4 Formal moduli of p-divisible groups

In this chapter we are going to study deformation functors of p-divisible groups with extra structure, such as a ring action and/or a principal quasi-polarization. From here on the ground field k is assumed to be perfect. As usual we work on the category Art_Λ where Λ is a fixed complete Noetherian local ring with $\Lambda/m_\Lambda = k$.

To illustrate our approach, consider a p-divisible group G/k and fix a $\mathbf{Z}_p$-subalgebra $\mathcal{O} \subset \mathrm{End}(G)$. Let $\mathcal{G}/A$ be a deformation of G to a ring $A \in Art_\Lambda$ and assume that the $\mathcal{O}$-action lifts to $\mathcal{G}$. Associated to $\mathcal{G}$ there is a filtration on the Lie algebra of the universal extension of $\mathcal{G}$,

$$V\mathcal{G} \subset M\mathcal{G}\,.$$

Here $M\mathcal{G}$ is a finite free A-module of rank equal to the height of G and $V\mathcal{G}$ is a direct summand. Moreover, by functoriality, the ring $\mathcal{O}$ acts on $V\mathcal{G}$ and $M\mathcal{G}$. So $V\mathcal{G} \subset M\mathcal{G}$ can be considered as a deformation of $VG \subset MG$ on the category of filtered modules with an $\mathcal{O}$-action. This gives a natural transformation of deformation functors (see 4.1.4, 4.3.1 for definitions)

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O}) \longrightarrow \mathcal{D}\mathrm{ef}(VG \subset MG,\mathcal{O})\,. \tag{13}$$

We are going study how the two functors are related.

We appeal to the Grothendieck-Messing deformation theory of p-divisible groups ([23], Ch. IV). Let $\mathcal{G}'/A'$ be a p-divisible group. If $A \twoheadrightarrow A'$ is a surjection in Art_Λ, whose kernel has a nilpotent divided power structure, one can relate deformations of $\mathcal{G}'$ to A to the deformations of the universal extension filtration. This relies, in particular, on the "crystalline" nature of MG. Namely, for any two deformations $\mathcal{G}_1/A, \mathcal{G}_2/A$ of $\mathcal{G}'/A'$, there is a *canonical* isomorphism

$$M\mathcal{G}_1 \cong M\mathcal{G}_2$$

which reduces to the identity on $M\mathcal{G}'$. It follows that there is a universal A-module $M_A\mathcal{G}'$, which can be canonically identified with every $M\mathcal{G}$ for $\mathcal{G}/A$ deforming $\mathcal{G}'/A'$. Hence, associated to $\mathcal{G}/A$ there is a deformation of the filtration $V\mathcal{G}' \subset M\mathcal{G}'$ to a filtration $V\mathcal{G}$ of a *fixed* A-module, namely $M_A\mathcal{G}'$. By the result of Messing ([23], V, 1.6) this association is a bijection.

This is a powerful method of studying deformations of p-divisible groups with extra data. For example if $\mathcal{G}'/A'$ admits an $\mathcal{O}$-action, then $M_A\mathcal{G}'$ is an $\mathcal{O}$-module (by functoriality) and the deformations G/A which inherit the $\mathcal{O}$-action correspond precisely to the $\mathcal{O}$-stable filtrations of $M_A\mathcal{G}'$. One does need to know, however, what is the structure of $M_A\mathcal{G}'$ as an $\mathcal{O}$-module. The difficulty is that although $M\mathcal{G}$ "does not change" over divided power extensions, it does change over arbitrary extensions $A \to k$. It is easy to give an example of rings $A \twoheadrightarrow A' \twoheadrightarrow k$ and two deformations $\mathcal{G}_1'/A', \mathcal{G}_2'/A'$ of G/k such that $M_A\mathcal{G}_1'$ and $M_A\mathcal{G}_2'$ are not isomorphic as $\mathcal{O}$-modules. So it is much easier to study a functor such as $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ on the category $Art_{W,pd}$ of divided power extensions $A\to k$ than on the full category Art_Λ.

The idea is to target the situations when MG is “rigid” as an $\mathcal{O}$-module, meaning that for any $A\in Art_\Lambda$ any two deformations M_1, M_2 of MG to A are isomorphic as $\mathcal{O}\otimes_{\mathbf{Z}_p} A$-modules. In this case for any deformation $\mathcal{G}/A$ there is an $\mathcal{O}\otimes_{\mathbf{Z}_p} A$-module isomorphism

$$M\mathcal{G}\cong \mathbf{D}(G)\otimes_{W(k)} A\,.$$

Here $\mathbf{D}(G)$ is a covaraint Dieudonné module of G, which is a finite free $W(k)$-module with an $\mathcal{O}$-action. In such a “rigid” situation it turns out that the natural transformation (13) is formally smooth: any deformation of the pair $VG\subset MG$ is induced by that of G. Note that this implies that the functor $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ can be determined in terms of pure linear algebra. It is namely pro-represented by a formal power series ring over the hull of $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$.

As it seems difficult to actually determine the hull of $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$, we are going to appeal instead to the strategy described in Section 1.5. Namely, we are going to produce another formally smooth natural transformation $\mathcal{F}\to\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ with $\mathcal{F}$ pro-representable and one which can be calculated explicitly. We get a diagram

$$\begin{array}{ccc}\mathcal{F} & & \mathcal{D}\mathrm{ef}(G,\mathcal{O})\\ & \searrow\quad\swarrow & \\ & \mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O}) & \end{array}\,.$$

If the tangent spaces of $\mathcal{F}$ and $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ happen to be of the same dimension, then the two functors are isomorphic by 1.5.4. There is in fact a natural candidate for $\mathcal{F}$. We can rigidify $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ by studying deformations $\mathcal{V}\subset\mathcal{M}$ over A given *together* with an isomorphism $\mathcal{M}\cong\mathbf{D}(G)\otimes_W A$ of $\mathcal{O}\otimes_{\mathbf{Z}_p} A$-modules. Then the corresponding deformation functor $\mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG)$ is easily seen to be pro-representable (cf. 2.3.2) and formally smooth over $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$. Moreover, its tangent space is isomorphic to that of $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ by crystalline theory. Hence

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O})\cong\mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG)\,.$$

This gives a way to determine $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ explicitly, provided the rigidity assumption on MG is satisfied.

The structure of this chapter is as follows:

First we recall the basic facts concerning p-divisible groups: rigidity of homomorphisms, duality and deformation theory (Section 4.1). For a subring $\mathcal{O}\subset\mathrm{End}(G)$ and a principal quasi-polarization $\lambda: G\xrightarrow{\sim}G^t$ we define the deformation functors $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G,\mathcal{O},\lambda)$.

In Section 4.2 we define the notion of *deformation data* $\mathcal{D}$, in order to generalize our method to any situation when the rigidity of MG applies. We also define deformation functors $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ of p-divisible groups with a given deformation data and the notion of rigidity in this context.

In Section 4.3 we prove the pro-representability of the functors $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ in general (4.3.5) and the main comparison theorem (4.3.8). We also present an example to illustrate that such a comparison result does not hold if the rigidity assumption is omitted (4.3.10).

Then we apply the result to the functors $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G,\mathcal{O},\lambda)$ (Sections 4.4, 4.5). In order to do that, it is necessary to determine for which $\mathcal{O}\subset\mathrm{End}(G)$ the $\mathcal{O}$-module MG is rigid. This turns out to hold whenever $\mathcal{O}$ is a *hereditary* (e.g. maximal) order in a semi-simple subalgebra of $\mathrm{End}(G)$ (Theorems 4.4.1,4.5.3). An interesting by-product is that in the case of a hereditary order the deformation functor $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{O})$ is isomorphic to the deformation functor of the tangent space representation $\rho_\tau:\mathcal{O}\longrightarrow\mathrm{End}(TG)$. In view of the formal smoothness of (13), a necessary and sufficient condition of deforming the pair $(G,\mathcal{O})$ to some $A\in Art_\Lambda$ is being able to deform this tangent space representation. This generalizes some known results on deformations with a restricted tangent space representation (cf. [8], [18], [33]).

Finally, we show that a deformation functor $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ with an arbitrary deformation data is isomorphic to a functor of the form $\mathcal{D}\mathrm{ef}(H,\mathcal{O},\lambda)$ for some p-divisible group H, a subring $\mathcal{O}\subset\mathrm{End}(G)$ (not necessarily a maximal order) and a principal quasi-polarization λ on H. This explains why in Chapter 5 we consider only deformation problems with one p-divisible group G/k.

As an illustration, we present another standard example, namely that of a chain of maps between p-divisible groups,

$$G_0\longrightarrow G_1\longrightarrow\cdots\longrightarrow G_{n-1}\longrightarrow G_n=G_0,$$

whose composition is multiplication by p (Section 4.7). The required rigidity condition is also satisfied in this case (but not, for example, if the composition is p^2) and our comparison theorem applies.

Our references for the deformation theory of p-divisible groups are Messing [23] (Chapters IV, V) and Berthelot, Breen, Messing [2] (especially 3.3, 4.2 and 5.3). We have chosen to follow the *covariant* Dieudonné module convention, as in the Cartier theory.

4.1 Deformations of p-divisible groups

For the definition of p-divisible groups and Serre duality we refer to [38], 2.1, 2.3. We work on the category Art_Λ of Artinian local Λ-algebras with residue field k, perfect of characteristic p. By G/k we denote a p-divisible group over k and $\mathcal{G}/A$ or $\mathcal{G}_A/A$ denotes a p-divisible group over $A\in Art_\Lambda$. We use $\mathcal{G}^t$ for the Serre dual of $\mathcal{G}$ and similarly for morphisms. Recall that $G\cong G^{tt}$ canonically, as follows from the corresponding result for finite group schemes.

In order to study the deformations of p-divisible groups, we rely on the Grothendieck-Messing approach ([23], Ch. IV).

Notation. For a p-divisible group $\mathcal{G}/A$ denote

$T\mathcal{G}$ — the tangent space (or the Lie algebra) of $\mathcal{G}$,
$M\mathcal{G}$ — the Lie algebra of the universal extension of $\mathcal{G}$,
$V\mathcal{G}$ — the canonical filtration on $M\mathcal{G}$.

There is an exact sequence (of finite free A-modules)

$$0 \longrightarrow V\mathcal{G} \longrightarrow M\mathcal{G} \longrightarrow T\mathcal{G} \longrightarrow 0, \tag{14}$$

Moreover, $T\mathcal{G}, M\mathcal{G}, V\mathcal{G}$ and the above sequence are compatible with base change and are functorial in $\mathcal{G}$. We have

$$\dim_A T\mathcal{G} = n, \quad \dim_A M\mathcal{G} = h, \quad \dim_A V\mathcal{G} = n' .$$

where n, n' denote the dimensions of $\mathcal{G}$ and $\mathcal{G}^t$ and $h = n + n'$ is the height. The sequence (14) for $\mathcal{G}^t$ is canonically isomorphic to the (A-linear) dual of the corresponding sequence for $\mathcal{G}$.

Finally, for G/k there are canonical isomorphisms ([2], 4.2.14)

$$MG = \mathbf{D}(G[p]) = \mathbf{D}(G) \otimes_{W(k)} k ,$$

functorial in G. Here $\mathbf{D}(-)$ denotes the covariant Dieudonné module.

We need the following rigidity result for morphisms of p-divisible groups:

Theorem 4.1.1. *Let $\mathcal{G}, \mathcal{H}$ be p-divisible groups over A and $A \to B$ a ring homomorphism in Art_Λ. Then*

$$\mathrm{Hom}(\mathcal{G}, \mathcal{H}) \hookrightarrow \mathrm{Hom}(\mathcal{G} \otimes_A B, \mathcal{H} \otimes_A B) .$$

Proof. Compose the map $A \to B$ with the augmentation to k,

$$A \longrightarrow B \longrightarrow k .$$

To show that $\mathrm{Hom}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}(\mathcal{G} \otimes_A B, \mathcal{H} \otimes_A B)$ is injective, it suffices to verify that the composition $\mathrm{Hom}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}(\mathcal{G} \otimes_A k, \mathcal{H} \otimes_A k)$ is injective. In other words we can reduce to the case of a map $A \to k$. From here we can also reduce to the case when $A \twoheadrightarrow B$ is a small extension, in particular an extension with divided powers.

Thus let $A \twoheadrightarrow B$ be a divided power extension. Then the Grothendieck-Messing theory identifies $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ with a *subset* of $\mathrm{Hom}(\mathcal{G} \otimes_A B, \mathcal{H} \otimes_A B)$ of those homomorphisms which preserve the filtrations. Hence the injectivity follows.

Definition 4.1.2. A *quasi-polarization* on a p-divisible group $\mathcal{G}_A/A$ is an isogeny

$$\lambda : \mathcal{G}_A \longrightarrow \mathcal{G}_A^t$$

which satisfies $\lambda^t = -\lambda$. We say that the quasi-polarization is *principal* if λ is an isomorphism.

Remark. It follows from 4.1.1 that a morphism $\lambda : \mathcal{G}_A \to \mathcal{G}_A^t$ is a (principal) quasi-polarization on $\mathcal{G}_A$ if and only if $\lambda \otimes_A k$ is a (principal) quasi-polarization on $\mathcal{G}_A \otimes_A k$.

Our primary goal is to study the structure of the following deformation functors:

Definition 4.1.3. Let $A \twoheadrightarrow A'$ be a morphism in Art_Λ and $\mathcal{G}'/A'$ a p-divisible group. A *deformation* of $\mathcal{G}'$ to A is a p-divisible group $\mathcal{G}/A$ together with an isomorphism

$$i : \mathcal{G} \otimes_A A' \cong \mathcal{G}' . \tag{15}$$

Definition 4.1.4. Let G/k be a p-divisible group. Define the deformation functor of G,

$$\begin{array}{rccc} \mathcal{D}\mathrm{ef}(G): & Art_\Lambda & \longrightarrow & Sets \\ & A & \longmapsto & \left\{ \begin{array}{c} \text{deformations} \\ \text{of } G \text{ to } A \end{array} \right\} / \cong \end{array} .$$

Given a subring $j : \mathcal{O} \hookrightarrow \mathrm{End}(G)$, we let

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}) : Art_\Lambda \longrightarrow Sets$$

to be the functor of deformations $\mathcal{G}/A$ together with the action of $\mathcal{O}$ which reduces to j on G (under (15)). Similarly, given a subring $\mathcal{O} \subset \mathrm{End}(G)$ and a quasi-polarization λ on G, we define

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda) : Art_\Lambda \longrightarrow Sets$$

to be the functor of deformations $\mathcal{G}/A$ together with the action of $\mathcal{O}$ and a quasi-polarization which reduce to those of G.

Remark. It is well-known that $\mathcal{D}\mathrm{ef}(G)$ is pro-representable and

$$\mathcal{D}\mathrm{ef}(G) \cong \mathrm{Hom}_\Lambda(\Lambda[[t_1, ..., t_d]], -), \quad d = \dim G \cdot \dim G^t .$$

From the rigidity theorem (4.1.1) it follows that $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$ are subfunctors of $\mathcal{D}\mathrm{ef}(G)$. These subfunctors are pro-representable (4.3.5 below), so the pro-representing rings are of the form

$$\Lambda[[t_1, ..., t_d]]/J .$$

These rings are often singular and our goal is to describe them in some cases.

4.2 Deformation data

In order to generalize 4.1.4 to a potentially larger class of situations, we define the notion of a deformation data. Such a deformation data can be of the form "an object with an action of a ring $\mathcal{O}$" or "an object with an action of a ring $\mathcal{O}$ and a quasi-polarization" or, most generally, a finite collection of objects together with certain morphisms between them and their duals. For such a deformation data $\mathcal{D}$, it is clear how to define a $\mathcal{D}$-object of pDiv_A or any other additive $\mathbf{Z}_p$-linear category with duality (such as finite free modules over a given $\mathbf{Z}_p$-algebra). We also define deformation functors of $\mathcal{D}$-objects and give the examples that we have in mind.

Notation 4.2.1. For $A \in Art_\Lambda$ let $\mathcal{C} = \mathcal{C}_A$ denote one of the following categories:

1. The category pDiv_A of p-divisible groups $\mathcal{G}_A$ over A.
2. The category Mod_A of finite free A-modules $\mathcal{M}_A$.
3. The category FMod_A of filtrations $\mathcal{F}_A \subset \mathcal{M}_A$, where $\mathcal{M}_A$ is a finite free A-module and $\mathcal{F}_A$ a direct A-summand.

In each case $\mathcal{C}_A$ is an additive $\mathbf{Z}_p$-linear category with a duality, an anti-equivalence of categories $t : \mathcal{C}_A^\circ \to \mathcal{C}_A$. We have namely the Serre duality for p-divisible groups, the A-linear duals for modules over A and

$$(F \subset M) \longmapsto ((M/F)^t \subset M^t)$$

for the filtrations. In any case, we denote the dual object of X by X^t and similarly for morphisms.

A morphism $A \to A'$ in Art_Λ induces a $\mathbf{Z}_p$-linear "base change" functor

$$- \otimes_A A' : \mathcal{C}_A \longrightarrow \mathcal{C}_{A'} .$$

There are also some obvious forgetful functors, such as

$$\mathrm{FMod}_A \longrightarrow \mathrm{Mod}_A \qquad \text{(forget the filtration)} .$$

These are $\mathbf{Z}_p$-linear, commute with base change and preserve duality.

Definition 4.2.2. An arbitrary self-dual $\mathbf{Z}_p$-linear category $\mathcal{D}$ is called a *deformation data* if it has finitely many objects and all $\mathrm{Hom}(X, Y)$ are finitely generated $\mathbf{Z}_p$-modules.

In the following list of basic definitions, let $\mathcal{C} = \mathcal{C}_A$ be as in 4.2.1 and $\mathcal{D}$ a deformation data. The term functor will refer to a $\mathbf{Z}_p$-linear duality-preserving functor.

Definition 4.2.3. A $\mathcal{D}$*-object* X_A of a category $\mathcal{C}$ is a covariant functor $X_A : \mathcal{D} \to \mathcal{C}$. By a *morphism* $X \to Y$ of $\mathcal{D}$-objects we mean a natural transformation as functors.

Notation 4.2.4. For a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and a $\mathcal{D}$-object X of $\mathcal{C}$ we let $\mathcal{F}(X)$ to be the $\mathcal{D}$-object of $\mathcal{C}'$ given by the composition $\mathcal{F}(X) = \mathcal{F} \circ X$. In particular, this defines the base change of $\mathcal{D}$-objects (let $\mathcal{F} = - \otimes_A A' : \mathcal{C}_A \to \mathcal{C}_{A'}$).

Remark. With the notations of Section 4.1, the following associations give duality-preserving $\mathbf{Z}_p$-linear covariant functors (cf. [2], 5.3.6).

$$\begin{array}{rccc} U: & \mathrm{pDiv}_A & \longrightarrow & \mathrm{FMod}_A \\ & \mathcal{G} & \longmapsto & (V\mathcal{G} \subset M\mathcal{G}) \\ \mathbf{D}(-): & \mathrm{pDiv}_k & \longrightarrow & \mathrm{Mod}_{W(k)} \\ & G & \longmapsto & \mathbf{D}(G) \\ \mathbf{D}(-[p]): & \mathrm{pDiv}_k & \longrightarrow & \mathrm{Mod}_k \\ & G & \longmapsto & \mathbf{D}(G[p]) \end{array}$$

Following 4.2.4, for a deformation data $\mathcal{D}$ and a $\mathcal{D}$-object $\mathcal{G}_A$ of pDiv_A we can speak of $U(\mathcal{G}_A)$. In case $A = k$ and G/k we can also define $\mathbf{D}(G)$ and $\mathbf{D}(G[p])$.

Definition 4.2.5. Let $A \twoheadrightarrow A'$ be a homomorphism in Art_Λ. Given a $\mathcal{D}$-object $X_{A'}$ of $\mathcal{C}_{A'}$, a *deformation* of $X_{A'}$ to $\mathcal{C}_A$ is a $\mathcal{D}$-object X_A of $\mathcal{C}_A$ given together with an isomorphism

$$X_A \otimes_A A' \cong X_{A'} \,.$$

For a $\mathcal{D}$-object X_k of $\mathcal{C}_k$, let the *deformation functor of* X_k to be

$$\begin{array}{rccc} \mathcal{D}\mathrm{ef}(X_k, \mathcal{D}): & Art_\Lambda & \longrightarrow & Sets \\ & A & \longmapsto & \{\text{deformations of } X_k \text{ to } \mathcal{C}_A\}/\cong \ . \end{array}$$

Keeping in mind the deformation functors that we are interested in (cf. 4.1.4), we have the following examples:

Example 4.2.6. (Endomorphisms.) Let $\mathcal{O}$ be a $\mathbf{Z}_p$-algebra. Let $\mathcal{D}$ consist of two objects, X and its dual X^t with

$$\mathrm{End}(X) = \mathcal{O}, \quad \mathrm{End}(X^t) = \mathcal{O}^{\mathrm{op}}, \quad \mathrm{Hom}(X, X^t) = 0, \quad \mathrm{Hom}(X^t, X) = 0 \,.$$

We let duality interchange X and X^t and act as identity $\mathrm{End}(X) \to \mathrm{End}(X^t)$. Then $\mathcal{D}$ defines the data "an object with an $\mathcal{O}$-action". For instance, a $\mathcal{D}$-object of pDiv_A can be identified with a p-divisible group $\mathcal{G}_A/A$ together with an action of $\mathcal{O}$. In particular, for a $\mathcal{D}$-object G/k we have (cf. 4.1.4)

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \cong \mathcal{D}\mathrm{ef}(G, \mathcal{O}) \,.$$

Example 4.2.7. (Endomorphisms, principal quasi-polarization.) Let $\mathcal{O}$ be a $\mathbf{Z}_p$-algebra with a $\mathbf{Z}_p$-linear anti-involution $r : \mathcal{O} \cong \mathcal{O}^{\mathrm{op}}$. Again take $\mathcal{D} = \{X, X^t\}$ and let

$$\mathrm{End}(X) = \mathcal{O}, \quad \mathrm{End}(X^t) = \mathcal{O}^{\mathrm{op}}, \quad \mathrm{Hom}(X, X^t) = \mathbf{Z}_p\lambda, \quad \mathrm{Hom}(X^t, X) = \mathbf{Z}_p\lambda^{-1} \,.$$

Here λ and λ^{-1} are formal symbols and

$$\lambda\lambda^{-1} = id = \lambda^{-1}\lambda, \qquad \lambda^t = -\lambda, \qquad \lambda^{-1}o^t\lambda = r(o) \ \ (o \in \mathcal{O}) \,.$$

Then $\mathcal{D}$ defines the data "an object with an $\mathcal{O}$-action and a principal quasi-polarization". For instance, a $\mathcal{D}$-object of pDiv_A is a p-divisible group $\mathcal{G}_A/A$ together with an action of $\mathcal{O}$ and a self-dual isomorphism $\lambda : G \to G^t$ whose Rosati involution on $\mathcal{O}$ is r. So for a $\mathcal{D}$-object G/k we have (cf. 4.1.4)

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \cong \mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda) \,.$$

Example 4.2.8. (p-chain.) Take $n \geq 1$. Let $\mathcal{D}$ consist of objects X_i indexed by $i \in \mathbf{Z}/n\mathbf{Z}$ and their duals X_i^t. Let

$$\mathrm{Hom}(X_i, X_{i+1}) = \mathbf{Z}_p f_i, \quad \mathrm{Hom}(X_{i+1}^t, X_i^t) = \mathbf{Z}_p f_i^t$$

and define the compositions of all the f_i to be multiplication by p,

$$f_{i-1}f_{i-2}\cdots f_{i-n} = p \in \mathbf{Z}_p = \mathrm{End}(X_i), \qquad i \in \mathbf{Z}/n\mathbf{Z}\,.$$

As in 4.2.6 we let all the homomorphisms between X_i and X_j^t to be 0. Then $\mathcal{D}$ defines the data "p-chain of length n". For instance, a $\mathcal{D}$-object of pDiv_A is a collection of p-divisible groups $\mathcal{G}_i/A$ (with $i \in \mathbf{Z}/n\mathbf{Z}$) and maps $f_i : G_i \to G_{i+1}$ every of whose cyclic compositions equals p. In particular this forces the G_i to have the same height and the f_i to be isogenies.

4.3 The comparison theorem

Notation. Let G be a $\mathcal{D}$-object of pDiv_k. Let $\mathcal{M} = \mathbf{D}(G) \otimes_W \Lambda$, which is a $\mathcal{D}$-object of Mod_Λ. Denote by $VG \subset MG$ the $\mathcal{D}$-object $U(G)$. Thus, canonically, $MG = \mathcal{M} \otimes_\Lambda k$. Following 4.2.2, we can define the deformation functors $\mathcal{D}\mathrm{ef}(G, \mathcal{D})$ and $\mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{D})$. We also define the "rigidified" version of the latter deformation functor, $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{D})$:

Definition 4.3.1. Let $\mathcal{N}$ be a deformation of MG to Λ. Define

$$\mathcal{D}\mathrm{ef}_{\mathcal{N}}(VG \subset MG, \mathcal{D}) : Art_\Lambda \longrightarrow Sets$$

to be the covariant functor which associates to a ring $A \in Art_\Lambda$ the set of isomorphism classes of elements $(\mathcal{V}_A \subset \mathcal{M}_A) \in \mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{D})$ given together with an isomorphism $\mathcal{M}_A \cong \mathcal{N} \otimes_\Lambda A$.

Lemma 4.3.2. *Let $\mathcal{D}$ be a deformation data and G a $\mathcal{D}$-object of* pDiv_k. *For any deformation $\mathcal{N}$ of the $\mathcal{D}$-object MG to Λ, the functor $\mathcal{D}\mathrm{ef}_{\mathcal{N}}(VG \subset MG, \mathcal{D})$ is pro-representable.*

Proof. Apply the same argument as in the proof of Theorem 2.3.2. It is easy to see that the $\mathbf{Z}_p$-generators of the $\mathrm{Hom}(X, Y)$ for varying $X, Y \in \mathcal{D}$ plus the duality constraints give finitely many equations for the deformation functor.

Remark 4.3.3. The crystalline theory establishes a canonical bijection

$$\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{D})(A) = \mathcal{D}\mathrm{ef}(G, \mathcal{D})(A)$$

for every k-algebra $A \in Art_\Lambda$ whose augmentation $A \to k$ is a divided power extension. In particular, this applies to $A = k[V]$ for any finite-dimensional k-vector space V. So the tangent spaces of the two functors are isomorphic. In particular, $\mathcal{D}\mathrm{ef}(G, \mathcal{D})$ does have a (finite-dimensional) tangent space.

Example 4.3.4. Let G/k be a p-divisible group and $\mathcal{O} \subset \mathrm{End}(G)$ a $\mathbf{Z}_p$-subalgebra. Then the tangent space to $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ equals to the Hochschild cohomology group $\mathrm{H}^1(\mathcal{O}-\mathcal{O}, TG \otimes_k TG^t)$ as it is the tangent space to the deformation functor of the filtration (Theorem 2.3.2).

Lemma 4.3.5. *Let $\mathcal{D}$ be a deformation data and G a $\mathcal{D}$-object of* pDiv_k. *Then the functor $\mathcal{D}\mathrm{ef}(G,\mathcal{D}) : Art_\Lambda \to Sets$ is pro-representable.*

Proof. We apply the Schlessinger's criterion (Theorem 1.4.3). By 4.3.3, $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ has a finite-dimensional tangent space. So it suffices to prove that

$$\mathcal{D}\mathrm{ef}(G,\mathcal{D})(A \times_{A'} B') \longrightarrow \mathcal{D}\mathrm{ef}(G,\mathcal{D})(A) \times_{\mathcal{D}\mathrm{ef}(G,\mathcal{D})(A')} \mathcal{D}\mathrm{ef}(G,\mathcal{D})(B') \tag{16}$$

is a bijection whenever $A \twoheadrightarrow A'$ is a small extension and $B' \to A'$ a morphism in Art_Λ. Let $\mathcal{G}_{B'} \in \mathcal{D}\mathrm{ef}(G,\mathcal{D})(B')$ be a deformation of G to B'.

Associated to $\mathcal{G}_{B'}$ there is a universal extension filtration $V\mathcal{G}_{B'} \subset M\mathcal{G}_{B'}$. Moreover, since

$$B = A \times_{A'} B' \twoheadrightarrow B'$$

is a small (in particular, a divided power) extension, we can also define $M_B\mathcal{G}_{B'}$, the value of the universal extension crystal of $\mathcal{G}_{B'}$ on the ring B. This is a $\mathcal{D}$-object of Mod_B. Moreover, by Grothendieck-Messing, there is a bijection between the deformations of $\mathcal{G}_{B'}$ to B (as a $\mathcal{D}$-object) and deformations of the filtration $V\mathcal{G}_{B'} \subset M\mathcal{G}_{B'}$ to a filtration of $M_B\mathcal{G}_{B'}$ (again, as a $\mathcal{D}$-object). However, the functor

$$\mathcal{F} = \mathcal{D}\mathrm{ef}_{M_B\mathcal{G}_{B'}}(VG \subset MG) : Art_B \longrightarrow Sets$$

is pro-representable by Lemma 4.3.2. In particular, it commutes with fibre products, so

$$\mathcal{F}(A \times_{A'} B') \longrightarrow \mathcal{F}(A) \times_{\mathcal{F}(A')} \mathcal{F}(B')$$

is a bijection. It follows that (16) is a bijection as well. ∎

Remark. In order to prove the pro-representability of $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ we have used the Grothendieck-Messing theory together with the pro-representability of $\mathcal{D}\mathrm{ef}_{\mathcal{N}}(VG \subset MG,\mathcal{D})$ for various choices of $\mathcal{N}$. However, non-isomorphic p-divisible groups over A might have non-isomorphic $M\mathcal{G}$'s, as $\mathcal{D}$-objects. Consequently, one should not expect the full deformation functor $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ to be isomorphic to $\mathcal{D}\mathrm{ef}_{\mathcal{N}}(VG \subset MG,\mathcal{D})$ for any particular choice of $\mathcal{N}$. In some cases, however, the $\mathcal{D}$-object MG is "rigid" in the sense that it can be uniquely deformed to any $A \in Art_\Lambda$. Then $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ could be expected to be (non-canonically) isomorphic to $\mathcal{D}\mathrm{ef}_{\mathcal{M}_\Lambda}(VG \subset MG,\mathcal{D})$ where $\mathcal{M}_\Lambda$ is the unique deformation of M to Λ. Such a non-canonical isomorphism in fact exists, as we show in 4.3.8 below.

Definition 4.3.6. Let $\mathcal{C}$ be as in 4.2.1 and let $\mathcal{D}$ be a deformation data. A $\mathcal{D}$-object X_k of $\mathcal{C}_k$ is said to be *rigid* if there is a "universal" $\mathcal{D}$-object X_Λ of $\mathcal{C}_\Lambda$ such that

$$\mathcal{D}\mathrm{ef}(X_k,\mathcal{D})(A) = \{X_\Lambda \otimes_\Lambda A\}, \qquad A \in Art_\Lambda\,,$$

and such that the automorphism functor

$$\mathcal{A}ut(X_\Lambda):\ A \longmapsto \mathrm{Aut}(X_\Lambda \otimes_\Lambda A)$$

is formally smooth.

Remark 4.3.7. It is not difficult to show that a $\mathcal{D}$-object X_k of $\mathcal{C}_k$ is rigid if and only if the following holds. First, X_k can be deformed to any $A \in Art_\Lambda$. Second, given a surjection $A \twoheadrightarrow A'$ in Art_Λ and a deformation $\mathcal{X}_{A'}$ of X_k to A', any two deformations $\mathcal{X}_A^{(1)}, \mathcal{X}_A^{(2)}$ of $\mathcal{X}_{A'}$ to A are isomorphic *over* $\mathcal{X}_{A'}$. In other words, there is an isomorphism of $\mathcal{D}$-objects $\mathcal{X}_A^{(1)} \cong \mathcal{X}_A^{(2)}$ which becomes identity on $\mathcal{X}_{A'}$ after applying $\otimes_A A'$.

Theorem 4.3.8. *Let $\mathcal{D}$ be a deformation data and G a $\mathcal{D}$-object of* pDiv_k. *Let $\mathcal{M} = \mathbf{D}(G)\otimes_{W(k)}\Lambda$. Consider a diagram of functors*

$$\begin{array}{ccc} \mathcal{D}\mathrm{ef}(G,\mathcal{D}) & & \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{D}) \\ {\scriptstyle q_1}\searrow & & \swarrow{\scriptstyle q_2} \\ & \mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{D}) & \end{array} \quad . \tag{17}$$

Assume that the $\mathcal{D}$-object $\mathcal{M}\otimes_\Lambda k = \mathbf{D}(G[p])$ of Mod_k *is rigid. Then q_1 and q_2 are formally smooth and there is a (non-canonical) isomorphism of functors $i: \mathcal{D}\mathrm{ef}(G,\mathcal{D}) \to \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{D})$ which makes (17) commute.*

Proof. The strategy is to apply the comparison theorem 1.5.4. First note that both $\mathcal{D}\mathrm{ef}(G,\mathcal{D})$ and $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{D})$ are pro-representable (4.3.2, 4.3.5). Moreover, their tangent spaces are isomorphic by 4.3.3 and this isomorphism commutes with the projections to $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{D})$. In order to conclude that the two functors are isomorphic over $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{D})$ it suffices to prove that the projections q_1 and q_2 are formally smooth. We begin with q_2.

Let $A \twoheadrightarrow A'$ be a surjection in Art_Λ. Let $\mathcal{V}_{A'} \subset \mathcal{M}_{A'} = \mathcal{M}\otimes_\Lambda A'$ be a $\mathcal{D}$-object of $\mathrm{FMod}_{A'}$, considered as an element of $\mathcal{D}\mathrm{ef}(VG\subset MG,\mathcal{D})(A')$. Assume that we are given a deformation $\mathcal{V}_A \subset \mathcal{M}_A$ of this element to A. In particular, $\mathcal{M}_A$ is a deformation of $\mathcal{M}_{A'}$. However, $\mathcal{M}\otimes_\Lambda A$ is also a deformation of $\mathcal{M}_{A'}$. So, by the rigidity assumption, this two deformations are isomorphic. Moreover, by 4.3.7, we can choose as identification $\mathcal{M}_A = \mathcal{M}\otimes_\Lambda A$ which reduces to the identity map on $\mathcal{M}_{A'}$. Then $\mathcal{V}_A \subset \mathcal{M}_A = \mathcal{M}\otimes_\Lambda A$ is a required deformation.

To show that q_1 is formally smooth we apply a similar argument. Let $A \twoheadrightarrow A'$ be a small extension in Art_Λ. Let $\mathcal{G}_{A'} \in \mathcal{D}\mathrm{ef}(G,\mathcal{D})(A')$. Denote by $\mathcal{V}_{A'} \subset \mathcal{M}_{A'}$ the associated universal filtration object and let $\mathcal{V}_A \subset \mathcal{M}_A^{(1)}$ be a deformation of it to A (as a $\mathcal{D}$-object of $\mathrm{FMod}_{A'}$). Since $A \twoheadrightarrow A'$ has divided powers, we can also define the value of the universal extension crystal of $\mathcal{G}_{A'}$ on A. Denote it by $\mathcal{M}_A^{(2)}$. Both $\mathcal{M}_A^{(1)}$ and $\mathcal{M}_A^{(2)}$ are deformations of $\mathcal{M}_{A'}$. Hence, by rigidity, they are isomorphic over $\mathcal{M}_{A'}$. Using such as isomorphism, $\mathcal{V}_{A'}$ can be considered as a filtration on $\mathcal{M}_A^{(2)}$. An application of crystalline theory shows that this filtration comes from a $\mathcal{D}$-object $\mathcal{G}_A$ of pDiv_A. Then $\mathcal{G}_A$ is a required deformation of $\mathcal{G}_{A'}$ to A. Hence q_2 is formally smooth. ∎

Remark 4.3.9. The isomorphisms established in Theorems 4.3.8 are in no way canonical. For example, we do not claim that they are functorial in G. The following example shows that such a functorial isomorphism can not exist in general, even in the case $\mathcal{O}=\mathbf{Z}_p$.

Example 4.3.10. Let $k=\mathbf{F}_p$ and G be the p-divisible group of an ordinary elliptic curve over k. Assume for a moment that for any $A\in Art_\Lambda$ and any deformation $\mathcal{G}/A$ of G/k, there is a canonical isomorphism

$$\mathbf{D}(G)\otimes_\Lambda A = M\mathcal{G},$$

which is compatible with base change, functorial in $\mathcal{G}$ and which coincides with the Grothendieck-Messing isomorphism at least when $A=k[W]$ for a finite-dimensional k-vector space W. Construct a natural transformation of functors

$$\mathcal{D}\mathrm{ef}(G) \longrightarrow \mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG)$$

by letting

$$\mathcal{G}/A \longmapsto V\mathcal{G}\subset E\mathcal{G}=\mathbf{D}(G)\otimes_\Lambda A\,.$$

This natural transformation is in fact an isomorphism, since both functors are pro-represented by the ring $\Lambda[[t]]$ and the map is an isomorphism on the tangent spaces. Moreover, by functoriality, we get an induced inclusion of functors,

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O}) \hookrightarrow \mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG,R),$$

for an *arbitrary* subring $\mathcal{O}\subset\mathrm{End}(G)$ and $R=\mathcal{O}\otimes_{\mathbf{Z}_p}\Lambda$. Now denote by $\varphi\in\mathrm{End}(G)$ the (geometric) Frobenius on G and let

$$\mathcal{O}_n=\mathbf{Z}_p[p^n\varphi]\subset\mathrm{End}(G),\quad n\geq 0\,.$$

From the Serre-Tate theory, it follows that

$$\mathcal{D}\mathrm{ef}(G,\mathcal{O}_n)\cong\mathrm{Hom}_\Lambda\Big(W[[t]]/((1+t)^{p^n}-1),-\Big)\,.$$

However, it is easy to see that

$$\mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG,R)\cong\mathrm{Hom}_\Lambda\Big(W[[t]]/(p^nt),-\Big)\,.$$

Remark. In Chapter 5 we compute the functors $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G,\mathcal{O},\lambda)$ in some cases using the above isomorphisms. Recall the basic steps of our method to determine the above functors.

Say we are interested in $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$. Then firstly, we have found a functor (not necessarily pro-representable) over which $\mathcal{D}\mathrm{ef}(G,\mathcal{O})$ is formally smooth. Then we have rigidified this functor to get a pro-representable functor ($\mathcal{D}\mathrm{ef}_{\mathbf{D}(G)}(VG\subset MG,R)$ in this case) which is relatively easy to compute. Then we have applied the comparison theorem.

It might be interesting to apply the same method in a different setting. For example, a theorem of Grothendieck-Illusie ([13], Thm. 4.4) asserts that

$$q:\mathcal{D}\mathrm{ef}(G)\longrightarrow\mathcal{D}\mathrm{ef}(G[p])$$

is formally smooth. Here $\mathcal{D}\mathrm{ef}(G[p])$ is the deformation functor of the p-torsion of G as a truncated p-divisible group. Suppose we could rigidify this functor to get

$$r:\mathcal{F}\longrightarrow\mathcal{D}\mathrm{ef}(G[p])$$

with r formally smooth as well. Assume also that $\mathcal{F}$ is pro-representable and explicit enough. By "explicit enough" we mean that one can determine, say, the filtrations of $\mathcal{F}$ determined by the p-rank filtration on $\mathcal{D}\mathrm{ef}(G[p])$. Then using the comparison theorem as in the theorems above, one could deduce the corresponding information about the deformation functor of G.

4.4 The maximal order case

To discuss the applications of our results, we consider the case of a p-divisible group G with an action of a ring $\mathcal{O}$. We show that in this case, the rigidity required for 4.3.8 is satisfied if the Dieudonné module $\mathbf{D}(G)$ is $\mathcal{O}\otimes_{\mathbf{Z}_p}W(k)$-projective. This, in turn, is true whenever $\mathcal{O}$ is a hereditary (e.g. maximal) order in a semi-simple $\mathbf{Q}_p$-algebra.

We fix the following notations. Let G/k be a p-divisible group over a perfect field of characteristic p and $\mathcal{O}\subset\mathrm{End}(G)$ a $\mathbf{Z}_p$-subalgebra. Let $\mathcal{D}$ be the deformation data of Example 4.2.6 with $\mathcal{O}$ as the acting ring. We let $R=\mathcal{O}\otimes_{\mathbf{Z}_p}\Lambda$ and

$$\rho_\tau : R \;\longrightarrow\; \mathrm{End}(TG)$$

denote the tangent space representation. Note that TG is a $\mathcal{D}$-object and (cf. 2.2.3)

$$\mathcal{D}\mathrm{ef}(TG,\mathcal{D}) \;\cong\; \mathcal{D}\mathrm{ef}(\rho_\tau)\,.$$

Let $VG\subset MG=\mathbf{D}(G[p])$ as usual denote the universal extension filtration. We let $\mathcal{M}$ denote $\mathbf{D}(G)\otimes_W\Lambda$. Clearly (cf. 2.3.1)

$$\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,\mathcal{D}) \;\cong\; \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG\subset MG,R)\,.$$

Finally, by 4.2.6 we have $\mathcal{D}\mathrm{ef}(G,\mathcal{O})=\mathcal{D}\mathrm{ef}(G,\mathcal{D})$. This is a subfunctor of $\mathcal{D}\mathrm{ef}(G)$, the full deformation functor of the p-divisible group G.

Theorem 4.4.1. *Let G/k be a p-divisible group. Let $\mathcal{O} \subset \mathrm{End}(G)$ be a $\mathbf{Z}_p$-subalgebra which is isomorphic to a hereditary order in a semi-simple $\mathbf{Q}_p$-algebra. Consider the diagram of functors*

$$\begin{array}{ccc} \mathcal{D}\mathrm{ef}(G,\mathcal{O}) & & \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, R) \\ & {}_{q_1}\searrow \quad \swarrow_{q_2} & \\ & \mathcal{D}\mathrm{ef}(\rho_\tau) & \end{array} \tag{18}$$

Here q_1 and q_2 are the obvious maps given by $\mathcal{G}_A \mapsto T\mathcal{G}_A$ and $(V_A \subset M_A) \mapsto (M_A/V_A)$. Then

1. *$\mathbf{D}(G)$ is a projective $R = \mathcal{O} \otimes_{\mathbf{Z}_p} W(k)$-module.*
2. *$\mathbf{D}(G[p])$ is rigid as a $\mathcal{D}$-object of Mod_k.*
3. *q_1 and q_2 are formally smooth.*
4. *There is a (non-canonical) isomorphism of functors $i : \mathcal{D}\mathrm{ef}(G,\mathcal{O}) \to \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, R)$ which completes (18) to a commutative diagram.*

Proof. 1. Since hereditary orders stay hereditary after an unramified base change over a complete field (Theorem 3.2.11), R is a hereditary order in a semi-simple algebra over the fraction field of $W(k)$. Thus (by definition, cf. 3.2.8), every R-module which is free over W is projective.
2. From the first part of the theorem it follows that $\mathcal{M}$ is a projective R-module. The assertion follows from the fact that projective modules satisfy the rigidity condition (cf. 2.2.5).
3, 4. This follows from Theorem 4.3.8 once we show that

$$\begin{array}{ccc} \mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{D}) & \longrightarrow & \mathcal{D}\mathrm{ef}(\rho_\tau) \\ V_A \subset M_A & \longmapsto & M_A/V_A \end{array} \tag{19}$$

is an isomorphism. We start with surjectivity. Let $A \in Art_\Lambda$ and $\wp_A \in \mathcal{D}\mathrm{ef}(\rho_\tau)(A)$,

$$\wp_A : R \longrightarrow \mathrm{End}_A(\mathcal{T}_A) .$$

Here $\mathcal{T}_A$ is a finite free A-module and $\mathcal{T}_A \otimes_A k = TG$. Let $\mathcal{M}_A = \mathcal{M} \otimes_\Lambda A$. We have a diagram of $R \otimes_\Lambda A$-modules,

$$\begin{array}{ccccc} \mathcal{M}_A & \xrightarrow{\otimes k} & MG & \twoheadrightarrow & TG \\ & & & & \uparrow \otimes k \\ & & & & \mathcal{T}_A \end{array} \tag{20}$$

where the map $MG \twoheadrightarrow TG$ comes from the canonical isomorphism $MG/VG \cong TG$. Since $\mathcal{M}_A$ is a projective $R \otimes_\Lambda A$-module, there exists a $R \otimes_\Lambda A$-module map $\mathcal{M}_A \to \mathcal{T}_A$

which makes (20) commute. It is surjective by Nakayama's lemma and its kernel $\mathcal{V}_A \subset \mathcal{M}_A$ is the required deformation of the filtration $VG \subset MG$.

It remains to show that (19) is injective. Let $\mathcal{V}_{1,A} \subset \mathcal{M}_{1,A}$ and $\mathcal{V}_{2,A} \subset \mathcal{M}_{2,A}$ be two elements of $\mathcal{D}\mathrm{ef}(VG \subset MG, \mathcal{D})(A)$. Suppose

$$\mathcal{M}_{1,A}/\mathcal{V}_{1,A} \cong \mathcal{T}_A \cong \mathcal{M}_{2,A}/\mathcal{V}_{2,A} \,. \tag{21}$$

as $R \otimes_\Lambda A$-modules. By the second part of the theorem, there are isomorphisms

$$\mathcal{M}_{1,A} \cong \mathcal{M} \otimes_\Lambda A \cong \mathcal{M}_{2,A} \,.$$

In particular, $\mathcal{M}_{i,A}$ are projective $R \otimes_\Lambda A$-modules. We have to show that there is an isomorphism $\mathcal{M}_{1,A} \cong \mathcal{M}_{2,A}$ which reduces to the identity map on MG and which takes $\mathcal{V}_{1,A}$ to $\mathcal{V}_{2,A}$.

Consider the set $\mathcal{T}_A \times_{TG} MG$. It can be naturally given an $R \otimes_\Lambda A$-module structure (via that of its components). We have the following diagram of $R \otimes_\Lambda A$-modules

$$\begin{array}{ccc} \mathcal{M}_{1,A} & \cdots\cdots\cdots> & \mathcal{M}_{2,A} \\ {}_{\pi_1}\searrow & & \swarrow_{\pi_2} \\ & \mathcal{T}_A \times_{TG} MG & \end{array} \tag{22}$$

Here π_i have the maps induced by (21) as their first components and the natural projections $\mathcal{M}_{1,A} \to MG$ as their second components. In particular π_2 (and π_1) is surjective. By projectivity of $\mathcal{M}_{1,A}$, there exists a dotted map which makes (22) commute. Such a map has both of the required properties (look at its components). ■

4.5 The polarized maximal order case

Our next case is that of a p-divisible group G with a fixed subring $\mathcal{O} \subset \mathrm{End}(G)$ and given together with a principal quasi-polarization, an isomorphism

$$\lambda : G \longrightarrow G^t$$

which is anti-symmetric, $\lambda^t = -\lambda$. We assume that $\mathcal{O}$ is stable under the Rosati involution

$$\varphi \longmapsto i(\varphi) = \lambda^{-1}\varphi^t\lambda, \quad \varphi \in \mathrm{End}(G) \,.$$

To simplify our considerations, we only study the case $p \neq 2$. So we use the blanket assumption $\mathrm{char}\, k \neq 2$ is used throughout this section.

Notation 4.5.1. Denote $R = \mathcal{O} \otimes_{\mathbf{Z}_p} \Lambda$ and $\mathcal{M} = \mathbf{D}(G) \otimes_{W(k)} \Lambda$ as in the previous section. The quasi-polarization induces an isomorphism, which we also denote by λ,

$$\lambda : \mathcal{M} \longrightarrow \mathcal{M}^t \,.$$

This is an isomorphism of (left) R-modules if we give $\mathcal{M}^t$ the left R-module structure via the Rosati involution,

$$(r \cdot f)(x) = f(i(r) \cdot x), \quad f \in \mathcal{M}^t \,. \tag{23}$$

Notation 4.5.2. Denote by $\mathcal{D}$ the deformation data of Example 4.2.7 with our given ring $\mathcal{O}$ and the involution i induced by λ. Note that (G, G^t) becomes a $\mathcal{D}$-object of pDiv_k. By abuse of notation we will denote this object just by G. Hence we will also refer to $VG \subset MG$ as a $\mathcal{D}$-object of FMod_k and adopt similar notations for the deformations $\mathcal{G}_A$ of G to some $A \in Art_\Lambda$.

Remark. The deformation functor $\mathcal{D}\mathrm{ef}(G, \mathcal{D})$ is canonically isomorphic to the one defined in 4.1.4 (cf. 4.2.7),

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) = \mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda) \,.$$

Denote also

$$\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, R, \lambda) = \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{D}) \,.$$

Remark. An element of $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)(A)$ is thus a p-divisible group $\mathcal{G}_A/A$ deforming G, which admits an $\mathcal{O}$-action and a principal quasi-polarizations reducing to those of G. An element of $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, R, \lambda)(A)$ is an $\mathcal{O}$-stable (eqivalently R-stable) totally isotropic filtration $\mathcal{V}_A \subset \mathcal{M}_A = \mathcal{M} \otimes_\Lambda A$ which reduces to $VG \subset MG$.

Theorem 4.5.3. *Let $\lambda : G \to G^t$ be a principal quasi-polarization. Let $\mathcal{O} \subset \mathrm{End}(G)$ be a Rosati-invariant $\mathbf{Z}_p$-subalgebra which is isomorphic to a hereditary order in a semi-simple $\mathbf{Q}_p$-algebra. Then there is a (non-canonical) isomorphism*

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda) \cong \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, R, \lambda) \,.$$

Proof. Let $\mathcal{D}$ be as above (see 4.5.2). We show that

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \cong \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{D})$$

by applying the main comparison theorem (4.3.8) to this situation. In order to do this, it suffices to prove that $MG = \mathbf{D}(G[p])$ is rigid as a $\mathcal{D}$-object. Let $A \twoheadrightarrow A'$ be a surjection in Art_Λ and $\mathcal{M}_{A'}$ a deformation of MG to A' as a $\mathcal{D}$-object. Hence $\mathcal{M}_{A'}$ is a finite free A'-module with an R-action and given together with a self-dual (left) R-module isomorphism $\mathcal{M}_{A'} \cong \mathcal{M}_{A'}^t$ (as in 4.5.1).

Let $\mathcal{M}_A^{(1)}$ and $\mathcal{M}_A^{(2)}$ be two deformations of $\mathcal{M}_{A'}$ to A (as a $\mathcal{D}$-object). We claim that they are isomorphic. Let

$$\Lambda^{(1)} : \mathcal{M}_A^{(1)} \longrightarrow (\mathcal{M}_A^{(1)})^t, \qquad \Lambda^{(2)} : \mathcal{M}_A^{(2)} \longrightarrow (\mathcal{M}_A^{(2)})^t \quad .$$

be the quasi-polarizations. By 2.2.5, there is an R-module isomorphism $\varphi : \mathcal{M}_A^{(1)} \to \mathcal{M}_A^{(2)}$ which reduces to the identity map on $\mathcal{M}_{A'}$. If, moreover, φ commutes with the Λ's, that is, if $\Lambda^{(1)} = \varphi^t \Lambda^{(2)} \varphi$, then $\mathcal{M}_A^{(1)}$ and $\mathcal{M}_A^{(2)}$ are indeed isomorphic as $\mathcal{D}$-objects, as required. Otherwise, consider the maps

$$\varphi \qquad \text{and} \qquad (\Lambda^{(2)})^{-1}(\varphi^t)^{-1}\Lambda^{(1)} .$$

Both are R-module isomorphisms $\mathcal{M}_A^{(1)} \to \mathcal{M}_A^{(2)}$ which reduce to the identity on $\mathcal{M}_{A'}$. Hence their "average",

$$\psi = \frac{1}{2}\left(\varphi + (\Lambda^{(2)})^{-1}(\varphi^t)^{-1}\Lambda^{(1)}\right)$$

is also an R-module map which reduces to the identity on $\mathcal{M}_{A'}$. In particular (Nakayama's lemma), it is an isomorphism as well. Using the self duality of $\Lambda^{(1)}$ and $\Lambda^{(2)}$, it is easy to check that $\Lambda^{(1)} = \psi^t \Lambda^{(2)} \psi$. Hence ψ is the required isomorphism of $\mathcal{D}$-objects. This completes the proof. ▪

Remark. Given a triple $(G, \mathcal{O}, \lambda)$, it might be interesting to compare the structure of the deformation functors $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$. One can describe the latter functor as a fibre product functor

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda) = \mathcal{D}\mathrm{ef}(G, \mathcal{O}) \times_{\mathcal{D}\mathrm{ef}(G)} \mathcal{D}\mathrm{ef}(G, \lambda),$$

which presents the pro-representing ring of $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$ as a (completed) tensor product of the corresponding rings. Here $\mathcal{D}\mathrm{ef}(G)$ is the full deformation functor of the p-divisible group G, pro-represented by the formal power series ring $\Lambda[[t_1, ...t_{n^2}]]$ with $n = \dim G$. The functor $\mathcal{D}\mathrm{ef}(G, \lambda)$ of deformations of G which respect λ is pro-represented by $\Lambda[[t_1, ...t_{n(n+1)/2}]]$. Note however, that knowing abstractly the pro-representing ring of $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ does not by itself give that of $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$. In fact, one needs to know how exactly the subfunctors $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G, \lambda)$ "intersect" inside $\mathcal{D}\mathrm{ef}(G)$. For example assume that $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ is formally smooth, i.e. it is pro-represented by a formal power series ring (in some number of variables) over Λ. It is not clear then that $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$ is formally smooth as well, as two regular subschemes of a regular scheme can have a singular intersection. Surprisingly, the formal smoothness of $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ does imply that of $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$, as we show in the theorem below. The proof makes use of the fact that $\mathcal{O}$ is Rosati invariant (although not the actual involution on $\mathcal{O}$) and the "averaging" trick used in the proof of 4.5.3.

Theorem 4.5.4. *Let $\lambda : G \to G^t$ be a principal quasi-polarization. Let $\mathcal{O} \subset \mathrm{End}(G)$ be a Rosati-invariant $\mathbf{Z}_p$-subalgebra and assume that $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ is formally smooth. Then $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$ is formally smooth as well.*

Proof. Let $A \twoheadrightarrow A'$ be a small extension in Art_Λ and $\mathcal{G}_{A'} \in \mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)(A')$. Hence $\mathcal{G}_{A'}/A'$ a deformation of G/k to which the quasi-polarization and the ring action lift. We have to show that there is a $\mathcal{G}_A \in \mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)(A)$ which deforms $\mathcal{G}_{A'}$. The formal smoothness of $\mathcal{D}\mathrm{ef}(G, \lambda)$ implies the existence of a deformation $\mathcal{G}_A^\lambda/A$ of $\mathcal{G}_{A'}$ which inherits the quasi-polarization. On the other hand, by formal smoothness of $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$

there is also (another) deformation $\mathcal{G}_A^{\mathcal{O}}/A$ of $\mathcal{G}_{A'}$ to which the $\mathcal{O}$-action lifts. We are going to use the relation between λ and $\mathcal{O}$ (namely, the Rosati invariance) to show that there is a third lifting, $\mathcal{G}_A = \mathcal{G}_A^{\mathcal{O},\lambda}/A$ to which both λ and $\mathcal{O}$-action lift. In some sense, $\mathcal{G}_A^{\mathcal{O},\lambda}$ is going to be a combination of $\mathcal{G}_A^{\mathcal{O}}$ and $\mathcal{G}_A^{\lambda}$.

As in the proof of pro-representability 4.3.5, we use the results of Grothendieck-Messing. Associated to $\mathcal{G}_{A'}$ there is a universal extension filtration $V\mathcal{G}_{A'} \subset M\mathcal{G}_{A'}$. Since $A \twoheadrightarrow A'$ has divided powers, we can also define $M_A\mathcal{G}_{A'}$, the value of the universal extension crystal of $\mathcal{G}_{A'}$ on the ring A. This is a $\mathcal{D}$-object of Mod_A. There is a bijection between the deformations of $\mathcal{G}_{A'}$ to A (as a $\mathcal{D}$-object) and deformations of the filtration $V\mathcal{G}_{A'} \subset M\mathcal{G}_{A'}$ to a filtration of $M_A\mathcal{G}_{A'}$ (again, as a $\mathcal{D}$-object).

Fix an identification of Λ-modules $\ker(A \twoheadrightarrow A') \cong k$. By Theorem 2.3.2, the set of all deformations of $V\mathcal{G}_{A'} \subset M\mathcal{G}_{A'}$ to a filtration of $M_A\mathcal{G}_{A'}$ is a principal homogeneous space under $TG \otimes TG^t$. Thus, for any $\mathcal{G}_A$ deforming $\mathcal{G}_{A'}$, we can formally write

$$\mathcal{G}_A = \mathcal{G}_A^{\lambda} + \xi \tag{24}$$

for some $\xi \in TG \otimes TG^t$. Since it is easy to characterize the filtrations to which either λ or the $\mathcal{O}$-action lifts, the same is true for the deformations $\mathcal{G}_A$ of $\mathcal{G}_{A'}$ to A. Consider the composition s of the maps

$$TG \otimes TG^t \xrightarrow{d\lambda \otimes d\lambda^{-1}} TG^t \otimes TG \xrightarrow{i} TG \otimes TG^t .$$

Here i interchanges the two factors and $d\lambda : TG \longrightarrow TG^t$ is induced by $\lambda : G \to G^t$ on the tangent spaces. Then λ lifts to the deformation $\mathcal{G}_A$ as in 24 if and only if ξ is symmetric under s,

$$s(\xi) = \xi .$$

On the other hands, the liftings $\mathcal{G}_A$ which inherit the $\mathcal{O}$-action can be written as

$$\mathcal{G}_A = \mathcal{G}_A^{\mathcal{O}} + \eta, \qquad \eta \in \mathrm{H}^0(R\text{-}R, TG \otimes TG^t) \subset TG \otimes TG^t .$$

Write

$$\mathcal{G}_A^{\mathcal{O}} = \mathcal{G}_A^{\lambda} + \theta, \qquad \theta \in TG \otimes TG^t$$

From the relation 23, it follows that

$$s(\theta) - \theta \in \mathrm{H}^0(R\text{-}R, TG \otimes TG^t) \subset TG \otimes TG^t .$$

Thus both $\mathcal{G}_A^{\lambda} + \theta$ and $\mathcal{G}_A^{\lambda} + s(\theta)$ give deformations which inherit the $\mathcal{O}$-action. Hence so does

$$\mathcal{G}_A^{\mathcal{O},\lambda} = \mathcal{G}_A^{\lambda} + \frac{\xi + s(\xi)}{2}$$

It is also clear that $(\xi + s(\xi))/2$ is symmetric under s, so $\mathcal{G}_A^{\mathcal{O},\lambda}$ inherits both λ and the $\mathcal{O}$-action, as asserted. ∎

4.6 Non-rigid deformation problems

Let G/k be a p-divisble group. We have studied the deformation functors of G with an action of a maximal order $\mathcal{O}$ and/or a principal quasi-polarization λ. In many applications to abelian varieties, such as the study of CM-liftings or construction of abelian varieties with a given endomorphism ring, one is led to study a more general situation. This leads to consider a functor of the type

$$\mathcal{D} = \mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda_1, ..., \lambda_n, \mu_1, ..., \mu_m), \quad \lambda_i : G \to G^t,\ \mu_j : G^t \to G \tag{25}$$

where $\mathcal{O} \subset \mathrm{End}(G)$ is an arbitrary subring and λ_i, μ_j are quasi-polarizations, not necessarily principal. It is possible to reduce the study of such functors to a simpler case. Namely, there is an isomorphism

$$\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda_1, ..., \lambda_n, \mu_1, ..., \mu_m) \;\cong\; \mathcal{D}\mathrm{ef}(H, \mathcal{O}_H, \lambda_H) \tag{26}$$

for a certain choice of $H, \mathcal{O}_H$ and λ_H. In fact, take $H = G \times G^t$ and let i be the map $H = G \times G^t \to G^t \times G = H^t$ which interchanges the two factors. Then $\lambda_H = i \circ (1, -1)$ is a (principal) quasi-polarization on H. If the subring $\mathcal{O}_H \subset \mathrm{End}(H)$ happens to be a hereditary order in a semi-simple $\mathbf{Q}_p$-algebra, we can apply our previous results. Unfortunately, this is far from the case in general.

The ring $\mathcal{O}_H$ and the isomorphism (26) are established as follows. Let $\mathcal{D}$ be as in (25). Let $H = G \times G^t$ and decompose

$$\mathrm{End}(H) = \left(\begin{array}{c|c} \mathrm{End}(G) & \mathrm{Hom}(G, G^t) \\ \hline \mathrm{Hom}(G^t, G) & \mathrm{End}(G^t) \end{array}\right)$$

Define $\mathcal{O}_H \subset \mathrm{End}(H)$ to be

$$\mathcal{O}_H = \left\langle p_G, p_{G^t}, \left(\begin{array}{c|c} \varphi & 0 \\ \hline 0 & \varphi^t \end{array}\right)_{\varphi \in \mathcal{O}}, \left(\begin{array}{c|c} 0 & \lambda_i \\ \hline 0 & 0 \end{array}\right)_{1 \le i \le n}, \left(\begin{array}{c|c} 0 & 0 \\ \hline \mu_j & 0 \end{array}\right)_{1 \le j \le m} \right\rangle .$$

In other words, $\mathcal{O}_H$ is generated by the data defining $\mathcal{D}$ plus the projections p_G and p_{G^t} of H on the two factors. Take λ_H as above and consider

$$\mathcal{F} = \mathcal{D}\mathrm{ef}(H, \mathcal{O}_H, \lambda_H) .$$

We claim that $\mathcal{D} \cong \mathcal{F}$. Clearly a deformation $\mathcal{G} \in \mathcal{D}(A)$ for some $A \in Art_\Lambda$ gives also an element of $\mathcal{F}(A)$.

Conversely, take $\mathcal{H} \in \mathcal{F}(A)$. Define

$$\mathcal{G}_1 = \ker(p_G : \mathcal{H} \to \mathcal{H}), \quad \mathcal{G}_2 = \ker(p_{G^t} : \mathcal{H} \to \mathcal{H}) .$$

These are p-divisible groups over A which deform G and G^t. Since $p_G, p_{G^t} \in \mathrm{End}(\mathcal{H})$ are orthogonal idempotents, $\mathcal{H} \cong \mathcal{G}_1 \times \mathcal{G}_2$.

From the relation

$$p_{G^t} = \lambda_H^{-1} p_G^t \lambda_H$$

it follows that the lift of λ_H to a principal quasi-polarization on $\mathcal{H}$ identifies $\mathcal{G}_1^t$ with $\mathcal{G}_2$. So $H \cong \mathcal{G}_1 \times \mathcal{G}_1^t$.

Finally, from the commutation relation of the elements of $\mathcal{O}_H$ with the projections p_G, p_{G^t} it follows that every $\varphi \in \mathcal{O}$ lifts indeed to an endomorphism of $\mathcal{G}_1$, rather than just an endomorphism of $\mathcal{H}$. The same hold for λ_i and μ_j. So $\mathcal{G}_1 \in \mathcal{D}(A)$ as asserted.

Remark 4.6.1. The above argument clearly generalizes to a deformation problem with an arbitrary deformation data (cf. 4.2.2). Thus, given a deformation data $\mathcal{D}$ and a $\mathcal{D}$-object G of pDiv_k, there is an isomorphism

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \cong \mathcal{D}\mathrm{ef}(H, \mathcal{O}_H, \lambda_H)$$

for some p-divisible group H/k, a subring $\mathcal{O} \subset \mathrm{End}(H)$ and a principal quasi-polarization λ_H on H. This is, however, mostly of theoretical interest, as the deformation functors $\mathcal{D}\mathrm{ef}(H, \mathcal{O}_H, \lambda_H)$ can be extremely complicated in case $\mathcal{O}_H$ is not a maximal order.

4.7 The p-chain case

Although our computations in Chapter 4 concern primarily the deformation functors $\mathcal{D}\mathrm{ef}(G, \mathcal{O})$ and $\mathcal{D}\mathrm{ef}(G, \mathcal{O}, \lambda)$, it is interesting to give an example of a slightly different kind. Here is a well-known deformation problem which involves more than one p-divisible group.

Let $\mathcal{D}$ be the deformation data of 4.2.8, a "p-chain of length n". As we already remarked, a $\mathcal{D}$-object of pDiv_A can be identified with a collection of p-divisible groups $\{\mathcal{G}_i/A\}$ indexed by $i \in \mathbf{Z}/n\mathbf{Z}$ and maps $f_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$, such that

$$f_{i-1} f_{i-2} \cdots f_{i-n} = p \in \mathrm{End}(\mathcal{G}_i), \qquad i \in \mathbf{Z}/n\mathbf{Z}\,. \tag{27}$$

In particular, f_i are isogenies. We claim that in this situation the comparison theorem 4.3.8 applies:

Proposition 4.7.1. *Let G be a $\mathcal{D}$-object of pDiv_k. Then $\mathbf{D}(G[p])$ is a rigid $\mathcal{D}$-object of Mod_k. There is a non-canonical isomorphism of functors*

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \;\cong\; \mathcal{D}\mathrm{ef}_{\mathcal{M}}(VG \subset MG, \mathcal{D})\,.$$

Proof. Let G_i denote the p-divisible groups which form the p-chain and $f_i : G_i \to G_{i+1}$ the connecting maps. As usual, let $\mathcal{M}_i = \mathbf{D}(G_i)$ and $M_i = \mathbf{D}(G_i[p]) = \mathcal{M}_i \otimes_W k$. For any $i \in \mathbf{Z}/n$ choose a subspace $K_i \subset M_i$ such that $M_i = f_{i-1} M_{i-1} \oplus K_i$. The compositions

$$f_{i-1} \cdots f_{i-j} : \;\; M_{i-j} \longrightarrow M_i$$

map K_{i-j} injectively into M_i for $1 \leq j < n$. Letting $f_{i-1} \cdots f_{i-j}$ denote the identity map for $j = 0$, we have

$$M_i = \bigoplus_{j=0}^{n-1} f_{i-1} \cdots f_{i-j} K_{i-j} .$$

If $\{\mathcal{M}_i\}$ is a deformation of M to $A \in Art_\Lambda$ as a $\mathcal{D}$-object, one obtains a similar decomposition: let $\mathcal{K}_i \subset \mathcal{M}_i$ be a finite free A-module which lifts $K_i \subset M_i$. Then, using Nakayama's lemma, one shows that

$$\mathcal{M}_i = \bigoplus_{j=0}^{n-1} f_{i-1} \cdots f_{i-j} \mathcal{K}_{i-j} .$$

It follows that every two $\mathcal{D}$-deformations of M to A are isomorphic. The second assertion of the proposition follows from Theorem 4.3.8. ■

Remark 4.7.2. It is interesting to note that the condition (27) is essential for the rigidity. In fact, if one takes a chain of p-divisible groups (G_i, f_i) with, for instance, $\prod f_j = p^2$ instead, the corresponding deformation data is not rigid and the statement corresponding to 4.7.1 does not hold.

Remark 4.7.3. Proposition 4.7.1 allows to write down equations for the deformation functor $\mathcal{D}\mathrm{ef}(G, \mathcal{D})$ of a p-chain of p-divisible groups. There is, however, a different approach to study this functor. As in the previous section, it is possible to find an isomorphism

$$\mathcal{D}\mathrm{ef}(G, \mathcal{D}) \cong \mathcal{D}\mathrm{ef}(H, \mathcal{O}) \tag{28}$$

for certain p-divisible group H and a hereditary order $\mathcal{O} \subset \mathrm{End}(H)$.

Namely, let the p-chain in question be given by

$$G_n \xrightarrow{f_n} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_{n-1} \xrightarrow{f_{n-1}} G_n .$$

Define $H = G_1 \oplus \cdots \oplus G_n$. Let $e_i \in \mathrm{End}(H)$ be the projector on the i-th factor and $f = (f_1, ..., f_n) \in \mathrm{End}(H)$. Let $\mathcal{O} \subset \mathrm{End}(H)$ be the $\mathbf{Z}_p$-subalgebra generated by the e_i and f. Then (28) holds.

The structure of $\mathcal{O}$ can be also easily determined. It is isomorphic to the subring of $\mathrm{Mat}_{n\times n}(\mathbf{Z}_p)$ given by

$$\mathcal{O} = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ b_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ b_{n-1,1} & b_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n-1} & a_{n,n} \end{pmatrix}, \quad a_{i,j} \in \mathbf{Z}_p, b_{i,j} \in p\mathbf{Z}_p \right\} . \tag{29}$$

We refer to Section 5.3 for the proof of this statement (cf. 5.3.1) and a study of the deformation problem $\mathcal{D}\mathrm{ef}(H, \mathcal{O})$. There we discuss the case of a p-divisible group with an action of a maximal order a central division algebra over $\mathbf{Q}_p$, which leads to the same functor. In fact, these considerations also give an alternative proof of 4.7.1.