Taming the cosmological constant in 2D causal quantum gravity with topology change

R. Loll[†], W. Westra[†] and S. Zohren[†,‡]

†*Institute for Theoretical Physics, Utrecht University*
Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands

‡*Institut für Theoretische Physik E, RWTH-Aachen*
D-52056 Aachen, Germany

E-Mail:
r.loll@phys.uu.nl, w.westra@phys.uu.nl and zohren@physik.rwth-aachen.de

Abstract

As shown in previous work, there is a well-defined nonperturbative gravitational path integral including an explicit sum over topologies in the setting of Causal Dynamical Triangulations in two dimensions. In this paper we derive a complete analytical solution of the quantum continuum dynamics of this model, obtained uniquely by means of a double-scaling limit. We show that the presence of infinitesimal wormholes leads to a decrease in the effective cosmological constant, reminiscent of the suppression mechanism considered by Coleman and others in the four-dimensional Euclidean path integral. Remarkably, in the continuum limit we obtain a finite spacetime density of microscopic wormholes without assuming fundamental discreteness. This shows that one can in principle make sense of a gravitational path integral which includes a sum over topologies, provided suitable causality restrictions are imposed on the path integral histories.

1 Introduction

Despite recent progress [1, 2], little is known about the ultimate configuration space of quantum gravity on which its nonperturbative dynamics takes place. This makes it difficult to decide which (auxiliary) configuration space to choose as starting point for a quantization. In the context of a path integral quantization of gravity, the relevant question is which class of geometries one should be integrating over in the first place. Setting aside the formidable difficulties in "doing the integral", there is a subtle balance between including too many geometries – such that the integral will simply fail to exist (nonperturbatively) in any meaningful way, even after renormalization – and including too few geometries, with the danger of not capturing a physically relevant part of the configuration space.

A time-honoured part of this discussion is the question of whether a sum over different spacetime topologies should be included in the gravitational path integral. The absence to date of a viable theory of quantum gravity in four dimensions has not hindered speculation on the potential physical significance of processes involving topology change (for reviews, see [3, 4]). Because such processes necessarily violate causality, they are usually considered in a Euclidean setting where the issue does not arise. Even if one believes that Euclidean quantum gravity *without* a sum over topologies exists nonperturbatively as a fundamental theory of nature – something for which there is currently little evidence –, insisting on including such a sum makes it all the more difficult to perform the path integral and extract physical information from it. This happens because the number of ways in which one can cut and reglue a manifold to obtain manifolds with a different topology is very large and leads to uncontrollable divergences in the path integral.[1]

Formal path integrals involving nontrivial topologies are either semiclassical and assume that a (usually small and hand-picked) class of such configurations dominates the path integral, without being able to check that they are indeed saddle points of a full, nonperturbative formulation (see [7] for a recent example), or otherwise postulate a separation of scales between the fundamental quantum excitations of the geometry and a (larger) length scale characteristic for the topology changes, for example, the size of wormholes [8]. We are not aware of any evidence from nonperturbative approaches that would support either of these assumptions. On the contrary, as already mentioned, path integrals including a sum over topologies tend not to exist at all. Furthermore, as is clear from studies of both Euclidean [6] and Lorentzian [9] sums over geometries in two dimensions, once topology changes are allowed dynamically, they occur everywhere and at all scales.[2]

A new idea to tame the divergences associated with topology changes in the path inte-

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[1]Even in the simplest case of two-dimensional geometries, the number of possible configurations grows faster than exponentially with the volume of the geometry. A well-known manifestation of this problem is the non-Borel summability of the genus expansion in string theory. This does not necessarily mean that there is no underlying well-defined theory, but even in the much-studied case of two-dimensional Euclidean quantum gravity no physically satisfactory, unambiguous solution has been found [6].

[2]In the Lorentzian case cited, the standard sum over causal geometries of a *fixed* spacetime topology [10] is extended by allowing "baby universes" which branch off the main universe, causing changes in the topology of spatial slices as a function of time, but not changing the spacetime topology.

gral was advanced in [11] and implemented in a model of two-dimensional nonperturbative Lorentzian quantum gravity. The idea is to include a sum over topologies, or over some subclass of topologies, in the state sum, but to restrict this class further by certain *geometric* (as opposed to topological) constraints. These constraints involve the causal (and therefore Lorentzian) structure of the spacetimes and thus would have no analogue in a purely Euclidean formulation. In the concrete two-dimensional model considered in [11], the path integral is taken over a geometrically distinguished class of spacetimes with arbitrary numbers of "wormholes", which violate causality only relatively mildly (see also [12]). As a consequence, the nonperturbative path integral turns out to be well defined. This is an extension of the central idea of the approach of causal dynamical triangulations, namely, to use physically motivated causality restrictions to make the gravitational path integral better behaved (see [13] for a review).

In this paper, we will present a complete analytical solution of the statistical model of two-dimensional Lorentzian random geometries introduced in [11], whose starting point is a regularized sum over causal triangulated geometries *including* a sum over topologies. For a given genus (i.e. number of (worm)holes in the spacetime) not all possible triangulated geometries are included in the sum, but only those which satisfy certain causality constraints. As shown in [11], this makes the statistical model well defined, and an unambiguous continuum limit is obtained by taking a suitable double-scaling limit of the two coupling constants of the model, the gravitational or Newton's constant and the cosmological constant. The double-scaling limit presented here differs from the one found in [11, 12], where only the partition function for a single spacetime strip was evaluated. We will show that when one includes the boundary lengths of the strip explicitly – as is necessary to obtain the full spacetime dynamics – the natural renormalization of Newton's constant involves the boundary "cosmological" coupling constants conjugate to the boundary lengths. Although the holes we include exist only for an infinitesimal time, and we do not keep track of them explicitly in the states of the Hilbert space, their integrated effect is manifest in the continuum Hamiltonian of the resulting gravity theory. As we will see, their presence leads to an effective lowering of the cosmological constant and therefore represents a concrete and nonperturbative implementation of an idea much discussed in the late eighties in the context of the ill-defined continuum path integral formulation of Euclidean quantum gravity (see, for example, [8, 14]).

The remainder of the paper is structured as follows. In the next section, we briefly describe how a nonperturbative theory of two-dimensional Lorentzian quantum gravity can be obtained by the method of causal dynamical triangulation (CDT), and how a sum over topologies can be included. For a more detailed account of the construction of topology-changing spacetimes and the geometric reasoning behind the causality constraints we refer the reader to [11, 12]. The main result of Sec. 3 is the computation of the Laplace transform of the one-step propagator of the discrete model for arbitrary boundary geometries. In Sec. 4 we make a scaling ansatz for the coupling constants and show that just one of the choices for the scaling of Newton's constant leads to a new and physically sensible continuum theory. We calculate the corresponding quantum Hamiltonian and its spectrum, as well as the full propagator of the theory. Using these results, we compute several observables of

the continuum theory in Sec. 5, most importantly, the expectation value of the number of holes and its spacetime density. In Sec. 6, we summarize our results and draw a number of conclusions. In Appendix A, we discuss the properties of alternative scalings for Newton's constant which were discarded in the main text. This also establishes a connection with a previous attempt [15] to generalize the original Lorentzian model without topology changes. In Appendix B, we calculate the spacetime density of holes from a single infinitesimal spacetime strip.

2   Lorentzian sum over topologies

Our aim is to calculate the (1+1)-dimensional gravitational path integral

$$Z(G_N, \Lambda) = \sum_{topol.} \int D[g_{\mu\nu}] e^{iS(g_{\mu\nu})} \tag{1}$$

nonperturbatively by using the method of Causal Dynamical Triangulations (CDT).[3]. The sum in (1) denotes the inclusion in the path integral of a specific, causally preferred class of fluctuations of the manifold topology. The action $S(g_{\mu\nu})$ consists of the usual Einstein-Hilbert curvature term and a cosmological constant term. Since we work in two dimensions, the integrated curvature term is proportional to the Euler characteristic $\chi = 2 - 2\mathfrak{g} - b$ of the spacetime manifold, where $\mathfrak{g}$ denotes the genus (i.e. the number of handles or holes) and b the number of boundary components. Explicitly, the action reads

$$S = 2\pi\chi K - \Lambda \int d^2 x \sqrt{|\det g_{\mu\nu}|} \tag{2}$$

where $K = 1/G_N$ is the inverse Newton's constant and Λ the cosmological constant (with dimension of inverse length squared).

Just like in the original CDT model [10] (which from now on we will also refer to as the "pure" model, i.e. without topology changes), we will first regularize the path integral (1) by a sum over piecewise flat two-dimensional spacetimes, whose flat building blocks are identical Minkowskian triangles, all with one space-like edge of squared length $+a^2$ and two time-like edges of squared length $-\alpha a^2$, where α is a real positive constant. The CDT path integral takes the form of a sum over triangulations, with each triangulation consisting of a sequence of spacetime strips of height $\Delta t = 1$ in the time direction. A single such strip is a set of l_{in} triangles pointing up and l_{out} triangles pointing down (Figure 1). Because the geometry has a sliced structure, one can easily Wick-rotate it to a triangulated manifold of Euclidean signature by analytically continuing the parameter α to a real negative value [17]. For simplicity, we will set $\alpha = -1$ in evaluating the regularized, real and Wick-rotated version of the path integral (1).

In the pure CDT model the one-dimensional spatial slices of constant proper time t are usually chosen as circles, resulting in cylindrical spacetime geometries. For our present

[3]For an introduction to CDT the reader is referred to [10, 16, 17].
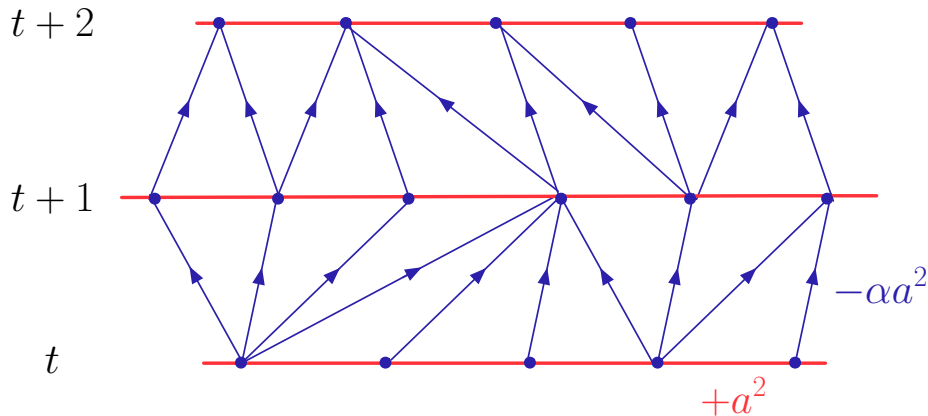
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Figure 1: A section of a sequence $[t, t + 2]$ of two spacetime strips of a triangulated two-dimensional spacetime contributing to the regularized path integral without topology changes.

purposes, we will enlarge this class of geometries by allowing the genus to be variable. We define the sum over topologies by performing surgery moves directly on the triangulations to obtain regularized versions of higher-genus manifolds [11, 12]. They are generated by adding tiny wormholes that connect two regions of the same spacetime strip. Starting from a regular strip of topology $[0, 1] \times S^1$ and height $\Delta t = 1$, one can construct a hole by identifying two of the strip's time-like edges and subsequently cutting open the geometry along this edge (Fig. 2). By applying this procedure repeatedly (obeying certain causality constraints [11, 12]), more and more wormholes can be created. Once the regularized path integral has been performed, including a sum over geometries with wormholes, one takes a continuum limit by letting $a \to 0$ and renormalizing the coupling constants appropriately, as will be described in the following sections.

A similar type of wormhole has played a prominent role in past attempts to devise a mechanism to explain the smallness of the cosmological constant in the Euclidean path integral formulation of *four*-dimensional quantum gravity in the continuum [8, 14]. The wormholes considered there resemble those of our toy model in that both are non-local identifications of the spacetime geometry of infinitesimal size. The counting of our wormholes is of course different since we are working in a genuinely Lorentzian setup where certain causality conditions have to be fulfilled. This enables us to do the sum over topologies completely explicitly. Whether a similar construction is possible also in higher dimensions is an interesting, but at this stage open question.
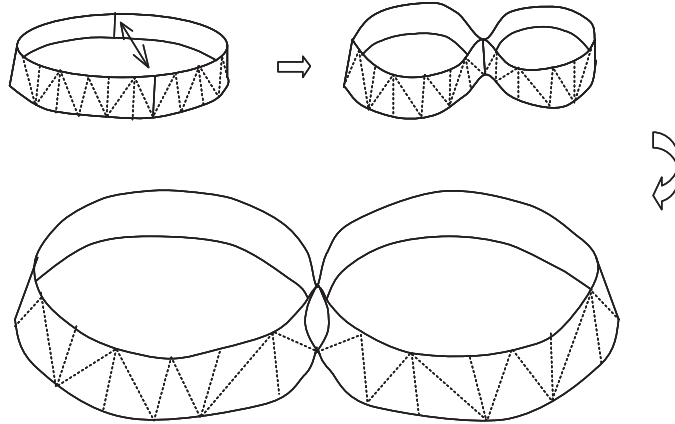
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Figure 2: Construction of a wormhole by identifying two time-like edges of a spacetime strip and cutting open the geometry along the edge.

3   Discrete solution: the one-step propagator

For the (1+1)-dimensional Lorentzian gravity model including a sum over topologies, the partition function of a single spacetime strip of infinitesimal duration with summed-over boundaries was evaluated in [11] and [12]. In the present paper, we will extend this treatment by calculating the full one-step propagator, or, equivalently, the generating function for the partition function of a single strip with given, fixed boundary lengths. This opens the way for investigating the full dynamics of the model.

The discrete set-up described above leads to the Wick-rotated one-step propagator

$$G_{\lambda,\kappa}(l_{in}, l_{out}, t = 1) = e^{-\lambda(l_{in}+l_{out})} \sum_{T|l_{in},l_{out}} e^{-2\kappa\mathfrak{g}}, \tag{3}$$

where κ is the bare inverse Newton's constant and λ the bare (dimensionless) cosmological constant, and we have omitted an overall constant coming from the Gauss-Bonnet integration. The sum in (3) is to be taken over all triangulations with l_{in} space-like links in the initial and l_{out} space-like links in the final boundary. Note that the number of holes does not appear as one of the arguments of the one-step propagator since we only consider holes that exist within one strip. Consequently, the number of holes does not appear explicitly as label for the quantum states, and the Hilbert space coincides with that of the pure CDT model. Nevertheless, the integrated effect of the topologically non-trivial configurations changes the dynamics and the quantum Hamiltonian, as we shall see.

The one-step propagator (3) defines a transfer matrix $\hat{T}$ by

$$G_{\lambda,\kappa}(l_{in}, l_{out}, 1) = \langle l_{out}|\hat{T}|l_{in}\rangle, \tag{4}$$

from which we obtain the propagator for t time steps as usual by iteration,

$$G_{\lambda,\kappa}(l_{in}, l_{out}, t) = \langle l_{out}|\hat{T}^t|l_{in}\rangle. \tag{5}$$

6

For simplicity we perform the sum in (3) over triangulated strips with periodically identified boundaries in the spatial direction and one marked time-like edge. By virtue of the latter, $G_{\lambda,\kappa}(l_{in}, l_{out}, t)$ satisfies the desired composition property of a propagator,

$$G_{\lambda,\kappa}(l_{in}, l_{out}, t_1 + t_2) = \sum_l G_{\lambda,\kappa}(l_{in}, l, t_1) G_{\lambda,\kappa}(l, l_{out}, t_2), \tag{6}$$

$$G_{\lambda,\kappa}(l_{in}, l_{out}, t + 1) = \sum_l G_{\lambda,\kappa}(l_{in}, l, 1) G_{\lambda,\kappa}(l, l_{out}, t), \tag{7}$$

where the sums on the right-hand sides are performed over an intermediate constant-time slice of arbitrary discrete length l.

Performing the fixed-genus part of the sum over triangulations in (3) yields

$$G_{\lambda,\kappa}(l_{in}, l_{out}, 1) = e^{-\lambda N} \sum_{\mathfrak{g}=0}^{[N/2]} \binom{N}{l_{in}} \binom{N}{2\mathfrak{g}} \frac{(2\mathfrak{g})!}{\mathfrak{g}!(\mathfrak{g}+1)!} e^{-2\kappa\mathfrak{g}}, \tag{8}$$

with $N = l_{in} + l_{out}$. To simplify calculations we will use the generating function formalism with

$$G(x, y, g, h, 1) = \sum_{l_{in}, l_{out}=0}^{\infty} G_{\lambda,\kappa}(l_{in}, l_{out}, 1) x^{l_{in}} y^{l_{out}}, \tag{9}$$

where we have defined $g = e^{-\lambda}$ and $h = e^{-\kappa}$. The quantities x and y can be seen as purely technical devices, or alternatively as exponentiated bare boundary cosmological constants

$$x = e^{-\lambda_{in}}, \quad y = e^{-\lambda_{out}}. \tag{10}$$

Upon evaluating the sum over l_{in} and l_{out} one obtains the generating function of the one-step propagator

$$G(x, y, g, h, 1) = \frac{1}{1 - g\,(x+y)} \frac{2}{1 + \sqrt{1 - 4u^2}}, \tag{11}$$

with

$$u = \frac{h}{\frac{1}{g\,(x+y)} - 1}. \tag{12}$$

Note that in order to arrive at the final result (11), we have performed an explicit sum over all topologies! The fact that this infinite sum converges for appropriate values of the bare couplings has to do with the causality constraints imposed on the model, which were geometrically motivated in [11], and which effectively reduce the number of geometries in the genus expansion.

In (11) one recognizes the generating function $\mathrm{Cat}(u^2)$ for the Catalan numbers,

$$\mathrm{Cat}(u^2) = \frac{2}{1 + \sqrt{1 - 4u^2}}. \tag{13}$$

For $h = 0$ one has $\text{Cat}(u^2) = 1$ and expression (11) reduces to the one-step propagator without topology changes,

$$G(x, y, g, h = 0, 1) = \frac{1}{1 - g\ (x + y)}.\tag{14}$$

Furthermore, one recovers the one-step partition function with summed-over boundaries of [11, 12] by setting $x = y = 1$,

$$Z(g, h, 1) = \frac{1}{1 - 2g}\frac{2}{1 + \sqrt{1 - 4(\frac{2gh}{1-2g})^2}}.\tag{15}$$

4   Taking the continuum limit

Taking the continuum limit in the case without topology changes is fairly straightforward [10]. The joint region of convergence of (14) is given by

$$|x| < 1, \quad |y| < 1, \quad |g| < \frac{1}{2}.\tag{16}$$

One then tunes the couplings to their critical values according to the scaling relations

$$g = \frac{1}{2}(1 - a^2\,\Lambda) + \mathcal{O}(a^3),\tag{17}$$

$$x = 1 - a\,X + \mathcal{O}(a^2), \quad y = 1 - a\,Y + \mathcal{O}(a^2).\tag{18}$$

Up to additive renormalizations, x, y and λ scale canonically, with corresponding renormalized couplings X, Y and Λ. In the case with topology change we have to introduce an additional scaling relation for h. Since Newton's constant is dimensionless in two dimensions, there is no preferred canonical scaling for h. We make the multiplicative ansatz[4]

$$h = \frac{1}{\sqrt{2}}h_{ren}(ad)^{\beta},\tag{19}$$

where h_{ren} depends on the renormalized Newton's constant G_N according to

$$h_{ren} = e^{-2\pi/G_N}.\tag{20}$$

In order to compensate the powers of the cut-off a in (19), d must have dimensions of inverse length. The most natural ansatz in terms of the dimensionful quantities available is

$$d = (\sqrt{\Lambda}^{\alpha}(X + Y)^{1-\alpha}).\tag{21}$$

The constants β and α in relations (19) and (21) must be chosen such as to obtain a physically sensible continuum theory. By this we mean that the one-step propagator should

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[4]Here the factor $\frac{1}{\sqrt{2}}$ is chosen to give a proper parametrization of the number of holes in terms of Newton's constant (see Section 5).

yield the Dirac delta-function to lowest order in a, and that the Hamiltonian should be bounded below and not depend on higher-order terms in (17), (18), in a way that would introduce a dependence on new couplings without an obvious physical interpretation.

To calculate the Hamiltonian operator $\hat{H}$ we use the analogue of the composition law (6) for the Laplace transform of the one-step propagator [10],

$$G(x, y, t + 1) = \oint \frac{dz}{2\pi i z} G(x, z^{-1}; 1) G(z, y, t). \tag{22}$$

In a similar manner we can write the time evolution of the wave function as

$$\psi(x, t + 1) = \oint \frac{dz}{2\pi i z} G(x, z^{-1}; 1) \psi(z, t). \tag{23}$$

When inserting the scaling relations (17), (18) and $t = \frac{T}{a}$ into this equation it is convenient to treat separately the first factor in the one-step propagator (11), which is nothing but the one-step propagator without topology changes (14), and the second factor, the Catalan generating function (13). Expanding both sides of (23) to order a gives

$$\left(1 - a\hat{H} + \mathcal{O}(a^2)\right) \psi(X) = \int_{-i\infty}^{i\infty} \frac{dZ}{2\pi i} \left\{ \left(\frac{1}{Z - X} + a \frac{2\Lambda - XZ}{(Z - X)^2} \right) \mathrm{Cat}(u^2) \right\} \psi(Z), \tag{24}$$

where we have used

$$\psi(X, T + a) = e^{-a\hat{H}} \psi(X, T), \tag{25}$$

with $\psi(X) \equiv \psi(x = 1 - aX)$. Note that the first term on the right-hand side of (24), $\frac{1}{Z-X}$, is the Laplace-transformed delta-function. The interesting new behaviour of the Hamiltonian is contained in the expansion of the Catalan generating function. Combining (13) and (12), and inserting the scalings (17), (18), yields

$$\mathrm{Cat}(u^2) = 1 + \frac{2 \, d^{2\beta} \, h_{ren}^2}{(Z - X)^2} a^{2\beta - 2} + \text{h.o.}, \tag{26}$$

where h.o. refers to terms of higher order in a. In order to preserve the delta-function and have a non-vanishing contribution to the Hamiltonian one is thus naturally led to $\beta = 3/2$. For suitable choices of α it is also possible to obtain the delta-function by setting $\beta = 1$, but the resulting Hamiltonians turn out to be unphysical or at least do not have an interpretation as gravitational models with wormholes, as we will discuss in Appendix A.[5]

For $\beta = 3/2$ the right-hand side of (24) becomes

$$\int_{-i\infty}^{i\infty} \frac{dZ}{2\pi i} \left\{ \frac{1}{Z - X} + a \left(\frac{2\Lambda - XZ}{(X - Z)^2} - \frac{2\sqrt{\Lambda}^{3\alpha} h_{ren}^2}{(X - Z)^{3\alpha}} \right) \right\} \psi(Z). \tag{27}$$

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[5]One might also consider scalings of the form $h \to c_1 h_{ren}(ad) + c_2 h_{ren}(ad)^{3/2}$, but they can be discarded by arguments similar to those of Appendix A.

We observe that for $\alpha \leqslant 0$ the last term in (27) does not contribute to the Hamiltonian. Performing the integration for $\alpha > 0$ and discarding the possibility of fractional poles the Hamiltonian reads

$$\hat{H}(X, \frac{\partial}{\partial X}) = X^2 \frac{\partial}{\partial X} + X - 2\Lambda \frac{\partial}{\partial X} + 2\Lambda^{\frac{3\alpha}{2}} h_{ren}^2 \frac{(-1)^{3\alpha}}{\Gamma(3\alpha)} \frac{\partial^{3\alpha-1}}{\partial X^{3\alpha-1}}, \quad \alpha = \frac{1}{3}, \frac{2}{3}, 1, ... \quad (28)$$

For all α's, these Hamiltonians do not depend on higher-order terms in the scaling of the coupling constants. One can check this by explicitly introducing a term quadratic in a (which can potentially contribute to $\hat{H}$) in the scaling relations (18), namely,

$$x = 1 - aX + \frac{1}{2}\gamma a^2 X^2 + \mathcal{O}(a^3),$$
$$y = 1 - aY + \frac{1}{2}\gamma a^2 Y^2 + \mathcal{O}(a^3), \quad (29)$$

and noticing that (28) does not depend on γ. After making an inverse Laplace transformation $\psi(L) = \int_0^\infty dX e^{X L} \psi(X)$ to obtain a wave function in the "position" representation (where it depends on the spatial length L of the universe), and introducing $m = 3\alpha - 1$ the Hamiltonian reads

$$\hat{H}(L, \frac{\partial}{\partial L}) = -L\frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + 2\Lambda L - \frac{2\Lambda^{\frac{m+1}{2}} h_{ren}^2}{\Gamma(m+1)} L^m, \quad m = 0, 1, 2, ... \quad (30)$$

Since $\hat{H}$ is unbounded below for $m \geqslant 2$, we are left with $m=0$ and $m=1$ as possible choices for the scaling. However, setting $m=0$ merely has the effect of adding a constant term to the Hamiltonian, leading to a trivial phase factor for the wave function. We conclude that the only new and potentially interesting model corresponds to the scaling with $m=1$ and

$$h^2 = \frac{1}{2} h_{ren}^2 \Lambda (X + Y) a^3, \quad (31)$$

with the Hamiltonian given by

$$\hat{H}(L, \frac{\partial}{\partial L}) = -L\frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + (1 - h_{ren}^2) 2\Lambda L. \quad (32)$$

Note that for all values $G_N \geq 0$ of the renormalized Newton's constant (19) the Hamiltonian is bounded from below and therefore well defined. It is self-adjoint with respect to the natural measure $d\mu(L) = dL$ and has a discrete spectrum, with eigenfunctions

$$\psi_n(L) = \mathcal{A}_n e^{-\sqrt{2\Lambda(1-h_{ren}^2)}L} L_n(2\sqrt{2\Lambda(1-h_{ren}^2)}), \quad n = 0, 1, 2, ..., \quad (33)$$

where L_n denotes the n'th Laguerre polynomial. Choosing the normalization constants as

$$\mathcal{A}_n = \sqrt[4]{8\Lambda(1-h_{ren}^2)}, \quad (34)$$

10

the eigenvectors $\{\psi_n(L),\ n=0,1,2,...\}$ form an orthonormal basis, and the corresponding eigenvalues are given by

$$E_n = \sqrt{2\Lambda(1 - h_{ren}^2)}\,(2n + 1)\,, \quad n = 0, 1, 2, ... \tag{35}$$

Having obtained the eigenvalues one can easily calculate the Euclidean partition function for finite time T (with time periodically identified)

$$Z_T(G_N, \Lambda) = \sum_{n=0}^{\infty} e^{-T\,E_n} = \frac{e^{-\sqrt{2\Lambda(1-h_{ren}^2)}T}}{1 - e^{-2\sqrt{2\Lambda(1-h_{ren}^2)}T}}\,, \quad h_{ren} = e^{-2\pi/G_N}\,. \tag{36}$$

For completeness we also compute the finite-time propagator

$$G_{\Lambda,G_N}(L_1, L_2, T) \equiv \langle L_2 | e^{-T\hat{H}} | L_1 \rangle = \sum_{n=0}^{\infty} e^{-T E_n} \psi_n^*(L_2)\psi_n(L_1). \tag{37}$$

Inserting (33) into (37) and using known relations for summing over Laguerre polynomials [19] yields

$$G_{\Lambda,G_N}(L_1, L_2, T) = \omega\,\frac{e^{-\omega(L_1+L_2)\coth(\omega T)}}{\sinh(\omega T)}\,I_0\left(\frac{2\omega\sqrt{L_1 L_2}}{\sinh(\omega T)}\right), \tag{38}$$

where we have used the shorthand notation $\omega = \sqrt{2\Lambda(1 - h_{ren}^2)}$. As expected, for $h_{ren} \to 0$ the results reduce to those of the pure two-dimensional CDT model.

5   Observables

Due to the low dimensionality of our quantum-gravitational model, it has only a few observables which characterize its physical properties. Given the eigenfunctions (33) of the Hamiltonian (32) one can readily calculate the average spatial extension $\langle L \rangle$ of the universe and all higher moments

$$\langle L^m \rangle_n = \int_0^\infty dL\, L^m |\psi_n(L)|^2. \tag{39}$$

Using integral relations for the Laguerre polynomials [19] one obtains[6]

$$\langle L^m \rangle_n = \left(\frac{1}{8\Lambda(1 - h_{ren}^2)}\right)^{\frac{m}{2}} \frac{\Gamma(n - m)\Gamma(m + 1)}{\Gamma(n + 1)\Gamma(-m)} \times$$

$$\times\ _3F_2(-n, 1 + m, 1 + m; 1, 1 + m - n; 1), \tag{40}$$

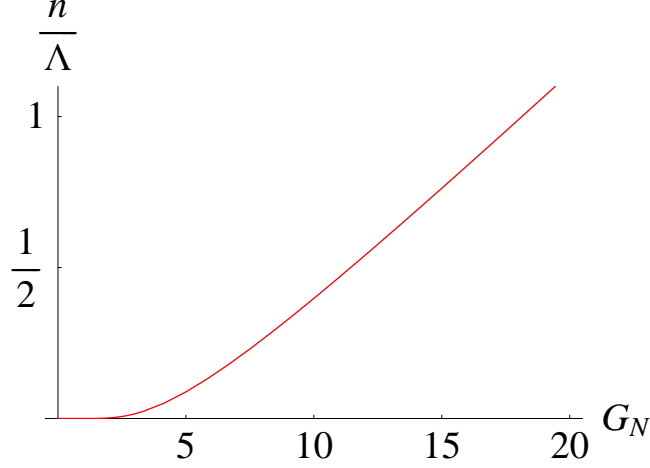[6]Note that the poles of $\Gamma(-m)$ cancel with those of the hypergeometric function.

Figure 3: The density of holes n in units of Λ as a function of Newton's constant G_N.

where $_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is the generalized hypergeometric function defined by

$$_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k \, z^k}{(b_1)_k (b_2)_k k!}. \tag{41}$$

Observe that the moments scale as $\langle L^m \rangle_n \sim \Lambda^{-\frac{m}{2}}$ which indicates that the effective Hausdorff dimension is given by $d_H = 2$, just like in the pure CDT model [16].

In addition to these well-known geometric observables, the system possesses a new type of "topological" observable which involves the number of holes $N_{\mathfrak{g}}$, as already anticipated in [11, 12]. As spelled out there, the presence of holes in the quantum geometry and their density can be determined from light scattering. An interesting quantity to calculate is the average number of holes in a piece of spacetime of duration T, with initial and final spatial boundaries identified. Because of the simple dependence of the action on the genus this is easily computed by taking the derivative of the partition function Z_T with respect to the corresponding coupling, namely,

$$\langle N_{\mathfrak{g}} \rangle = \frac{1}{Z_T} \frac{h_{ren}}{2} \frac{\partial Z_T}{\partial h_{ren}}. \tag{42}$$

Upon inserting (36) this yields

$$\langle N_{\mathfrak{g}} \rangle = T \, h_{ren}^2 \Lambda \frac{\coth\left(\sqrt{2\Lambda(1 - h_{ren}^2)} \, T\right)}{\sqrt{2\Lambda(1 - h_{ren}^2)}}. \tag{43}$$

In an analogous manner we can also calculate the average spacetime volume

$$\langle V \rangle = -\frac{1}{Z_T} \frac{\partial Z_T}{\partial \Lambda}, \tag{44}$$

leading to

$$\langle V \rangle = T \frac{\sqrt{(1 - h_{ren}^2)}}{\sqrt{2\Lambda}} \coth\left(\sqrt{2\Lambda(1 - h_{ren}^2)}\, T\right). \tag{45}$$

Dividing (43) by (45) we find that the spacetime density n of holes is constant,

$$n = \frac{\langle N_{\mathfrak{g}} \rangle}{\langle V \rangle} = \frac{h_{ren}^2}{1 - h_{ren}^2} \Lambda. \tag{46}$$

The density of holes in terms of the renormalized Newton's constant is given by

$$n = \frac{1}{e^{\frac{4\pi}{G_N}} - 1} \Lambda. \tag{47}$$

The behaviour of n in terms of the renormalized Newton's constant is shown in Fig. 3. The density of holes vanishes as $G_N \to 0$ and the model reduces to the case without topology change. – An alternative calculation of the density of holes from an infinitesimal strip, which leads to the same result, is presented in Appendix B.

We can now rewrite and interpret the Hamiltonian (32) in terms of physical quantities, namely, the cosmological scale Λ and the density of holes in units of Λ, i.e. $\eta = \frac{n}{\Lambda}$, resulting in

$$\hat{H}\left(L, \frac{\partial}{\partial L}\right) = -L \frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + \frac{1}{1 + \eta} 2\,\Lambda\, L. \tag{48}$$

One sees explicitly that the topology fluctuations affect the dynamics since the effective potential depends on η, as illustrated by Fig. 4.

It should be clear from expressions (48) and (47) that the model has two scales instead of the single one of the pure CDT model. As in the latter, the cosmological constant defines the global length scale of the two-dimensional "universe" through $\langle L \rangle \sim \frac{1}{\sqrt{\Lambda}}$. The new scale in the model with topology change is the relative scale η between the cosmological and topological fluctuations, which is parametrized by Newton's constant G_N.

6 Conclusions

In this paper, we have presented the complete analytic solution of a previously proposed model [11] of two-dimensional Lorentzian quantum gravity including a sum over topologies. The presence of causality constraints imposed on the path-integral histories – physically motivated in [11, 12] – enabled us to derive a new class of continuum theories by taking an unambiguously defined double-scaling limit of a statistical model of simplicially regularized spacetimes. After computing the Laplace transform of the exact one-step propagator of the discrete model, we investigated a two-parameter family, defined by (19) and (21), of possible scalings for the gravitational (or Newton's) coupling, from which physical considerations singled out a unique one. For this case, we computed the quantum Hamiltonian, its spectrum and eigenfunctions, as well as the partition function and propagator. Using these continuum results, we then calculated a variety of physical observables, including the
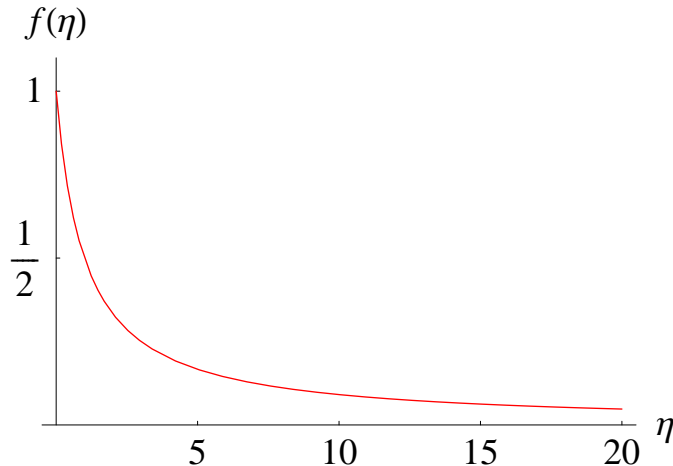
Figure 4: The coefficient of the effective potential, $f(\eta) = 1/(1 + \eta)$, as function of the density of holes in units of Λ, $\eta = \frac{n}{\Lambda}$.

average spacetime density of holes and the expectation values of the spatial volume and all its moments.

This should be contrasted with the previous treatment in [11, 12], in which only the one-step partition function with summed-over boundaries was evaluated. Because of the lack of boundary information, no explicit Hamiltonian was obtained there. Moreover, it turns out that the scaling of the couplings which in the current work led to the essentially unique Hamiltonian (32) could *not* have been obtained or even guessed in the previous work. This is simply a consequence of the fact that the dimensionful renormalized boundary cosmological constants make an explicit appearance in the scaling relation (31) for h, and thus for Newton's constant. We conclude that – unlike in the case of the original Lorentzian model – for two-dimensional causal quantum gravity with topology changes one cannot obtain the correct scalings for the "bulk" coupling constants from the one-step partition function with boundaries summed over (which is easier to compute than the full one-step propagator).

In contrast with what was extrapolated from the single-strip model in [12], the total number of holes in a finite patch of spacetime turns out to be a finite quantity determined by the cosmological and Newton's constants. Note that this finiteness result has been obtained dynamically and without invoking any fundamental discreteness. Since the density of holes is finite and every hole in the model is infinitesimal, this implies – and is confirmed by explicit calculation – that the expectation value of the number of holes in a general spatial slice of constant time is also infinitesimal. The fact that physically sensible observables are obtained in this toy model reiterates the earlier conclusion [11] that causality-inspired methods can be a useful tool in constructing gravitational path integrals which include a sum over topologies.

From the effective potential displayed in Fig. 4 one observes that the presence of worm-

holes in our model leads to a decrease of the "effective" cosmological constant $f(\eta)\Lambda$. In Coleman's mechanism for driving the cosmological constant Λ to zero [8, 14], an additional sum over different baby universes is performed in the path integral, which leads to a distribution of the cosmological constant that is peaked near zero. We do not consider such an additional sum over baby universes, but instead have an explicit expression for the effective potential which shows that an increase in the number of wormholes is accompanied by a decrease of the "effective" cosmological constant. A first step in establishing whether an analogue of our suppression mechanism also exists in higher dimensions would be to try and understand whether one can identify a class of causally preferred topology changes which still leaves the sum over geometries exponentially bounded. We will return to this issue in a future publication.

Acknowledgments

A Scalings with $\beta = 1$

In this appendix we discuss the scalings with $\beta = 1$ which we discarded as unphysical in Sec. 4 above. We proceed as before by inserting the scaling relations (17) and (29) into the composition law (23). Instead of using $\beta = \frac{3}{2}$ we set $\beta = 1$, leading to the scaling

$$h = \frac{1}{4} h_{ren} \, a \, \sqrt{\Lambda}^{\alpha} (X + Y)^{1-\alpha}, \tag{49}$$

where the normalization factor on the right-hand side has been chosen for later convenience. Up to first order in a one obtains

$$(1 - a\hat{H} + \mathcal{O}(a^2))\psi(X) = \int_{-i\infty}^{i\infty} \frac{dZ}{2\pi i} \left\{ A(X, Z) + B(X, Z)a + \mathcal{O}(a^2) \right\} \psi(Z), \tag{50}$$

where the leading-order contribution is given by

$$A(X, Z) = \frac{2}{(Z - X)(1 + C(X, Z))} \tag{51}$$

with

$$C(X, Z) = \sqrt{1 - h_{ren}^2 (X - Z)^{-2\alpha} \Lambda^{\alpha}}. \tag{52}$$

For the Laplace transform of $A(X, Z)$ to yield a delta-function, the scaling should be chosen such that $\alpha \leqslant 0$. Considering now the terms of first order in a,

$$B(X, Z) = \frac{h_{ren}^2 (X + Z - 4Z\gamma)\Lambda^\alpha}{(X - Z)^{1+2\alpha}C(X, Z)\,(1 + C(X, Z))^2}$$
$$- 2\frac{XZ - 2\Lambda + \gamma(X - Z)^2}{(X - Z)^2 C(X, Z)\,(1 + C(X, Z))}, \tag{53}$$

one finds that for $\alpha \leqslant -1$ the continuum limit is independent of any "hole contribution" (i.e. terms depending on h_{ren}) and therefore leads to the usual Lorentzian model. This becomes clear when one expands the last term of (53) in $(X - Z)$, resulting in

$$\frac{XZ - 2\Lambda}{(X - Z)^2 C\,(1 + C)} = \frac{1}{2}\frac{XZ - 2\Lambda}{(X - Z)^2}\left(1 + \frac{3}{4}h_{ren}^2\Lambda^\alpha(X - Z)^{-2\alpha} + \mathcal{O}((X - Z)^{-4\alpha})\right). \tag{54}$$

For $\alpha \leqslant -1$ the term depending on h_{ren} does not have a pole and therefore does not contribute to the Hamiltonian. Since we are only interested in non-fractional poles, this leaves as possible α-values only $\alpha = 0$ and $\alpha = -\frac{1}{2}$.

A.1 The case $\beta = 1$, $\alpha = 0$

For $\alpha = 0$ the Hamiltonian retains a γ-dependence contained in the first line of (53). Since there is no immediate physical interpretation of γ in our model, it seems natural to choose $\gamma = 0$, although strictly speaking this does not resolve the problem of explaining the γ-dependence of the continuum limit. Setting this question aside, one may simply look at the resulting model as an interesting integrable model in its own right. In order to obtain a delta-function to leading order, one still needs to normalize the transfer matrix by a constant factor $2/(1 + s)$, with $s := \sqrt{1 - h_{ren}^2}$. After setting $\gamma = 0$ and performing an inverse Laplace transformation, the Hamiltonian reads

$$\hat{H}(L, \frac{\partial}{\partial L}) = \frac{1}{s}\left(-L\frac{\partial^2}{\partial L^2} - s\frac{\partial}{\partial L} + 2\Lambda L\right). \tag{55}$$

It is self-adjoint with respect to the measure $d\mu(L) = L^{s-1}dL$. Further setting $L = \frac{\varphi^2}{2\,s}$ one encounters the one-dimensional Calogero Hamiltonian

$$\hat{H}(\varphi, \frac{\partial}{\partial\varphi}) = -\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{2}\omega^2\varphi^2 - \frac{1}{8}\frac{A}{\varphi^2}, \tag{56}$$

with $\omega = \frac{\sqrt{2\Lambda}}{s}$ and $A = 1 - 4(1 - s)^2$, which implies that the model covers the parameter range $-3 \leqslant A \leqslant 1$. The maximal range for which the Calogero Hamiltonian is self-adjoint is $-\infty < A \leqslant 1$. The usual Lorentzian model without holes corresponds to $A = 1$. The Hamiltonian (56) has already appeared in a causal dynamically triangulated model where the two-dimensional geometries were decorated with a certain type of "outgrowth" or small "baby universes" [15]. This model covered the parameter range $0 \leqslant A \leqslant 1$.

The eigenvectors of the Hamiltonian (55) are given by

$$\psi_n(L) = \mathcal{A}_n e^{-\sqrt{2\Lambda}L} {}_1F_1(-n, s, 2\sqrt{2\Lambda}L), \quad d\mu(L) = L^{s-1}dL, \tag{57}$$

where ${}_1F_1(-n, a, b)$ is the Kummer confluent hypergeometric function. The eigenvectors form an orthonormal basis with the normalization factors

$$\mathcal{A}_n = (8\Lambda)^{\frac{s}{4}} \sqrt{\frac{\Gamma(n+s)}{\Gamma(n+1)\Gamma(s)^2}} \tag{58}$$

and the corresponding eigenvalues

$$E_n = \frac{\sqrt{2\Lambda}}{s}(2n+s), \quad n = 0, 1, 2, \dots. \tag{59}$$

One sees explicitly that the case $s = 1$ or, equivalently, $A = 1$ corresponds to the pure two-dimensional CDT model.

A.2   The case $\beta = 1$, $\alpha = -\frac{1}{2}$

For $\alpha = -\frac{1}{2}$ the result does not depend on γ and therefore on the detailed manner in which we approach the critical point. However, the Hamiltonian

$$\hat{H}(L, \frac{\partial}{\partial L}) = -L\frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + 2\Lambda L - \frac{3}{4}h_{ren}^2\Lambda^{-1/2}\frac{\partial^2}{\partial L^2} \tag{60}$$

cannot be made self-adjoint with respect to any measure $d\mu(L)$ because the boundary part of the partial integration always gives a nonvanishing contribution. We therefore discard this possibility.

B   The density of holes of an infinitesimal strip

In this appendix we give an alternative derivation of the spacetime density n of holes and explicitly show that the number of holes in a spacetime strip of infinitesimal time duration a is also infinitesimal. The operator in the L-representation of the number of holes per infinitesimal strip with fixed initial boundary L can be calculated by

$$\hat{N}_{\mathfrak{g},a\to 0} = \hat{T}^{-1}\frac{h_{ren}}{2}\frac{\partial \hat{T}}{\partial h_{ren}}, \tag{61}$$

where $\hat{T}$ is the transfer matrix defined in (4). Using $\hat{T} = 1 - a\hat{H} + \mathcal{O}(a^2)$ and evaluating (61) to leading order in a gives

$$\hat{N}_{\mathfrak{g},a\to 0} = -a\frac{h_{ren}}{2}\frac{\partial \hat{H}}{\partial h_{ren}} + \mathcal{O}(a^2) = 2\Lambda h_{ren}^2 L a + \mathcal{O}(a^2). \tag{62}$$

Similarly, the volume operator of the same infinitesimal spacetime strip in the L-representation is given by

$$\hat{V}_{a\to 0} = -\hat{T}^{-1}\frac{\partial \hat{T}}{\partial \Lambda} = a\,\frac{\partial \hat{H}}{\partial \Lambda} + \mathcal{O}(a^2) = 2\,(1 - h_{ren}^2)\,L\,a + \mathcal{O}(a^2). \tag{63}$$

Although both expressions (62) and (63) vanish in the limit as $a \to 0$ (and therefore the number of holes and the strip volume are both "infinitesimal"), their quotient evaluates to a finite number independent of L, namely,

$$n = \frac{N_{\mathfrak{g},a\to 0}}{V_{a\to 0}} = \frac{h_{ren}^2}{1 - h_{ren}^2}\,\Lambda. \tag{64}$$

This is the exactly the same result for the spacetime density n of holes as we obtained earlier from the continuum partition function (46).

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