2 Cohomology of R-R bimodules

It is typical, that tangent spaces and obstruction spaces to moduli functors are certain cohomology groups. For example, the tangent space to the deformation space of a regular variety X/k is $\mathrm{H}^1(X,\tau_X)$ and the obstruction lies in $\mathrm{H}^2(X,\tau_X)$, where τ_X is the tangent sheaf. Deforming a morphism $f:X\to Y$ of (say, regular) varieties gives $\mathrm{H}^0(X,f^*\tau_Y)$ and $\mathrm{H}^1(X,f^*\tau_Y)$ respectively. Mazur's deformation theory of a Galois representation $\rho:G\to\mathrm{Aut}(V)$ gives $\mathrm{H}^1(G,\mathrm{End}(V))$ as the tangent space and $\mathrm{H}^2(G,\mathrm{End}(V))$ as an obstruction space, cf. [22], §1.2, §1.6. Illusie [12] has shown that in general the tangent space of a functor can be identified with a certain Ext^1 and the obstructions lie in Ext^2.

Our primary interest lies in deformations of p-divisible group with an $\mathcal{O}$-action as well as those of ring representations and R-stable filtrations on an R-module. In all three cases the corresponding tangent and obstruction spaces turn out to be the Hochschild cohomology groups.

In Section 2.1 we recall the basic properties of these cohomology groups and prove that $\mathrm{H}^1(R\text{-}R,\mathrm{End}_A(R))=0$ if the ring R is a finite free A-module (2.1.12). The corollaries (2.1.13, 2.1.14) are used later to show that the deformation functor of a projective module has a trivial tangent space (2.2.5).

Section 2.2 is devoted to the deformation functor $\mathcal{D}\mathrm{ef}(\bar\rho)$ of a ring representation $\bar\rho:R\to\mathrm{End}(V)$ on the category Art_Λ. This basically follows the work of Mazur on deformations of group representations [22]. The deformation functor is pro-representable under the appropriate finiteness condition on R and the tangent space (respectively an obstruction space) is the Hochschild cohomology group $\mathrm{H}^1(R\text{-}R,\mathrm{End}(V))$ (respectively $\mathrm{H}^2(R\text{-}R,\mathrm{End}(V))$). In case $R=\Lambda[G]$, the group algebra of a group G, we recover Mazur's results. In fact it is easy to see that the Hochschild cohomology groups are isomorphic with the usual group cohomology in this case.

In Section 2.3 we study the case of filtrations. The pro-representability result 2.3.2 serves for us primarily as a tool to study the deformations of p-divisible groups later (Chapter 4).

As in the previous chapter, k is an arbitrary field and Λ is a complete Noetherian local ring given with an augmentation $\eta:\Lambda/m_\Lambda\cong k$. Throughout this chapter R denotes a Λ-algebra which is not necessarily commutative.

2.1 Hochschild cohomology

Throughout this section A is a commutative ring and R a not necessarily commutative A-algebra which is finite and free as an A-module. In particular $A\subset R$. Note that $ar=ra$ for all $a\in A\subset R$ and $r\in R$ (by definition of an A-algebra, see [14], p.44). We recall some basic results on the Hochschild cohomology of R-R bimodules. See [14], Section 6.11 for details. We also prove that $\mathrm{H}^1(R\text{-}R,\mathrm{End}_A(R))=0$ and deduce some corollaries, which are going to be used later on.

Definition 2.1.1. An R-R bimodule is an abelian group M together with left and right actions of R (denoted $r \cdot m$ and $m \cdot r$) such that for all $r_1, r_2 \in R$, $m \in M$ and $a \in A$,

$$\begin{aligned} r_1 \cdot (m \cdot r_2) &= (r_1 \cdot m) \cdot r_2 && \text{(actions commute)}\,, \\ a \cdot m &= m \cdot a && \text{(and coincide on } A)\,. \end{aligned}$$

Example 2.1.2. An R-algebra homomorphism $R \to S$ gives an R-R bimodule structure on S via the left and the right multiplication ($r \cdot s = rs, s \cdot r = sr$). In particular, R itself can be considered an R-R bimodule.

Example 2.1.3. If M is a left R-module and N a right R-module, then $M \otimes_A N$ is in a natural way an R-R bimodule.

Example 2.1.4. If M, N are left R-modules, then $\mathrm{Hom}_A(M, N)$ is an R-R bimodule: let $r \cdot f$ and $f \cdot r$ to be $(r \cdot f)(x) = r \cdot f(x)$ and $(f \cdot r)(x) = f(r \cdot x)$. This applies notably to the endomorphism ring of a left R-module.

Definition 2.1.5. A *homomorphism* of R-R bimodules is a homomorphism as abelian groups commuting with both actions. An *exact sequence* is a chain of R-R bimodule homomorphisms which is exact as a sequence of abelian groups (or A-modules).

Remark 2.1.6. To give an R-R bimodule M is equivalent to giving an A-module M together with left R and R^{op} actions. This is equivalent to giving a left $R \otimes_A R^{\mathrm{op}}$ action on M. Hence there is an equivalence of categories

$$\{R\text{-}R \text{ bimodules}\} \;\sim\; \{\text{left } R \otimes_A R^{\mathrm{op}}\text{-modules}\}\,.$$

Definition 2.1.7. Given an R-R bimodule M, let

$$\mathrm{H}^0(R\text{-}R, M) = \{m \in M \mid r \cdot m = m \cdot r, \text{ all } r \in R\}$$

Note that this an A-submodule of M, although *not* in general an R-module.

Remark 2.1.8. If we let R to be an R-R bimodule via the left and the right multiplication, then for any R-R bimodule M we have a canonical isomorphism of A-modules

$$\mathrm{H}^0(R\text{-}R, M) = \mathrm{Hom}_{R\text{-}R}(R, M)$$

In particular (use Remark 2.1.6), the functor $\mathrm{H}^0(R\text{-}R, -)$ is left exact.

Definition 2.1.9. The right derived functors of $\mathrm{H}^0(R\text{-}R, -)$, denoted $\mathrm{H}^n(R\text{-}R, -)$, are called *Hochschild cohomology groups of R with values in M*.

Example 2.1.10. $A=\mathbf{Z}$, $R=\mathbf{Z}[G]$ with a finite group G. If M is a G-module, define an R-R bimodule structure on M by letting G to act naturally on the left and trivially on the right,

$$\begin{aligned} g\cdot m &= {}^g m \\ m\cdot g &= m \end{aligned}$$

and extending by $\mathbf{Z}$-linearity. Then $H^0(R\text{-}R, M)$ becomes the usual 0-th cohomology group,

$$\mathrm{H}^0(R\text{-}R, M) = \{m \in M \mid {}^g m = m, \text{ all } g \in G\} = M^G = \mathrm{H}^0(G, M)\,.$$

Consequently $\mathrm{H}^n(R\text{-}R, M) = \mathrm{H}^n(G, M)$.

Example 2.1.11. One can show that $\mathrm{H}^1(R\text{-}R, M) \cong \mathrm{Z}^1(R\text{-}R, M)/\mathrm{B}^1(R\text{-}R, M)$ with

$$\begin{aligned} \mathrm{Z}^1(R\text{-}R, M) &= \{\alpha \quad \in \mathrm{Hom}_A(R, M) \mid \alpha(r_1 r_2) = r_1\cdot\alpha(r_2) + \alpha(r_1)\cdot r_2\} \\ \mathrm{B}^1(R\text{-}R, M) &= \{\alpha_m \in \mathrm{Hom}_A(R, M) \mid \alpha_m(r) = r\cdot m - m\cdot r, \text{ for some } m \in M\}\,. \end{aligned}$$

Proposition 2.1.12. *Consider R as a left module over itself and define the R-R-bimodule structure on $\mathrm{End}_A(R)$ as in 2.1.4. Then*

$$\mathrm{H}^1(R\text{-}R, \mathrm{End}_A(R)) = 0\,.$$

Proof. An element $r \in R$ acts on R via left multiplication. Thus it defines an element in $\mathrm{End}_A(R)$ which we denote by r_l. Let $\alpha \in \mathrm{Z}^1(R\text{-}R, \mathrm{End}_A(R))$, so

$$\alpha : R \longrightarrow \mathrm{End}_A(R)$$

is an A-module homomorphism, such that

$$\alpha(rs) = r_l\alpha(s) + \alpha(r)s_l\,.$$

We claim that $\alpha \in \mathrm{B}^1(R\text{-}R, \mathrm{End}_A(R))$, so $\alpha = \alpha_\xi$, the coboundary defined by an element $\xi \in \mathrm{End}_A(R)$. Here ξ can be explictly given by

$$\xi(r) = -(\alpha(r))(1)\,,$$

the value of the endomorphism $\alpha(r) \in \mathrm{End}_A(R)$ on $1 \in R$. Indeed, for all $r, s \in R$,

$$\begin{aligned} \alpha_\xi(r)(s) &= (r_l\xi - \xi r_l)(s) \\ &= r_l(\xi(s)) - \xi(r_l(s)) \\ &= -r_l(\alpha(s)(1)) + \alpha(rs)(1) \\ &= -r_l(\alpha(s)(1)) + (r_l\alpha(s))(1) + (\alpha(r)s_l)(1) \\ &= \alpha(r)(s)\,. \end{aligned}$$

Hence $\mathrm{H}^1(R\text{-}R, \mathrm{End}_A(R)) = 0$. ∎

Corollary 2.1.13. *If M is a projective left $R \otimes_A R^{\mathrm{op}}$-module considered as an R-R bimodule, then* $\mathrm{H}^1(R\text{-}R, M) = 0$.

Proof. Since $R \otimes_A R^{\mathrm{op}} \cong \mathrm{End}_A(R)$ as an R-R bimodule, this statement is a reformulation of the above proposition in case M is free of rank 1. For an arbitrary projective M, it follows from the fact that M is a direct summand of a direct sum of free rank 1 modules and the fact that cohomology commutes with direct sums.

Corollary 2.1.14. *If M is a projective left R-module and N a projective right R-module, then* $\mathrm{H}^1(R\text{-}R, M \otimes_A N) = 0$.

2.2 Deforming ring representations

Definition 2.2.1. Let $A \in Art_\Lambda$ and let $\mathcal{V}$ be a finite free A-module. A representation of R on $\mathcal{V}$ is a Λ-algebra homomorphism

$$\wp : R \longrightarrow \mathrm{End}_A(\mathcal{V}) .$$

If $\pi : A \to B$ is a homomorphism in Art_Λ, then $\wp \otimes_A B$ is a representation of R on the B-module $\mathcal{V} \otimes_A B$.

Definition 2.2.2. A representation ρ of R on a finite-dimensional k-vector space V (i.e. in case $A = k$) is called *residual.* Define a *deformation of ρ to $A \in Art_\Lambda$* to be a representation $\wp$ on an A-module $\mathcal{V}$ given together with an isomorphism $i : \wp \otimes_A k \cong \rho$.

Definition 2.2.3. Let $\rho : R \to \mathrm{End}(V)$ be a residual representation. Define the *deformation functor of ρ,*

$$\begin{array}{rcl} \mathcal{D}\mathrm{ef}(\rho) : Art_\Lambda & \longrightarrow & Sets \\ A & \longmapsto & \{\text{deformations of } \rho \text{ to } A\}/\cong \end{array}$$

A representation $\wp : R \to \mathrm{End}_A(\mathcal{V})$ gives an R-R bimodule structure on $\mathrm{End}_A(\mathcal{V})$ via the left and the right multiplication (cf. 2.1.2). The associated Hochschild cohomology groups are responsible for the behaviour of the deformation functor:

Theorem 2.2.4. *Assume R is finitely presented over Λ. Let $\rho : R \to \mathrm{End}(V)$ be a residual representation. Then*

1. $\mathrm{H}^2(R\text{-}R, \mathrm{End}(V))$ *is an obstruction space for* $\mathcal{D}\mathrm{ef}(\rho)$.

2. $\mathrm{H}^1(R\text{-}R, \mathrm{End}(V))$ *is the tangent space of* $\mathcal{D}\mathrm{ef}(\rho)$.

3. $\mathcal{D}\mathrm{ef}(\rho)$ *has a hull.*

Proof. 1. Let $A \twoheadrightarrow A'$ be a surjection with kernel I, such that $m_A I = 0$ Take

$$\wp' : R \longrightarrow \mathrm{End}_{A'}(\mathcal{V}') ,$$

a deformation of ρ to A'. Choose a basis $v_1', \ldots, v_n'$ of $\mathcal{V}'/A'$ and let

$$V = Av_1 + \ldots + Av_n$$

be a finite free A-module (so $\mathcal{V} \otimes_A A' = \mathcal{V}'$). We try to lift ρ' to a Λ-homomorphism $\rho : R \to \mathrm{End}(\mathcal{V})$. Denote for every $r \in R$,

$$\alpha_r' = \rho'(r) \in \mathrm{End}(V') = \mathrm{Mat}_{n\times n}(A') .$$

Choose a basis $\{r_i\}$ for R over Λ and lift each of the α_{r_i}' to an element $\alpha_{r_i} \in \mathrm{End}(\mathcal{V})$. Defining α_r for all $r \in R$ by linearity results in the map of Λ-modules

$$R \xrightarrow{\alpha} \mathrm{End}(\mathcal{V}) .$$

To measure the extent to which α fails to be a ring homomorphism, let

$$\beta_{r,s} = \alpha_{rs} - \alpha_r \alpha_s .$$

When all $\beta_{r,s} = 0$, then $\rho(r) = \alpha_r$ is the required deformation. In general, however,

$$\beta_{r,s} = \ker(\mathrm{End}(\mathcal{V}) \twoheadrightarrow \mathrm{End}(\mathcal{V}')) .$$

By assumption, the kernel I of $A \twoheadrightarrow A'$ can be considered as a k-vector space, so

$$\beta_{r,s} \in \mathrm{End}(V) \otimes_k I .$$

Also, from

$$\begin{aligned} \beta_{rs,t} &= \alpha_{rst} - \alpha_{rs}\alpha_t = \alpha_{rst} - (\alpha_r\alpha_s + \beta_{r,s})\alpha_t \\ \beta_{r,st} &= \alpha_{rst} - \alpha_r\alpha_{st} = \alpha_{rst} - \alpha_r(\alpha_s\alpha_t + \beta_{s,t}) \end{aligned}$$

it follows that

$$\beta_{rs,t} - \beta_{r,st} = \alpha_r\beta_{s,t} - \beta_{r,s}\alpha_t = (\bar{\rho}(r)\otimes 1)\beta_{s,t} - \beta_{r,s}(\bar{\rho}(t)\otimes 1) .$$

Hence β is an element of $\mathrm{Z}^2(R\text{-}R, \mathrm{End}(V)) \otimes_k I$. Replacing α_{r_i} by different lifts $\tilde{\alpha}_{r_i}$ of α_{r_i}' changes $\beta_{r,s}$ by an element in $\mathrm{B}^2(R\text{-}R, \mathrm{End}(V)) \otimes_k I$,

$$\tilde{\alpha}_r = \alpha_r + m_r \quad \Rightarrow \quad \tilde{\beta}_{r,s} = \beta_{r,s} + \Big(m_{rs} - (\bar{\rho}(r)\otimes 1)m_s - m_r(\bar{\rho}(s)\otimes 1)\Big) .$$

Thus, the obstruction to deforming ρ' to A lies in $\mathrm{H}^2(R\text{-}R, \mathrm{End}(V)) \otimes_k I$. Since our construction is clearly functorial (Definition 1.3.4), the vector space $\mathrm{H}^2(R\text{-}R, \mathrm{End}(V))$ is an obstruction space for $\mathcal{D}\mathrm{ef}(\rho)$.

2. Let $A=k[I]\to k=A'$ for some k-vector space I. In this case there is a section $A'\to A$, so there is a canonical deformation $\wp=\rho\otimes_k k[I]$. It is given by $\alpha_r=\alpha'_r$. Any other deformation is given by

$$\tilde{\alpha}_r=\alpha_r+w_r,\quad w_r\in \mathrm{M}_n(k)\otimes_k I$$

The condition $\tilde{\alpha}_{rs}=\tilde{\alpha}_r\tilde{\alpha}_s$ yields (as in 2.)

$$m_{rs}=\rho(r)m_s+m_r\rho(s)\ .$$

Hence $m\in Z^1(R\text{-}R,\mathrm{End}(V))\otimes_k I$. Moreover, $m^{(1)}$ and $m^{(2)}$ give isomorphic deformations if and only if there is a basis transformation $Q\in\mathrm{M}_n(I)$ which transforms one into the other. This implies that

$$m^{(2)}(r)=m^{(1)}(r)+(\rho(r)Q-Q\rho(r)),$$

i.e. $m^{(2)}-m^{(1)}\in B^1(R\text{-}R,\mathrm{End}(V))\otimes_k I$. It follows that $\mathrm{H}^1(R\text{-}R,\mathrm{End}(V))$ is tangent space of the functor $\mathcal{D}\mathrm{ef}(\rho)$.
3. We have already shown that $\mathcal{D}\mathrm{ef}(\rho)$ has a tangent space. The idea is that one can rigidify $\mathcal{D}\mathrm{ef}(\rho)$ by fixing a basis of the module. This yields a pro-representable functor $\mathcal{R}$ of which $\mathcal{D}\mathrm{ef}(\rho)$ is a quotient by a $\widehat{GL}_n$-action. Choose a basis $\{\bar{v}_1,...,\bar{v}_n\}$ of V and consider the finite free Λ-module

$$\mathcal{V}_\Lambda=\Lambda v_1+\ldots+\Lambda v_n$$

with an identification $\mathcal{V}_\Lambda\otimes_\Lambda k=V$ given by $v_i\mapsto\bar{v}_i$. For $A\in Art_\Lambda$ let

$$\mathcal{R}(A)=\{\varphi\in\mathrm{Hom}(R,\mathrm{End}(\mathcal{V}_\Lambda\otimes_\Lambda A)\mid\varphi\otimes_A k=\rho\}\ .$$

Here Hom denotes homomorphisms of (non-commutative) Λ-algebras. This gives a functor $\mathcal{R}:Art_\Lambda\to Sets$. Let

$$\widehat{GL}_n(A)=\ker\Big(GL_n(A)\to GL_n(k)\Big)$$

This gives a pro-representable group functor $\widehat{GL}_n$, smooth on n^2 parameters. If we let $\widehat{GL}_n(A)$ act on $\mathrm{End}(\mathcal{V}_\Lambda\otimes_\Lambda A)$ by conjugation, then clearly $\mathcal{D}\mathrm{ef}(\rho)=\mathcal{R}/\widehat{GL}_n$. So, by theorem 1.7.2, it suffices to show that $\mathcal{R}$ is pro-representable. Let $\{x_1,...,x_m\}$ be the set of generators of R over Λ. Then $\varphi\in\mathcal{R}(A)$ is determined by $\varphi(x_i)\in\mathrm{Mat}_{n\times n}(A)$. Here the isomorphism $\mathrm{End}(\mathcal{V}_\Lambda\otimes_\Lambda A)\cong\mathrm{Mat}_{n\times n}(A)$ is fixed by the choice of the v_i. In other words φ is determined by the coefficients $\alpha_{ijk}\in m_A$ of $\varphi(x_i)$ in the basis $\{v_j\otimes v_k^*\}$ of $\mathrm{End}(\mathcal{V})$. It follows that

$$\mathcal{R}\ \cong\ \mathrm{Hom}(\Lambda[[t_{ijk}]]/J,-)$$

where J is the ideal generated by the relations among the x_i's in R. Hence $\mathcal{R}$ is pro-representable and $\mathcal{D}\mathrm{ef}(\rho)$ has a hull. ∎

Corollary 2.2.5. *Assume that R is finite and free as a Λ-module. Let $\mathcal{V}_\Lambda$ be a projective R-module and $\mathcal{W}_\Lambda$ any R-module which is finite and free over Λ. Then*

$$\mathcal{V}_\Lambda \cong \mathcal{W}_\Lambda \quad \Longleftrightarrow \quad \mathcal{V}_\Lambda \otimes_\Lambda k \cong \mathcal{W}_\Lambda \otimes_\Lambda k\,.$$

Proof. The implication from left to right is trivial. Conversely, assume that $\mathcal{V}_\Lambda \otimes_\Lambda k \cong \mathcal{W}_\Lambda \otimes_\Lambda k$. Then $\mathcal{V}_\Lambda$ and $\mathcal{W}_\Lambda$ are two deformations of the same residual representation. Since $\mathcal{V}_\Lambda$ is R-projective, $V_k = \mathcal{V}_\Lambda \otimes_\Lambda k$ is projective over $R_k = R \otimes_\Lambda k$. By 2.1.13 we have

$$\mathrm{H}^1(R\text{-}R\,, \mathrm{End}(V_k)) \cong \mathrm{H}^1(R_k\text{-}R_k, \mathrm{End}(V_k)) = 0\,.$$

By the above theorem, the deformation functor $\mathcal{D}\mathrm{ef}(\rho : R_A \to \mathrm{End}(V_k))$ has trivial tangent space. It follows that every two deformations of V_k to $A \in Art_\Lambda$ are isomorphic. Thus $\mathcal{V}_\Lambda \cong \mathcal{W}_\Lambda$. ∎

2.3 Deforming filtrations on R-modules

Let $\Lambda \twoheadrightarrow k$ and R be as before. As the proof of Theorem 2.2.4 shows, one way to rigidify the deformation functor of a ring representation $\rho : R \to \mathrm{End}(V)$ is to fix a basis of liftings of V. As we will see in Section 4.4, in case R is a finite free Λ-module, another way is to represent ρ as a quotient representation of a "free representation of R". In this section we study deformations of an R-stable filtration, which are directly related to quotient representations.

The following defines such a deformation functor in general.

Notation. Let $\wp : R \to \mathrm{End}(\mathcal{M})$ be a fixed representation of R on a finite free Λ-module $\mathcal{M}$. Denote $M = \mathcal{M} \otimes_\Lambda k$, $\rho = \wp \otimes_\Lambda k$ and let

$$V \subset M$$

be an R-stable submodule, i.e. a subrepresentation of ρ.

Definition 2.3.1. For $A \in Art_\Lambda$ denote $\mathcal{M}_A = \mathcal{M} \otimes_\Lambda A$. Let

$$\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)(A) = \{\mathcal{V}_A \subset \mathcal{M}_A \mid \mathcal{V}_A \otimes_A k = V\}\,,$$

the set of direct A-submodules $\mathcal{V}_A$ deforming V in $\mathcal{M}$, such that $\mathcal{V}_A$ is R-stable. We call $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)$ *the deformation functor of V in $\mathcal{M}$*. Note that this functor depends on the R-module structure of $\mathcal{M}$, rather than just on V and M.

In the following theorem we consider the k-vector space $\mathrm{Hom}_k(V, M/V)$ an R-R bimodule via 2.1.8.

Theorem 2.3.2. *Assume R is finitely generated over Λ. Then*

1. *$\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)$ is pro-representable.*

2. *The tangent space of* $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V{\subset}M, R)$ *is* $\mathrm{H}^0(R\text{-}R\,, \mathrm{Hom}_k(V, M/V))$.

3. $\mathrm{H}^1(R\text{-}R\,, \mathrm{Hom}_k(V, M/V))$ *is an obstruction space for* $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V{\subset}M, R)$.

Proof. 1.

Let $\{e_1, ..., e_n, f_1, ..., f_m\}$ be a basis of $\mathcal{M}$ over Λ which reduces to bases of V and M/V. This also gives a basis $\{e_1^A, ..., e_n^A, f_1^A, ..., f_m^A\}$ of $\mathcal{M}_A$ for any $A \in Art_\Lambda$. It is easy to see that any filtration $\mathcal{V}_A \subset \mathcal{M}_A$ which deforms $V \subset M$ has a unique basis of the form

$$\begin{array}{lllll} e_1^A & & + \, u_{11} f_1^A + \cdots + u_{m1} f_m^A \\ & e_2^A & + \, u_{12} f_1^A + \cdots + u_{m2} f_m^A \\ & & \vdots \\ & & e_n^A + u_{1n} f_1^A + \cdots + u_{mn} f_m^A \end{array} \tag{9}$$

with $u_{ij} \in m_A$. Incidentally, this shows that the functor $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V{\subset}M)$ of all (not necessary R-stable) deformations of V in $\mathcal{M}$ is pro-represented by $\Lambda[[t_{ij}]]$ with $1 \le i \le n, 1 \le j \le m$. Clearly $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V{\subset}M, R) \subset \mathcal{D}\mathrm{ef}_{\mathcal{M}}(V{\subset}M)$ is a subfunctor. To show that it is indeed pro-represented by a quotient of $\Lambda[[t_{ij}]]$, we describe explicitly the equations. This computation will be used in chapter 5.

Take $A \in Art_\Lambda$ and a filtration $\mathcal{V}_A \subset \mathcal{M}_A$, described by (9). We put the coefficients u_{ij} into an $n \times m$ matrix U. Thus the basis elements (9) make columns of the block matrix

$$\begin{pmatrix} I \\ U \end{pmatrix}$$

where I denotes the identity matrix ($n \times n$ in this case).

The action of an element $r \in R$ on $\mathcal{M}$ can be described by a block matrix

$$r \quad \mapsto \quad \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \in \mathrm{End}_\Lambda(\mathcal{M})$$

in the basis $\{e_1, ..., e_n, f_1, ..., f_m\}$. The condition that r maps the filtration $\mathcal{V}$ into itself is given by

$$\begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \begin{pmatrix} I \\ U \end{pmatrix} = \begin{pmatrix} I \\ U \end{pmatrix} N, \qquad \text{for some } N \in \mathrm{Mat}_{n\times n}(A)$$

This gives two matrix equations, from which we eliminate N and get

$$UA_r + UB_rU - D_rU - C_r = 0, \quad 1 \le i \le k\,. \tag{10}$$

Note that this equation can be also written in a matrix form,

$$\begin{pmatrix} U & -I \end{pmatrix} \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \begin{pmatrix} I \\ U \end{pmatrix} = 0\,. \tag{11}$$

Let $\{r_1, ..., r_k\}$ be a set of generators of R as a Λ-algebra. Then the conditions (10) for $r = r_1, ..., r_k$ are necessary and sufficient for the R-stability of $\mathcal{V}$. Replacing u_{ij} by indeterminants t_{ij} yields exactly the equations (nmk of them) which determine $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)$. Hence this functor is pro-representable. The pro-representing ring is $\Lambda[[t_{ij}]]/J$ where J is the ideal generated by the above equations.

2. Let $A = k[\epsilon]$. Following the above reasoning, an R-stable filtration $\mathcal{V}_A \subset \mathcal{M}_A$ which deforms V is given by a matrix U for which the equations (10) hold. Since the entries of U lie in m_A and $m_A^2 = 0$, the term UB_iU vanishes. Also the fact that the original filtration V is R-stable implies $C_i = 0$. So the equations read

$$UA_r - D_rU = 0, \quad r \in R\,.$$

Write $U = \epsilon \bar{U}$ with $\bar{U} \in \mathrm{Mat}_{n\times m}(k)$. Then this equation can be re-written as

$$\bar{U} \cdot r - r \cdot \bar{U} = 0, \quad r \in R\,.$$

Here $\bar{U}$ is considered as an element in $\mathrm{Hom}_k(V, M/V)$ and $r\cdot$ and $\cdot r$ denote the left and the right action of r on this vector space. This identifies the tangent space of $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)$ with $\mathrm{H}^0(R\text{-}R, \mathrm{Hom}_k(V, M/V))$.

3. To simplify the notation, we let $\pi : A \twoheadrightarrow A'$ be a small extension, i.e. assume $\ker \pi \cong k$ as a Λ-module. Let $\mathcal{V}' \in \mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)(A')$ be an R-stable filtration of $\mathcal{M}_{A'}$. We try to deform it to an R-stable filtration of $\mathcal{M}_A$.

Let $\mathcal{V}$ be any filtration of $\mathcal{M}$ which deforms $\mathcal{V}'$. Denote by U and U' the matrices defining $\mathcal{V}$ and $\mathcal{V}'$. To measure the failure of $\mathcal{V}$ being R-stable, consider

$$UA_r + UB_rU - D_rU - C_r = \epsilon E_r \;\in\; \epsilon\cdot \mathrm{Mat}_{n\times m}(k)\,.$$

Here (ϵ) is the kernel of $A \twoheadrightarrow A'$. Consider again $\mathrm{Mat}_{n\times m}(k) = \mathrm{Hom}_k(V, M/V)$ an an R-R bimodule. Then a direct computation shows

$$\begin{aligned}
\epsilon E_{rs} &= (U\;\; {-I}) \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix} \begin{pmatrix} I \\ U \end{pmatrix} \\
&= (U\;\; {-I}) \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix} \begin{pmatrix} I \\ U \end{pmatrix} \\
&= (U\;\; {-I}) \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \left[\begin{pmatrix} I \\ U \end{pmatrix} (I\;\; 0) + \begin{pmatrix} 0 \\ -I \end{pmatrix} (U\;\; {-I}) \right] \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix} \begin{pmatrix} I \\ U \end{pmatrix} \qquad (12) \\
&= \epsilon E_r (A_s + B_s U) + (-UB_r + D_r)\epsilon E_s \\
&= \epsilon E_r A_s + D_r \epsilon E_s \\
&= \epsilon E_r \cdot s + r \cdot \epsilon E_s \quad .
\end{aligned}$$

The last but one equality uses the fact that the maximal ideal of A is annihilated by ϵ, thus $\epsilon U = 0$. So $E : r \mapsto E_r$ is a 1-cocycle for the R-R bimodule cohomology with coefficients in $\mathrm{Hom}_k(V, M/V)$. Also note that $E_r = 0$ for all $r \in R$ if and only if the chosen filtration $\mathcal{V}$ is R-stable.

If we change the filtration $\mathcal{V}$ by a different lifting $\tilde{\mathcal{V}}$ of $\mathcal{V}'$, then we can write

$$\tilde{U} = U + N, \quad N \in \epsilon\,\mathrm{Mat}_{n\times m}(k)\,.$$

Then the relation between the cocycles $\tilde{E}$ and E is given by

$$\tilde{E}_r = E_r + (r \cdot N - N \cdot r)\,.$$

Hence a different choice of $\mathcal{V}$ changes the cocycle E by a 1-coboundary. Thus the obstruction to the existence of an R-stable filtration $\mathcal{V}$ lies in the cohomology group $\mathrm{H}^1(R\text{-}R\,, \mathrm{Hom}(V, M/V))$, as asserted.

If $A \twoheadrightarrow A'$ is an arbitrary small surjection with kernel I, an identical argument (everything has to be tensored with I) shows that the corresponding obstruction lies in $\mathrm{H}^1(R\text{-}R\,, \mathrm{Hom}(V, M/V)) \otimes_k I$. Moreover, our construction is clearly functorial in the sense of Definition 1.3.4. Hence $\mathrm{H}^1(R\text{-}R\,, \mathrm{Hom}(V, M/V))$ is an obstruction space for $\mathcal{D}\mathrm{ef}_{\mathcal{M}}(V \subset M, R)$, as asserted. ∎