3 Modules over maximal orders

In this chapter we recall the basic structure theorems for maximal and hereditary orders in (not necessarily commutative) semisimple algebras. These results are used in Section 4.4.

More specifically let G/k be a p-divisible group. Assume k is perfect and denote by $W = W(k)$ its Witt vector ring. If $\mathcal{O} \subset \mathrm{End}(G)$ is a $\mathbf{Z}_p$-subalgebra, we will need to single out the situations when the representation of $\mathcal{O}$ on the Dieudonné module $\mathbf{D}(G[p])$ has a trivial deformation functor. Using the results of the previous section, we show that this is the case whenever the Dieudonné module $\mathbf{D}(G)$ is $\mathcal{O} \otimes_{\mathbf{Z}_p} W(k)$-projective. This turns out to be the case whenever $\mathcal{O}$ is a *maximal* order in a semi-simple $\mathbf{Q}_p$-subalgebra of $\mathrm{End}(G)$. Indeed, maximal orders over a complete field are *hereditary*, so torsion-free modules over them are projective (3.2.9). In order to prove that if $\mathcal{O}$ is hereditary then so is $\mathcal{O} \otimes_{\mathbf{Z}_p} W(k)$, we show that hereditary orders stay hereditary after an unramified base ring extension (3.2.11). This is a simple extension of [15], Theorem 1.

All algebras considered in this chapter are *finite-dimensional* and *separable* over a field K. In our applications (Chapter 4) K will be the fraction field of $W(k)$, hence separability will be automatic. The word module stands for a left module. We refer to Reiner [34] for the proofs of most of the statements.

3.1 Semi-simple algebras

In this section the ground field K is arbitrary.

Definition 3.1.1. A K-algebra D is *simple* if D has no non-trivial two-sided ideals. A K-algebra whose radical (intersection of all maximal left ideals) is zero is called *semisimple*. A K-algebra D is called *central* if the center $Z(D)$ equals K.

Example. A finite field extension L of K is a simple K-algebra (non-central, unless $L = K$). A finite-dimensional division algebra D over K is simple. A matrix ring $\mathrm{Mat}_{n\times n}(K)$ and, more generally, a matrix ring $\mathrm{Mat}_{n\times n}(D)$ over a (central) division K-algebra D is a (central) simple K-algebra.

These are in fact the only examples:

Structure Theorem 3.1.2. *A semisimple K-algebra D decomposes as a product of matrix algebras over division algebras,*

$$D = \mathrm{Mat}_{n_1\times n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_k\times n_k}(D_k)\ .$$

Each of the D_i's is central over a finite field extension K_i of K.

Proof. [34], Theorems 7.1,7.4.

If D is a (semi)simple K-algebra and L/K a finite field extension, then $D \otimes_K L$ is easily seen to be a (semi)simple L-algebra. Moreover, central K-algebras become central L-algebras after such a base change. It is not true, however, that division algebras stay division. For example if a division algebra D/K contains a non-trivial field extension L/K, then $D \otimes_K L$ contains $L \otimes_K L$, which has zero divisors. If, moreover, $L \subset D$ is maximal commutative, then $D \otimes_K L \cong \mathrm{Mat}_{n\times n}(L)$. The converse to this is the following theorem.

Theorem 3.1.3. *Let D be a division algebra, central over K and L/K a finite field extension. Then*

1. *$D \otimes_K L$ is a division algebra if and only if L/K and D/K have no isomorphic intermediate subfields (except for K itself).*

2. *$D \otimes_K L \cong \mathrm{Mat}_{n\times n}(L)$ if and only if L can be embedded into D as a maximal commutative subalgebra.*

Proof. [14], Theorem 4.8; [34], Theorem 7.15. ∎

Definition 3.1.4. In the situation of (2.) we say that L *splits* D.

Finally, we discuss the structure of (left) modules over semisimple algebras.

Theorem 3.1.5. *Let D be a semisimple K-algebra. Then every finitely generated D-module is projective.*

Proof. If D is a division algebra over K, then every finitely generated D-module is free (easy induction argument, using that D has no two-sided ideals). If D is a matrix algebra over a division algebra, the result follows from Morita equivalence. Finally, if D is a product of simple K-algebras, every D-module decomposes as a direct sum of modules over the factors of D and, hence, is projective. ∎

3.2 Maximal and hereditary orders

Throughout this section K is a field, which is complete with respect to a discrete valuation v and A its valuation ring. Again, K-algebras are assumed to be separable and finite-dimenional.

Definition 3.2.1. Let D be a semi-simple K-algebra. An *order of D* is a finitely generated A-subring $\mathcal{O}$ of D such that $\mathcal{O} \otimes_A K = D$.

Definition 3.2.2. An order $\mathcal{O}$ of D is said to be *maximal* if there is no order $\mathcal{O}'$ of D which strictly contains $\mathcal{O}$.

Over a complete field, the structure of maximal orders in a semi-simple K-algebra is summarized in the following theorems ([34], Theorems 12.8, 17.3, 10.5).

Theorem 3.2.3. *Let D/K be a division algebra* (recall that K is complete by assumption). *Then D has a unique maximal order, the integral closure of A in D.*

Theorem 3.2.4. *Let $D = \mathrm{Mat}_{n\times n}(D_0)$, a matrix ring over a division algebra D_0. Denote the unique maximal order of D_0 by $\mathcal{O}_0$. Then $\mathrm{Mat}_{n\times n}(\mathcal{O}_0)$ is a maximal order of D and every other maximal order of D is conjugate to it.*

Theorem 3.2.5. *Every maximal order of a product $D = D_1 \times \cdots \times D_n$ of simple K-algebras is conjugate to a product of (some) maximal orders of the D_i's.*

Remark 3.2.6. Let $L \supset K$ be a finite field extension. Denote by B the integral closure of A in L. We have already remarked that if D is a central semi-simple K-algebra, then $D \otimes_K L$ is central semi-simple. It is also clear that if $\mathcal{O} \subset D$ is an order, then $\mathcal{O} \otimes_R S \subset D \otimes_K L$ is again an order. However, if $\mathcal{O}$ is maximal, this does *not* imply that $\mathcal{O} \otimes_A B$ is maximal, even if L/K is unramified. Consider the following example:

Example 3.2.7. Let $p \neq 2$ be a prime, $K = \mathbf{Q}_p$ and L the unique unramified quadratic extension of K. Let $A = \mathbf{Z}_p$, $B = \mathbf{Z}_p \oplus \mathbf{Z}_p\xi$ be the rings of integers of K and L respectively. Denote by σ the unique non-trivial automorphism of L over K. Consider

$$\mathcal{O} = \left\{ \mathbf{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{Z}_p \begin{pmatrix} \xi & 0 \\ 0 & \xi^\sigma \end{pmatrix} + \mathbf{Z}_p \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} + \mathbf{Z}_p \begin{pmatrix} 0 & \xi^\sigma \\ \xi p & 0 \end{pmatrix} \right\} \subset \mathrm{Mat}_{2\times 2}(L)\,.$$

It is easy to see that $D = \mathcal{O} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a division algebra, in fact the unique quaternion algebra over $\mathbf{Q}_p$. The subring $\mathcal{O} \subset D$ is the maximal order of D. The field L splits D. (L is contained in D as a maximal commutative subfield.) Consider the order

$$O \otimes_A B \subset D \otimes_K L \cong \mathrm{Mat}_{2\times 2}(L)$$

It is easy to see by looking at the given generators that

$$O \otimes_A B \;\cong\; \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \;\middle|\; a, b, c, d \in B \right\}\,,$$

which is *not* a maximal order in $D \otimes_K L$, as it is contained in $\mathrm{Mat}_{2\times 2}(B)$. It is fortunate for the applications in Chapter 4 that the orders which can be obtained by a base change from a maximal order by an unramified ring extension do inherit the following important property of maximal orders:

Definition 3.2.8. An order $\mathcal{O}$ of D is *(left) hereditary* if every $\mathcal{O}$-module, which is finitely generated and free as an A-module is $\mathcal{O}$-projective.

Theorem 3.2.9. *([34], 18.1) A maximal order in a K-algebra D is hereditary.*

Theorem 3.2.10. *(Structure theorem; [34], 39.14)*

1. *A division algebra D/K has a unique hereditary order, namely the maximal order of D.*

2. *Let D/K be a division algebra. Let $\mathcal{O}$ denote the unique maximal order of D_0 and r denote the radical of $\mathcal{O}_0$. Let $E \cong \mathrm{Mat}_{n\times n}(D)$. Then for every hereditary order H of E, there are positive integers $\{n_1, ..., n_k\}$ with sum n and an identification $E = \mathrm{Mat}_{n\times n}(D)$, such that H takes the form*

$$H = \begin{pmatrix} \mathrm{Mat}_{n_1\times n_1}(\mathcal{O}) & \mathrm{Mat}_{n_1\times n_2}(\mathcal{O}) & \mathrm{Mat}_{n_1\times n_3}(\mathcal{O}) & \cdots & \mathrm{Mat}_{n_1\times n_k}(\mathcal{O}) \\ \mathrm{Mat}_{n_2\times n_1}(r) & \mathrm{Mat}_{n_2\times n_2}(\mathcal{O}) & \mathrm{Mat}_{n_2\times n_3}(\mathcal{O}) & \cdots & \mathrm{Mat}_{n_2\times n_k}(\mathcal{O}) \\ \mathrm{Mat}_{n_3\times n_1}(r) & \mathrm{Mat}_{n_3\times n_2}(r) & \mathrm{Mat}_{n_3\times n_3}(\mathcal{O}) & \cdots & \mathrm{Mat}_{n_3\times n_k}(\mathcal{O}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathrm{Mat}_{n_k\times n_1}(r) & \mathrm{Mat}_{n_k\times n_2}(r) & \mathrm{Mat}_{n_k\times n_3}(r) & \cdots & \mathrm{Mat}_{n_k\times n_k}(\mathcal{O}) \end{pmatrix}.$$

Conversely, every order of this form is hereditary.

3. *If D is semi-simple with simple components D_i, then the hereditary orders of D are exactly the direct sums of hereditary orders in the D_i.*

Theorem 3.2.11. *Let D/K be a (finite-dimensional) semi-simple algebra and $\mathcal{O} \subset D$ a hereditary order. Let L/K be an unramified extension of complete fields and let B denote the ring of integers of L. Then $\mathcal{O} \otimes_A B$ is a hereditary order in $D \otimes_K L$.*

Proof. In case $[L:K] < \infty$ this is Janusz [15], Theorem 1. Now let L/K be arbitrary. Using the Structure theorem 3.2.10, one reduces to the case D is division. In this case $\mathcal{O} \subset D$ is the maximal order. If $D \otimes_K L$ happens to be a division algebra, then $\mathcal{O} \otimes_A B$ is easily seen to be the maximal order of $D \otimes_K L$, hence it is hereditary. If $D \otimes_K L$ is not division, then there is a finite extension K_1 of K in L such that already $D \otimes_K L$ is not division. Let B_1 denote the ring of integers of K_1. Replace K by K_1, D by $D_1 = D \otimes_K K_1$ and $\mathcal{O}$ by $O_1 = O \otimes_A B$. The order O_1 is hereditary in D_1 (again Janusz [15], Theorem 1) and we can apply the same procedure until on some step $D_n \otimes_{K_n} L$ is a division algebra. This happens necessarily after finitely many ($< rk_L D$) steps. ∎

Remark. We will use this theorem in 4.4.1 with $A = \mathbf{Z}_p$ and $B = W(k)$, the ring of Witt vectors of a perfect field k of characteristic p. We show namely that the Dieudonné module $D(G)$ of a p-divisible group G/k is $\mathcal{O} \otimes_{\mathbf{Z}_p} W(k)$-projective whenever $\mathcal{O} \subset \mathrm{End}(G)$ is a hereditary order in a semi-simple $\mathbf{Q}_p$-algebra.