P}^3 \\ (x_0 : x_1 : x_2) & \longmapsto & (x_0^2 x_1 : x_1^2 x_2 : x_0 x_1 x_2 : x_2^2 x_0), \end{array}$$

determined by the basis.

Let $F = x_3 x_0 x_1 - x_2^3$. We have

$$
\begin{aligned}
F(x_0^2 x_1, x_1^2 x_2, x_0 x_1 x_2, x_2^2 x_0) \ &= (x_2^2 x_0)(x_0^2 x_1)(x_1^2 x_2) - (x_0 x_1 x_2)^3 \\
&= x_0^3 x_1^3 x_2^3 - x_0^3 x_1^3 x_2^3, \\
&= 0.
\end{aligned}
$$

Since F is irreducible, the csurface of $\mathcal{Q}$ is defined by F. This means that the csurface of $\mathcal{Q}$ is of type $3\mathcal{A}_2$ and therefore is isomorphic to the surface X_x.

We end this section with Table 3.1 containing the information about the number of lines, the number of triple intersections on semi-stable cubic surfaces.

Num.	$\mathbb{P}^{19} - \Delta$	$\mathcal{A}_1$	$2\mathcal{A}_1$	$\mathcal{A}_2$	$3\mathcal{A}_1$	$\mathcal{A}_1\mathcal{A}_2$	$4\mathcal{A}_1$	$2\mathcal{A}_1\mathcal{A}_2$	$2\mathcal{A}_2$	$\mathcal{A}_1 2\mathcal{A}_2$	$3\mathcal{A}_2$
Lines	27	21	16	15	12	11	9	8	7	5	3
Triple Int.	45	30	20	17	14	11	11	8	6	5	4
Star points	0	0	1	1	3	1	6	2	∞	∞	∞

Table 3.1: Information about lines, triple intersections and star points on semi-stable cubic surfaces; note that the star points mentioned in the table are star points which the corresponding surface of every element point *at least has*.

For each class $i\mathcal{A}_1 j\mathcal{A}_2$ for $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$, we compute the number of star points, which the corresponding surface of every element of the class *at least has*; this number of star points is realized on a dense open subset in that class. As we have seen, every of these star points is either an A_2 singularity or any point on a line containing two A_2 singularities or the specific point on a line containing two A_1 singularities.

3.4 On the boundaries of the moduli spaces of non-singular cubic surfaces with star points

Let $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ be the quotient space with respect to the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. We know from Section 3.2 that $\overline{M}$ is projective. Let $M := \mathrm{PGL}(3)\backslash(\mathbb{P}^{19} - \Delta)$, where Δ is the locus of singular cubic surfaces in $\mathbb{P}^{19}$. Then M can be viewed as the coarse moduli space of the non-singular cubic surfaces. The space $\overline{M}$ is a compactification of M. Let $N := \mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^{s}$.

In this section, we study the boundaries of the subsets $H_i^{(k)}$ inside the set of semi-stable cubic surfaces (see Chapter 2 for the definition of $H_i^{(k)}$); we also study the boundaries of moduli spaces $\phi(H_i^{(k)})$ inside $\overline{M}$.

Let $(\Delta)^{ss} = (\mathbb{P}^{19})^{ss} \cap \Delta$. We denote by $\Delta H_i^{(k)}$ the intersection of the closure of $H_i^{(k)}$ with $(\Delta)^{ss}$, which is called the boundary of $H_i^{(k)}$ in $(\mathbb{P}^{19})^{ss}$. Denote $\overline{i\mathcal{A}_1 j\mathcal{A}_2}$ for the closure of $i\mathcal{A}_1 j\mathcal{A}_2$ in $(\mathbb{P}^{19})^{ss}$ for $2i + 3j \leq 9$, $i \leq 4$ and $(i, j) \neq (3, 1)$.

A. Some basic facts

Lemma 3.4.1. *The subsets $i\mathcal{A}_1 j A_2$ are irreducible in $\mathbb{P}^{19}$.*

Proof. Consider the case $i > 0$. Each $x \in i\mathcal{A}_1 j\mathcal{A}_2$ for $i > 1$ is given by a polynomial

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 1, 2$ is a homogeneous polynomial of degree i; the rank of f_2 is 3 and $V_{\mathbb{P}^2}(f_2, f_3)$ corresponds to a partition $(2^{i-1}3^j1^k)$ with $2(i-1) + 3j + k = 6$.

By choosing coordinates, we can assume that $f_2 = x_1^2 - x_0 x_2$ and

$$f_3 = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^3,$$

see [B-W], p. 248.

The points of $V_{\mathbb{P}^2}(f_2, f_3)$ are determined by the solutions of the equation:

$$f_3(\theta^2, \theta\psi, \psi^2) := a_0\theta^6 + a_1\theta^5\psi + a_2\theta^4\psi^2 + a_3\theta^3\psi^3 + a_4\theta^2\psi^4 + a_5\theta\psi^5 + a_6\psi^6 = 0.$$

We prove that the set of all homogeneous polynomials of degree 6 in 2 variables which possess solutions corresponding to a given partition of 6 is irreducible.

Let T be the projective space parameterizing all homogeneous polynomials of degree 6 in two variables. Let a be the partition $(2^{i-1}3^j1^k)$ of 6. Let T_a be the subset of T consisting of all homogeneous polynomials of degree 6 in 2 variables which possess solutions corresponding to the partition a. Consider the morphism:

$$\begin{array}{ccc} (\mathbb{P}^1)^{i+j+k-1} & \longrightarrow & T \\ (a_1 : b_1; \ldots ; a_{i+j+k-1} : b_{i+j+k-1}) & \longmapsto & F, \end{array}$$

where

$$F = \prod_{t=1}^{i-1}(b_t\psi - a_t\theta)^2 \prod_{r=i}^{i+j-1}(b_r\psi - a_r\theta)^3 \prod_{s=i+j}^{i+j+k-1}(b_s\psi - a_s\theta).$$

The image of $(\mathbb{P}^1)^{i+j+k-1} - \Delta_1$ is T_a, where Δ_1 is the diagonal of $(\mathbb{P}^1)^{i+j+k-1}$. So T_a is irreducible. This implies that the subset K_{ij} of $i\mathcal{A}_1 j\mathcal{A}_2$ consisting of points which are given by

$$F = x_3(x_1^2 - x_0 x_2) + a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^3,$$

is irreducible. Consider $\varphi : \mathrm{PGL}(3) \times K_{ij} \longrightarrow \mathbb{P}^{19}$ which is induced from the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. The subset $i\mathcal{A}_1 j\mathcal{A}_2$ is the image of φ, so it is irreducible.

Consider the case $i = 0$. An element in $j\mathcal{A}_2$ for $j = 1, 2, 3$ can be determined by a polynomial

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 1, 2$ is a homogeneous polynomial of degree i; the rank of f_2 is 2; the singular point of $V_{\mathbb{P}^2}(f_2)$ is not contained in $V_{\mathbb{P}^2}(f_3)$; the points of $V_{\mathbb{P}^2}(f_2, f_3)$ correspond to a partition $(3^{j-1}1^k)$ with $3(j-1) + k = 6$.

By choosing coordinates, we can assume that $f_2 = x_0 x_1$ and

$$f_3 = x_0(a_0 x_0^2 + a_1 x_0 x_2 + a_2 x_2^2) + x_1(a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2) + x_2^3; \qquad (3.6)$$

see [B-W], p. 249. Let N_0 be the set of all cubic forms in type (3.6).

j=1. Let N_1 be the subset of N_0 such that $V_{\mathbb{P}^2}(x_0 x_1, f_3)$ consists of 6 distinct points. Consider the morphism:

$$\begin{array}{ccc} \mathbb{A}^6 & \stackrel{\pi}{\longrightarrow} & N_0 \\ (a_0, \dots, a_5) & \longmapsto & f_3, \end{array}$$

where f_3 is a cubic form in type (3.6). We see that the image of the set

$$B_1 = \{(a_0 \dots, a_5) \in \mathbb{A}^6 \mid a_1^2 - 4a_0 a_2 \neq 0, a_4^2 - 4a_3 a_5 \neq 0\}$$

is the set N_1. So N_1 is irreducible. By the same argument as in the case of $i > 0$, we see that $\mathcal{A}_2$ is irreducible.

j=2. We can choose coordinates such that $V_{\mathbb{P}^2}(x_0 x_1, f_3)$ has a triple point at $(1 : 0 : 0)$. This means that in type (3.6), we have $a_0 = a_1 = a_2 = 0$. Let N_2 be the subset of all cubic forms in type (3.6), such that $a_0 = a_1 = a_2 = 0$ and $a_4^2 - 4a_3 a_5 \neq 0$. Then N_2 is the image via π of the set

$$B_2 = \{(a_0, \dots, a_5) \in \mathbb{A}^6 \mid a_0 = a_1 = a_2 = 0, a_4^2 - 4a_3 a_5 \neq 0\},$$

which is an open subset of $\mathbb{A}^3 \subset \mathbb{A}^6$. So N_2 is irreducible. With the same argument as in the case of $i > 0$, we see that $2\mathcal{A}_2$ is irreducible.

j=3. We can choose coordinates such that two triple points of $V_{\mathbb{P}^2}(x_0 x_1, f_3)$ are $(1 : 0 : 0)$ and $(0 : 1 : 0)$. This means that in type (3.6), we have $a_i = 0$ for all $0 \leq i \leq 5$. Let N_3 be the one-point set consisting of the point determined by $F = x_3 x_0 x_1 + x_2^3$. The set $3\mathcal{A}_2$ is the image of the morphism $\mathrm{PGL}(3) \times N_3 \longrightarrow \mathbb{P}^{19}$ which is induced from the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. So it is irreducible.

$\square$

Recall that $\overline{M} := \mathrm{PGL}(3) \backslash (\mathbb{P}^{19})^{ss}$ and $N := \mathrm{PGL}(3) \backslash (\mathbb{P}^{19})^s$.

Proposition 3.4.2. *Let* $x, y \in (\mathbb{P}^{19})^{ss}$ *such that the corresponding cubic surfaces contain at least one* A_2 *singularity. Let* $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ *be the quotient space with respect to the natural action of* $\mathrm{PGL}(3)$ *on* $\mathbb{P}^{19}$. *Then* $\phi(x) = \phi(y)$. *Consequently the set* $\overline{M} - N$ *consists of the singleton* s, *which is the image of all non-stable points.*

Proof. First of all, we note that, by choosing coordinates, any $z \in 3\mathcal{A}_2$ can be given by a polynomial $F = x_3 x_0 x_1 + x_2^3$. This means that $\phi(3\mathcal{A}_2)$ consists of one point. We denote this point by s. Let $x \in (\mathbb{P}^{19})^{ss}$ such that the corresponding cubic surface X_x contains at least one A_2 singularity. Let $\mathcal{O}(x)$ be the orbit of x with respect to the action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. Our task is now to prove that $\overline{\mathcal{O}(x)}$ contains at least one element of $3\mathcal{A}_2$, then by (3.2.5, (iii)), we have $\phi(x) = s$.

Since X_x contains one A_2 singularity, the surface can be given by a polynomial

$$F = x_3 x_0 x_1 + x_0(a_0 x_0^2 + a_1 x_0 x_2 + a_2 x_2^2) + x_1(a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2) + x_2^3,$$

see [B-W], p. 249. Consider the subset T of $\mathrm{PGL}(3)$ consisting of elements $A_{(a,b)}$ given by:

$$A_{(a,b)} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1/b^2 \end{pmatrix}$$

where $a, b \in k^*$. Let $\varphi : \mathrm{PGL}(3) \times \mathbb{P}^{19} \longrightarrow \mathbb{P}^{19}$ be the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. The set $T(x) := \varphi(T \times \{x\})$ consists of elements in $\mathcal{O}(x)$ given by

$$F_{(a,b)} = x_3 x_0 x_1 + b x_0 (b^2 a_0 x_0^2 + a b a_1 x_0 x_2 + a^2 a_2 x_2^2) + b x_1 (b^2 a_3 x_1^2 + a b a_4 x_1 x_2 + a^2 a_5 x_2^2) + $$
$$+ a^3 x_2^3.$$

Then $\overline{\mathcal{O}(x)}$ contains the point x_0 given by

$$F_{(a,0)} = x_3 x_0 x_1 + a^3 x_2^3.$$

It is clear that $x_0 \in 3\mathcal{A}_2$. [1] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

B. On the boundaries

B.1. On the boundary of H_1

Recall that H_1 is the subvariety of $\mathbb{P}^{19} - \Delta$ parameterizing all cubic surfaces with at least one star point.

Proposition 3.4.3. *The subset $2\mathcal{A}_1$ is contained in the closure of H_1. Consequently the star point on the line with multiplicity 4 of any cubic surface parametrized by a point of $2\mathcal{A}_1$ is a proper star point.*

Proof. Let $x \in 2\mathcal{A}_1$. We know from the previous section that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{Q} = \sum_{i=1}^{6} Q_i$ where 3 points Q_2, Q_3, Q_6 lie on a line l_1; three points Q_4, Q_5, Q_6 lie on another line l_2; no 3 of the five points $Q_1, \ldots, Q_5$ are collinear (see the Figure 3.11).
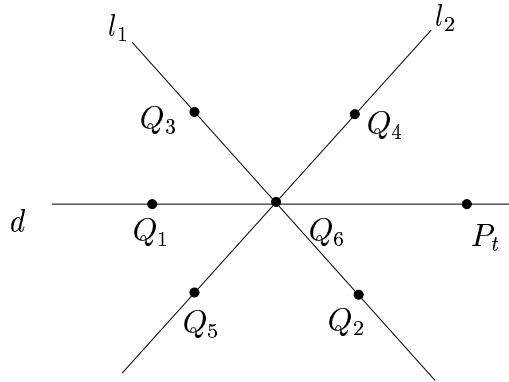

Figure 3.11: 6-point schemes giving points in $2\mathcal{A}_1$

Let P_t be a moving point on the line $d = \overline{Q_1 Q_6}$. At a general position of P_t on d, the 6-point scheme $\mathcal{P}_t = \sum_{i=1}^{6} P_i$ where $P_i = Q_i$ for $1 \le i \le 5$ and $P_6 = P_t$, gives

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[1]The proof is completed using a suggestion from Prof. Dr. E. Looijenga. I would like to thank him very much.

a non-singular cubic surface with at least one star point. Except for a finite number of positions, when P_t moves on the line d, we have a family in H_1. This implies that x lies on the closure of H_1. Moreover, we see that the section of star point over the family is defined by the tritangent plane $H_t = (\tilde{l}_{23}, \tilde{l}_{45}, \tilde{l}_{16})$ where $\tilde{l}_{ij}$ is the line on the csurface of a 6-point scheme in the family determined by the linear subspace S_{ij}. In the specialization position, the linear subspaces S_{23}, S_{45}, S_{C_1} and S_{Q_6} coincide. This means that $\tilde{Q}_6$ is the line connecting the 2 singular points and the section of tritangents H_t contains the triple $(2\tilde{Q}_1, \tilde{l}_{16})$. So the section of star points contains the star point on the line $\tilde{Q}_6$ with multiplicity 4. $\qquad\square$

Corollary 3.4.4. *The closure of* $2\mathcal{A}_1$ *in* $(\mathbb{P}^{19})^{ss}$ *is an irreducible component of* ΔH_1.

Proposition 3.4.5. *Any* $x \in \mathcal{A}_2$ *lies on the closure of* H_1. *Consequently, the* A_2 *singularity of* X_x, *as a star point, is a proper star point.*

Proof. Let $\mathcal{Q}$ be a 6-point scheme where $c(\mathcal{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$, such that three points Q_1, Q_2, Q_3 are contained in a line l; the direction at double point $2Q_1$ does not contain any Q_i for $i = 4, 5$; the four points Q_1, Q_2, Q_4, Q_5 as well as 4 points Q_1, Q_3, Q_4, Q_5 are in general position (see Figure 3.12, (a)). Consider the quadratic transformation with respect to Q_1, Q_4, Q_5. Then the image of $\mathcal{Q}$ is a 6-point scheme $\mathcal{P}$ consisting of 6 distinct points $P_1, \ldots, P_6$ such that 3 points P_1, P_2, P_3 as well as 3 points P_4, P_5, P_6 are collinear. We know that the csurface of $\mathcal{P}$ is isomorphic to a cubic surface with exactly one $\mathcal{A}_2$ singularity (see Figure 3.12, (b)).

Figure 3.12: 6-point schemes giving points in $\mathcal{A}_2$

Let $x \in \mathcal{A}_2$. The surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{Q}$ where $c(\mathcal{Q}) = 2Q_1 + \sum_{i=2}^{5} Q_i$ described as above (see Figure 3.12 (a)).

Let O be the intersection point of l and $\overline{Q_4Q_5}$. Let d be the direction at the double point $2Q_1$. Let m be a fixed line which contains Q_3 and does not contain any other point of $\mathrm{Supp}(\mathcal{Q})$. Let (P_6, P_3) be a pair of moving points such that $P_6 \in d$ and $P_3 \in m$ such that $\overline{P_3P_6}$ contains O. It is clear that, except for a finite number of positions, when moving (P_6, P_3), the csurfaces of 6-point schemes $\mathcal{P} = \sum_{i=1}^{6} P_i$, where $P_i = Q_i$ for $i \in \{1, 2, 4, 5\}$, are isomorphic to non-singular cubic surfaces with at least one star point. This defines a family in H_1. When $(P_6, P_3) = (Q_1, Q_3)$, we arrive at the 6-point

scheme $\mathcal{Q}$ whose csurface is isomorphic to X_x. So x lies on the closure of H_1. Moreover, the star section over the family is defined by the tritangent planes $(\tilde{l}_{12}, \tilde{l}_{45}, \tilde{l}_{36})$, where the line $\tilde{l}_{ij}$ on a surface of the family is determined by the linear subspace S_{ij}. In the specialization position, the linear subspaces S_{12}, S_{26} and S_{Q_3} coincide; the linear subspaces S_{36}, S_{13} and S_{Q_2} coincide. Note that the 6 lines $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{l}_{45}, \tilde{l}_{14}$ and $\tilde{l}_{15}$ have multiplicity 3 and they contain the A_2 singularity. It is clear that the section of star point gives a specialization to the intersection of $\tilde{Q}_2, \tilde{Q}_3$ and $\tilde{l}_{45}$, which is the A_2 singularity point. $\qquad\square$

Corollary 3.4.6. *The closure of $\mathcal{A}_2$ in $(\mathbb{P}^{19})^{ss}$ is an irreducible component of ΔH_1.*

Consider the set K_1 consisting of all 6-point schemes $\mathcal{P}$ where $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ such that $\mathcal{P}$ is contained in an irreducible conic and $l_{12} \cap l_{34} \cap l_{56} = \{O\}$ (see Figure 3.13).
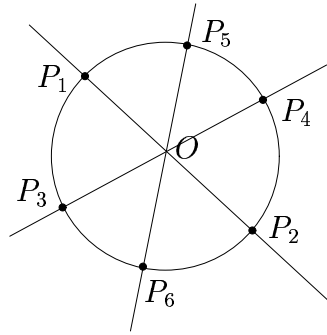

Figure 3.13: 6-point schemes of K_1

Let D_1 be the subset of $\mathbb{P}^{19}$ consisting of all points corresponding to the cubic surfaces, each of them is isomorphic to the csurface of some element in K_1. It is clear that $D_1 \subset \mathcal{A}_1$. On the other hand, we see that D_1 is contained in the closure of H_1 also. For this, let $x \in D_1$. The corresponding surface X_x is isomorphic to the csurface of some 6-point scheme $\mathcal{P} \in K$. Let $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$. Fix $P_1, \ldots, P_5$ and let P_6 move on the line $\overline{P_5 O}$. This defines a family of 6-point schemes whose csurfaces are isomorphic to non-singular cubic surfaces with at least one star point. When P_6 is contained in the conic defined by $P_1, \ldots, P_5$, we get $\mathcal{P}$. This implies that x lies on the closure of H_1. Moreover, we prove that:

Lemma 3.4.7. *The subset D_1 is irreducible in $(\mathbb{P}^{19})^{ss}$.*

Proof. Let $x \in D_1$. Since $D_1 \subset \mathcal{A}_1$, by choosing coordinates, we can assume that x is given by

$$F = x_3(x_1^2 - x_0 x_2) + f_3, \tag{3.7}$$

where

$$f_3 = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_2^2,$$

such that the scheme $V_{\mathbb{P}^2}(x_1^2 - x_0 x_2, f_3)$ consists of 6 distinct points. Furthermore there exists a numbering of 6 points $P_1, \ldots, P_6$ of $V_{\mathbb{P}^2}(x_1^2 - x_0 x_2, f_3)$ such that $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$ (see Figure 3.13).

The 6 points of $V_{\mathbb{P}^2}(x_1^2 - x_0x_2, f_3)$ are determined by the solutions of the equation

$$a_0\theta^6 + a_1\theta^5\psi + a_2\theta^4\psi^2 + a_3\theta^3\psi^3 + a_4\theta^2\psi^4 + a_5\theta\psi^5 + a_6\psi^6 = 0. \qquad (3.8)$$

Let T be the projective space parameterizing all homogeneous polynomials of degree 6 in two variables.

Consider the morphism

$$\begin{array}{ccc} (\mathbb{P}^1)^6 & \longrightarrow & T \\ (a_1 : b_1; \cdots ; a_6 : b_6) & \longmapsto & \prod_{i=1}^6 (b_i\theta - a_i\psi). \end{array}$$

Note that a solution $(\theta_i : \psi_i)$ of (3.8) corresponds to a point $P_i = (\theta_i^2 : \theta_i\psi_i : \psi_i^2)$ for $1 \leq i \leq 6$ contained in the conic $V_{\mathbb{P}^2}(x_1^2 - x_0x_2)$. The set of all elements of $(\mathbb{P}^1)^6 - \Delta$ such that $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$ is irreducible. This implies that the subset $D_1' \subset \mathbb{P}^{19}$ consisting of all elements which correspond to polynomials of the form (3.7) and satisfy the above condition is irreducible. We see that D_1 is the image of the morphism $\varphi : \mathrm{PGL}(3) \times D_1' \longrightarrow \mathbb{P}^{19}$ which is induced from the natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$. So the set D_1 is irreducible. $\qquad\square$

Proposition 3.4.8. $\Delta H_1 = \overline{D_1} \cup \overline{2\mathcal{A}_1} \cup \overline{\mathcal{A}_2}$.

Proof. It is clear that the sets $\overline{D_1}, \overline{2\mathcal{A}_1}$ and $\overline{\mathcal{A}_2}$ are irreducible components of ΔH_1. Conversely, let x be the generic point of an irreducible component W of ΔH_1. Suppose that $W \neq \overline{2\mathcal{A}_1}$ and $W \neq \overline{\mathcal{A}_2}$. Since $\dim W = 17$, we have $x \in \mathcal{A}_1$. So the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^6 P_i$ where 6 distinct points $P_1, \ldots, P_6$ are contained in an irreducible conic. The 21 lines of X_x are $\tilde{P}_i$ and $\tilde{l}_{ij}$ for $1 \leq i < j \leq 6$. Note that the A_1 singularity is not a star point. Therefore the star point of X_x is determined by a triple intersection of the form $(\tilde{l}_{ij}, \tilde{l}_{hk}, \tilde{l}_{mn})$. This implies that the 6 points $P_1, \ldots, P_6$ satisfy $l_{ij} \cap l_{hk} \cap l_{mn} \neq \emptyset$. This means that $x \in D_1$ and $W = \overline{D_1}$. $\qquad\square$

Recall that $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ is the quotient space with respect to the action of $\mathrm{PGL}(3)$ on $\mathbb{P}^{19}$.

Corollary 3.4.9. $\phi(\Delta H_1) = \phi(\overline{D_1}) \cup \phi(\overline{2\mathcal{A}_1})$. *Moreover, the components $\phi(\overline{D_1})$ and $\phi(\overline{2\mathcal{A}_1})$ contain the singleton s.*

Proof. Note that since $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ is a good quotient, the sets $\phi(\overline{D_1})$ and $\phi(\overline{2\mathcal{A}_1})$ are closed. The first conclusion follows from the proposition.

Since $\overline{2\mathcal{A}_2} \subset \overline{2\mathcal{A}_1}$, we have $s \in \phi(\overline{2\mathcal{A}_1})$.

Let $\mathcal{P} \in K$. By definition, we have $c(\mathcal{P}) = \sum_{i=1}^6 P_i$ where 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic and $l_{12} \cap l_{34} \cap l_{56} = \{O\}$. Consider the quadratic transformation φ with respect to P_1, P_3, P_5; we see that the 6-point scheme $\varphi(\mathcal{P})$ consists of 6 distinct points $Q_1, \ldots, Q_6$ such that Q_2, Q_4, Q_6 are collinear and $\overline{Q_1Q_2} \cap \overline{Q_3Q_4} \cap \overline{Q_5Q_6} = \{\varphi(O)\}$ (see Figure 3.14).

Consider a family in D_1 given by fixing 5 points $Q_2, \ldots, Q_6$ and moving Q_1 on the line $\overline{Q_2\varphi(O)}$, where $\varphi(O) = \overline{Q_3Q_4} \cap \overline{Q_5Q_6}$. When Q_1 coincides with the intersection point of $\overline{Q_2\varphi(O)}$ and $\overline{Q_3Q_5}$, we get a 6-point scheme whose csurface is isomorphic to a surface with exactly one A_2 singularity. This implies that $s \in \phi(\overline{D_1})$. $\qquad\square$
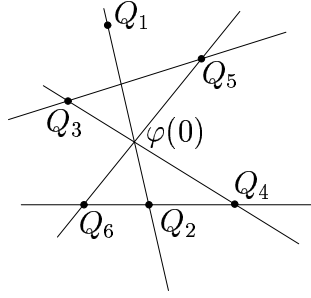
Figure 3.14: 6-point schemes giving points in D_1

Proposition 3.4.10. *Let X be a semi-stable cubic surface. Any star point of X is a proper star point.*

Proof. Let P be a star point of X. The result is clear if P is formed by a triple intersection whose lines are of multiplicity 1. If P is an A_2 singularity, then the result follows from (3.4.5). Consider the case X has two A_1 singularities. Let d be the line containing two A_1 singularities. Let $(2d, l)$ be the triple intersection which factors into $2d$ and another line l on X. Suppose that $P = d \cap l$, then the result follows from (3.4.3).

Consider the case that X has at least two A_2 singularities and P is a point in the line connecting two A_2 singularities. We only consider the case that P is not singular. Choose coordinates such that X is given by the polynomial (see [B-W], p. 249):

$$F_0 = x_3 x_0 x_1 + x_1(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2) + x_2^3;$$

moreover, if X has another singularity then the point $(0 : 1 : 0)$ is a multiple point of $V_{\mathbb{P}^2}(x_0, f_3)$ where $f_3 = x_1(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2) + x_2^3$. The surface X contains two A_2 singularities, namely $S_1 = (0 : 0 : 0 : 1)$ and $S_2 = (1 : 0 : 0 : 0)$. The line $d = V(x_1, x_2)$ contains two A_2 singularities. Let $P = (\lambda, 0 : 0 : 1) \in d$ where $\lambda \neq 0$.

Consider the family given by

$$F_t = x_3(x_0 x_1 + \lambda t x_2^2) + x_1(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2) + x_2^3 - t x_0 x_2^2, \qquad (3.9)$$

where $t \in k$. Let $f_2^t = x_0 x_1 + \lambda t x_2^2$ and $f_3^t = x_1(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2) + x_2^3 - t x_0 x_2^2$. For $t \neq 0$, the polynomial f_2^t has rank 3. Consider $\mathcal{P} = V_{\mathbb{P}^2}(f_2^t, f_3^t)$. We see that the line $V_{\mathbb{P}^2}(x_1)$ is tangent to $V_{\mathbb{P}^2}(f_2^t)$ and $V_{\mathbb{P}^2}(f_3^t)$ at $(1 : 0 : 0)$. Moreover $(1 : 0 : t) \in V(x_1, f_3^t)$. This implies that the point $(1 : 0 : 0)$ is a double point of $V(f_2^t, f_3^t)$. Other points of $\mathcal{P}$ are determined by $(-\lambda t b^2 : 1 : b)$ where b is a solution of the following equation

$$a_1 + a_2 x_2 + a_3 x_2^2 + x_2^3 - \lambda t x_2^4 = 0. \qquad (3.10)$$

The above equation has a multiple solution for an infinite number of values of t if and only if $(a_1, a_2) = (0, 0)$.

(i) If X defines a point in $2\mathcal{A}_2$. This is equivalent to say $a_3^2 - 4a_1 a_2 \neq 0$. Except for a finite number of values of t, we see that $\mathcal{P}$ has only one double point. This means that (3.9) defines a family Γ_t in $2\mathcal{A}_1$. Each corresponding cubic surface X_t of any element in Γ_t consists of two A_1 singularities, namely $S_1 = (0 : 0 : 0 : 1)$ and

$S_2 = (1 : 0 : 0 : 1)$. We see that $H = V(x_1) \cap X_t = 2d + l_t$, where $d = V(x_1, x_2)$ and $l_t = V(x_1, \lambda t x_3 + x_2 + t x_0)$. The surface X_t contains a proper star point $P_t = (\lambda : 0 : 0 : 1) = d \cap l_t$. When the family Γ_t gives a specialization to X_0, the sections of A_1 singularities contain the A_2 singularities of X_0. Moreover, the section of proper star points over Γ_t contains $P = (\lambda : 0 : 0 : 1)$ in X_0.

(ii) If X defines a point in $\mathcal{A}_1 2\mathcal{A}_2$ and $S_3 = (0 : 1 : 0 : 0)$ is the A_1 singular point of X. It is equivalent to say $(a_1, a_2) = (0, 0)$ and $a_3 \neq 0$. We see that (3.10) has one double solution at $x_2 = 0$ and two other solutions with multiplicity 1 except for a finite number of values of t. This means that (3.9) defines a family Γ_t in $3\mathcal{A}_1$. Each element in Γ_t corresponds to a surface X_t consisting of three A_1 singularities, namely $S_1 = (0 : 0 : 0 : 1), S_2 = (1 : 0 : 0 : 0)$ and $S_3 = (0 : 1 : 0 : 0)$. We see that $H = V(x_1) \cap X_t = 2d + l_t$, where $d = V(x_1, x_2)$ and $l_t = V(x_1, \lambda t x_3 + x_2 + t x_0)$. The surface X_t contains a proper star point $P_t = (\lambda : 0 : 0 : 1)$. When the family Γ_t gives a specialization to X_0, the sections of A_1 singularities S_1 and S_2 contain the two A_2 singularities S_1 and S_2 respectively of X_0. Moreover, the section of proper star points over Γ_t contains $P = (\lambda : 0 : 0 : 1)$ in X_0.

(iii) If X defines a point in $3\mathcal{A}_2$ and the third A_2 singularity is $S_3 = (0 : 1 : 0 : 0)$. This means that $a_1 = a_2 = a_3 = 0$. It is clear that (3.9) defines a family Γ_t in $2\mathcal{A}_1\mathcal{A}_2$. Each corresponding surface X_t of an element in Γ_t contains two A_1 singularities, namely $S_1 = (0 : 0 : 0 : 1), S_2 = (1 : 0 : 0 : 0)$ and one A_2 singularity at $S_3 = (0 : 1 : 0 : 0)$. The surface X_t contains a proper star point $P_t = (\lambda : 0 : 0 : 1)$. When the family Γ_t gives a specialization to X_0, the sections of A_1 singularities S_1 and S_2 contain the two A_2 singularities S_1 and S_2 respectively of X_0. Moreover, the section of proper star points over Γ_t contains $P = (\lambda : 0 : 0 : 1)$ in X_0.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

B.2. On the boundary of $H_2^{(3)}$

Recall that $H_2^{(3)}$ is the subvariety of $\mathbb{P}^{19} - \Delta$ whose corresponding cubic surfaces contain at least one star-Steiner set. The set $H_2^{(3)}$ generically consists of the points corresponding to non-singular cubic surfaces with exactly 3 star points which are collinear.

Proposition 3.4.11. *The set $3\mathcal{A}_1$ is contained in the closure of $H_2^{(3)}$. Consequently, the 3 star points on 3 lines of multiplicities 4 of a cubic surface corresponding to any point of $3\mathcal{A}_1$ are collinear.*

Proof. Let $x \in 3\mathcal{A}_1$. We know from the previous section that the corresponding surface X_x can be considered as the csurface of some 6-point scheme $\mathcal{Q}$ consisting of 6 distinct points $Q_1, \ldots, Q_6$ such that Q_1, Q_3, Q_6 as well as Q_3, Q_5, Q_2 and Q_4, Q_5, Q_6 are collinear; moreover $Q_1 \notin \overline{Q_2 Q_4}$ (see Figure 3.15).

Consider a family of 6-point schemes $\mathcal{P} = \sum_{i=1}^6 P_i$ where $P_i = Q_i$ for $i \in \{2, 3, 4, 6\}$ and P_1, P_5 move on the line $\overline{Q_1 Q_5}$ such that $\overline{P_1 Q_3} \cap \overline{Q_4 P_5} \cap \overline{Q_2 Q_6} \neq \emptyset$. Except for a finite number of positions, each position of (P_1, P_5) gives a 6-point scheme such that its csurface is isomorphic to a cubic surface in $H_2^{(3)}$, see Chapter 2, Section 2.3. This gives a family in $H_2^{(3)}$. When $P_1 = Q_1$ then $P_5 = Q_5$. This implies that $x \in \Delta H_2^{(3)}$. $\quad\square$
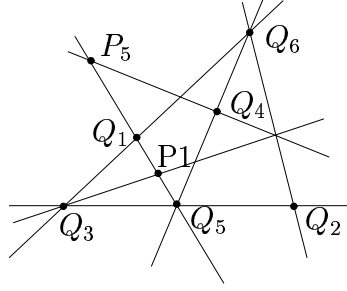
Figure 3.15: 6-point schemes giving points in $3\mathcal{A}_1$

Corollary 3.4.12. *The closure of $3\mathcal{A}_1$ in $(\mathbb{P}^{19})^{ss}$ is an irreducible component of $\Delta H_2^{(3)}$.*

Lemma 3.4.13. *Let x be the generic point of an irreducible component of $\Delta H_2^{(3)}$. Then $x \notin 2\mathcal{A}_1$.*

Proof. Suppose that $x \in 2\mathcal{A}_1$. Let S_1, S_2, S_3 be the three star points of X_x. By choosing coordinates, we can assume that the corresponding cubic surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{Q}$ such that $c(\mathcal{Q}) = \sum_{i=1}^{6} Q_i$ where Q_6, Q_2, Q_3 as well as Q_6, Q_4, Q_5 are collinear, and no 3 of the five points $Q_1, \ldots, Q_5$ are collinear (see Figure 3.11). The line $\tilde{Q}_6$ contains the two singular points and has multiplicity 4. The line $\tilde{l}_{16}$ intersects $\tilde{Q}_6$ but does not contain any singular point. Note that the 9 lines of 3 star triples of a non-singular cubic surface in $H_2^{(3)}$ are mutually different. Since the line $\tilde{l}_{16}$ has multiplicity 1, the line $\tilde{l}_{16}$ does not contain any two of S_1, S_2, S_3. So there exists a star point formed by a triple intersection T whose lines are different from $\tilde{l}_{16}$. From the configuration of lines on X_x, we see that the triple intersection T possesses 2 lines, each of them passes through one singular point of X_x. A contradiction appears from the fact that an A_1 singularity is not a star point. $\square$

Definition: Let K_0 be the subset of $(\mathbb{P}^2)^4$ consisting of 4-tuple (P_1, P_2, P_3, P_4) such that 4 point P_1, P_2, P_3, P_4 are in general position. Let

$$K_1 = \left\{ (P_1, P_2, P_3, P_4, P_5) \in (\mathbb{P}^2)^5 \mid (P_1, P_2, P_3, P_4) \in K_0; P_5 \in l_{12}; P_5 \notin l_{34}; \right.$$

$$\left. P_5 \neq P_i, \quad \forall i = 1, 2 \right\};$$

$$B_2^{(3)} = \left\{ (P_1, P_2, P_3, P_4, P_5, P_6) \in (\mathbb{P}^2)^6 \mid (P_1, P_2, P_3, P_4, P_5) \in K_1; \right.$$

$$\left. l_{16} \cap l_{24} \cap l_{35} \neq \emptyset; \quad l_{14} \cap l_{23} \cap l_{56} \neq \emptyset \right\} \quad \text{(see Figure 3.16).}$$

Let $D_2^{(3)}$ be the subset of $\mathbb{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of one 6-point scheme determined by some 6-tuple in $B_2^{(3)}$. It is easy to see that $D_2^{(3)} \subset \mathcal{A}_1$.
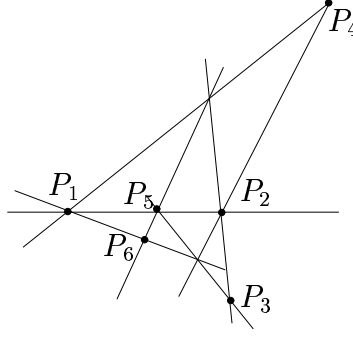
Figure 3.16: 6-point schemes giving points in $D_2^{(3)}$.

Proposition 3.4.14. *The closure of $D_2^{(3)}$ is an irreducible component of $\Delta H_2^{(3)}$.*

Proof. First of all, we prove that $D_2^{(3)}$ is irreducible. The set K_0 is an open subset of $(\mathbb{P}^2)^4$ so that it is irreducible. Each fiber of the projection $p : K_1 \longrightarrow K_0$ is isomorphic to an open set of $\mathbb{P}^1$. This implies that K_1 is irreducible. Since K_1 is isomorphic to $B_2^{(3)}$, the set $B_2^{(3)}$ is irreducible.

Let

$$L = \left\{ (\mathcal{P}, F_1, F_2, F_3, F_4) \mid \mathcal{P} \in B_2^{(3)}; \quad F_i \text{ for } 1 \leq i \leq 4 \text{ is a cubic form in } \mathcal{L}_{\mathcal{P}} \right\},$$

$$U = \left\{ (\mathcal{P}, F_1, F_2, F_3, F_4) \in L \mid \{F_1, F_2, F_3, F_4\} \text{ is a basis of } \mathcal{L}_P \right\}.$$

Consider the following diagram:

$$
\begin{array}{ccccc}
U & \xrightarrow{\text{open}} & L & \xrightarrow{\text{closed}} & B_2^{(3)} \times (\mathbb{P}^9)^4 \\
& \searrow & \Big\downarrow{g} & \swarrow{p} & \\
& & B_2^{(3)} & &
\end{array}
$$

where p is the projection. The map g is surjective and every fiber is isomorphic to $(\mathbb{P}^3)^4$. So L is irreducible. This implies that U is irreducible.

Let D_1 be the subset of $B_2^{(3)} \times \mathbb{P}^{19}$ consisting of all pairs $(\mathcal{P}, x)$ where cubic surface corresponding to x is isomorphic to the csurface of the 6-point scheme determined by $\mathcal{P}$. Given $(\mathcal{P}, F_1, F_2, F_3, F_4) \in U$, the closure of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the basis $\{F_1, F_2, F_3, F_4\}$ is a cubic surface in $\mathbb{P}^3$. We then have a morphism $\tau : U \longrightarrow D_1$ which is surjective. This implies that D_1 is irreducible. Consider the projection $D_1 \longrightarrow D_2^{(3)}$ which is surjective. Consequently $D_2^{(3)}$ is irreducible.

Let $x \in D_2^{(3)}$. The corresponding surface X_x can be considered as the csurface of a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^6 P_i$ where $(P_1, \ldots, P_6) \in B_2^{(3)}$. Fix P_1, P_2, P_6, P_4 and let P_3, P_5 move on the lines l_{23}, l_{56}, respectively such that $l_{16} \cap l_{24} \cap l_{35} \neq \emptyset$. We obtain a family in $H_2^{(3)}$ such that x is a specialization position. Moreover, we see that

when $P_5 = P_6$ then P_3 moves to the intersection point of l_{16} and l_{23}. This gives a 6-point scheme whose csurface is isomorphic to a cubic surface with exactly one A_2 singularity. For this, use a similar quadratic transformation as in Figure 3.20. $\qquad\square$

Corollary 3.4.15. *Let* $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ *be the quotient space with respect to the action of* $\mathrm{PGL}(3)$ *on* $\mathbb{P}^{19}$. *Then* $\phi(\Delta H_2^{(3)}) = \phi(\overline{D_2^{(3)}}) \cup \phi(\overline{3\mathcal{A}_1})$. *Moreover, the components* $\phi(\overline{D_2^{(3)}})$ *and* $\phi(\overline{3\mathcal{A}_1})$ *contain the singleton s.*

Proof. By the end of the proof of the previous proposition, we see that the boundary of $D_2^{(3)}$ contains a point of $\mathcal{A}_2$. So $s \in \phi(\overline{D_2^3})$. Since $3\mathcal{A}_2 \subset \overline{3\mathcal{A}_1}$, we also have $s \in \phi(\overline{3\mathcal{A}_1})$.

Let x be the generic point of an irreducible component W of $\phi(\Delta H_2^{(3)})$. Suppose that $W \neq \phi(\overline{3\mathcal{A}_1})$. By (3.4.13), we see that $x \in \mathcal{A}_1$. By choosing coordinates, we can assume that the surface X_x corresponding to x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ where $c(\mathcal{P}) = \sum_{i=1}^6 P_i$ such that the 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic. Since the singular point of X_x is not a star point, the 3 star points of X_x are determined by 3 triple intersections $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$, $(\tilde{l}_{im}, \tilde{l}_{jk}, \tilde{l}_{nh})$ and $(\tilde{l}_{mk}, \tilde{l}_{ih}, \tilde{l}_{jn})$, where $\{i, j, m, n, h, k\} = \{1, 2, 3, 4, 5, 6\}$. This implies that the 6 points $P_1, \ldots, P_6$ in $\mathbb{P}^2$ satisfy the corresponding conditions, namely $l_{ij} \cap l_{mn} \cap l_{kh} \neq \emptyset$, $l_{im} \cap l_{jk} \cap l_{nh} \neq \emptyset$ and $l_{mk} \cap l_{ih} \cap l_{jn} \neq \emptyset$. Consider the quadratic transformation with respect to P_i, P_n, P_k then the image of $\mathcal{P}$ is a 6-point scheme $\mathcal{Q}$ where $c(\mathcal{Q}) = \sum_{i=1}^6 Q_i$ such that 6 points $Q_1, \ldots, Q_6$, up to a permutation of 6 letters, form an element of $B_2^{(3)}$. This implies that $W = \phi(\overline{D_2^{(3)}})$. $\qquad\square$

B.3. On the boundary of $H_2^{(2)}$

Recall that $H_2^{(2)}$ is the subvariety of $\mathbb{P}^{19}$ consisting of the points whose cubic surfaces contain (at least) two star triples with one line in common.

Definition: Let

$$B_2^{(2)} = \Big\{ (P_1, \ldots, P_5, O) \in (\mathbb{P}^2)^6 \mid P_1, P_2, P_3, P_4, P_5 \text{ are in general position;}$$

$$l_{24} \cap l_{35} = \{O\}; \text{the conic determined by } P_1, \ldots, P_5 \text{ are tangent to } \overline{P_1 O} \text{ at } P_1 \Big\},$$

see Figure 3.17 (a).

Note that each element in $B_2^{(2)}$ defines uniquely a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = 2P_1 + \sum_{i=2}^5 P_i$, where the direction at the double point $2P_1$ is determined by $\overline{P_1 O}$. Moreover, we know that the csurface of $\mathcal{P}$ has exactly two A_1 singularities. Let $D_2^{(2)}$ be the subset of $\mathbb{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of one 6-point scheme determined by some element of $B_2^{(2)}$. It is easy to see that $D_2^{(2)} \subset 2\mathcal{A}_1$.

Let

$$C_2^{(2)} = \Big\{ (P_1, \ldots, P_6) \in (\mathbb{P}^2)^6 \mid P_i \neq P_j \, \forall i \neq j; l_{12} \cap l_{34} \cap l_{56} \neq \emptyset; l_{14} \cap l_{23} \cap l_{56} \neq \emptyset;$$

$$P_1, P_2, P_3, P_4, P_5, P_6 \text{ are contained in an irreducible conic} \Big\},$$

see Figure 3.17 (b))

Let $E_2^{(2)}$ be the subset of $\mathbb{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of one 6-point scheme determined by some 6-tuple in $C_2^{(2)}$. It is easy to see that $E_2^{(2)} \subset \mathcal{A}_1$.
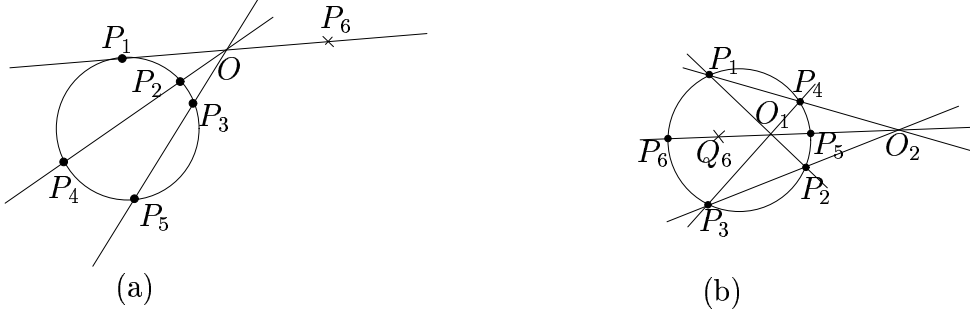
Figure 3.17: 6-point schemes of $B_2^{(2)}$ and $C_2^{(2)}$ respectively

Proposition 3.4.16. *The closures of $D_2^{(2)}$ and $E_2^{(2)}$ are irreducible components of $\Delta H_2^{(2)}$.*

Proof. Let $\mathbb{P}^5$ be the projective space parameterizing the non-zero quadratic forms in three variables. Let

$$K_0 = \Big\{ (P_1, P_2, P_3, P_4,) \in (\mathbb{P}^2)^4 \mid P_1, P_2, P_3, P_4 \text{ are in general position} \Big\};$$

$$K_1 = \Big\{ (P_1, P_2, P_3, P_4, O, \mathcal{C}) \in (\mathbb{P}^2)^5 \times \mathbb{P}^5 \mid (P_1, P_2, P_3, P_4) \in K_0; O \in l_{24}; O \notin \{P_2, P_4\};$$

$$O \notin l_{13}; \mathcal{C} \text{ is the conic containing } P_1, P_2, P_3, P_4 \text{ and tangent to } \overline{P_1 O} \text{ at } P_1 \Big\};$$

$$K_2 = \Big\{ (P_1, P_2, P_3, P_4, P_5, O, \mathcal{C}) \in (\mathbb{P}^2)^6 \times \mathbb{P}^5 \mid (P_1, P_2, P_3, P_4, O, \mathcal{C}) \in K_1; P_5 \in \mathcal{C} \cap \overline{P_3 O} \Big\}.$$

It is clear that the set K_0 is irreducible. Every fiber of the projection $p_1 : K_1 \longrightarrow K_0$ is isomorphic to an open set of $\mathbb{P}^1$. This implies that K_1 is irreducible. The projections $K_2 \longrightarrow K_1$ and $K_2 \longrightarrow B_2^{(2)}$ are isomorphisms. Therefore $B_2^{(2)}$ is irreducible.

Similarly, we prove that $C_2^{(2)}$ is irreducible. For this, let

$$K_3 = \Big\{ (P_1, P_2, P_3, P_4, P_5, O_1, O_2) \in (\mathbb{P}^2)^7 \mid P_1, \dots, P_5 \text{ are in general position};$$

$$l_{12} \cap l_{34} = \{O_1\}; l_{14} \cap l_{23} = \{O_2\}; P_5 \in \overline{O_1 O_2} \Big\};$$

$$K_4 = \Big\{ (P_1, \dots, P_6, O_1, O_2, \mathcal{C}) \in (\mathbb{P}^2)^7 \times \mathbb{P}^5 \mid (P_1, \dots, P_5, O_1, O_2) \in K_3;$$

$$\mathcal{C} \text{ is the conic determined by } P_1, \dots, P_5; P_6 \in \mathcal{C} \cap \overline{O_1 O_2} \Big\}.$$

Consider the projection $p : K_3 \longrightarrow K_0$. We see that every fiber is isomorphic to an open set of $\mathbb{P}^1$. So K_3 is irreducible. The projections $K_4 \longrightarrow K_3$ and $K_4 \longrightarrow C_2^{(2)}$ are isomorphisms. So $C_2^{(2)}$ is irreducible.

By the same argument used in the proof of (3.4.14), we see that $D_2^{(2)}$ and $E_2^{(2)}$ are irreducible.

Suppose $x \in D_2^{(2)}$. The corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}_0$ determined by an element $(P_1, \ldots, P_5, O) \in B_2^{(2)}$. Consider a family of $H_2^{(2)}$ given by 6-point schemes $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where P_6 is a moving point on the line $\overline{P_1 O}$ (see Figure 3.17 (a)). This implies that $x \in \Delta H_2^{(2)}$.

Let x be an element in $E_2^{(2)}$. The cubic surface X_x corresponding to x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}_0$ determined by a 6-tuple $(P_1, \ldots, P_6)$ in $C_2^{(2)}$. Consider a family of $H_2^{(2)}$ given by 6-point schemes $\mathcal{Q}$ such that $c(\mathcal{Q}) = \sum_{i=1}^{6} Q_i$ where $Q_i = P_i$ for $1 \le i \le 5$ and Q_6 is a moving point on the line $\overline{O_1 O_2}$ (see Figure 3.17 (b)). This implies that $x \in \Delta H_2^{(2)}$. $\square$

Proposition 3.4.17. *Let x be the generic point of an irreducible component W of $\Delta H_2^{(2)}$. If $x \in \mathcal{A}_1$, then $W = \overline{E_2^{(2)}}$.*

Proof. By choosing coordinates, we can assume that the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$ where 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic. Since the A_1 singularity is not a star point, the two star points of X_x are defined by triple intersections $(\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$ and $(\tilde{l}_{ij}, \tilde{l}_{mk}, \tilde{l}_{nh})$ where $\{i, j, m, n, h, k\} = \{1, 2, 3, 4, 5, 6\}$. This implies that the 6 points $P_1, \ldots, P_6$ in $\mathbb{P}^2$ satisfy the corresponding conditions, namely $l_{ij} \cap l_{mn} \cap l_{kh} \ne \emptyset$ and $l_{ij} \cap l_{mk} \cap l_{nh} \ne \emptyset$. Up to a permutation of 6 letters, the six points $P_1, \ldots, P_6$ define a 6-tuple in $C_2^{(2)}$. This means that $x \in E_2^{(2)}$ and therefore $W = \overline{E_2^{(2)}}$. $\square$

B.4. On the boundary of $H_4^{(6)}$

Recall that $H_4^{(6)}$ be the subvariety of $\mathbb{P}^{19} - \Delta$ parameterizing non-singular cubic surfaces, each of them possesses a pair (S, T) where S is a star-Steiner set and T is another star triple with exactly one line in common with S.

Proposition 3.4.18. *The set $4\mathcal{A}_1$ is contained in the closure of $H_4^{(6)}$.*

Proof. Let $x \in 4\mathcal{A}_1$. By choosing coordinates, the corresponding surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ consisting of 6 distinct points $P_1, \ldots, P_6$ such that these points form a complete tetragon (see Figure 3.18, (a)). Consider the quadratic transformation φ with respect to P_1, P_2, P_4. We see that $\mathcal{Q} = \varphi(\mathcal{P})$ where $c(\mathcal{Q}) = 2Q_1 + 2Q_4 + Q_2 + Q_5$ such that Q_1, Q_2, Q_3, Q_4 are in general position and the directions at Q_1 and Q_4 contain Q_5 (see Figure 3.18, (b)).

Therefore the surface X_x is isomorphic to the csurface of $\mathcal{Q}$. Let C be the conic containing Q_1, Q_2, Q_4 and being tangent to the directions at Q_1 and Q_4. Consider a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_6$ where $P_i = Q_i$ for $i \in \{1, 2, 4, 5\}$, two points P_3, P_6 are contained in C and $\overline{P_1 P_4} \cap \overline{P_2 P_5} \cap \overline{P_3 P_6} \ne \emptyset$ (see Figure 3.19).

Let P_3 and P_6 move on the conic C such that $\overline{P_1 P_4} \cap \overline{P_2 P_5} \cap \overline{P_3 P_6} \ne \emptyset$. It is clear that we have a family of cubic surfaces in $H_4^{(6)}$. When $(P_3, P_6) = (P_1, P_4)$, we get the 6-point scheme $\mathcal{Q}$. This implies that $x \in \Delta H_4^{(6)}$. $\square$
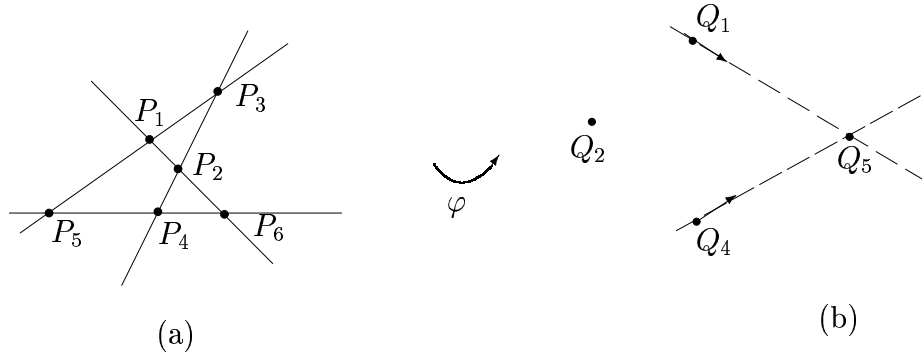
Figure 3.18: 6-point schemes giving points in $4\mathcal{A}_1$
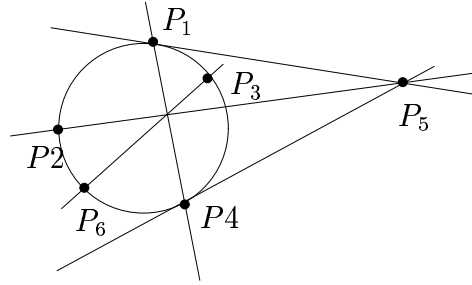
Figure 3.19: 6-point schemes giving points in $H_4^{(6)}$

Corollary 3.4.19. *The closure of $4\mathcal{A}_1$ in $(\mathbb{P}^{19})^{ss}$ is an irreducible component of $\Delta H_4^{(6)}$.*

Remark 3.4.20. We know that four A_1 singularities on a cubic surface in $4\mathcal{A}_1$ form a tetrahedron. Each edge of the tetrahedron contains one star point. Two star points on opposite edges lie on another line of X, which has multiplicity one. So the 6 star points of X lie on a hyperplane generated by the three lines of multiplicity 1.

Recall that $\phi : (\mathbb{P}^{19})^{ss} \longrightarrow \overline{M}$ be the quotient space with respect to the action of PGL(3) on $\mathbb{P}^{19}$.

Proposition 3.4.21. *The set $\phi(\Delta H_4^{(6)})$ consists of two points, one is the singleton s and another is the image of $4\mathcal{A}_1$ in $\overline{M}$.*

Proof. Let $K_4^{(6)}$ be the set consisting of all 6-point schemes $\mathcal{P}$ in general position such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_6$ where the conic C_5 is tangent to l_{15} and l_{45} at P_1 and P_4 respectively; the lines l_{14}, l_{25} and l_{36} have one point in common (see Figure 3.19).

We know from the previous chapter that the blowing-up of $\mathbb{P}^2$ at $\mathcal{P} \in K_4^{(6)}$ is isomorphic to a non-singular cubic surface in $H_4^{(6)}$. Conversely, for any $x \in H_4^{(6)}$, the corresponding cubic surface X_x is isomorphic to the csurface of some 6-point scheme in $K_4^{(6)}$. Let $\mathcal{P} \in K_4^{(6)}$. Fix P_1, P_2, P_4, P_5 and let P_3, P_6 move on the conic C_5 such that $\overline{P_3 P_6}$ contains the intersection point of l_{14} and l_{25}. Except for two positions determined

when $\overline{P_3P_6} = l_{25}$ and $\overline{P_3P_6} = l_{14}$, the 6 points $P_1, \ldots, P_6$ define 6-point schemes in $K_4^{(6)}$. This defines a surjective morphism from an open set of $\mathbb{P}^1$ to $\phi(H_4^{(6)})$. This extends to a surjective morphism $\xi : \mathbb{P}^1 \longrightarrow \phi(\overline{H_4^{(6)}})$. It is clear that when $\overline{P_3P_5} = l_{14}$, we get a point $t_1 \in \mathbb{P}^1$ such that $\xi(t_1) = \phi(4\mathcal{A}_1)$ (see Figure 3.18, (b)). When $\overline{P_3P_5} = l_{25}$, we get a point $t_2 \in \mathbb{P}^1$. The point $\xi(t_2)$ corresponds to the csurface of a 6-point scheme $\mathcal{P}_0$ such that $c(\mathcal{P}_0) = 2P_2 + P_1 + P_4 + P_3 + P_5 + P_6$ where P_2, P_3, P_5 are collinear (see Figure 3.20, (a)).
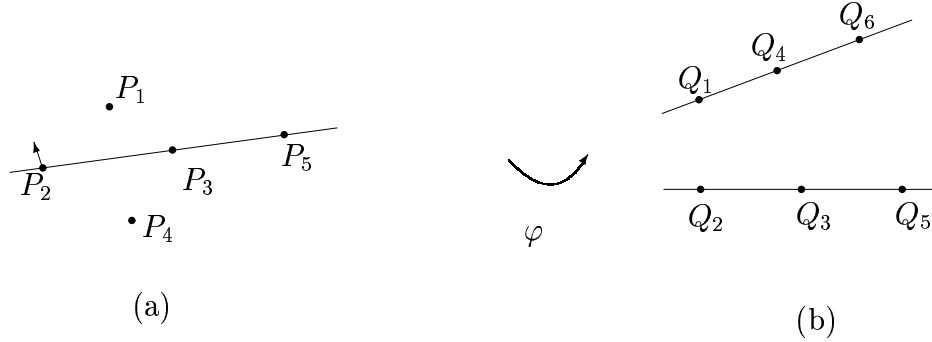

Figure 3.20: 6-point schemes giving points in $\mathcal{A}_2$

We prove that the csurface of $\mathcal{P}_0$ is isomorphic to a cubic surface in $\mathcal{A}_2$. For this, consider the quadratic transformation φ with respect to P_1, P_2, P_4. We see that $\mathcal{Q}_0 := \varphi(\mathcal{P}_0)$ consists of 6 distinct points $Q_1, \ldots, Q_6$ such that Q_1, Q_4, Q_6 as well as Q_2, Q_3, Q_5 are collinear (see Figure 3.20, (b)). As we know, the csurface of $\mathcal{Q}_0$ is isomorphic to a cubic surface with exactly one A_2 singularity. $\square$

Since $H_4^{(6)} \subset H_2^{(2)}$, we have:

Corollary 3.4.22. *The set $\phi(\Delta H_2^{(2)})$ contains the singleton s.*

B.5. On the boundary of $H_4^{(4)}$

Recall that $H_4^{(4)}$ is the subvariety of $\mathbb{P}^{19} - \Delta$ parameterizing non-singular cubic surfaces, each of them possesses a pair (S, T) where S is a star-Steiner set and T is another star triple with all three lines in common with S.

Recall that $N = \mathrm{PGL}(3)\backslash(\mathbb{P}^{19})^s$. We know that $\overline{M} = N \cup \{s\}$. Let

$$B_4^{(4)} = \Big\{ (P_1, \ldots, P_6, O_1) \in (\mathbb{P}^2)^6 \mid P_1, P_2, P_3, P_4 \text{ are in general position ;}$$

$$l_{12} \cap l_{34} = \{O_1\}; P_5 \in l_{13} \cap l_{24}; P_6 \in l_{23} \cap \overline{P_5O_1} \Big\},$$

see Figure 3.21 (a). Note that each element in $B_4^{(4)}$ defines uniquely a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=2}^6 P_i$. Moreover, we know that the csurface of $\mathcal{P}$ is isomorphic to a cubic surface with exactly three A_1 singularities. Let $D_4^{(4)}$ be the subset of $\mathbb{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the
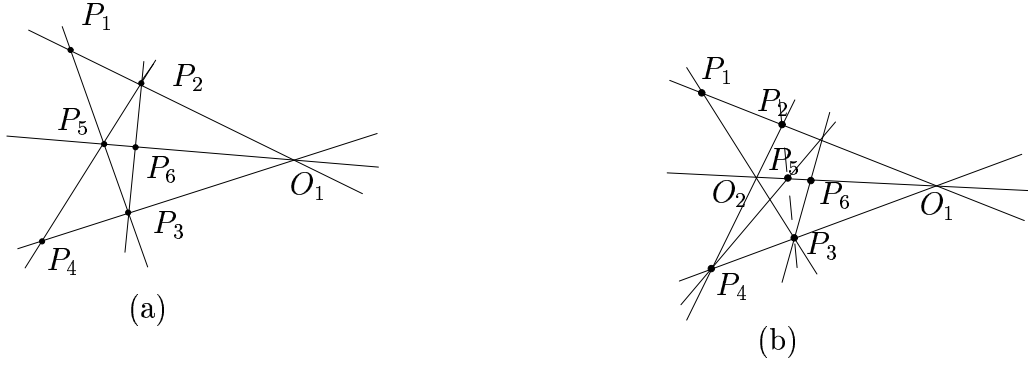
Figure 3.21: 6-point schemes of $B_4^{(4)}$ and $C_4^{(4)}$ respectively

csurface of one 6-point scheme determined by some element in $B_4^{(4)}$. It is easy to see that $D_4^{(4)} \subset 3\mathcal{A}_1$.

Let

$$C_4^{(4)} = \Big\{ (P_1, \ldots, P_6, O_1, O_2) \in (\mathbb{P}^2)^6 \mid P_1, P_2, P_3, P_4 \text{ are in general position ;}$$

$$l_{12} \cap l_{34} \cap l_{56} = \{O_1\}; l_{13} \cap l_{24} \cap l_{56} = \{O_2\}; l_{12} \cap l_{36} \cap l_{45} \neq \emptyset; P_5 \in l_{23} \Big\},$$

see Figure 3.21 (b). Each element in $C_4^{(4)}$ defines uniquely a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^{6} P_i$. Moreover, we know that the csurface of $\mathcal{P}$ is isomorphic to a cubic surface with exactly one A_1 singularity. Let $E_4^{(4)}$ be the subset of $\mathbb{P}^{19}$ consisting of all points such that each corresponding cubic surface is isomorphic to the csurface of one 6-point scheme determined by some element in $C_4^{(4)}$. It is easy to see that $E_4^{(4)} \subset \mathcal{A}_1$.

Proposition 3.4.23. *The closures of $D_4^{(4)}$ and $E_4^{(4)}$ in $(\mathbb{P}^{19})^{ss}$ are irreducible components of $\Delta H_4^{(4)}$.*

Proof. Let $K_0 = \big\{ (P_1, P_2, P_3, P_4) \in (\mathbb{P}^2)^4 \mid P_1, P_2, P_3, P_4 \text{ are in general position} \big\}$. Consider the projection $p : B_4^{(4)} \longrightarrow K_0$. We see that p is an isomorphism. So $B_4^{(4)}$ is irreducible. Similarly the projection $C_4^{(4)} \longrightarrow K_0$ is an isomorphism. So the set $C_4^{(4)}$ is irreducible. By the same argument used in the proof of (3.4.14), we see that $D_4^{(4)}$ and $E_4^{(4)}$ are irreducible.

Let $x \in D_4^{(4)}$. By definition, the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ determined by an element $(P_1, \ldots, P_6, O_1)$ of $B_4^{(4)}$. Fix 4 points P_1, P_2, P_3, P_4. Let $l_{13} \cap l_{24} = \{O_2\}$. Let P_5^t, P_6^t move on the line $\overline{O_1 O_2}$ such that $l_{45} \cap l_{36} \cap l_{12} \neq \emptyset$. Except for a finite number of positions of (P_5^t, P_6^t), the 6 points $P_1, \ldots, P_4, P_5^t, P_6^t$ form a 6-point scheme $\mathcal{P}$ such that the csurface of $\mathcal{P}$ is isomorphic to a cubic surface in $H_4^{(4)}$. We obtain a family in $H_4^{(4)}$. It is clear that x is a specialization position of this family.

Similarly, if $x \in E_4^{(4)}$, we consider a family defined as above. The point x is a specialization position which is determined when $l_{23} \cap \overline{O_1 O_2} = \{P_5\}$. $\qquad\square$

Corollary 3.4.24. *The set $\phi(\Delta H_4^{(4)}) \cap N$ consists of two points which are the image of $\overline{D_4^{(4)}}$ and $\overline{E_4^{(4)}}$.*

Proof. Since $\dim H_4^{(4)} = 16$ and the sets $D_4^4, E_4^{(4)}$ are irreducible, the image of $D_4^{(4)}$ as well as $E_4^{(4)}$ is one point in $\overline{M}$. Let $x \in \overline{H_4^{(4)}}$. Suppose that x is stable. It is sufficient to prove that $x \in D_4^{(4)}$ or $x \in E_4^{(4)}$. The corresponding cubic surface X_x contains 4 star points $S_1, \ldots, S_4$. Note that there exist 3 star points in $\{S_1, \ldots, S_4\}$ corresponding to a star-Steiner set. Therefore, we can apply the same argument as in (3.4.13) to show that $x \notin 2\mathcal{A}_1$.

Suppose that $x \in \mathcal{A}_1$. We can assume that the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=i}^6 P_i$ where 6 points $P_1, \ldots, P_6$ are contained in an irreducible conic. Since the A_1 singularity is not a star point, the 4 star points of X_x are determined by star triples $S_1 = (\tilde{l}_{ij}, \tilde{l}_{mk}, \tilde{l}_{nh}), S_2 = (\tilde{l}_{ih}, \tilde{l}_{mn}, \tilde{l}_{jk}), S_3 = (\tilde{l}_{jn}, \tilde{l}_{im}, \tilde{l}_{kh})$ and $T = (\tilde{l}_{ij}, \tilde{l}_{mn}, \tilde{l}_{kh})$. Note that the triple T has 3 lines in common with 3 triples S_1, S_2, S_3. This implies that the 6 points $P_1, \ldots, P_6$ satisfy the corresponding conditions, namely $l_{ij} \cap l_{mk} \cap l_{nh} \neq \emptyset, l_{ih} \cap l_{mn} \cap l_{jk} \neq \emptyset, l_{jn} \cap l_{im} \cap l_{kh} \neq \emptyset$ and $l_{ij} \cap l_{mn} \cap l_{kh} \neq \emptyset$. Consider the quadratic transformation with respect to P_i, P_n, P_k then we see that the image of $\mathcal{P}$ is a 6-point scheme $\mathcal{Q}$ such that $c(\mathcal{Q}) = \sum_{i=1}^6 Q_i$ where the 6 points $Q_1, \ldots, Q_6$, up to a permutation of 6 letters, form an element of $C_4^{(4)}$. This implies that $x \in E_4^{(4)}$.

If $x \in 3\mathcal{A}_1$. We can assume that the surface X_x is isomorphic to the csurface of a 6-point scheme $\mathcal{P}$ such that $c(\mathcal{P}) = \sum_{i=1}^6 P_i$ where P_1, P_3, P_5 are collinear, the points P_2, P_4, P_5 are collinear, the points P_2, P_3, P_6 are collinear and $P_6 \notin \overline{P_1 P_4}$ (see Figure 3.21 (a)). Let S_1, S_2, S_3 be the three star triples of X_x which correspond to 3 star triples in a star-Steiner set. Let T be another star triple of X_x which has 3 lines in common with S_1, S_2, S_3. The surface X_x has 3 lines of multiplicity 3. They are $\tilde{P}_2, \tilde{P}_3$ and $\tilde{P}_5$. The surface X_x has 3 star points formed by triples $T_1 = (2\tilde{P}_2, \tilde{l}_{12}), T_2 = (2\tilde{P}_3, \tilde{l}_{34})$ and $T_3 = (2\tilde{P}_5, \tilde{l}_{56})$. Suppose that T is one of these three star triples, say $T = (2\tilde{P}_2, \tilde{l}_{12})$. Since T has 3 lines in common with star triples S_1, S_2, S_3, we see that $\tilde{P}_2$ is contained in another star triple. It is impossible since any A_1 singularity is not a star point. This means that $T \notin \{T_1, T_2, T_3\}$. There exists a unique triple in X_x which is different from T_i for $1 \leq i \leq 3$ and able to form a star point, namely $(\tilde{l}_{12}, \tilde{l}_{34}, \tilde{l}_{56})$. This implies that the 6 points $P_1, \ldots, P_6$ satisfy $l_{12} \cap l_{34} \cap l_{56} \neq \emptyset$. This means that the 6 points $P_1, \ldots, P_6$ define an element in $B_4^{(4)}$. Therefore $x \in E_4^{(4)}$. $\square$

Bibliography

[A-F]   D. Allcock, E. Freitag, *Cubic surfaces and Borcherds products*, epreprint Math. AG/0002066.

[Ar1]   V.I. Arnol'd, *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, Funk. Anal. Appl., 6 (1972), 254-272.

[Ar2]   V.I. Arnol'd, *Normal forms of functions near degenerate critical points*, Russian Math. Surveys, 29 (2) (1974), 11-49.

[B]   A. Borel, *Linear Algebraic Groups* (second Enlarged Edition), Springer-Verlag, 1991.

[Ba]   H. F. Baker, *Notes on the theory of the cubic surface*, Proc. London Math. Soc. Ser. 2, Vol. IX. Parts II and III (1910), 205-230.

[Bar]   F. Bardelli, *Osservazioni sui moduli delle superfici cubiche generali*, Atti Accad. Naz. Lincei 64 (1978), 137-141.

[B-D]   F. Bardelli, A. Del Centina, *Nodal cubic surfaces and the rationality of the moduli space of curves of genus two*, Math. Ann. 270 (1985), 599-602.

[Be]   N.D. Beklemishev, *Invariants of cubic forms in four variables*, Vestnik Moskovkogo Universiteta. Matematika, Vol. 37, No. 2 (1982), 42-49.

[B-L]   M. Brundu, A. Logar, *Parametrization of the orbits of cubic surfaces*, Transformation Groups, Vol. III (1998), 1-31.

[Bl1]   W.H. Blythe, *On the construction of models of cubic surfaces*, Quart. Journ. Vol. XXIX (1898), 206-223.

[Bl2]   W.H. Blythe, *On the construction of models of cubic surfaces*, Quart. Journ. Vol. XXXIII (1901), 266-270.

[B-W]   J. W. Bruce, C. T. C. Wall. *On the classification of cubic surfaces*, J. London Math. Soc. (2), 19 (1979), 245-256.

[C1]   A. Cayley, *A memoir on cubic surfaces*, Phil. Trans. Roy. Soc. 159 (1869), 231-326.

[C2]   A. Cayley, *On the triple tangent planes of surfaces of the third order*, Camb. and Dublin Math. Journal, Vol. IV. (1849), 118-132.

[Cr]   L. Cremona, *Mémoire de géométrie pure sur les surfaces du troisième ordre*, Crelle's Journal, Vol. LXVIII (1868), 1-133.

[D-O]   I. Dolgachev, D. Ortland, *Point sets in projective space and theta functions*, Soc. Mathématique de France, Astérisque 165, 1988.

[E-H]   D. Eisenbud, J. Harris, *Schemes: The Language of Modern Algebraic Geometry*, Wadsworth & Brooks/ Cole Advanced Books & Software, 1992.

[Ec]   F. E. Eckardt, *Ueber diejenigen Flächen dritten Grades, auf denen sich drei gerade Linien in einem Punkte schneiden*, Math. Ann. Vol. 10 (1876), 227-272.

[F]   J. Forgaty. *Algebraic families on an algebraic surface*, Amer. J. Math. 90 (1968), 511-521.

[G]   A. Grothendieck. *Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert*, Sém. Bourbaki, n. 221 (1960/61).

[G-H]   P. Griffiths, J. Harris. *Principles of Algebraic Geometry*, John Willey & Sons, Inc., 1994.

[Ge]   A Geramita, *Lectures on the non-singular cubic surfaces in $\mathbb{P}^3$*, Queen's Papers in Pure and Applied Mathematics 83 (1989).

[Gei]   C. F. Geiser, *Ueber die Doppeltangenten einer ebenen Curve vierten Grades*, Math. Ann. Bd. I. (1869), 129-138.

[GIT]   D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory* (Third Enlarged Edition), Springer-Verlag, 1994.

[H]   R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer-Verlag, 1977.

[Ha]   J. Harris, *Algebraic Geometry*, Grad. Texts in Math. 133, Springer-Verlag, 1992.

[He]   A. Henderson, *The Twenty-seven Lines upon the Cubic Surfaces*, Hafner Publishing Co. New York, 1911.

[Hu1]   B. Hunt, *The Geometry of some Special Arithmetic Quotients*, LNM 1637, Springer-Verlag Berlin Heidelberg 1996.

[Hu2]   B. Hunt, *A gem of the modular universe*, epreprint Math. AG/9503018.

[K]   F. Klein, *Lectures on Mathematics*, Evanston Colloquium. Macmillan and Co., N. Y. 1894.

[Ma1]   Yu. I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic*, North-Holland, Amsterdam, 1974.

[Ma2]   Yu. I. Manin, *Cubic hypersurfaces I. Quasigroups of classes of points*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 1223-1224 (in Russian).

[Ma3]   Yu. I. Manin, *Hypersurfaces cubiques II. Automorphismes birationnels en dimension deux*, Invent. Math. 6 (1969), 334-352.

[Ma4]   Yu. I. Manin, *Cubic hypersurfaces III. Moufang loops and Brauer equivalence*, Mat. Sb. 79 (121) (1969), 155-170 (in Russian).

[Mu1]   D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer-Verlag, 1976.

[Mu2]   D. Mumford, *The Red Book of Varieties and Schemes*, LNM 1358, Springer-Verlag, 1988.

[Mu3]   D. Mumford, *Stability of projective varieties*, Enseign. Math., 23 (1977), 39-110.

[N]   P. E. Newstead, *Lectures on Introduction to Moduli Problems and Orbit Spaces*, Tata Inst. Lecture Notes, Springer-Verlag, 1978.

[Na]   I. Naruki, *Cross ratio varieties as a moduli space of cubic surfaces*, Proc. London Math. Soc. (3), 45 (1982), 1-80.

[N-M]   M. Nagata, T. Miyata, *Note on semi-reductive groups*, J. Math. Kyoto Univ. 3 (1963/64), 379-382.

[Na-Se] I. Naruki, J. Sekiguchi, *A modification of Cayley's family of cubic surfaces and birational action of $W(E_6)$ over it*, Proc. Japan Acad., 56, Ser. A (1980), 122-125.

[R]   M. Reid, *Undergraduate Algebraic Geometry*, London Mathematical Society Student Texts 12, 1990.

[S1]   B. Segre, *The Non-Singular Cubic Surfaces*, Oxford, at the Clarendon Press, 1942.

[S2]   B. Serge, *The rational solutions of homogeneous cubic equation in four variables*, Math. Notae Univ. Rosario anno II, fasc. 1-2 (1951), 1-68.

[Sa]   G. Salmon, *On the triple tangent planes to a surface of the third order*, Cambridge and Dublin Math. J. 4 (1849), 252-260.

[Sch1]   L. Schläfli, *On the distribution of the surfaces of the third order into species*, Phil. Trans. Roy. Soc. 153 (1864), 193-247.

[Sch2]   L. Schläfli, *An attemp to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface*, Quart. J. Math. 2 (1858), 56-65.

[Se1]   J. Sekiguchi, *The configuration space of 6 points in $\mathbb{P}^2$, the moduli spaces of cubic surfaces and the Weyl group of type E_6*, RIMS Kokyuroku 848 (1993), 74-85.

[Se2]   J. Sekiguchi, *The versal deformation of the E_6-singularity and a family of cubic surfaces*, J. Math. Soc. Japan 46, n. 2 (1994), 355-383.

[Sp]   T. A. Springer, *Linear Algebraic Groups*, Progress in Math. 9, Birkhäuser, 1981.

[St]   J. Steiner, *The twenty-seven real straight lines on the cubic surface*, Monatsberichte der K. Preuss. Akademie der Wissenschaften, Berlin (1856), 50-58

[T1]   N. C. Tu, *Non-singular cubic surfaces with at least 1,2 or 3 star points*, C.F.C.A. Vol. 2 (1998), 30-45.

[T2]   N. C. Tu, *Non-singular cubic surfaces with star points*, preprint nr. 1082, Department of Mathematics, Utrecht University, 12/1998.

[Ze]   H. G. Zeuthen, *Sur les différentes formes des courbes planes du quatrième ordre*, Math. Ann. Vol. VI (1874), 410-432.

[Y]   M. Yoshida, *A $W(E_6)$-equivariant projective embedding of the moduli space of cubic surface*, epreprint Math. AB/0002102.

Samenvatting

Een niet-singulier kubisch oppervlak bevat 27 lijnen. Dit is een klassiek onderwerp in de wiskunde. De configuratie van deze 27 lijnen is een fascinerende structuur. F. E. Eckardt onderzocht in 1876 niet-singuliere kubische oppervlakken waar drie lijnen op het oppervlak door één punt gaan; een dergelijk punt noemen we een *sterpunt* op een kubisch oppervlak (het wordt ook wel een *Eckardt punt* genoemd).

Een kubisch oppervlak wordt gegeven als nulpuntenverzameling van een homogeen polynoom in 4 variabelen van graad 3. Een dergelijke veelterm heeft 20 coëfficienten. Zo zien we dat $\mathbb{P}^{19}$ een parametrizatie geeft van de verzameling van alle kubische oppervlakken. In Hoofdstuk 2 bestuderen we voor elk natuurlijk getal k de deelvariëteit H_k van $\mathbb{P}^{19}$ die gedefiniëerd wordt als de verzameling van alle niet-singuliere kubische oppervlakken met tenminste k sterpunten. De stratificatie verkregen door de H_k wordt beschreven.

Een kubisch oppervlak kan worden verkregen door het opblazen van 6 punten in een projectief vlak. De configuratie van de 27 lijnen op dat oppervlak kan eenvoudig en elegant worden afgelezen uit eigenschappen van de configuratie van die 6 punten. Het hele proefschrift wordt gedacht vanuit de structuur van 6-punt schema's in $\mathbb{P}^2$ en het daardoor gedefiniëerde kubische oppervlak.

Vervolgens bestuderen we in het derde hoodstuk van dit proefschrift singuliere kubische oppervlakken, moduli-ruimten daarvan, de configuratie van lijnen op een kubisch oppervlak, en de "multipliciteit" van een lijn op een (singulier) kubisch oppervlak. Hierdoor krijgen we inzicht in eigenschappen van de rand van de H_k in een compactificatie, van multipliciteiten van drie-raakvlakken aan semi-stabiele kubische oppervlakken en de relatie daarvan met corresponderende 6-punt schema's.

Curriculum Vitae

23 January 1965:	born in Da Nang, Vietnam.
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