1 Infinitesimal deformation theory

In this chapter we study infinitesimal deformation theory, that is, properties of (covariant) functors of Artin rings. In our applications later, these will be deformation functors of some kind. Let k be an arbitrary ground field and fix a complete Noetherian local ring Λ with $\Lambda/m_\Lambda \cong k$. Following Schlessinger [35], we work on the category Art_Λ of Artinian local Λ-algebras A given together with an isomorphism $A/m_A \cong k$.

There are several points which make infinitesimal deformation theory usually far more accessible than a general moduli study on the full category of rings.

First, any surjection in Art_Λ can be split into a finite sequence of *small surjections*. A surjection $\pi : A \twoheadrightarrow A'$ is small if m_A annihilates $I = \ker \pi$. In this case I is a finite-dimensional k-vector space. So any ring $A \in Art_\Lambda$, however singular and complicated, can be obtained from the ground field k by a finite sequence of extensions by k-vector spaces. This often allows to reduce some questions in the study of (difficult) deformation functors to (hopefully simpler) linear algebra. Consider, for example, a pro-representable functor $\mathcal{F} : Art_\Lambda \to Sets$, and take an element $\xi' \in \mathcal{F}(A')$. Then the size of a fiber of the map $\mathcal{F}(A) \to \mathcal{F}(A')$ above ξ' is controlled by two finite-dimensional k-vector spaces, the obstruction space $O\mathcal{F}$ and the tangent space $T\mathcal{F}$. If $\mathcal{F}$ is a deformation functor of some kind, these are usually some kind of cohomology groups. In practice they can often be determined, yielding some amount of information about the functor in question.

Second, another attractive characteristic of working on Art_Λ is the simple nature of formal smoothness. While there exist plenty of smooth morphisms on the category of rings (or schemes), the analogous infinitesimal notion of formal smoothness is far more restrictive. In fact, any formally smooth natural transformation of pro-representable functors $\mathcal{F} \to \mathcal{G}$ is given in terms of the pro-representing rings by

$$G \longrightarrow G[[t_1, \ldots, t_n]] \cong F$$

for some $n \geq 0$. In particular the only formally smooth pro-representable functors on Art_Λ are the ones whose pro-representing ring is isomorphic to $\Lambda[[t_1, \ldots, t_n]]$ for some n.

Third useful feature of Art_Λ is that it is usually quite easy to determine whether a functor is pro-representable. This is again in constrast with the difficulties of solving the analogous representability questions on the category of rings. Schlessinger's theorem ([35], Theorem 2.11) asserts that $\mathcal{F} : Art_\Lambda \to Sets$ is pro-representable if and only if $\mathcal{F}$ commutes with fibre products,

$$\mathcal{F}(A \times_B C) = \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C) \tag{3}$$

plus $\mathcal{F}(k)$ consists of one point and the tangent space $T\mathcal{F}$ is finite-dimensional. Moreover, it is enough to test (3) when, say, $C \to B$ is a small surjection. This gives a practically effective criterion to show that a functor is pro-representable.

It should be noted, however, that not all deformation problems give rise to functors which are pro-representable. For example, very often one is led to study the functors which can be represented as quotients of a pro-representable functor by an action of a formal group, such as $\widehat{GL}_n$ for some n. These are not in general pro-representable, although they do have a weaker property of possessing a hull. Hence it is natural to ask whether the three points mentioned above generalize to a larger class of functors than just that of the pro-representable ones. Roughly speaking, the goal of this chapter is to give some answers to this question.

More precisely, our aim is threefold:

First, we axiomatize the notion of an obstruction space for an arbitrary covariant functor $\mathcal{F}: Art_\Lambda \to Sets$ (Section 1.3). This follows the ideas of Artin ([1], 2.6). We show (1.3.8) that the minimal obstruction space $O\mathcal{F}$ exists when $\mathcal{F}$ commutes with products,

$$\mathcal{F}(A \times_k B) \xrightarrow{\sim} \mathcal{F}(A) \times \mathcal{F}(B)\,.$$

This condition is satisfied for most of the deformation functors which occur in practice, since those can be usually represented as a quotient of a pro-representable functor by a formal group action (1.7.3). In the studies of concrete deformation functors, the technical point of the existence (and functoriality etc.) of an obstruction space is often ignored. Note that very similar results to those presented here have been obtained recently by Fantechi and Manetti [9].

Our second object of study is formally smooth natural transformations $\mathcal{F} \to \mathcal{D}$ where $\mathcal{D}$ is not necessarily pro-representable. More precisely, given a diagram

$$\mathrm{Hom}_\Lambda(F_1, -) = \mathcal{F}_1 \quad \xrightarrow{f} \quad \mathcal{D} \quad \xleftarrow{g} \quad \mathcal{F}_2 = \mathrm{Hom}_\Lambda(F_2, -),$$

with f, g formally smooth, we ask ourselves how are F_1 and F_2 related. For example, if $\mathcal{D}$ is pro-representable, it is clear that one of the rings F_1, F_2 is a formal power series ring over the other. The same statement holds if $\mathcal{D}$ is only assumed to have a tangent space (1.5.5). More generally, for an arbitrary $\mathcal{D}$ we prove a comparison theorem (1.5.3) which relates F_1 and F_2 purely from the tangent space information. This comparison theorem serves as a main tool for our study of deformation functors of p-divisible groups in Chapter 4.

The third part of this chapter addresses a question whether a given functor can be written as a quotient of a pro-representable one by a group action. If Γ is a group which acts on a pro-representable functor $\mathcal{F}$, it is easy to determine whether the question $\mathcal{F}/\Gamma$ has a hull (1.6.2). Conversely, if $\mathcal{D}: Art_\Lambda \to Sets$ has a hull $\mathcal{F} \to \mathcal{D}$, we show that $\mathcal{D}$ can be represented as $\mathcal{F}/\Gamma$ for some Γ if and only if the natural map

$$\mathcal{D}(A \times_B C) \longrightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$$

is surjective for all $A \to B \leftarrow C$ in Art_Λ(1.6.3). We also conjecture the analogous criterion for quotients by a formal group action (1.7.5).

To keep the presentation self-contained, we recall the basic facts about the category Art_Λ and Schlessinger's criterion (Sections 1.1, 1.2, 1.4).

1.1 Artinian local algebras

Let Λ be a complete Noetherian local ring with residue field k and fix an augmentation isomorphism $\eta_\Lambda : \Lambda/m_\Lambda \to k$. In practice one often has either $\Lambda = k$ (equal characteristics case) or k perfect of positive characteristic and Λ the ring of Witt vectors of k.

Definition 1.1.1. The category Art_Λ consists of Artinian local Λ-algebras A together with an augmentation isomorphism $\eta_A : A/m_A \cong k$. Morphisms in the category are local homomorphisms of Λ-algebras, commuting with the augmentation. The set of such homomorphisms is denoted $\mathrm{Hom}_\Lambda(F,G)$.

Remark. Note that Art_Λ has a final object (k with $\eta_k = \mathrm{id}$). Also note that every surjection $A \twoheadrightarrow A'$ in the category has a nilpotent kernel, so it can be split into a sequence

$$A = A_n \twoheadrightarrow A_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow A_1 \twoheadrightarrow A_0 = A'$$

of small surjections in the sense of [31]:

Definition. A *small surjection* (sometimes called an infinitesimal extension) is a morphism $\pi : A \twoheadrightarrow A'$ in Art_Λ such that $I = \ker \pi$ satisfies $m_A I = 0$.

Remark. The kernel I of a small surjection is a module over $A/m_A = k$. Hence it is a (finite-dimensional) k-vector space. Schlessinger's *small extension* ([35], 1.2) is a small surjection with an additional property that this vector space is one-dimensional. A small surjection (and, hence, any surjection in Art_Λ) can be split into a sequence of small extensions.

To study representability questions, one extends Art_Λ to a larger category $\widehat{Art}_\Lambda$, of which Art_Λ is a full subcategory. The following well-known lemma (cf. [3], Chap. 9, §2, No. 5, Lemme 3b) characterizes the rings of $\widehat{Art}_\Lambda$.

Lemma 1.1.2. *Let F be a complete local Λ-algebra with $F/m_F \cong k$. Then the following conditions are equivalent.*

1. *The vector space $m_F/(m_F^2 + m_\Lambda F)$ is finite-dimensional.*
2. *The ring F is Noetherian.*
3. *For all $n \geq 1$ we have $F/m_F^n \in Art_\Lambda$.*
4. *The algebra F is isomorphic to one of the form $\Lambda[[t_1, ..., t_n]]/J$.*

Proof.
$2 \Rightarrow 1$. If $m_F = (a_1, ..., a_n)F$, then every $x \in m_F$ can be written $x = r_1 a_1 + \cdots + r_n a_n$. Taking r_i modulo m_F and a_i modulo m_F^2, we see that a_i form generators for the F/m_F-vector space $m_F/(m_F^2 + m_\Lambda F)$.
$4 \Rightarrow 2$. Since Λ is Noetherian, $\Lambda[[t_1, ..., t_n]]$ is Noetherian as well.

$1 \Rightarrow 4$. Let $a_1, ..., a_n$ be representatives of a basis for $m_F/(m_F^2 + m_\Lambda F)$. Define a Λ-algebra homomorphism $\Lambda[[t_1, ..., t_n]] \to F$ by letting $t_i \mapsto a_i$. We claim that it is surjective. In other words, for every $x \in F$ there is $f \in \Lambda[[t_1, ..., t_n]]$ with $f(a_1, ..., a_n) = x$. To prove this, we construct inductively a compatible system $f_k \in \Lambda[[t_1, ..., t_n]]$ with total degree of f_k at most k and such that $f_k(a_1, ..., a_n) \equiv x \bmod m_F^{k+1}$. The constant f_0 exists since $\Lambda/m_\Lambda \to F/m_F$ is an isomorphism.

Now assume that f_{k-1} is constructed. Let $y = f_{k-1}(a_1, ..., a_n) - x \in m_F^k$. Firstly, the multiplication map

$$(m_F/m_F^2) \otimes_k \cdots \otimes_k (m_F/m_F^2) \longrightarrow m_F^k/m_F^{k+1}$$

is surjective (by definition of m_F^k). Secondly, m_F/m_F^2 is generated, as a k-vector space, by the a_i and the image of Λ. So there is a homogeneous polynomial $g_k(t_1, ..., t_n)$ of degree k with coefficients in Λ such that $g_k(a_1, ..., a_n) \equiv y \bmod m_F^{k+1}$. Here we again use that the composition $\Lambda \to F \to k$ is surjective. Now $f_k = f_{k-1} + g_k$ satisfies the required property.

$3 \Rightarrow 1$. Use that F/m_F^2 (and hence m_F/m_F^2) has finite length as a Λ-module.

$2 \Rightarrow 3$. The ring F/m_F^n is Noetherian, local and its maximal ideal is nilpotent. It follows that F/m_F^n is Artinian ([26], 9.1). ▪

Definition 1.1.3. The category $\widehat{Art}_\Lambda$ consists of Noetherian local Λ-algebras A given together with an augmentation isomorphism $\eta_A : A/m_A \cong k$. Morphisms in the category are local homomorphisms of Λ-algebras, commuting with the augmentation. Again we denote by $\mathrm{Hom}_\Lambda(F, G)$ the set of such homomorphisms.

Remark. Our Art_Λ is Schlessinger's C_Λ and our $\widehat{Art}_\Lambda$ is $\hat{C}_\Lambda$ ([35], 1). Note that by the above lemma the condition that A is Noetherian in Schlessinger's definition of $\hat{C}_\Lambda$ can be removed, since it follows from the other assumptions.

1.2 Pro-representable functors

This subsection describes the basic properties of pro-representable functors $\mathcal{F} : Art_\Lambda \to Sets$. We define the obstruction space (1.2.4) and show how the behaviour of $\mathcal{F}$ under small surjections is determined by the tangent and the obstruction space (1.2.7). All results presented here are well-known, but we recall them to keep the presentation self-contained and due to the lack of suitable reference.

Remark. Let $\mathcal{F} : Art_\Lambda \to Sets$ be a covariant functor. Then $\mathcal{F}$ can be canonically extended to a functor $\widehat{Art}_\Lambda \to Sets$ by letting

$$\mathcal{F}(G) = \varprojlim \mathcal{F}(G/m_G^n), \qquad G \in \widehat{Art}_\Lambda\,,$$

and similarly for morphisms.

Definition 1.2.1. A covariant functor $\mathcal{F}: Art_\Lambda \to Sets$ is said to be *pro-representable* if the extended functor on $\widehat{Art}_\Lambda$ is representable. In other words, $\mathcal{F}$ is pro-representable if there is a complete Noetherian local Λ-algebra F with $F/m_A = k$ and

$$\mathcal{F}(A) = \mathrm{Hom}_\Lambda(F, A)\ , \qquad A \in Art_\Lambda\ ,$$

functorially in A. We will usually denote the pro-representing ring by the corresponding Latin letter.

Definition 1.2.2. For a complete Noetherian local Λ-algebra F with an augmentation, define *the tangent space of F over* Λ to be the k-vector space

$$TF = \left(\frac{m_F}{m_F^2 + m_\Lambda F}\right)^* .$$

Here $*$ denotes k-linear dual. Equivalently, $TF = \mathrm{Der}_\Lambda(F, k)$, the set of Λ-linear derivations of F into k ([35], 1.0).

Remark 1.2.3. A homomorphism $\alpha: F \to G$ induces a k-linear map $d\alpha: TG \to TF$. It is easy to show that α is surjective if and only if $d\alpha$ is injective ([35], Lemma 1.1). Note also that TF is finite-dimensional by Lemma 1.1.2.

Definition 1.2.4. Let $F \in \widehat{Art}_\Lambda$. Let $n = \dim TF$ and write $F = S/J$ with $S = \Lambda[[t_1, \ldots, t_n]]$. This is possible by the proof of $(1 \Rightarrow 4)$ of Lemma 1.1.2. Define *the obstruction space OF of F over* Λ to be the k-vector space

$$OF = (J/m_S J)^* .$$

Here $*$ denotes k-linear dual.

Remark 1.2.5. It is easy to show that OF does not depend on the choice of a representation of F as S/J. Moreover, OF is contravariantly functorial in F. Note also that $O(F) = 0$ if and only if F is a power series ring over Λ. It is also clear that an inclusion $F \to F[[t_1, ...t_m]]$ induces an isomorphism on the obstruction spaces.

Remark 1.2.6. If F is a complete Noetherian local ring and M is an F-module, then $x_1, ..., x_n \in M$ generate M if and only if their residue classes generate $M/m_F M$ as a F/m_F-vector space ([26], 5.1). In particular, J is generated by $\dim_k OF$ elements. So $\dim_k TF$ is the smallest number of generators of F as a complete Λ-algebra and $\dim_k OF$ is the smallest number of relations.

The following theorem describes the behaviour of $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$ under a small surjection $A \twoheadrightarrow A'$ with kernel I. The vector space $OF \otimes_k I$ contains the obstruction elements to lifting points of $\mathcal{F}$ under a small extension with kernel I. The space $TF \otimes_k I$ measures how many liftings there are, provided the obstruction is zero.

Theorem 1.2.7. *Let $\mathcal{F} \cong \mathrm{Hom}_\Lambda(F,-)$ be a pro-representable functor from Art_Λ to Sets. Let $\pi : A \twoheadrightarrow A'$ be a small surjection in Art_Λ with kernel I. Take $\xi' \in \mathcal{F}(A')$. Then*

1. *There exists an element $\Theta \in OF \otimes_k I$ whose vanishing is necessary and sufficient for the existence of $\xi \in \mathcal{F}(A)$ such that $\pi(\xi) = \xi'$.*
2. *The obstruction element is functorial: assume given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\ A' & \xrightarrow{\varphi'} & B' \end{array} .$$

   where $B \twoheadrightarrow B'$ is a small surjection with kernel K. Then the obstruction element $\Theta \in OF \otimes_k I$ of $(A \twoheadrightarrow A', \xi')$ is related to the corresponding obstruction element $\Sigma \in OF \otimes_k K$ of $(B \twoheadrightarrow B', \varphi'(\xi'))$ by the formula $\Sigma = (1 \otimes \varphi)\Theta$.

3. *If $\Theta = 0$, then the set of all $\xi \in \mathcal{F}(A)$ with $\pi(\xi) = \xi'$ is a principal homogeneous space under $TF \otimes_k I$.*

Proof. We consider ξ' as a homomorphism $F \to A'$. Choose a surjection $p : S = \Lambda[[t_1, ..., t_n]] \to F$ with kernel J as in Definiton 1.2.4.

1. Define $a_i' \in m_{A'}$ to be the images of t_i under the composite morphism $\xi f_0 : S \to F \to A$. In order to lift ξ' to a homomorphism $\xi : F \to A$, choose arbitrary $a_i \in m_A$ such that $\pi(a_i) = a_i'$. This defines a Λ-homomorphism $\alpha : S \to A$ by letting $t_i \mapsto a_i$.

If $\ker \alpha \supset \ker f_0$, then α descends to a $\xi : F \to A$. In general, however, $\alpha : J \to I$ is non-zero. In any case, α vanishes on $m_S J$, and hence descends to an element $\Theta \in OF \otimes_k I$. Recall that I has a structure of a $A/m_A = k$-vector space by the assumption that $m_A I = 0$. The element Θ does not depend on the choice of a_i. Indeed, a different choice $\tilde{a}_i = a_i + \epsilon_i$ with $\epsilon_i \in I$ gives a map $\tilde{\alpha}$ which is the same on J. This follows from the fact that $I m_A = 0$ (so $a_i a_j = \tilde{a}_i \tilde{a}_j$ etc.) and $J \subset m_S^2 + m_\Lambda S$. The element $\Theta \in OF \otimes_k I$ is the required obstruction.

2. Immediate from the construction.

3. Let $\xi, \tilde{\xi} : F \to A$ be two liftings of $\xi' : F \to A'$. Consider the homomorphism (of Λ-modules) $t = \tilde{\xi} - \xi : F \to A$. Then $\mathrm{Im}\, t \subset I$ and $t(m_\Lambda F) = t(m_F^2) = 0$, since $I m_A = 0$. So $t \in T\mathcal{F} \otimes_k I$. Conversely, given ξ and $t \in T\mathcal{F} \otimes_k I$, the Λ-module map $\tilde{\xi} = \xi + t$ is easily verified to be a Λ-algebra homomorphism $F \to A$. ∎

Remark. In practice, given a functor $\mathcal{F}$, one can often prove that $\mathcal{F}$ is pro-representable (e.g. using Schlessinger's criterion, see Theorem 1.4.3). To determine the pro-representing ring F of $\mathcal{F}$ is, however, generally much harder. It is often possible, though, to determine TF and some vector space V containing OF in terms of $\mathcal{F}$ itself. In some cases, for example if $V = 0$ (and hence $OF = 0$), this suffices to determine the ring F. Otherwise, one has at least the following dimension estimate.

Lemma 1.2.8. *For any* $F \in \widehat{Art}_\Lambda$,

$$\dim \Lambda + \dim_k TF - \dim_k OF \;\leq\; \dim F \;\leq\; \dim \Lambda + \dim_k TF\,. \tag{4}$$

Proof. A Noetherian local ring has finite (Krull) dimension ([26], 9.4–9.6), so all the terms of (4) are finite. The second inequalitity follows from the fact that F can be written as $\Lambda[[t_1,...,t_n]]/J$ with $n = \dim_k TF$ (cf. 1.1.2). For the first inequality, use that J is generated by $\dim_k OF$ elements (by 1.2.6) and use ([26], 9.7)

$$\dim F/(x) \;\leq\; \dim F \;\leq\; \dim F/(x) + 1, \qquad x \in m_F\,.$$

This proves the lemma. ∎

Finally, let us recall the notion of formal smoothness:

Definition 1.2.9. A natural transformation of functors $\mathcal{F} \to \mathcal{D}$ is said to be *formally smooth* if for every surjection $A \twoheadrightarrow A'$ in Art_Λ, the natural map

$$\mathcal{F}(A) \longrightarrow \mathcal{F}(A') \times_{\mathcal{D}(A')} \mathcal{D}(A)$$

is surjective.

Remark 1.2.10. If both $\mathcal{F}$ and $\mathcal{D}$ are pro-representable, then the formal smoothness of $\mathcal{F} \to \mathcal{D}$ is equivalent to the fact that the corresponding map $D \to F$ of Λ-algebras makes F into a formal power series over D,

$$D \longrightarrow F \cong D[[t_1, \ldots, t_n]]\,.$$

See e.g. [35], Proposition 2.5(i). We will show later (1.5.3) that more generally, whenever $\mathcal{F}_1, \mathcal{F}_2 \to \mathcal{D}$ are formally smooth with $\mathcal{F}_1, \mathcal{F}_2$ pro-representable and $\mathcal{D}$ has a tangent space, one of the pro-representing rings F_1, F_2 is a formal power series ring over the other one.

1.3 The tangent space and the obstruction space

In this section we show how to define the tangent space $T\mathcal{F}$ and an obstruction space $O\mathcal{F}$ of a functor $\mathcal{F}$ which is not necessarily pro-representable. For the tangent space this is well-known (cf. [35], Lemma 2.10). The definition of an obstruction space is suggested by Theorem 1.2.7 and Artin's obstruction theory for a groupoid ([1], 2.6).

If $\mathcal{F}$ happens to be pro-representable, then both the tangent space $T\mathcal{F}$ and the (minimal) obstruction space $O\mathcal{F}$ exist and coincide with those of the pro-representing ring F (1.3.2,1.3.9).

We also show that the (minimal) obstruction space $O\mathcal{F}$ exists when $\mathcal{F}$ commutes with products over k. This applies to most of the deformation functors which come up in practice. Note, however, that the obstruction spaces which one gets in practice (usually some cohomology groups) are rarely minimal. Our result 1.3.8 has been recently obtained independently by Fantechi and Manetti ([9], 2.10, 2.11).

Notation. For a finite-dimensional k-vector space V and $A \in Art_\Lambda$ we let $A[V] \in Art_\Lambda$ denote the ring $A \oplus V$ with $V^2 = m_A V = 0$ and the augmentation determined by that of A. If $A = k$ and $V = k$, we denote the resulting ring by $k[\epsilon]$.

Remark. The association $V \mapsto k[V]$ embeds the category of finite-dimensional k-vector spaces as a full subcategory of Art_Λ.

If V_1 and V_2 are finite-dimensional k-vector spaces, then there are natural projections $k[V_1 \times V_2] \to k[V_1]$ and $k[V_1 \times V_2] \to k[V_2]$. Thus for any $\mathcal{F}$, we have a map

$$\mathcal{F}(k[V_1 \times V_2]) \longrightarrow \mathcal{F}(k[V_1]) \times \mathcal{F}(k[V_2]) . \tag{5}$$

Remark. If the above map is bijective for any V_1 and V_2, then $\mathcal{F}(k[\epsilon])$ has a structure of a k-vector space given by (cf. [35], Lemma 2.10):

addition: The (k-linear) addition map $k \times k \to k$ induces

$$\alpha : k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$$

and thus

$$T\mathcal{F} \times T\mathcal{F} = \mathcal{F}(k[\epsilon]) \times \mathcal{F}(k[\epsilon]) = \mathcal{F}(k[\epsilon] \times_k k[\epsilon]) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(k[\epsilon]) = T\mathcal{F} .$$

k-action: The action of $a \in k$ on $\mathcal{F}(k[\epsilon])$ is induced by the map $\epsilon \mapsto a\epsilon$ on $k[\epsilon]$.

Definition 1.3.1. We say that $\mathcal{F}$ has a *tangent space* if (5) is bijective for all V_1 and V_2. In that case we call $T\mathcal{F} = \mathcal{F}(k[\epsilon])$ the tangent space of $\mathcal{F}$.

Remark 1.3.2. A pro-representable functor $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$, has a finite-dimensional tangent space, since

$$\mathrm{Hom}_\Lambda(F, k[V \times W]) = \mathrm{Hom}_\Lambda(F, k[V]) \times \mathrm{Hom}_\Lambda(F, k[W])$$

and there are canonical k-vector space isomorphisms

$$T\mathcal{F} = \mathrm{Hom}_\Lambda(F, k[\epsilon]) = \mathrm{Hom}_k(m_F/(m_F^2 + m_\Lambda F), k) = TF .$$

Definition 1.3.3. Let $\mathcal{F} : Art_\Lambda \to Sets$ be a covariant functor. An *obstruction* Θ is a triple (A, I, ξ') where $A \in Art_\Lambda$ is a ring, $I \subset A$ an ideal for which $m_A I = 0$ and $\xi' \in \mathcal{F}(A/I)$. We say that Θ is *trivial* if there exists $\xi \in \mathcal{F}(A)$ such that $\mathcal{F}(A \to A/I)(\xi) = \xi'$.

Definition 1.3.4. Let $\mathcal{F} : Art_\Lambda \to Sets$ be a covariant functor. We say that (V, o) is an *obstruction space* for $\mathcal{F}$ if V is a k-vector space and

$$(I, A, \xi') = \Theta \longmapsto o(\Theta) \in V \otimes_k I$$

is a rule which associates to an obstruction (I, A, ξ') an element of $V \otimes_k I$, satisfying

1. (functoriality) If $\Theta = (I, A, \xi'_A)$ and $\Sigma = (J, B, \eta'_B)$ and there exists a map $f : A \to B$ with $f(I) \subset J$ and $\mathcal{F}(A/I \to B/J)(\xi') = \eta'$, then
$$(1 \otimes f)\, o(\Theta) = o(\Sigma)\ ,$$
2. (vanishing)
$$\Theta \text{ trivial} \quad \Longleftrightarrow \quad o(\Theta) = 0\ .$$

We will sometimes denote an obstruction space just by V, dropping o from the notation.

Remark 1.3.5. If (V, o) is an obstruction space and $i : V \hookrightarrow \tilde{V}$ an inclusion of k-vector spaces, then we can let
$$\tilde{o}(\Theta) = (i \otimes 1)(o(\Theta))$$
and thus get an obstruction space $(\tilde{V}, \tilde{o})$. The requirement that i must be injective follows from the vanishing condition of 1.3.4. If we weaken the vanishing condition by replacing " $\Longleftrightarrow$ " by "$\Rightarrow$", then any linear map i will do. In defining the notion of a universal obstruction space we allow this larger class of pairs $(\tilde{V}, \tilde{o})$ as test objects. This choice has an advantage that it gives the functoriality in $\mathcal{F}$ for free (cf. 1.3.7).

Definition 1.3.6. We say that the obstruction space (V, o) for $\mathcal{F}$ is *minimal* or *universal* if it satisfies the following universal property: let $(\tilde{V}, \tilde{o})$ satisfy the functoriality condition and the "$\Rightarrow$" part of the vanishing condition of 1.3.4. Then there is a unique k-linear map $V \to \tilde{V}$ which makes $\tilde{o}$ factor via o.

Notation. If a universal obstruction space of $\mathcal{F}$ exists, we denote it by $O\mathcal{F}$. This makes sense as it is clearly unique up to a (canonical) isomorphism.

Theorem 1.3.7. *The association $O : \mathcal{F} \mapsto O\mathcal{F}$ gives a covariant functor from the category of functors $Art_\Lambda \to Sets$ which have a universal obstruction space to the category of vector spaces over k.*

Proof. We have already defined O on objects. To define O on morphisms, let $t : \mathcal{F} \to \mathcal{G}$ be a natural transformation of functors. Denote by $(O\mathcal{F}, o_\mathcal{F})$, $(O\mathcal{G}, o_\mathcal{G})$ the universal obstruction spaces of $\mathcal{F}$ and $\mathcal{G}$ respectively. Take an obstruction for $\mathcal{F}$,
$$\Theta = \Big(A, I, \xi' \in \mathcal{F}(A/I)\Big)\ .$$
Let
$$l(\Theta) = o_\mathcal{G}\Big(A, I, t(\xi') \in \mathcal{G}(A/I)\Big) \in O\mathcal{G} \otimes_k I\ .$$
Clearly $(O\mathcal{G}, l)$ satisfies the functoriality axiom and the "$\Rightarrow$" part of the vanishing axiom of 1.3.4. Define
$$Ot : O\mathcal{F} \longrightarrow O\mathcal{G}$$
to be the factoring map of l which exists and is unique by the universal property of $O\mathcal{F}$. This defines O for morphisms. From the universal property it also follows that O takes composition to composition, so O is a covariant functor. ∎

Remark. Clearly not every functor $\mathcal{F}$ has an obstruction space. The pro-representable ones do (see 1.3.9), but for example the condition that $\mathcal{F}$ has a hull (see 1.4) alone does not guarantee the existence of $O\mathcal{F}$. The functors which arise in practice, however, can be usually written as quotients of a pro-representable functor by a smooth formal group action. Those satisfy the following condition which does imply the existence of $O\mathcal{F}$ (cf. 1.7.2, 1.7.3).

Theorem 1.3.8. *Let $\mathcal{F} : Art_\Lambda \to Sets$ be a covariant functor. Assume that the natural map*

$$\mathcal{F}(A \times_k B) \longrightarrow \mathcal{F}(A) \times \mathcal{F}(B) \tag{6}$$

is bijective for all $A, B \in Art_\Lambda$. Then $\mathcal{F}$ has a universal obstruction space.

Proof. First note that $\mathcal{F}(k)$ consists of one element (take $A = B = k$).

Consider the set S of tuples (A, I, ξ', s) where $\Theta = (A, I, \xi')$ is an obstruction for $\mathcal{F}$ for which $A \to A/I$ is a small surjection and $s : k \cong I$ an isomorphism of W-modules. By abuse of notation, we will denote such a 4-tuple again by Θ. As a set, $O\mathcal{F}$ is supposed to consist of elements of S modulo equivalence, so we define it this way:

Let $\Theta_1 = (A_1, I_1, \xi_1', s_1)$ and $\Theta_2 = (A_2, I_2, \xi_2', s_2)$ be elements of S. Denote $A_1' = A_1/I_1, A_2' = A_2/I_2$. Define the difference $\Theta_1 - \Theta_2 \in S$ as follows. The product map $A_1 \times_k A_2 \longrightarrow A_1' \times_k A_2'$ is a small extension whose kernel is a 2-dimensional k-vector space, generated by $\epsilon_1 = (s_1(1), 0)$ and $\epsilon_2 = (0, s_2(1))$. The map

$$A_1 \times_k A_2/(\epsilon_1 + \epsilon_2) \longrightarrow A_1' \times_k A_2'$$

is a small surjection. Define an isomorphism u between k and the kernel of this small surjection by letting $u(1) = \epsilon_2$. Finally, define $\eta' \in \mathcal{F}(A' \times_k B')$ to be the unique element which maps to (ξ_1', ξ_2') via 6. Let

$$\Theta_1 - \Theta_2 = (A \times_k B/(\epsilon_1 + \epsilon_2), (\epsilon_1), \eta', u)\,.$$

Let

$$\Theta_1 \sim \Theta_2 \qquad \Longleftrightarrow \qquad \Theta_1 - \Theta_2 \ \text{ is trivial}\,.$$

It is easy to check that "$\sim$" is an equivalence relation. Moreover, the subtraction operation defined above respects the equivalence and gives the set $S/\sim$ a structure of an abelian group. We let $O\mathcal{F} = S/\sim$ and give it a k-vector space structure by letting

$$\Theta = (A, I, \xi', \beta \mapsto s(\beta)),\ \ \alpha \in k^* \quad \Rightarrow \quad \alpha \cdot \Theta = (A, I, \xi', \beta \mapsto s(\alpha\beta))\,.$$

It is easy to see that $O\mathcal{F}$ indeed becomes a k-vector space and that it satisfies the required universal property. ∎

Theorem 1.3.9. *Let* $\mathcal{F}: Art_\Lambda \to Sets$ *be pro-representable. Then the obstruction space* OF *of the pro-representing ring* F *is the universal obstruction space for* $\mathcal{F}$*.*

Proof. Theorem 1.2.7 shows that OF is, indeed, an obstruction space for $\mathcal{F}$. Recall the construction: write $F \cong S/J$,

$$S \cong \Lambda[[t_1, ..., t_n]], \quad J \subset m_S^2 + m_\Lambda S\,.$$

We let $OF = \mathrm{Hom}(J/m_S J, k)$, as a k-vector space. Given $\Theta = (I, A, \xi')$, the element $o(\Theta)$ is constructed as follows. Denote $A' = A/I$ and let $a_i' = \xi'(t_i)$. Then lift a_i' arbitrarily to $a_i \in A$. The homomorphism $S \to A$ defined by $t_i \mapsto a_i$ maps J to I and descends to a linear map $J/m_S J \to I$, hence an element of $OF \otimes_k I$. We denote this element by $f(\Theta)$.

It remains to prove that (OF, o) is universal. Since pro-representable functors commute with fibred products, $\mathcal{F}$ has a universal obstruction space $O\mathcal{F}$. By Remark 1.3.5, the canonical map

$$i : O\mathcal{F} \longrightarrow OF$$

which exists by the universal property, is injective. Hence it suffices to show that it is surjective. Take $\varphi \in OF$, considered as a k-linear form on $J/m_S J$. Let I denote the kernel of the composition of maps of W-modules

$$J \longrightarrow J/m_S J \xrightarrow{\varphi} k\,.$$

Then $I \subset S$ is an ideal and we let $A = S/I$ and $A' = S/J = F$. The natural projection $A \twoheadrightarrow A'$ is a small extension. Finally, the identity map $F \longrightarrow A'$ gives an element $\xi' \in \mathcal{F}(A')$. The triple

$$\Theta = (I, A, \xi')$$

is an obstruction for which $o(\Theta) = \varphi$. Hence $i(\Theta) = \varphi$. This shows that i is surjective. ∎

Remark. In practice, if a $\mathcal{F}$ is a deformation functor of some kind, then often there are (co)homology groups playing a role of tangent and obstruction spaces for $\mathcal{F}$. This is for example the case for deformations of group representations [22], Lie algebras, subschemes of projective space, ring representations (Theorem 2.2.4), filtrations (Theorem 2.3.2), varieties, endomorphisms of p-divisible groups (4.3.4) and in many other situations. Knowing the tangent and an obstruction space either helps to determine the pro-representing ring (or a hull) of a functor itself, or at least to get some estimates on its dimension. It should be noted, however, that one rarely knows that a given obstruction space is actually minimal. So such a computation can be very often used to show that a functor is formally smooth, but does not help much in proving, for example, that a functor is *not* formally smooth.

1.4 Schlessinger's theory

For the sake of completeness, we recall the notion of a fibre product and state Schlessinger's necessary and sufficient conditions for a functor on Art_Λ to be pro-representable and to possess a hull. The results here are taken completely from Schlessinger [35].

Definition 1.4.1. Let $f: A \to B$ and $g: C \to B$ be two morphisms in Art_Λ. Define *the fibre product of A and C over B* to be the ring

$$A \times_B C = \{(x, y) \in A \times C \mid f(x) = g(y)\}$$

with an obvious Λ-structure and augmentation to k.

Remark. The fibre product $A \times_B C$ is the categorical fibre product in Art_Λ. In fact,

$$\mathrm{Hom}_\Lambda(F, A \times_B C) = \mathrm{Hom}_\Lambda(F, A) \times_{\mathrm{Hom}_\Lambda(F,B)} \mathrm{Hom}_\Lambda(F, C)$$

for *any* Λ-algebra F, not necessarily Artinian. In particular, pro-representable functors commute with fibre products, in the sense of the following definition.

Definition 1.4.2. We say that a functor $\mathcal{F}$ *commutes with fibre products* if for any $f: A \to B$ and $g: C \to B$, the natural map

$$\mathcal{F}(A \times_B C) \to \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C) \tag{7}$$

is bijective.

Remark. If $\mathcal{F}$ commutes with fibre products then, in particular, it has a tangent space (cf. 1.3.1). For a pro-representable $\mathcal{F}$ the tangent space is, moreover, finite-dimensional. The converse to this due to Schlessinger:

Theorem 1.4.3. *A functor $\mathcal{F}: Art_\Lambda \to Sets$ is pro-representable if and only if $\mathcal{F}(k)$ consists of one element, $\mathcal{F}$ commutes with fibre products and has a finite-dimensional tangent space.*

Proof. [35], Theorem 2.11. ∎

Many of the geometrically interesting functors are not pro-representable. For instance, the deformation functors of complete varieties, of group/ring representations and of group schemes are in general not pro-representable. These, however, can often be represented as quotient functors of a pro-representable functor by a group action of a smooth formal group, usually some GL_n. In particular, they satisfy a weaker condition of possessing a hull. (For instance, see Theorem 2.2.4 for the case of ring representations and [35], Prop. 3.10, 3.12 for the case of varieties.)

Theorem 1.4.4. *Assume a functor $\mathcal{F}$ (such that $\mathcal{F}(k)$ has one element) has a finite-dimensional tangent space. The following conditions are equivalent:*

1. *The map (7) is surjective for all $A \to B \leftarrow C$ in Art_Λ (it suffices to check this when $A \twoheadrightarrow B$ is a small extension).*
2. *There is a pro-representable functor $\mathcal{G}$ and a formally smooth map $\mathcal{G} \to \mathcal{F}$ which is an isomorphism on tangent spaces.*
3. *There is a pro-representable functor $\mathcal{G}$ and a formally smooth map $\mathcal{G} \to \mathcal{F}$.*

Proof. $1 \Leftrightarrow 2$ is Schlessinger's theorem [35], Theorem 2.11. $3 \Rightarrow 1$ follows from the fact that the map (7) for $\mathcal{G}$ is surjective and $\mathcal{G}(A) \twoheadrightarrow \mathcal{F}(A)$ for all A. The implication $2 \Rightarrow 3$ is trivial. ∎

Definition. If $\mathcal{F}$ satisfies the equivalent conditions of Theorem 1.4.4, we say that $\mathcal{F}$ has a hull. In fact, a *hull of $\mathcal{F}$* is a pro-representable functor $\mathcal{G}$ together with a formally smooth map $\mathcal{G} \to \mathcal{F}$ which is an isomorphism on tangent spaces.

Remark. A hull $\mathcal{F}$ of $\mathcal{D}$, if it exists, is unique up to a non-canonical isomorphism ([35], Proposition 2.9). This also follows from Corollary 1.5.4.

1.5 Comparing formally smooth extensions

Suppose a functor $\mathcal{D}$ possesses a hull $g : \mathcal{G} \to \mathcal{D}$ with $\mathcal{G} = \mathrm{Hom}_\Lambda(G, -)$. It turns out that any other formally smooth map $f : \mathrm{Hom}_\Lambda(F, -) = \mathcal{F} \to \mathcal{D}$ factors through $\mathcal{G}$,

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathcal{D} & \end{array} \quad .$$

Moreover the factoring map α is formally smooth, so $F \cong G[[t_1, ..., t_n]]$. In other words, the only rings which can be mapped to $\mathcal{D}$ in a formally smooth way, are formal power series over G. In practice, this can be used to determine the hull of a functor $\mathcal{D}$. Firstly, find *any* formally smooth map $\mathcal{F} \to \mathcal{D}$ with $\mathcal{F}$ pro-representable. For example, $\mathcal{D}$ is often given as a quotient of some $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$ by a smooth group action. Secondly, find an isomorphism

$$F \;\cong\; G[[t_1, ..., t_n]], \quad n = \dim_k T\mathcal{F} - \dim_k T\mathcal{D}\,,$$

for some G. Then (combine 1.5.6 with 1.5.7) the ring G pro-represents the hull of $\mathcal{D}$.

The main result of this section is Theorem 1.5.3, which compares formally smooth extensions of a functor $\mathcal{D}$. In order to give a formulation in case $\mathcal{D}$ does not necessarily have a tangent space, we need some preliminary definitions.

Remark. (cf. [35], 2.10) Suppose $\mathcal{G} \cong \mathrm{Hom}_\Lambda(G, -)$ is pro-representable. The cotangent space $T\mathcal{G}^*$ of G over Λ has the property that

$$\mathcal{G}(k[V]) = \mathrm{Hom}_\Lambda(G, k[V]) = \mathrm{Hom}_k(T\mathcal{G}^*, V)$$

for any (finite-dimensional) k-vector space V. In other words, $T\mathcal{G}^*$ represents the functor $V \mapsto \mathcal{G}(k[V])$ on the category of k-vector spaces.

In particular, given another pro-representable functor $\mathcal{F} \cong \mathrm{Hom}_\Lambda(F, -)$, a natural transformation $\mathcal{F} \to \mathcal{G}$ induces a k-linear map $T\mathcal{G}^* \to T\mathcal{F}^*$ and, hence, gives an element of $\mathcal{G}(k[T\mathcal{F}^*])$.

For example, the identity map $\mathcal{G} \to \mathcal{G}$ corresponds to an element which we denote by $1 \in \mathcal{G}(k[T\mathcal{G}^*])$. It is the image of $1 \in \mathcal{G}(G) = \mathrm{Hom}_\Lambda(G, G)$ under the natural projection $G \to k[T\mathcal{G}^*]$.

Definition 1.5.1. Assume given a diagram of natural transformations of functors

$$\begin{tikzcd} \mathcal{F} \arrow[dr, "f"'] & & \mathcal{G} \arrow[dl, "g"] \\ & \mathcal{D} & \end{tikzcd} \tag{8}$$

with $\mathcal{F}$ and $\mathcal{G}$ pro-representable. We say that a k-linear map

$$t : T\mathcal{F} \longrightarrow T\mathcal{G}$$

lies above $\mathcal{D}$, if the corresponding element in $\mathcal{G}(k[T\mathcal{F}^*])$ and the element $1 \in \mathcal{F}(k[T\mathcal{F}^*])$ project via g and f to the same element of $\mathcal{D}(k[T\mathcal{F}^*])$. By a *lift* of such a t, we mean a natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$, which makes (8) commute (i.e. $g\alpha = f$) and which induces t on the tangent spaces.

Remark 1.5.2. Conversely, given $\alpha : \mathcal{F} \to \mathcal{G}$ with $g\alpha = f$, it induces a map $t : T\mathcal{F} \to T\mathcal{G}$ which lies above $\mathcal{D}$, and α is a lift of t. This, perhaps, explains the meaning of these notions.

Remark. If $\mathcal{D}$ has a tangent space, then $t : T\mathcal{F} \to T\mathcal{G}$ lies above $\mathcal{D}$ if and only if it commutes with projections to $T\mathcal{D}$.

Theorem 1.5.3. *Assume given $\mathcal{F} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{G}$ with $\mathcal{F}, \mathcal{G}$ pro-representable and a k-linear $t : T\mathcal{F} \to T\mathcal{G}$ which lies above $\mathcal{D}$. Then*

1. *If g is formally smooth then a lift of t exists.*
2. *If f is formally smooth and t is surjective, then any lift of t is formally smooth.*
3. *If f is formally smooth and t is bijective, then any lift of t is an isomorphism of functors.*

Proof. **1.** Constructing a natural transformation $\alpha:\mathcal{F}\to\mathcal{G}$ which lies above $\mathcal{D}$ (i.e. $g\alpha=f$) and induces t on the tangent spaces is equivalent to giving an element $\alpha\in\mathcal{G}(F)$ such that $g(\alpha\in\mathcal{G}[F])=f(1\in\mathcal{F}[F])\in\mathcal{D}(F)$ and such that α maps to an element which corresponds to t under the natural projection $\mathcal{G}(F)\to\mathcal{G}(k[T\mathcal{F}^*])$. In other words, we are looking for a pre-image of $(f(1),t)$ under the map

$$\mathcal{G}(F)\to\mathcal{D}(F)\times_{\mathcal{D}(k[T\mathcal{F}^*])}\mathcal{G}(k[T\mathcal{F}^*]),$$

This map is surjective by the assumption that $g:\mathcal{G}\to\mathcal{D}$ is formally smooth.
2. Let $\alpha:\mathcal{F}\to\mathcal{G}$ induce a surjection $t:T\mathcal{F}\to T\mathcal{G}$. We have to show that given $\pi:A\twoheadrightarrow A'$,

$$\mathcal{F}(A)\ \twoheadrightarrow\ \mathcal{F}(A')\times_{\mathcal{G}(A')}\mathcal{G}(A),$$

in other words, given $\varphi'\in\mathcal{F}(A')$ and $\gamma\in\mathcal{G}(A)$ with $\alpha(\varphi')=\pi(\gamma)(=\gamma')$, we have to construct $\varphi\in\mathcal{F}(A)$ with $\pi(\varphi)=\varphi'$ and $\alpha(\varphi)=\gamma$.

By induction over the length, it suffices to consider the case $A\twoheadrightarrow A'$ is a small extension ($\ker\pi=(\epsilon)\neq 0$, $m_A\epsilon=0$). In other words $(\epsilon)\cong k$ as a Λ-module, and we *fix* such an identification.

Let $\delta\in\mathcal{D}(A),\delta'\in\mathcal{D}(A')$ be the images of γ,γ' under g. Note that $\delta'=f(\varphi')$ and by the formal smoothness of f,

$$\mathcal{F}(A)\ \twoheadrightarrow\ \mathcal{F}(A')\times_{\mathcal{D}(A')}\mathcal{D}(A),$$

so there is $\tilde{\varphi}\in\mathcal{F}(A)$ such that $\pi(\tilde{\varphi})=\varphi'$.

Let $\tilde{\gamma}=\alpha(\tilde{\varphi})\in\mathcal{G}(A)$. If $\tilde{\gamma}=\gamma$, let $\varphi=\tilde{\varphi}$ and we are done. Otherwise, we adjust $\tilde{\varphi}$ (within the fibre above φ'), using the surjectivity of t as follows:

Both γ and $\tilde{\gamma}$ are ring homomorphisms $G\to A$. Consider them as just homomorphisms of Λ-modules. Then $\tau=\gamma-\tilde{\gamma}$ is a map $G\to A$ of Λ-modules which lands in $(\epsilon)=\ker\pi$. Restrict it to the map $m_G\to(\epsilon)$ and note that it descends to $m_G/(m_G^2+m_\Lambda G)\to(\epsilon)$, since $(\epsilon)\cong k$ as a Λ-module. Hence τ can be considered as an element of $T\mathcal{G}$.

Now lift τ to an element $\sigma\in T\mathcal{F}$ via the surjection $t:T\mathcal{F}\to T\mathcal{G}$. Extend it to a homomorphism of Λ-modules $F\to(\epsilon)\subset A$ by letting 1 map to 0 and define $\varphi=\tilde{\varphi}+\sigma$. Then one easily checks that φ is a local Λ-algebra homomorphism, $\pi(\varphi)=\varphi'$ and $\alpha(\varphi)=\gamma$ as required.
3. The same argument as in (2.) applies, except that now $t:T\mathcal{F}\to T\mathcal{G}$ is injective, so σ and, therefore, the desired φ is also unique. Hence

$$\mathcal{F}(A)\xrightarrow{\cong}\mathcal{F}(A')\times_{\mathcal{G}(A')}\mathcal{G}(A)$$

whenever $A\twoheadrightarrow A'$. In particular, taking $A'=k$, we see that $\alpha:\mathcal{F}(A)\to\mathcal{G}(A)$ is a bijection for all A. Hence α is an isomorphism of functors. ∎

Corollary 1.5.4. *If $\mathcal{F} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{G}$ with $\mathcal{F}, \mathcal{G}$ pro-representable and f, g formally smooth, then $\mathcal{F}$ is isomorphic to $\mathcal{G}$ over $\mathcal{D}$ if and only if the tangent spaces $T\mathcal{F}$ and $T\mathcal{G}$ are isomorphic over $\mathcal{D}$.*

Proof. Follows immediately from the theorem. It should be mentioned, however, that this corollary requires only the (easier) part 1. of the theorem. If $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{F}$ lift the given isomorphisms on the tangent spaces, then $\alpha\beta$ and $\beta\alpha$ are isomorphisms: an endomorphism of a complete Noetherian local Λ-algebra which is identity on the tangent space is an isomorphism. ▪

Corollary 1.5.5. *Assume given $\mathcal{F} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{G}$ with $\mathcal{F}, \mathcal{G}$ pro-representable and f, g formally smooth. Assume also that $\mathcal{D}$ has a tangent space. Then one of the pro-representing rings F, G is a formal power series ring over the other one.*

Proof. The tangent space maps $T\mathcal{F} \twoheadrightarrow T\mathcal{D}$ and $T\mathcal{G} \twoheadrightarrow T\mathcal{D}$ are both surjective by formal smoothness. Hence there is either a surjection $T\mathcal{F} \to T\mathcal{G}$ or a surjection $T\mathcal{G} \to T\mathcal{F}$ of k-vector spaces which commutes with the projections to $T\mathcal{D}$. The statement follows from parts 1 and 2 of the theorem. ▪

Corollary 1.5.6. *(Versal property of the hull.) Assume that a functor $\mathcal{D}$ has a hull $\mathcal{G} = \mathrm{Hom}_\Lambda(G, -)$, and let $f : \mathcal{F} \to \mathcal{D}$ be formally smooth with $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$. Then*

$$F \;\cong\; G[[t_1, ..., t_n]], \quad n = \dim_k T\mathcal{F} - \dim_k T\mathcal{D} \,,$$

Proof. Since $\mathcal{D}$ has a tangent space and $T\mathcal{G} \cong T\mathcal{D}$ (by definition of a hull), there is a unique map $t : T\mathcal{F} \to T\mathcal{G}$ which lies above $\mathcal{D}$, namely the one induced by f on tangent spaces. ▪

Remark. The corollary often allows to determine a hull G of $\mathcal{D}$, when given a formally smooth map $\mathrm{Hom}_\Lambda(F, -) \to \mathcal{D}$. Indeed, $F \cong G[[t_1, ..., t_n]]$ with $n = \dim T\mathcal{F} - \dim T\mathcal{D}$. So, if one finds a ring G' for which $F \cong G'[[s_1, ..., s_n]]$, then G' is isomorphic to the hull G of $\mathcal{D}$, by the following "cancellation theorem for complete local rings", due to A. J. de Jong.

Proposition 1.5.7. *If $F, G \in \widehat{Art}_\Lambda$ are complete Noetherian local Λ-algebras with an augmentation such that $F[[t]] \cong G[[t]]$, then $F \cong G$.*

Proof. [17], Lemma 4.7. ▪

1.6 Quotients by groups

One often obtains non-pro-representable functors which, nevertheless, possess a hull by taking quotients of pro-representable functors. One can do it either by taking quotients by one group of automorphisms or by taking an action of a formal group instead. The latter way is the one which mostly occurs in practice. This and the next section describe the respective properties of these constructions. In the constant group case we give necessary and sufficient conditions for a functor which has a hull to be represented as such a quotient (1.6.3). In the formal group case we conjecture the corresponding result (1.7.8).

Definition 1.6.1. Let $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$ and let $\Gamma \subset \mathrm{Aut}_\Lambda(F)$ be a subgroup. Then Γ acts on $\mathcal{F}(A)$ for all A by composing a homomorphism $F \to A$ with an element of Γ. Define the *quotient functor* $\mathcal{F}/\Gamma$ by letting $A \mapsto \mathcal{F}(A)/\Gamma$.

Remark. We have let $\Gamma \subset \mathrm{Aut}_\Lambda(F)$ act on $\mathcal{F}(A)$ for all A by composition. Equivalently, one can let an abstract group Γ act on $\mathcal{F}(A)$ for all A, in such a way that for $A \to B$, the maps $\mathcal{F}(A) \to \mathcal{F}(B)$ are Γ-equivariant.

Theorem 1.6.2. *Let $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$ and $\Gamma \subset \mathrm{Aut}_\Lambda(F)$. Denote by $\mathcal{D} = \mathcal{F}/\Gamma$ the quotient functor.*

1. *The quotient map $q : \mathcal{F} \to \mathcal{D}$ is formally smooth.*
2. *$\mathcal{D}$ has the property that $\mathcal{D}(A \times_B C) \twoheadrightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$ for all $A \to B \leftarrow C$.*
3. *$\mathcal{D}$ has a hull if and only if*

$$\Gamma \subset \ker\Big(\mathrm{Aut}_\Lambda(F) \to \mathrm{Aut}_\Lambda(F/m_F^2)\Big) ,$$

   in other words, if Γ acts trivially on the tangent space of F.

4. *$\mathcal{D}$ is pro-representable if and only if $\Gamma = \{1\}$.*

Proof. **1.** Let $\pi : A \twoheadrightarrow B$. We have to show that

$$\mathcal{F}(A) \twoheadrightarrow \mathcal{F}(B) \times_{\mathcal{D}(B)} \mathcal{D}(A) .$$

Take $a \in \mathcal{D}(A)$ and $\tilde{b} \in \mathcal{F}(B)$ such that $\pi(a) = q(\tilde{b})$ in $\mathcal{D}(B)$. Choose a representative $\tilde{a} \in \mathcal{F}(A)$ of a. If $\pi(\tilde{a}) = \tilde{b}$, then we are done. In any case,

$$g \cdot \pi(\tilde{a}) = \tilde{b}$$

for some $g \in \Gamma$. Then $g \cdot \tilde{a}$ is the required lift.
2. Let $\pi : A \to B$ and $\rho : C \to B$ in Art_Λ. Let $a \in \mathcal{D}(A)$ and $c \in \mathcal{D}(C)$ be such that $\pi(a) = \rho(c)$ in $\mathcal{D}(B)$. Choose representatives $\tilde{a} \in \mathcal{F}(A)$ and $\tilde{c} \in \mathcal{F}(C)$ of a and c respectively. The elements $\pi(\tilde{a})$ and $\rho(\tilde{c})$ in $\mathcal{F}(B)$ map to the same element in $\mathcal{D}(B)$, hence there is a $g \in \Gamma$ such that

$$g \cdot \pi(\tilde{a}) = \rho(\tilde{c}) .$$

Replace $\tilde{a}$ by $g \cdot \tilde{a}$. Then $\tilde{a}$ still maps to $a \in \mathcal{D}(A)$, but now we have $\pi(\tilde{a}) = \rho(\tilde{c})$. Since $\mathcal{F}$ commutes with fibre products, there is $\tilde{r} \in \mathcal{F}(A \times_B C)$ which projects to $\tilde{a} \in \mathcal{F}(A)$ and $\tilde{c} \in \mathcal{F}(C)$. Then the image r of $\tilde{r}$ in $\mathcal{D}(A \times_B C)$ is the required lift of (a, c).

3. The "if" part is clear: using part 2. and Schlessinger's criterion, it suffices to prove that $\mathcal{D}$ has a tangent space. But $\mathcal{F}$ has a tangent space and $\mathcal{F}(k[V]) \to \mathcal{D}(k[V])$ is bijective for all k-vector spaces V by our assumption on Γ. Hence $\mathcal{D}$ has a tangent space as well.

For the converse, assume that the action of Γ on TF is non-trivial but the quotient $\mathcal{D} = \mathcal{F}/\Gamma$ has a hull. In particular $\mathcal{D}$ has a tangent space. Let $V = T\mathcal{F} = \mathcal{F}(k[\epsilon])$. Let $k[\epsilon_1, \epsilon_2]$ denote the ring $k[t_1, t_2]/(t_1^2, t_2^2, t_1 t_2)$ and consider the map

$$\pi : \mathcal{D}(k[\epsilon_1, \epsilon_2]) \longrightarrow \mathcal{D}(k[\epsilon]) \times \mathcal{D}(k[\epsilon]).$$

whose components π_1 and π_2 are the natural projections. By assumption $\mathcal{D}$ has a tangent space, so π is a bijection. However,

$$\mathcal{D}(k[\epsilon_1, \epsilon_2]) = \mathcal{F}(k[\epsilon_1, \epsilon_2]/\Gamma = (V \oplus V)/\Gamma\,,$$

and

$$\mathcal{D}(k[\epsilon]) \times \mathcal{D}(k[\epsilon]) = (V/\Gamma) \times (V/\Gamma)\,.$$

Moreover, the action of Γ on $\mathcal{F}(k[\epsilon_1, \epsilon_2]) = V \oplus V$ is diagonal,

$$g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2), \qquad v_1, v_2 \in V\,, g \in \Gamma\,,$$

by compatibility of the action with the two inclusions $k[\epsilon] \hookrightarrow k[\epsilon_1, \epsilon_2]$. As we have assumed that the action of G on V is non-trivial, there are $v_1 \neq v_2 \in V$ such that $g \cdot v_1 = v_2$ for some $g \in \Gamma$. Then

$$h \cdot (v_1, v_1) = (h \cdot v_1, h \cdot v_1) \neq (v_1, v_2)$$

for any $h \in \Gamma$. Hence (v_1, v_1) and (v_1, v_2) give two *distinct* elements of $\mathcal{D}(k[\epsilon_1, \epsilon_2])$. However $\pi_1(v_1, v_2) = \pi_2(v_1, v_2)$ as elements of $\mathcal{D}(k[\epsilon]) \times \mathcal{D}(k[\epsilon])$. Hence π is not injective, a contradiction.

4. If $\Gamma = \{1\}$, then $\mathcal{D} = \mathcal{F}$ is pro-representable. Conversely, assume $\mathcal{D}$ is pro-representable. Since, in particular, $\mathcal{D}$ has a hull, $\Gamma \subset \ker(\mathrm{Aut}_\Lambda(F) \to \mathrm{Aut}_\Lambda(F/m_F^2))$ by part 3. of the theorem. Hence $\mathcal{F} \to \mathcal{D}$ is identity on the tangent spaces. Since it is also formally smooth, $\mathcal{F} = \mathcal{D}$ for example by uniqueness of the hull. ∎

Theorem 1.6.3. *Let $\mathcal{D} : Art_\Lambda \to Sets$. The following conditions are equivalent.*

1. *$\mathcal{D}$ possesses a hull and $\mathcal{D}(A \times_B C) \twoheadrightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$ for all $A \to B \leftarrow C$ (not only in case $C \twoheadrightarrow B$).*

2. *There exists a pro-representable functor $\mathcal{F} = \mathrm{Hom}_\Lambda(F, -)$ and a subgroup*

$$\Gamma \subset \ker\Big(\mathrm{Aut}_\Lambda(F) \to \mathrm{Aut}_\Lambda(F/m_F^2)\Big)$$

such that $\mathcal{D} \cong \mathcal{F}/\Gamma$.

Proof. $2 \Rightarrow 1$ is a part of Theorem 1.6.2. Now we prove $1 \Rightarrow 2$.

Let $\mathcal{F} \cong \mathrm{Hom}_\Lambda(F, -)$ be a hull of $\mathcal{D}$ and $q : \mathcal{F} \to \mathcal{D}$ the defining map. Let $\Gamma \subset \mathrm{Aut}_\Lambda(F)$ consist of those automorphisms g which, considered as elements of $\mathrm{Aut}(\mathcal{F})$ satisfy $qg = q$, as natural transformations. Since q is identity on tangent spaces, $\Gamma \subset \ker\Big(\mathrm{Aut}_\Lambda(F) \to \mathrm{Aut}_\Lambda(F/m_F^2)\Big)$ as required. It suffices to prove that $\mathcal{D} \cong \mathcal{F}/\Gamma$. Clearly q factors through $\mathcal{F}/\Gamma$ and

$$\mathcal{F}(A)/\Gamma \to \mathcal{D}(A)$$

is surjective for all $A \in Art_\Lambda$, since $\mathcal{F}(A) \twoheadrightarrow \mathcal{D}(A)$ by formal smoothness. To prove injectivity, assume $x, y \in \mathcal{F}(A)$ are such that $q(x) = q(y) \in \mathcal{D}(A)$. We have to prove that there is $g \in \Gamma$ for which $g \cdot x = y$.

Consider x and y as homomorphisms $F \to A$. We first want to reduce to the case that x, y are surjective. Let $A' \subset A$ be the Λ-subalgebra genereated by $\mathrm{Im}\, x$ and $\mathrm{Im}\, y$. Then both x and y factor via A',

$$x, y : F \longrightarrow A' \hookrightarrow A\,.$$

In other words $x, y \in \mathcal{F}(A)$ lie in the image of $\mathcal{F}(A') \hookrightarrow \mathcal{F}(A)$. Let $x', y' \in \mathcal{F}(A')$ be the same homomorphisms, considered as elements of $\mathcal{F}(A')$. We claim that $q(x') = q(y') \in \mathcal{D}(A')$.

We know that $q(x')$ and $q(y')$ have the same image in $\mathcal{D}(A)$. By the second assumption on $\mathcal{D}$, the map

$$\mathcal{D}(A') = \mathcal{D}(A' \times_A A') \to \mathcal{D}(A') \times_{\mathcal{D}(A)} \mathcal{D}(A')$$

is surjective. Equivalently, $\mathcal{D}(A') \hookrightarrow \mathcal{D}(A)$. So $\mathcal{D}$ takes injections to injections. Thus $q(x') = q(y')$. If we can find a $g \in \Gamma$ for which $g \cdot x' = y'$, then $g \cdot x = y$ as required. So we can replace A by A', in other words assume that A is generated by $\mathrm{Im}\, x$ and $\mathrm{Im}\, y$ as a Λ-algebra.

We claim that in this case both x and y have to be surjective.

Indeed, let $B = \mathrm{Im}(x)$ and $C = \mathrm{Im}(y)$. As A is generated by B and C as a Λ-algebra, the cotangent space $V = m_A/(m_A^2 + m_\Lambda A)$ is generated, as a vector space, by $m_B V$ and $m_C V$. Thus, if we show that $m_B V = m_C V$, then it follows that x, y are surjective on cotangent spaces, hence surjective (Remark 1.2.3).

Consider the projection $A \to A/m_A^2$, composed with x and y:

$$F \xrightarrow{x,y} A \twoheadrightarrow A/m_A^2\,.$$

The compositions $\bar{x}$ and $\bar{y}$ define elements $\bar{x}, \bar{y} \in \mathcal{F}(A/m_A^2)$. Since $q(\bar{x}) = q(\bar{y})$, and $q : \mathcal{F} \to \mathcal{D}$ is a bijection on the rings of the form $k[V]$ (such as A/m_A^2), it follows that $\bar{x} = \bar{y}$. Hence $m_B V = m_C V$ and x, y are both surjective.

In summary, we have surjections $x, y : F \to A$ and $q(x) = q(y) \in \mathcal{D}(A)$ if x, y are considered as elements of $\mathcal{F}(A)$. We have to prove that there is a $g \in \Gamma$ for which $g \cdot x = y$.

By the lemma below, there exists a homomorphism $g: F \to F$ such that the corresponding natural transformation $\alpha: \mathcal{F} \to \mathcal{F}$ commutes with q and such that $xg = y$. Since $q\alpha = q$, it follows that g is identity on the tangent space of F. In particular, it is an automorphism of F and $g \in \Gamma$. Also $g \cdot x = y$, as required. ∎

Lemma 1.6.4. *Let $q : \mathcal{F} \to \mathcal{D}$ be formally smooth with $\mathcal{F} \cong \mathrm{Hom}_\Lambda(F, -)$ pro-representable. Let $x, y \in \mathcal{F}(A)$ satisfy $q(x) = q(y)$ and assume that y is surjective, if considered as a homomorphism $F \to A$. Then there exists a natural transformation $\alpha: \mathcal{F} \to \mathcal{F}$ for which*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\alpha"] \arrow[dr, "q"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{D} & \end{tikzcd}$$

commutes and such that $\alpha(x) = y$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(F) & \xrightarrow{q} & \mathcal{D}(F) \ni q \\ {\scriptstyle \mathcal{F}(x)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}(x)} \\ y \in \mathcal{F}(A) & \xrightarrow{q} & \mathcal{D}(A) \end{array} .$$

Here q is seen both as a natural transformation and as an element of $\mathcal{D}(F)$. As x is surjective and $\mathcal{F} \to \mathcal{D}$ is formally smooth, there exists a $g \in \mathcal{F}(F)$ lifting (q, y) in the above diagram.

We claim that $g \in \mathrm{Hom}_\Lambda(F, F)$ is a homomorphism which gives the required natural transformation α. Firstly, $q(g) = q$ is the above diagram implies that $q\alpha = q$ as natural transformations. Secondly, $\mathcal{F}(x)(g) = y$ says precisely that $\alpha(x) = y$. ∎

1.7 Quotients by formal groups

Definition 1.7.1. Let $\mathcal{G}$ be a group functor $Art_\Lambda \to Groups$ and $\mathcal{F}: Art_\Lambda \to Sets$. An *action of $\mathcal{G}$ on $\mathcal{F}$* consists of group actions of $\mathcal{G}(A)$ on $\mathcal{F}(A)$ for all $A \in Art_\Lambda$, functorial in A. Recall also that a *formal group* is a formally smooth pro-representable group functor.

Theorem 1.7.2. *Let $\mathcal{F}$ be a pro-representable functor and $\mathcal{G}$ a formal group which acts on $\mathcal{F}$. Then $\mathcal{D} = \mathcal{F}/\mathcal{G}$, defined by $A \mapsto \mathcal{F}(A)/\mathcal{G}(A)$, has a hull.*

Proof. For $A, B \in Art_\Lambda$, the natural maps

$$\mathcal{F}(A \times_k B) \to \mathcal{F}(A) \times \mathcal{F}(B), \quad \mathcal{G}(A \times_k B) \to \mathcal{G}(A) \times \mathcal{G}(B)$$

are isomorphisms, since $\mathcal{F}, \mathcal{G}$ are pro-representable. Hence

$$\mathcal{F}(A \times_k B)/\mathcal{G}(A \times_k B) \longrightarrow \mathcal{F}(A)/\mathcal{G}(A) \times \mathcal{F}(B)/\mathcal{G}(B)$$

is bijective for any A, B. In particular, $\mathcal{F}/\mathcal{G}$ has a tangent space.

Next, we show that the natural map $\mathcal{F} \to \mathcal{F}/\mathcal{G}$ is formally smooth, so

$$\mathcal{F}(A) \twoheadrightarrow \mathcal{F}(B) \times_{\mathcal{F}(B)/\mathcal{G}(B)} \mathcal{F}(A)/\mathcal{G}(A)\,.$$

whenever $\pi : A \twoheadrightarrow B$. Take an element in the right-hand side, represented by a pair $\xi \in \mathcal{F}(B)$, $\eta \in \mathcal{F}(A)$ such that $\pi(\eta) = g_B \cdot \xi$ for some $g_B \in \mathcal{G}(B)$. Since $\mathcal{G}$ is formally smooth, $\mathcal{G}(A) \twoheadrightarrow \mathcal{G}(B)$, so g_B can be lifted to an element $g_A \in \mathcal{G}(A)$. Then $g_A^{-1} \cdot \eta \in \mathcal{F}(A)$ is the required lift.

Hence $\mathcal{D}$ has a hull (cf. definition 1.4). ■

Remark 1.7.3. From this proof it also follows that $\mathcal{D} = \mathcal{F}/\mathcal{G}$ has a universal obstruction space (Theorem 1.3.8).

Remark. Note that the pro-representability of $\mathcal{G}$ is used *only* to prove that $\mathcal{F}/\mathcal{G}$ has a tangent space and not for the formal smoothness of the quotient map.

Theorem 1.7.4. *Let $\mathcal{G} : Art_\Lambda \to Groups$ be an formally smooth group functor. Assume that $\mathcal{G}$ acts on a pro-representable functor $\mathcal{F}$ in such a way that for every k-vector space V, $\mathcal{G}(k[V])$ acts trivially on $\mathcal{F}(k[V])$. Then $\mathcal{F}/\mathcal{G}$ has a hull, namely $\mathcal{F}$ with the natural quotient map.*

Proof. By the remark above, formal smoothness of $\mathcal{G}$ implies that the quotient map is formally smooth. Since $\mathcal{F}(k[V]) \to (\mathcal{F}/G)(k[V])$ is bijective for all V, the quotient functor has a tangent space and the quotient map is bijective on the tangent spaces. Hence $\mathcal{F}$ is the hull of $\mathcal{F}/\mathcal{G}$. ■

Remark. Let $\mathcal{G} : Art_\Lambda \to Groups$ be as in the above theorem and let $\Gamma = \mathcal{G}(k)$. For any $A \in Art_\Lambda$ there is a surjective group homomorphism $\mathcal{G}(\eta_A) : \mathcal{G}(A) \to \Gamma$ induced by the augmentation $\eta : A \to k$ (see Definition 1.1.1). Moreover, for any $f : A \to B$ the induced homomorphism $\mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(B)$ commutes with these projections to Γ. So we have a natural transformation of group functors

$$\mathcal{G} \longrightarrow \underline{\Gamma}$$

where $\underline{\Gamma}$ denotes the constant group functor with value Γ on every $A \in Art_\Lambda$ (and taking every morphism in Art_Λ to the identity on Γ). So we get an exact sequence

$$1 \longrightarrow \mathcal{G}^{formal} \longrightarrow \mathcal{G} \longrightarrow \underline{\Gamma} \longrightarrow 1$$

of group functors on Art_Λ, where $\mathcal{G}^{formal}$ denotes the kernel. By definition $\mathcal{G}^{formal}(k)$ consists of one element. So $\mathcal{G}^{formal}$ is close to being a formal group, except that it is not necessary pro-representable. Theorem 1.6.3 characterizes in general quotients by constant groups, but it seems difficult to find to corresponding result for (even pro-representable) formally smooth group functors. Nevertheless, the following conjecture seems feasible.

Conjecture 1.7.5. *Let $\mathcal{D}$ be a functor which has a hull $\mathcal{F}$. Then there exists a formally smooth $\mathcal{G}: Art_\Lambda \to Groups$ (as in Theorem 1.7.4) and an action of $\mathcal{G}$ on $\mathcal{F}$ such that $\mathcal{D} \cong \mathcal{F}/\mathcal{G}$.*

Remark 1.7.6. One might also conjecture that $\mathcal{G}$ can be chosen to be an extension of a formal group by a constant group.

Remark 1.7.7. Let $\mathcal{F}$ be a pro-representable functor and let $\mathcal{G}$ act on $\mathcal{F}$. Assume for simplicity that the action on $\mathcal{F}(k[V])$ is trivial for all V. Let $\mathcal{D} = \mathcal{F}/\mathcal{G}$ and consider the following properties of $\mathcal{D}$:

(1) $\mathcal{D}(A \times_B C) \twoheadrightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$ for all $A \to B \twoheadleftarrow C$.

(2) $\mathcal{D}(A \times_B C) \xrightarrow{\cong} \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$ for $B = k$, all A, C.

(3) $\mathcal{D}(A \times_B C) \twoheadrightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{D}(C)$ for all $A \to B \leftarrow C$.

In any case, $\mathcal{D}$ has property (1) by the above theorem. If $\mathcal{G}$ is a constant group functor, then $\mathcal{G}$ satisfies (3) but not (2), unless it is trivial (take $A = C = F$ and $B = k$). If $\mathcal{G}$ is a formal group, then $\mathcal{D}$ satisfies (2) but seemingly never (3), unless again it is trivial. In this respect, the two quotient constructions are complementary to each other. Hence one might conjecture a criterion for quotients by formal groups analogous to Theorem 1.6.3.

Conjecture 1.7.8. *Let $\mathcal{D}$ be a functor which has a hull $\mathcal{F}$. Assume that $\mathcal{D}$ commutes with products,*

$$\mathcal{D}(A \times_k B) \xrightarrow{\sim} \mathcal{D}(A) \times \mathcal{D}(B) .$$

Then $\mathcal{D} \cong \mathcal{F}/\mathcal{G}$ for some formal group $\mathcal{G}$ acting on $\mathcal{F}$.