Introduction

The moduli spaces of p-divisible groups with a PEL-type structure have recently attracted considerable attention. One reason for this interest is the search for good integral models of Shimura varieties. Another one is a wish to have a better understanding of the moduli of abelian varieties. This thesis attempts to add to the knowledge of the structure of these moduli spaces.

Our moduli spaces are obtained by looking at those p-divisible groups that possess a given extra structure, which can be a polarization, a ring of endomorphisms and/or a fixed level structure (whence the PEL abbreviation). As Kottwitz has shown ([18], §5), if this extra structure is "prime-to-p", the resulting deformation functors are smooth over the base. If not, the spaces generally become singular. These singularities have been studied in many cases; see for example Deligne-Pappas [6], Rapoport-Zink [33] and Pappas [32] for the ramified ring of endomorphisms, Norman [27], de Jong [17] and Crick [5] for inseparable polarizations and Chai-Norman [4] for the p-level structure.

One of the difficulties in such studies is a lack of deformation theory of p-divisible groups, which would be both general enough to work over an arbitrary base and simple enough to do all the necessary computations. The crystalline approach (Messing [23]; Berthelot, Breen, Messing [2]) and that of Fontaine [10] have a disadvantage that they work only for divided power extensions. Consequently, they directly allow to determine the moduli space only in the cases of not too high ramification (cf. Norman [27]). On the other hand, the Cartier theory or the theory of displays (Norman-Oort [28], Zink [40]) does work over an arbitrary base. However, these theories require computations in σ-linear algebra, which are usually quite difficult.

A possible way out is to use the so-called *local models*. The idea is to find, étale-locally, a *non-canonical* isomorphism between the moduli space that one is interested in and a moduli space of a certain linear algebra problem. This has the advantage of allowing explicit computations. It is the approach used in [6], [17] and [33] for specific moduli problems. The unifying idea is that such an isomorphism is supposed to exist, whenever the deformation data in question is *rigid* on the Dieudonné modules. Our main goal is to give this idea a precise formulation and prove the existence of such an isomorphism (Theorem 4.3.8). To illustrate the possible applications we present some examples in Chapter 5.

Fix a perfect ground field k of characteristic $p>0$ and a complete Noetherian local ring Λ with $\Lambda/m_\Lambda \cong k$. Since we are interested primarily in the "very local" structure of the moduli spaces, we formulate our deformation problems in terms of functors on the category Art_Λ, Artinian local Λ-algebras with residue field k.

For example, let G/k be a p-divisible group and fix a finitely generated $\mathbf{Z}_p$-subalgebra $\mathcal{O} \subset \mathrm{End}(G)$. For simplicity take $\Lambda = W = W(k)$, the ring of Witt vectors of k. One can define the (covariant) functor

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O}) : Art_W \longrightarrow Sets\,,$$

which associates to a ring $A \in Art_W$ the set of pairs $(\mathcal{G}/A, \mathcal{O} \subset \mathrm{End}(\mathcal{G}))$ up to isomorphism. Here $\mathcal{G}/A$ is a deformation of G/k, a p-divisible group given together with an identification $\mathcal{G} \otimes_A k \cong G$. As for the inclusion $\mathcal{O} \subset \mathrm{End}(\mathcal{G})$, we require it to reduce to the chosen one on G. In other words we are interested in those deformations of G which inherit the given $\mathcal{O}$-action. It is not difficult to show that the functor $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ is pro-representable (4.3.5). Since $\mathcal{D}\mathrm{ef}(G)$ is well-known to be pro-represented by the ring $W[[t_1, ..., t_d]]$ with $d = \dim G \dim G^t$, it follows that $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ is pro-represented by a ring of the form

$$U = W[[t_1, ..., t_d]]/J$$

for some ideal J. We use here the rigidity of morphisms, which implies that the forgetful map $\mathcal{D}\mathrm{ef}(G, \mathcal{O}) \to \mathcal{D}\mathrm{ef}(G)$ is an inclusion of functors. The question is how to determine the pro-representing ring U.

Associated to deformation $\mathcal{G}/A$ of G/k there is a filtration of the Lie algebra of the universal extension of $\mathcal{G}$ (cf. Messing [23], Chapter IV),

$$V\mathcal{G} \subset M\mathcal{G} .$$

The A-modules $V\mathcal{G}$ and $M\mathcal{G}$ are functorial in G and the pair $V\mathcal{G} \subset M\mathcal{G}$ deforms (in the obvious sense) the corresponding pair $VG \subset MG$ for G. Further, if $\mathcal{G}$ admits an $\mathcal{O}$-action, then $V\mathcal{G}$ and $M\mathcal{G}$ are $\mathcal{O}$-modules. So there is a natural transformation of deformation functors (see 4.1.4, 4.3.1 for definitions)

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}) \longrightarrow \mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{O}) .$$

Thanks to the crystalline theory, we know that the deformation behaviour of the universal extension filtration determines, to a certain extent, that of G. Let us restrict our functors to the category $Art_{W,pd}$ of those $A \in Art_\Lambda$ for which the kernel of the structure map $A \to k$ has nilpotent divided powers. Then the $M\mathcal{G}$'s form a *crystal*; in other words, for any $A \in Art_{W,pd}$ and $\mathcal{G}_1, \mathcal{G}_2/A$ deforming G/k, there are canonical isomorphisms

$$M\mathcal{G}_1 \cong \mathcal{M} \otimes_W A \cong M\mathcal{G}_2,$$

where $\mathcal{M} = \mathbf{D}(G)$ is the covariant Dieudonné module of G. By functoriality, everything is compatible with the $\mathcal{O}$-action. Hence there is a canonical isomorphism of functors

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}) \longrightarrow \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{O}) . \tag{1}$$

Here $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{O})$ is the rigidified version of $\mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{O})$; an element of $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{O})(A)$ is an $\mathcal{O}$-stable filtration of finite free A-modules $V_A \subset M_A$ which deforms $VG \subset MG$ and an isomorphism

$$M_A \cong \mathcal{M} \otimes_W A ,$$

compatible with the $\mathcal{O}$-action.

Unfortunately, a canonical isomorphism such as (1) does not exist on the full category Art_W(cf. 4.3.10). Still, one can consider the following diagram of natural transformations of functors on Art_W:

$$\begin{array}{ccccc} \mathcal{D}\mathrm{ef}(G,\mathcal{O}) & & \cdots\cdots\!> & & \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{O}) \\ & {\scriptstyle q_1}\searrow & & \swarrow{\scriptstyle q_2} & \\ & & \mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O}) & & . \end{array}$$

Assume that the ring $\mathcal{O}$ has the property that the module $\mathcal{M}$ is *rigid*. By this we mean that any deformation of $\mathcal{M}\otimes_W k$ to a ring $A\in Art_W$ is isomorphic to $\mathcal{M}\otimes_W A$, as an $\mathcal{O}$-module. Then one can expect *every* element of $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ to be in the image from $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{O})$. Moreover, by crystalline theory, the same argument should hold for the functor $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$. Indeed, we will show that the transformations q_1 and q_2 are formally smooth (4.3.8, 4.4.1).

The consequence is that there is a non-canonical isomorphism (dotted arrow in the above diagram),

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O}) \;\cong\; \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{O})\,, \tag{2}$$

compatible with the projections to $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$. This is clear if the functor $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ is pro-representable. Then the formal smoothness of q_1 and q_2 implies that both the pro-representing rings of the functors above are formal power series over the pro-representing ring of $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$. By comparing the tangent spaces (crystalline theory again), it follows that the isomorphism (2) indeed exists. In fact, $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ is usually not pro-representable. However, a general comparison theorem (1.5.3) for formally smooth extensions implies the isomorphism (2) exists anyway.

Several comments are in order.

First, one has to determine what is the condition on the ring $\mathcal{O}$ which guarantees the required rigidity. It turns out, that whenever $\mathcal{O}$ is a *hereditary* (e.g. maximal) order in a semi-simple $\mathbf{Q}_p$-algebra, the Dieudonné module $\mathbf{D}$(G) is a projective $\mathcal{O}\otimes_{\mathbf{Z}_p}W$-module (4.4.1, part 1) and, hence, satisfies the rigidity condition (4.4.1, part 2).

Second remark is that the functor $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ is of interest in itself. In fact, let

$$\rho_\tau : \mathcal{O} \longrightarrow \mathrm{End}(TG)$$

be the tangent space representation of $\mathcal{O}$. Then it follows from our rigidity assumption that the natural map

$$\begin{array}{ccc} \mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O}) & \longrightarrow & \mathcal{D}\mathrm{ef}(\rho\tau) \\ V_A\subset M_A & \longmapsto & M_A/V_A \end{array}$$

gives an isomorphism of functors (cf. the proof of Theorem 4.4.1). In view of this, the formal smoothness of q_1 means the following: a necessary and sufficient condition to deform the pair $(G, \mathcal{O})$ to a ring $A \in Art_W$ is being able to deform the tangent space representation ρ_τ to A. Therefore, the geometric properties of the functor $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ (flatness, smoothness etc.) can be read off from those of $\mathcal{D}\mathrm{ef}(\rho_\tau)$.

This explains why in the search of good integral models of Shimura varieties, one is bound to restrict the tangent space representation. Indeed, the minimal requirement for these models is that they should be flat over $\operatorname{Spec} W$. However, the deformation functor $\mathcal{D}\mathrm{ef}(\rho_\tau)$ is definitely not flat in general; consider for example a supersingular elliptic curve E with $\mathcal{O} = \mathrm{End}(E)$. Then

$$\mathcal{D}\mathrm{ef}(E, \mathcal{O}) \cong \mathrm{Hom}_W(k, -)$$

is not flat over $\operatorname{Spec} W$. Kottwitz [18] has formulated a determinantal condition which does imply flatness in certain cases. In fact, Rapoport and Zink [33] have conjectured that under this condition, all local models are flat (in case $\mathcal{O}$ is a maximal order). This was disproved by Pappas [32] in case $\mathcal{O}$ is a quadratic extension of $\mathbf{Z}_p$. He has, moreover, conjectured flatness under a modified version of this condition. In any case, as we have seen above, such a flatness condition can be formulated purely in terms of the tangent space representation. If one provides such a condition and shows that (the hull of) the resulting restricted deformation functor $\mathcal{D}\mathrm{ef}'(\rho_\tau)$ is flat over $\operatorname{Spec} W$, the same holds for $\mathcal{D}\mathrm{ef}'(G, \mathcal{O})$.

The final remark is that the proof of the existence of an isomorphism (2) has little to do with the fact that we are looking at the case of endomorphisms. So we can prove the main comparison theorem (4.3.8) for a rather general deformation data.

The structure of the manuscript is as follows. We refer to the introductions of the chapters for a more extended outline.

Chapter 1 is dedicated to the infinitesimal deformation theory in general. It can be read independently of the rest of the thesis. Although infinitesimal methods form a basis of almost every deformation study, the basic statements and even definitions (obstruction space, for instance) seem to have been undocumented until recently (see [9]). So we decided to give a short consistent presentation of the basic results in the theory. We also prove the comparison theorem for formally smooth extensions (Section 1.5) and discuss quotient functors (Sections 1.6–1.7).

Chapters 2,3 form preliminaries needed for the main results in Chapters 4, 5. Chapter 2 is dedicated to the deformation functors of representations of R and of R-stable filtrations (Sections 2.2,2.3). The Hochschild cohomology groups which occur as tangent and obstruction spaces to these functors are recalled in Section 2.1. The ring representation case is similar to Mazur's study of group representations in [22], except that Hochschild cohomology replaces group cohomology.

Chapter 3 recalls the basic structure theorems of maximal and hereditary orders in semisimple algebras over a field K, which is complete with respect to a discrete valuation. We give a simple extension of the result of Janusz [15] on base change of hereditary orders in case of an infinite base extension.

In Chapter 4 we prove the main comparison result for the PEL-type moduli problems of p-divisible groups. To present the result as general as possible, we define the notion of a deformation data (Section 4.2) and formulate the main theorem (Section 4.3) in terms of it. As the theorem only applies when the deformation data is rigid on the Dieudonné modules, there is an obvious question in which situations this condition is satisfied. For the deformation functor $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ this turns out to be the case whenever $\mathcal{O}$ is a hereditary (e.g. maximal) order in a semi-simple $\mathbf{Q}_p$-algebra (Section 4.4); for the functor deformation functor $\mathcal{D}\mathrm{ef}(G,\mathcal{O},\lambda)$ when the order $\mathcal{O}$ is hereditary and λ is principal (Section 4.5). We show also how to reduce the more general deformation problems to the case of $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ or $\mathcal{D}\mathrm{ef}(G,\mathcal{O},\lambda)$ with λ principal (Section 4.6). As an illustration, we consider the case of the "p-chain" of p-divisible groups (Section 4.7).

In Chapter 5 we use the comparison theorem and the relation to the tangent space representation to determine the pro-representing ring of the functor $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ in some cases. We discuss the following examples:

1. $\mathcal{O}$ unramified.

2. $\mathcal{O}/\mathbf{Z}_p$ quadratic, G arbitrary.

3. $\mathcal{O}=\mathbf{Z}_p[\sqrt[h]{\pi}]$ and G of height $h\leq 4$.

4. $\mathcal{O}$ maximal order in a central division algebra over $\mathbf{Q}_p$ and G arbitrary.

5. $\mathcal{O}$ arbitrary, G one-dimensional.

In case 1 we get the result of Kottwitz; case 3 gives back a local result of Drinfeld ([8], Prop. 4.2) in case $\mathcal{O}$ is commutative. Case 4 generalizes the example of the so-called special formal $\mathcal{O}_D$-modules ([33], 3.69). Finally we discuss the canonical liftability of morphisms (Section 5.5).

Notations. We work over a ground field k which is arbitrary in Chapters 1–2 and perfect of positive characteristic p in Chapters 4, 5. We denote by Λ a fixed complete Noetherian local ring given together with an augmentation isomorphism $\eta_\Lambda:\Lambda/m_\Lambda \xrightarrow{\sim} k$. In Chapter 5 we let $\Lambda=W=W(k)$, the ring of Witt vectors.

A ring by definition contains 1.

To denote the duals, V^* is used for k-vector spaces in Chapters 1–2. From Chapter 4 on, we use the consistent notation G^t, M^t etc. for the Serre duals of p-divisible groups, A-linear duals for finite free A-modules etc.

The symbol Hom_k stands for morphisms in the category of k-vector spaces and Hom_Λ for morphisms in Art_Λ. The set of $m\times n$ matrices over A is denoted by $\mathrm{Mat}_{m\times n}(A)$.