# Termination of term rewriting

Well-foundedness, totality and transformations

Terminatie van termherschrijfsystemen
Welgefundeerdheid, totaalheid en transformaties
(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor
aan de Universiteit Utrecht
op gezag van de Rector Magnificus, Prof. dr. J. A. van Ginkel
ingevolge het besluit van het College van Decanen
in het openbaar te verdedigen
op maandag 20 november 1995 des middags te 14.30 uur

door
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geboren op 23 maart 1962
te Lissabon

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Co-Promotor: dr. H. Zantema

Faculteit Wiskunde en Informatica

CIP-GEGEVENS KONINKLIJKE BIBLIOTHEEK, DEN HAAG
Ferreira, Maria da Conceição Fernández
Termination of term rewriting: well-foundedness, totality
and transformations / Maria da Conceição Fernández Ferreira. -
Utrecht: Universiteit Utrecht, Faculteit Wiskunde en Informatica
Thesis Universiteit Utrecht. - With summary in Dutch.
ISBN 90-393-1231-1
Subject headings: term rewriting systems; termination

The research of this thesis was supported by the Netherlands
Organisation for Scientific Research (NWO) under grant
612-16-605 (612-316-041).

*A Manuela, Daniël*
*y Kiko*

# Contents

# Acknowledgements

I would like to acknowledge every person that in a way or another had some influence in the elaboration of this thesis.

To the Netherlands Organisation for Scientific Research (NWO) I thank the financial support I received from them. I thank the department of Computer Science at Utrecht University for providing such a nice working environment. I also thank my colleagues for their helpfulness: lending a dictionary, correcting my Dutch, helping with LaTeX, (thanks Piet) or simply listening to my ravings, there was always somebody there.

The meetings of the TeReSe group at Amsterdam and the meetings of the FIP group at Utrecht taught me things I wouldn't otherwise have learnt. They also gave me the chance to expose my work and get helpful comments; my thanks to all the participants.

The review comittee of this thesis was formed by Prof. dr. P. Lescanne, Prof. dr. J. A. Bergstra, Prof. dr. J. W. Klop and dr. A. Middeldorp. To all of them my thanks and appreciation for accepting this task and for the comments and suggestions given. I also want to thank my promotor Prof. dr. J.-J. Ch. Meyer for his advice and comments on earlier versions of this manuscript.

Hans Zantema is my co-promotor and has been my supervisor for the last (almost) four years. He is a very enthousiastic and encouraging researcher; we had countless fruitful discussions about our work. It is difficult to express what his supervision meant for the elaboration of this thesis; I just want to say that it was a privilege to work with him.

I also want to thank my friends for their help and support. Aart did a terrific job proof-reading the work (answering his remarks was sometimes quite funny); thanks also to Femke for her comments and her friendship (and the greetings to "de generaal"), thanks to Daniël for the translation work. The lunches with Márcia, the support of people like Adriana, Maribel and Delia, Margarida and Vasco, helped me keeping in touch with my roots. To them, and the unnamed ones, many thanks.

These acknowledgements would be quite incomplete without mentioning my family. Grandmother Manuela, and my parents Maria del Carmen and Victor, have always guided me. They and the rest of the family lent support and patience; my apologies to all for being away so long.

At home, I was lucky enough to have Daniël and Kiko. They made sure I kept my mental sanity through the bad periods, and that I enjoyed the simple pleasures of everyday life.

I had the good fortune of writing this thesis in a foreign country. It was, and is, an extremely enriching experience, full of both exciting moments and pitfalls. At the end, not only the final result is important but perhaps even more so, the whole way that led to it. This is why I would like to close these acknowledgements with an excerpt of the poem *Ítaca*, by the Greek poet Constantin Cavafy.

... ...
*Terás sempre Ítaca no teu espírito,*
*que lá chegar é o teu destino último.*
... ...
*Ítaca deu-te essa viagem esplêndida.*
*Sem Ítaca não terias partido.*
*Mas Ítaca não tem nada mais para dar-te.*

*Por pobre que a descubras, Ítaca não te traiu.*
*Sábio come és agora, senhor de tanta experiência,*
*terás compreendido o sentido de Ítaca.*

# Chapter 1

# Introduction

Rewriting is a relatively new technique developed in Computer Science; as its name indicates it is a technique concerned with replacement of some objects by other objects following certain rules. Rewriting arises very naturally when reasoning about equations. Consider for example the natural numbers represented as $0, s(0), s(s(0)), \ldots$, and the following equations specifying the maximum of two natural numbers

$$\begin{array}{rcl} \mathtt{max}(x,0) & = & x \\ \mathtt{max}(0,x) & = & x \\ \mathtt{max}(s(x),s(y)) & = & s(\mathtt{max}(x,y)) \end{array}$$

If we want to determine what $\mathtt{max}(s(s(0)), s(s(s(0))))$ is, a possible way of doing it is to reason as follows:

$$\begin{array}{rcl} \mathtt{max}(s(s(0)), s(s(s(0)))) & = & s(\mathtt{max}(s(0), s(s(0)))) \\ & = & s(s(\mathtt{max}(0, s(0)))) \\ & = & s(s(s(0))) \end{array}$$

What is peculiar about the above reasoning is that the equations specifying `max` have been used in one direction. This one-directional use of equations leads to rewriting. A rewrite system is then a set of oriented equations or rules describing some relation between some objects. If our objects are terms in a "term algebra", we talk about term rewriting systems.

Consider the next example of a rewriting system (where the symbol $\rightarrow$ denotes in an obvious way what the intended direction of the rule is)

$$\begin{array}{rcl} \mathtt{white\ red} & \rightarrow & \mathtt{red\ white} \\ \mathtt{blue\ white} & \rightarrow & \mathtt{white\ blue} \\ \mathtt{blue\ red} & \rightarrow & \mathtt{red\ blue} \end{array}$$

Our objects are sequences over the alphabet $\{\mathtt{blue},\ \mathtt{white},\ \mathtt{red}\}$, and the rules dictate what transformations can be applied to those sequences. This system represents the "Dutch National Flag" game and traces its origins back to Dijkstra [30] (see also Dershowitz and Manna [27]). We will use it to illustrate some of the main concepts involved in term rewriting.

Consider the term

```
blue white red white blue
```

In order to obtain a (one-step) reduction from the term, we have to identify a part of it that matches the lefthand-side of some rewrite rule and then replace that matched part of the term with the righthand-side of the rule matched (in the presence of variables this is a little more complicated but the basic idea is the same). In Figure 1.1, we show all possible reductions for the term considered.

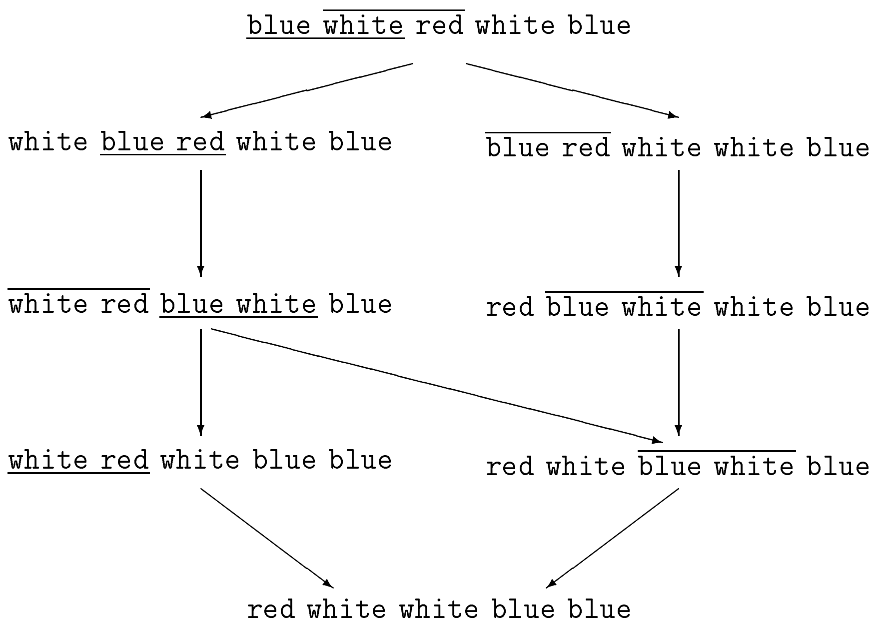



Figure 1.1: All possible reductions of the term `blue white red white blue`.

From this example we see that there are terms that may admit more than one reduction sequence. In our case, the term `blue white red white blue` admits the three reduction sequences shown in Figure 1.1 (in the leftmost sequence we always reduce the underlined terms, in the rightmost sequence we reduce the overlined terms and in the sequence going from left to right we start by reducing underlined terms and then go on reducing overlined terms). When several reduction sequences exist for a term, it may or may not be possible to join the different reductions, i. e., to find another term such that all reduction sequences end in that term. Systems in which this is possible are called *confluent* or said to have the *Church-Rosser* property. There are also terms that admit no reduction sequences, i. e., they are *normal forms*. Normal forms play an important role in the theory of rewriting and of particular interest are the systems in which each term admits a normal form (especially if the normal form is unique). Normal forms are related to another important property of rewriting systems, namely termination. A rewriting system is said to be *(uniformly) terminating*, *strongly normalizing* or *noetherian*, if no term admits an infinite reduction sequence.

Term rewriting systems (TRS's) constitute a simple formalism useful for the study of computational procedures. For equational reasoning, for example, term rewriting systems are used as interpreters for equational programs O'Donnell [84], for solving unification problems, Fay [31], in Knuth-Bendix completions, Knuth and Bendix [63] (where given an equational theory a TRS is constructed proving the same equations as the original theory, and providing a solution for the validity problem in the theory), proof by consistency procedures, Musser [80], abstract data types specifications and so forth.

Also of interest are several extensions of (first-order) rewriting. We mention some: graph-rewriting, Barendregt et al. [4], rewriting modulo equational theories, Jouannaud and Kirchner [50], Bachmair [1], Dershowitz and Jouannaud [26], term rewriting with priorities, Baeten, Bergstra and Klop [2], order-sorted rewriting, Goguen, Jouannaud and Meseguer [43], infinite rewriting, Kennaway, Klop, Sleep and de Vries [58], conditional rewriting, Brand, Darringer and Joyner Jr. [10], Bergstra and Klop [7], Dershowitz, Okada and Sivakumar [28], higher-order rewriting, Klop [60], Wolfram [105], Nipkow [83], van Oostrom [103]. Included in the last group are $\lambda$-Calculus, Church [12], Barendregt [3], used to formalize the notion of computability, and Combinatory Logic, Schönfinkel [98], Curry [13, 14], Curry and Feys [15], Curry, Hindley and Seldin [16], which has proved itself very useful in the implementation of programming languages.

## Overview of the thesis

This thesis is about termination of rewriting. Our setting is first-order term rewriting, extended in the last chapter to rewriting modulo equations. In general, given a finite TRS it is undecidable if the TRS terminates or not; there are nevertheless some techniques that are successful in particular cases. Generally speaking those techniques can be divided into two main groups: *syntactical* methods and *semantical* methods. In the first class only the syntactical structure of the terms is used to devise a proof of termination. The best known method of this type is probably the *recursive path order (rpo)* of Dershowitz [22]. In the second class, terms are interpreted compositionally in some (well-founded monotone) algebra in order to prove termination of the system. However, at the core of both classes of methods lies a *well-founded* order. Indeed, well-foundedness is the essential property needed for proving termination. Unfortunately checking if a particular order is well-founded is usually a difficult task. In chapter 3 we investigate this important property for orders on terms and give new criteria, in the style of Kruskal's theorem, that can be used to infer well-foundedness. The results obtained there can be applied, in particular, to prove well-foundedness of an extended version of the recursive path order that is defined in chapter 4. It turns out that recursive path order, though being a syntactic method, fits nicely in the semantic framework, as we see in chapter 4. In this chapter we also investigate the problem of defining path orders recursively and give a general framework for proving well-definedness of such orders. In chapter 5 we concentrate on the study of TRS's for which a proof of termination can be given by interpretation of terms in a *total well-founded monotone algebra*. TRS's having this property are said to be *totally terminating*. Interestingly enough, the class of totally terminating TRS's contains the class of TRS's for which a proof of termination using *rpo* can be given. Finally, in chapter 6 we present a method for simplifying the task of proving termination of TRS's. The method consists of transformations on TRS's induced by transformations on terms, that are sound with respect to termination (i. e., if the transformed system terminates, so does the original one). The method remains valid for rewriting modulo a set of equations as long as the equations satisfy some minor restrictions.

We summarize what we believe are the main results in this thesis:

- the new criteria for proving well-foundedness of term orderings; unlike Kruskal's theorem, our criteria cover all terminating TRS's.
- the characterization of the algebras associated to totally terminating TRS's and modularity of total termination.
- a new transformational method for proving termination; we emphasize that, unlike most syntactical methods, this approach remains valid in the presence of equations.

We review briefly the chapters and subjects not yet mentioned. In chapter 2 we will introduce the main concepts of the theory of rewriting and of orders. With respect to termination, we will concentrate on the semantical framework; well-founded monotone algebras will be defined and linked to termination of TRS's. Terminating TRS's can be divided in categories based on the properties of the algebras used for termination proofs; such a classification is presented and discussed. We will also discuss undecidability of termination. As mentioned before, in chapter 5 we will concentrate on total termination. A lot of the techniques used in this chapter come from the theory of ordinals. In order to keep the thesis self-contained, we will give a summary of the theory of ordinals needed.

In Appendix A we present a proof of a well-known result about extending well-founded orders to total well-founded ones. The reason why we include such a well-known result is that it is quite hard to find a proof of it in the literature. Finally in Appendix B contains a proof of undecidability of termination and a comparison between the proof given and the work of Huet and Lankford [47].

Most of the results presented in this thesis have already been published elsewhere. Chapter 3 was presented in Ferreira and Zantema [34], chapter 4 is based on Ferreira and Zantema [33] and chapter 5 on Ferreira and Zantema [32, 33]. Chapter 6 is unpublished (though it originates in the work presented at Ferreira and Zantema [35]).

As far as dependecies between the chapters are concerned, chapter 2 contains most of the notions needed to read the rest of the thesis so its reading should precede the reading of any other chapter. As for chapters 3, 4, 5 and 6, and appendices A and B, though obviously connected, they can be read as independent units, so no order is imposed. We suggest the order in which they are presented since it seemed the most natural to us.

# Chapter 2

# Preliminaries

In this chapter we introduce most of the notions over first-order term rewriting and partial orders needed in the sequel. More complete information about term rewriting and its applications can be found in the surveys of Klop [61], Dershowitz and Jouannaud [26], and Plaisted [89]. For more information on partial orders, see for example Fraïssé [37].

Throughout this chapter (and the rest of the thesis) we will use the following convention: whenever an object (relation, set, etc.) is defined inductively, we always have in mind the smallest object of the same type as the one being defined, satisfying the conditions specified in the definition, i. e., all other objects of the same type satisfying the conditions of the definition, will contain the object being defined.

## 2.1 Sets, Relations and Orders

Given sets $A_i$, with $1 \leq i \leq n$, for some fixed natural $n \geq 0$, their *cartesian product* is given by $A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) : \ a_i \in A_i \text{ for all } 1 \leq i \leq n\}$; if $n = 0$, the cartesian product contains only one element. If $A_i = A$ for all $1 \leq i \leq n$, we abbreviate this product to $A^n$. For any set $S$, the set of its parts or *powerset* is denoted by $\mathcal{P}(S)$ and defined by $\mathcal{P}(S) = \{A | \ A \subseteq S\}$.

Given $n$ sets $S_i$, $1 \leq i \leq n$, a $n$-ary relation $\Theta$ over $S_1 \times \ldots \times S_n$ is a subset of $S_1 \times \ldots \times S_n$. Most of the relations we will be concerned with are binary relations, i. e., subsets of $A \times B$, for some sets $A, B$. For this type of relation, we use the notation $(u, v) \in \Theta$ or $u \Theta v$ meaning that the elements $u$ and $v$ are in the relation $\Theta$. Given two (binary) relations $\Theta$ and $\Xi$ over $A \times B$ and $B \times C$, respectively, for some sets $A, B, C$, their composition is denoted by $\Theta \circ \Xi$ and is a (binary) relation on $A \times C$, defined as follows.

$$(u, v) \in \Theta \circ \Xi \iff \exists w \in B : \ (u, w) \in \Theta \wedge (w, v) \in \Xi$$

If $\Theta$ is a binary relation over a set $S$, we use the notation $\Theta^n$ to denote the composition of $\Theta$ with itself $n$ times (if $n = 0$ then $\Theta^0$ is the identity, i. e. the relation containing the pairs $(s, s)$, for any element $s$ in $S$). Composition of relations is an *associative* operation, i. e., $\Theta \circ (\Gamma \circ \Upsilon) = (\Theta \circ \Gamma) \circ \Upsilon$, whenever the compositions are well-defined.

Given a binary relation $\Theta$ over $A \times B$, its inverse is denoted by $\Theta^{-1}$ and is a binary relation on $B \times A$ given by $b\Theta^{-1}a \iff a\Theta b$.

A function $f : A \to B$ (where $A$ is said to be the *domain* of the function and $B$ its *codomain*) is a particular kind of relation, since it can be represented (and represents) the set $\{(a, f(a))|\ a \in A\} \subseteq A \times B$. Among the properties of functions we are interested in are the following:

- *injectivity*: $\forall x, y \in A :\ f(x) = f(y) \Rightarrow x = y$,
- *surjectivity*: $\forall y \in B\ \exists x \in A :\ f(x) = y$.

A function which is simultaneously injective and surjective is called *bijective* or said to be a *bijection*. In this case the sets $A$ and $B$ are in a one-to-one correspondence.

Composition of functions is also denoted by "$\circ$" and defined as follows. If $f : A \to B$ and $g : B \to C$ are functions, its composition is the function $g \circ f : A \to C$ with $(g \circ f)(x) = g(f(x))$. Composition of functions, when defined, is also an associative operation.

We will use a particular kind of function named *projection* . If $A = A_1 \times \ldots \times A_n$, $n \geq 1$, for some sets $A_i$, $1 \leq i \leq n$, the projection of order $i$, or $i^{th}$ projection is a function $\pi_i : A_1 \times \ldots \times A_n \to A_i$ given by $\pi_i(a_1, \ldots , a_n) = a_i$.

**Definition 2.1.** A binary relation $\Theta$ on a set $S$ is said to be

- *(ir)reflexive* if $(\neg)s\Theta s$, for all $s \in S$,
- *symmetric* if $\Theta$ satisfies $(u\Theta v \Rightarrow v\Theta u)$, for all $u, v \in S$,
- *anti-symmetric* if $\Theta$ satisfies $(u\Theta v \wedge v\Theta u \Rightarrow u = v)$, for all $u, v \in S$,
- *transitive* if $\Theta$ satisfies $(u\Theta v \wedge v\Theta w \Rightarrow u\Theta w)$, for all $u, v, w \in S$.

**Definition 2.2.** Given a binary relation $\Theta$, the *reflexive closure* of $\Theta$ is denoted by $\Theta^=$ and is the smallest reflexive binary relation containing $\Theta$. The *transitive closure* of $\Theta$ is denoted by $\Theta^+$ and is the smallest transitive binary relation containing $\Theta$. The *transitive-reflexive closure* of $\Theta$ is denoted by $\Theta^*$ and is the smallest transitive and reflexive binary relation containing $\Theta$.

**Remark 2.3.** It is not difficult to see that for a binary relation $\Theta \subseteq S \times S$, its transitive closure satisfies $\Theta^+ = \bigcup_{n\geq 1} \Theta^n$, and its transitive-reflexive closure satisfies $\Theta^* = \bigcup_{n\geq 0} \Theta^n$. Note also that $\Theta^+ = \Theta \circ \Theta^* = \Theta^* \circ \Theta$.

**Definition 2.4.** A binary relation $\sim$ on a set $S$ is said to be an *equivalence relation* if it is reflexive, symmetric and transitive. The set $S/{\sim} = \{\langle s\rangle_\sim|\ s \in S\}$ is the quotient of $S$ modulo $\sim$ and $\langle s\rangle_\sim$ is the $\sim$-equivalence class of the element $s \in S$, i. e., $\langle s\rangle_\sim = \{x \in S|\ x \sim s\}$.

Note that the equivalence class of an element does not depend on the element chosen for its representative. In the following when describing equivalence classes we will only explicitly indicate which equivalence we refer to if that is not clear from context.

**Definition 2.5.** A binary relation on a set $S$ is called a (strict) *partial order*, or simply *order*, over $S$ if it is a transitive and irreflexive relation on $S$. We use the terminology *poset* meaning a set with a partial order. The partial order is usually denoted by $>$ and its inverse by $<$.

Note that the inverse of a partial order is also a partial order. Furthermore, a partial order can also be characterized by reflexivity, anti-symmetry and transitivity. It is not difficult to see that there is a one-to-one correspondence between strict partial orders and partial orders. In the following we will use the terminology *partial order* in the strict sense.

**Definition 2.6.** A *quasi-order* over a set $S$ is a transitive and reflexive relation over $S$. We denote such relations in general by $\succeq$.

Quasi-orders also appear in the literature under the name *pre-orders*. Any quasi-order defines an equivalence relation, namely $\succeq \cap \preceq$, and a partial order, namely $\succeq \setminus \preceq$ (or its inverse $\preceq \setminus \succeq$). We usually denote the induced equivalence relation by $\sim$ and the induced partial order by $\succ$. But when need arises, we will also use the following notation:

**Definition 2.7.** If $\geq$ is a quasi-order over a set $S$ then $\mathtt{ord}(\geq) = \geq \setminus \leq$ and $\mathtt{eq}(\geq) = \geq \cap \leq$, i. e., $\mathtt{ord}(\geq)$ represents a partial order contained in $\geq$, and $\mathtt{eq}(\geq)$ represents the equivalence relation contained in $\geq$.

Conversely, given a partial order $\succ$ and an equivalence $\sim$, their union does not always define a quasi-order (the transitive closure of their union does). However if $\succ$ and $\sim$ satisfy

$$(\sim \ \circ \succ \ \circ \ \sim) \ \subseteq \ \succ \tag{2.1}$$

where $\circ$ represents composition, then $\succ \ \cup \ \sim$ is a quasi-order, of which $\succ$ is the strict part and $\sim$ the equivalence part.

**Remark 2.8.** From now on if we characterize a quasi-order via $\succ \cup \sim$, we assume that the condition (2.1) is satisfied. Also we take as partial order defined by a quasi-order $\succeq$ the relation $\succ \ = \ \succeq \ \setminus \ \preceq$. Note that if $\succ$ and $\sim$ satisfy condition 2.1, then $\succ \cap \sim = \emptyset$, as we want it to be: if this condition is not satisfied we have that $a \succ b \sim a$, for some elements $a, b$, and this conflicts either with irreflexivity or condition 2.1.

**Definition 2.9.** Given a quasi-order $\succeq$ over $S$ and the quotient $S/\!\sim$ consisting of the ($\sim$-) equivalence classes of $\sim$ (which are are denoted by $\langle \ \rangle$), we can extend $\succ$ to $S/\!\sim$ in a natural way, namely by defining $\langle s \rangle \sqsupset \langle t \rangle$ if and only if $s \succ t$.

The following lemma is not difficult to prove.

**Lemma 2.10.** *In the conditions of definition 2.9, the relation $\sqsupset$ on $S/\!\sim$ is well-defined. Furthermore $\sqsupset$ is a partial order over $S/\!\sim$.*

Note that well-definedness means that $\sqsupset$ does not depend on the class representative and is a consequence of the fact that $\succ$ and $\sim$ satisfy condition (2.1). When the extension $\sqsupset$ is well-defined we abusively write $\succ$ instead of $\sqsupset$.

**Definition 2.11.** Let $>$ (respectively $\geq$) be a partial order (respectively quasi-order) on a set $S$ and let $>'$ (respectively $\geq'$) be a partial order (respectively quasi-order) on a set $S'$. A function $f : S \to S'$ is said to be:

- *order-preserving* or *weakly monotone* if $\forall x, y \in S : \ x \geq y \Rightarrow f(x) \geq' f(y)$,
- *(strictly) monotone* or *(strictly) increasing* if $\forall x, y \in S : \ x > y \Rightarrow f(x) >' f(y)$,
- *order-isomorphism* if it is bijective and satisfies: $\forall x, y \in S : \ x \geq y \iff f(x) \geq' f(y)$. If $\geq$ is anti-symmetric then bijectivity is replaced by surjectivity.

**Definition 2.12.** Given two partial orders $>, >'$ (respectively quasi-orders $\succeq$ and $\succeq'$) over some set $S$, we say that $>$ *extends* $>'$ (respectively $\succeq$ *extends* $\succeq'$) if and only if $>' \subseteq >$ (respectively $\succ' \subseteq \succ$ and $\sim' \subseteq \sim$).

**Remark 2.13.** In the case of quasi-orders, we could define extension simply as $\succeq \subseteq \succeq'$. However this would not allow us to say anything about the relation between the respectives strict and equivalence parts associated with the quasi-orders. The following example illustrates what we mean. Suppose $S = \{a, b\}$ and that $\succeq$, $\succeq'$, in addition to the reflexive property, satisfy $a \succeq b$ while not $b \succeq a$, and $a \succeq' b$, $b \succeq' a$. Clearly $\succeq \subseteq \succeq'$ but we have $\succ = \{(a, b)\}, \sim = \{(a, a), (b, b)\}$ and $\succ' = \emptyset, \sim' = S \times S$. Since we want to avoid this situation we define extension as in 2.12.

Before introducing the definition of well-foundedness we make some comments on the notion of sequence and indexing. A *sequence* of elements of a set $S$ is just a function $\zeta : I \to S$, where $I$ represents a set of indexes. The function $\zeta$ is an indexing function; we will use indexing over the natural numbers (or parts of it) and ordinals. Usually a sequence is represent by $(s_i)_{i \in I}$, instead of giving the function $\zeta$ explicitly. Actually $s_i = \zeta(i)$, so the function is implicit in the notation. A *sub-sequence* $(t_i)_{i \in I}$ of a sequence $(s_i)_{i \in I}$ is given by a injective function $\phi : I \to I$ such that $t_i = \zeta(\phi(i)) = s_{\phi(i)}$, for all $i \in I$, and where $\zeta$ is the function giving the sequence $(s_i)_{i \in I}$. When the set $I$ is partially ordered by $>$, sub-sequences are assumed to be order-preserving, i. e., $i > j \Rightarrow \phi(i) > \phi(i)$.

When indexing sequences over $\mathbb{N}$ (or subsets of it) or ordinals, we sometimes use the notation $(s_i)_{i \geq k}$, for fixed $k$, meaning the domain of the indexing contains all elements greater or equal to $k$.

**Definition 2.14.** Given a partial order $\succ$ (respectively quasi-order $\succeq$) over some set $S$, we say that $\succ$ (respectively $\succeq$) is *well-founded* if and only if $\succ$ (respectively $\succeq$) has no infinite descending sequences, i. e., there are no sequences of the form $s_0 \succ s_1 \succ s_2 \succ \ldots$. We extend the terminology well-founded to the elements of $S$: $s \in S$ is well-founded (with respect to a given partial or quasi-order) if and only if $s$ does not occur in any infinite descending chain.

We will also need a concept similar to well-foundedness for arbitrary binary relations.

**Definition 2.15.** Given a binary relation $\theta$ over a set $S$, we say that $\theta$ is *well-founded* or *terminating* if there is no infinite sequence $(s_i)_{i \in \mathbb{N}}$ of elements of $S$, such that $s_i \ \theta \ s_{i+1}$, for all $i \in \mathbb{N}$. An element $s \in S$ is well-founded or terminating if and only if $s$ does not occur in any such infinite sequence.

**Definition 2.16.** Given a quasi-order $\succeq$ over some set $S$, we say that $\succeq$ is *total* if and only if for any elements $u, v \in S$ we have either $u \sim v$ or $u \succ v$ or $v \succ u$. A partial order $>$ is total if the above assertion holds with $\sim$ replaced by equality.

The following is also a standard result.

**Lemma 2.17.** *Let $(S, \geq)$ be a quasi-ordered set and let $\sqsupset$ be the extension of $>$ to the equivalence classes, i. e., to $S/\sim$. Then $\geq$ is total (respectively well-founded) on $S$ if and only if $\sqsupset$ is total (respectively well-founded) on $S/\sim$.*

We consider two useful extensions of partial orders, namely the *multiset* and *lexicographic* extensions. First we have to define the domain of these extensions.

**Definition 2.18.** Let $S$ be any set. A *finite multiset* over $S$ is a function $\rho : S \to \mathbb{N}$ such that the set $\{s \in S | \ \rho(s) \neq 0\}$ is finite. The set of all finite multisets over $S$ is denoted by $\mathcal{M}(S)$.

Intuitively a finite multiset is a finite set where elements can be repeated finitely many times. For any $s \in S$, $\rho(s)$ just gives the frequency (number of occurrences) of the element $s$ in the multiset.

**Notation 2.19.** We will use a set-like notation $\{\!\{ \ \}\!\}$ to denote a multiset. Operations similar to the ones applied on sets (e. g. $\in$, $\cup$, $\subseteq$ etc.) are also applied to multisets. We will use round symbols to denote operations on sets (e. g. $\subseteq$) and similar squared symbols for the same operation on multisets (e. g. $\sqsubseteq$), whenever possible. Some operations, like $\in$, $\setminus$, will be denoted ambiguously by the same symbol. In the following we abbreviate finite multiset to multiset.

**Definition 2.20.** Let $\rho, \pi$ be arbitrary multisets over a set $S$. The operations $\in$, $\sqcup$, $\sqsubseteq$, $\setminus$, $\sqcap$ on $\mathcal{M}(S)$, the set of finite multisets over $S$, are defined as follows:

- $\forall s \in S : \ s \in \rho \iff \rho(s) > 0$,
- $\rho \sqcup \pi$ is the multiset defined by $(\rho \sqcup \pi)(s) = \rho(s) + \pi(s)$, for all $s \in S$,
- $\rho \sqsubseteq \pi \iff \forall s \in S : \ \rho(s) \leq \pi(s)$. If the last inequality is strict for all $s \in S$ then we have strict inclusion of multisets, i. e., $\rho \sqsubset \pi$,
- $\rho \sqcap \pi$ is the multiset defined by $(\rho \sqcap \pi)(s) = min\{\rho(s), \pi(s)\}$, for all $s \in S$,

- $\rho \setminus \pi$ is the multiset defined by $(\rho \setminus \pi)(s) = max\{\rho(s) - \pi(s), 0\}$, for all $s \in S$.

**Example 2.21.** Consider $(\mathbb{N}, >)$, the natural numbers with the usual order, and let $\rho$ be the multiset given by $\rho(0) = 4$, $\rho(1) = \rho(2) = 0$, $\rho(3) = 2$ and $\rho(n) = 0$, for $n \geq 4$. We can also represent $\rho$ as $\{\!\{0, 0, 0, 0, 3, 3\}\!\}$.

In the following we will use multiset union indexed over finite multisets. We define what that is.

**Definition 2.22.** Let $\phi : A \to \mathcal{M}(A)$ be a function. We extend this function to a function $\overline{\phi} : \mathcal{M}(A) \to \mathcal{M}(A)$ as follows:

- $\overline{\phi}(\emptyset) = \emptyset$,
- $\overline{\phi}(\{\!\{a\}\!\}) = \phi(a)$,
- $\overline{\phi}(X \sqcup Y) = \overline{\phi}(X) \sqcup \overline{\phi}(Y)$.

We sometimes use the notation $\bigsqcup_{s \in S} \phi(s)$ instead of $\overline{\phi}(S)$.

In order to see that the previous definition makes sense, we have to show that it does not depend on the choices of $X$ and $Y$, i. e., if $X \sqcup Y = X' \sqcup Y'$ then

$$(\bigsqcup_{x \in X} \phi(x)) \bigsqcup (\bigsqcup_{y \in Y} \phi(x)) = (\bigsqcup_{x \in X'} \phi(x)) \bigsqcup (\bigsqcup_{y \in Y'} \phi(x))$$

This is cumbersome but not difficult to show if we use the relations $(X \sqcup Y) \setminus X' = Y'$, $(X \sqcup Y) \setminus Y' = X'$, symmetrical relations in the roles of $X, Y$ and $X', Y'$, and the properties of the operations on multisets.

Similarly we can extend functions with more than one argument to multisets.

**Definition 2.23.** Let $\phi : A^n \to \mathcal{M}(A)$ (with $n \geq 1$) be a function. We extend this function to a function $\overline{\phi} : \mathcal{M}(A)^n \to \mathcal{M}(A)$ as follows:

- $\overline{\phi}(\ldots, \emptyset, \ldots) = \emptyset$,
- $\overline{\phi}(\{\!\{a_1\}\!\}, \ldots, \{\!\{a_n\}\!\}) = \phi(a_1, \ldots, a_n)$,
- $\overline{\phi}(X_1, \ldots, X_i \sqcup Y, \ldots X_n) = \overline{\phi}(X_1, \ldots, X_i, \ldots X_n) \sqcup \overline{\phi}(X_1, \ldots, Y, \ldots, X_n)$.

We sometimes use the notation $\bigsqcup_{s_1 \in S_1} \ldots \bigsqcup_{s_n \in S_n} \phi(s_1, \ldots, s_n)$ instead of $\overline{\phi}(S_1, \ldots, S_n)$.

Note that similar observations as for definition 2.22 apply here.

**Definition 2.24.** Let $S$ be any set and $n \in \mathbb{N}$, fixed. Then $S^n$ represents the set of sequences of elements of $S$ of size exactly $n$. $S^* = \bigcup_{k \geq 0} S^k$ represents all possible sequences over $S$, where $S^0$ contains only the empty sequence $\epsilon$. We use the notation $S^{\leq n}$ for the set $\bigcup_{k=0}^{n} S^k$. Elements of $S^k$, for any $k$, are denoted by $s_1 \cdots s_k$, where "$\cdot$" denotes concatenation.

Note that we use the notation $S^n$ ambiguously: it either represents the cartesian product of $S$ with itself $n$ times or the set of sequences of elements of $S$ of size $n$. For any $n \geq 0$ those sets are in a one-to-one correspondence. For $n = 0$, both sets contain only one element. It should be clear from context what is the meaning associated to $S^n$.

We now consider posets and define the multiset and lexicographic extension of the orders. The following definition is due to Dershowitz and Manna [27].

**Definition 2.25.** Let $(S, >)$ be a poset. The multiset extension of $>$ over $\mathcal{M}(S)$ is denoted by $>_{mul}$ and defined as follows: $X >_{mul} Y$ if and only if there are multisets $X_0, Y_0 \in \mathcal{M}(S)$ satisfying

- $X_0 \neq \emptyset$ and $X_0 \subseteq X$,
- $Y = (X \setminus X_0) \cup Y_0$,
- $\forall y \in Y_0 \; \exists x \in X_0 : \; x > y$.

In [51], Jouannaud and Lescanne proved that the above definition is equivalent to the definition of Huet and Oppen [48] we present below.

**Definition 2.26.** Let $(S, >)$ be a poset. The multiset extension of $>$ over $\mathcal{M}(S)$ is defined as follows: $X >_{mul} Y$ if $X$ and $Y$ satisfy

- $X \neq Y$, and
- $(\exists y \in S : \; Y(y) >_{\mathbb{N}} X(y)) \Rightarrow (\exists x \in S : \; x > y$ and $X(x) >_{\mathbb{N}} Y(x))$.

We will also need to use a multiset extension of general binary relations. The intention behind the definition is that should the relation lifted be a partial order, then the multiset extension of the relation coincides with the multiset extension for partial orders. This property is enjoyed by the following definition taken from Middeldorp [76].

**Definition 2.27.** Let $S$ be a set and $\theta$ a binary relation on $S$. The multiset extension of $\theta$ is a binary relation on $\mathcal{M}(S)$, denoted by $\theta_{mul}$ and defined as follows: $X \; \theta_{mul} \; Y$ if and only if there are multisets $X_0, Y_0 \in \mathcal{M}(S)$ satisfying

- $X_0 \neq \emptyset$ and $X_0 \subseteq X$,
- $Y = (X \setminus X_0) \cup Y_0$,
- $\forall y \in Y_0 \; \exists x \in X_0 : \; x \theta y$.

The following lemma is proven in Dershowitz and Manna [27].

**Lemma 2.28.** *If $(S, >)$ is a poset then $(\mathcal{M}(S), >_{mul})$ is also a poset. Furthermore, $>$ is well-founded (respectively total) on $S$ if and only if $>_{mul}$ is well-founded (respectively total) on $\mathcal{M}(S)$.*

Recall the definition of $\overline{\phi}$ from 2.23. We have the following result.

**Lemma 2.29.** *Let $(A, >)$ be a poset and consider the poset $(\mathcal{A}, >_{mul})$. If $\phi : A^n \to \mathcal{M}(A)$, with $n \geq 1$, is strictly monotone in all arguments, then $\overline{\phi} : \mathcal{M}(A)^n \to \mathcal{M}(A)$ is also strictly monotone in all arguments provided that if $n > 1$, the arguments of $\overline{\phi}$ are non-empty multisets.*

**Proof** We see that $\overline{\phi}$ is strictly monotone in each argument. Suppose $X_i' >_{mul} X_i$, then we can write $X_i = (X_i' \setminus X) \sqcup Y$, for some multisets $X, Y$ satisfying

- $X \neq \emptyset$, $X \subseteq X_i'$,
- $\forall y \in Y\ \exists x \in X :\ x > y$.

Then

$$\overline{\phi}(X_1, \dots, X_i, \dots, X_n) \;=\; \overline{\phi}(X_1, \dots, X_i' \setminus X, \dots, X_n) \bigsqcup \overline{\phi}(X_1, \dots, Y, \dots, X_n)$$

But we can also write $X_i' = (X_i' \setminus X) \sqcup X$, so

$$\overline{\phi}(X_1, \dots, X_i', \dots, X_n) \;=\; \overline{\phi}(X_1, \dots, X_i' \setminus X, \dots, X_n) \;\sqcup\; \overline{\phi}(X_1, \dots, X, \dots, X_n).$$

So it suffices to see that for each element in $\overline{\phi}(X_1, \dots, Y, \dots, X_n)$ there is an element in $\overline{\phi}(X_1, \dots, X, \dots, X_n)$ bounding it. For that it is essential that this last multiset is never empty and that implies that we have to restrict ourselves to arguments which are themselves not empty. Take then $\phi(x_1, \dots, y, \dots, x_n) \in \overline{\phi}(X_1, \dots, Y, \dots, X_n)$. Since $y \in Y \Rightarrow \exists x \in X :\ x > y$ and $\phi$ is monotone in all arguments, we have that $\phi(x_1, \dots, x, \dots, x_n) > \phi(x_1, \dots, y, \dots, x_n)$, for that particular $x$.
Since $\phi(x_1, \dots, x, \dots, x_n) \in \overline{\phi}(X_1, \dots, X, \dots, X_n)$, we are done. □

**Definition 2.30.** Let $(S, >)$ be a poset. The *lexicographic extension* of $>$ over $S^n$, $S^{\leq n}$ (for some fixed $n \in \mathbb{N}$) or $S^*$ is defined as follows:

$$u_1 \cdots u_k >_{lex} v_1 \cdots v_m \iff \begin{cases} m < k \wedge \forall 1 \leq j \leq m :\ u_j = v_j, \text{ or} \\ \exists 1 \leq j \leq min\{m, k\} : (u_j > v_j) \wedge (\forall 1 \leq i < j : u_i = v_i)) \end{cases}$$

Note that when restricted to $S^n$, the first condition is irrelevant. As for multiset extensions, we will need to consider lexicographic extensions of arbitrary binary relations. The definition is similar to definition 2.30.

**Definition 2.31.** Let $S$ be a set and $\theta$ an arbitrary binary relation on $S$. The *lexicographic extension* of $\theta$ over $S^n$, $S^{\leq n}$ (for some fixed $n \in \mathbb{N}$) or $S^*$ is defined as follows:

$$u_1 \cdots u_k \ \theta_{lex} \ v_1 \cdots v_m \iff \begin{cases} m < k \wedge \forall 1 \leq j \leq m : \ u_j = v_j, \text{ or} \\ \exists 1 \leq j \leq min\{m, k\} : (u_j \ \theta \ v_j) \text{ and } (\forall 1 \leq i < j : u_i = v_i)) \end{cases}$$

We have a result similar to lemma 2.28.

**Lemma 2.32.** *If $(S, >)$ is a poset then $(S^n, >_{lex})$, $(S^{\leq n}, >_{lex})$ and $(S^*, >_{lex})$ are also posets. Furthermore, $>$ is well-founded on $S$ if and only if $>_{lex}$ is well-founded on $S^n$ or $S^{\leq n}$ and $>$ is total on $S$ if and only if $>_{lex}$ is total on $S^n$, $S^{\leq n}$ or $S^*$.*

Note that if $>$ is well-founded, $>_{lex}$ is not necessarily well-founded on $S^*$, as the following example shows.

**Example 2.33.** Let $S = \{a, b\}$ with $a > b$. Then we have the infinite descending chain

$$a >_{lex} ba >_{lex} bba >_{lex} bbba >_{lex} \ldots$$

This problem can easily be avoided if we take the length of the sequence into consideration, i. e., if we define

$$u_1 \cdots u_k >^*_{lex} v_1 \cdots v_m \iff \begin{cases} k > m, \text{or} \\ m = k \text{ and } u_1 \cdots u_k >_{lex} v_1 \cdots v_m \end{cases}$$

We have that $>^*_{lex}$ is a partial order whenever $S$ is a partial order. Furthermore $>^*_{lex}$ is well-founded (respectively total) if and only if $>$ is well-founded (respectively total).

Sometimes we are also interested in the lexicographic combination of orders over possibly different sets.

**Definition 2.34.** Given $n \geq 1$ posets $(A_i, >_i)$, then $\succ$, the lexicographic product of the orders $>_i$, $1 \leq i \leq n$, over the set $A_1 \times \ldots \times A_n$, is defined as

$$(u_1, \ldots, u_n) \succ (v_1, \ldots, v_n) \iff \begin{cases} \exists 1 \leq j \leq n : & (u_j >_j v_j \text{ and} \\ & (\forall 1 \leq i < j : u_i = v_i)) \end{cases}$$

This lexicographic product preserves totality and well-foundedness.

**Lemma 2.35.** *Let $(A_i, >_i)$ be posets, with $1 \leq i \leq n$, for some fixed natural $n \geq 1$. Then $\succ$ is a partial order over $A_1 \times \ldots \times A_n$. Furthermore $\succ$ is well-founded (respectively total) over $A_1 \times \ldots \times A_n$ if and only if $>_i$ is well-founded (respectively total) over $A_i$, for all $1 \leq i \leq n$.*

We can also define the multiset and lexicographic extensions and lexicographic product for quasi-orders. Direct definitions similar to the definitions 2.25, 2.30 and 2.34, can be given, but the simplest way of defining these concepts is, in our view, to consider the equivalence classes.

**Definition 2.36.** Let $\geq\ =\ >\cup\sim$ be a quasi-order over $S$ and let $\langle a\rangle$ denote the $\sim$-equivalence class of the element $a \in S$. Let $\sqsupset$ denote the extension of $>$ to the quotient $S/\sim$ of the $\sim$-equivalence classes, and $\sqsupset_{mul}$ its multiset extension on $\mathcal{M}(S/\sim)$. The multiset extension of $\geq$ is denoted by $\geq_{mul}$ and defined as follows:

$$\{\!\{a_1,\cdots,a_m\}\!\}\ \mathtt{eq}(\geq_{mul})\ \{\!\{b_1,\cdots,b_n\}\!\} \iff \{\!\{\langle a_1\rangle,\cdots,\langle a_m\rangle\}\!\} = \{\!\{\langle b_1\rangle,\cdots,\langle b_n\rangle\}\!\}$$

$$\{\!\{a_1,\cdots,a_m\}\!\}\ \mathtt{ord}(\geq_{mul})\ \{\!\{b_1,\cdots,b_n\}\!\} \iff \{\!\{\langle a_1\rangle,\cdots,\langle a_m\rangle\}\!\} \sqsupset_{mul} \{\!\{\langle b_1\rangle,\cdots\langle b_n\rangle\}\!\}$$

**Definition 2.37.** Let $\geq\ =\ >\cup\sim$ be a quasi-order over $S$ and let $\langle a\rangle$ denote the $\sim$-equivalence class of the element $a \in S$. Let $\sqsupset$ denote the extension of $>$ to the quotient $S/\sim$ of the $\sim$-equivalence classes, and $\sqsupset_{lex}$ its lexicographic extension on $(S/\sim)^*$ ($(S/\sim)^n$, $(S/\sim)^{\leq n}$, for some $n$). The lexicographic extension of $\geq$ is denoted by $\geq_{lex}$ and defined as follows:

$$\begin{array}{lcl} a_1\cdot\ldots\cdot a_m\ \mathtt{eq}(\geq_{lex})\ b_1\cdot\ldots\cdot b_n & \iff & \langle a_1\rangle\cdot\ldots\cdot\langle a_m\rangle = \langle b_1\rangle\cdot\ldots\cdot\langle b_n\rangle \\ a_1\cdot\ldots\cdot a_m\ \mathtt{ord}(\geq_{lex})\ b_1\cdot\ldots\cdot b_n & \iff & \langle a_1\rangle\cdot\ldots\cdot\langle a_m\rangle \sqsupset_{lex} \langle b_1\rangle\cdot\ldots\cdot\langle b_n\rangle \end{array}$$

It is important to note that both $\mathtt{ord}(\geq_{lex})$ and $\mathtt{ord}(\geq_{mul})$ are different from the lexicographic and multiset extensions, respectively, of $>$, the strict part of $\geq$. Consider the set $S = \{a, b\}$ and the quasi-order $\geq$ satisfying reflexivity and $a \geq b$ and $b \geq a$. Then $>$ is the empty relation. We have that $a\cdot a\ \mathtt{ord}(\geq_{lex})\ b$ and $\{\!\{a,a\}\!\}\ \mathtt{ord}(\geq_{mul})\ \{\!\{b\}\!\}$, while $a\cdot a \not>_{lex} b$ and $\{\!\{a,a\}\!\} \not>_{mul} \{\!\{b\}\!\}$.

The relations $\succeq_{lex}$ and $\succeq_{mul}$ are themselves quasi-orders, satisfying condition 2.1 and preserving both well-foundedness and totality. More precisely:

**Lemma 2.38.** *In the conditions of definition 2.36, $\geq_{mul}=\mathtt{ord}(\geq_{mul})\cup\mathtt{eq}(\geq_{mul})$ is a quasi-order satisfying condition 2.1. Furthermore $\geq$ is well-founded (respectively total) over a set $S$ if and only if $\geq_{mul}$ is well-founded (respectively total) over $\mathcal{M}(S)$.*

**Lemma 2.39.** *In the conditions of definition 2.37, $\geq_{lex}=\mathtt{ord}(\geq_{lex})\cup\mathtt{eq}(\geq_{lex})$ is a quasi-order satisfying condition 2.1. Furthermore $\geq$ is well-founded over a set $S$ if and only if $\geq_{lex}$ is well-founded over $S^n$ (or $S^{\leq n}$), for a fixed $n \geq 1$. Also $\geq$ is total over a set $S$ if and only if $\geq_{lex}$ is total over $S^n$ ($S^{\leq n}$ or $S^*$).*

The lexicographic product of $n \geq 1$ quasi-orders $(A_i, \succeq_i)$, $1 \leq i \leq n$, is defined similarly to the lexicographic product of partial orders, we only need to change equality in definition 2.34 to $\sim_i$, the equivalence relation contained in $\succeq_i$, while the equivalence relation associated to the lexicographic product is defined by using equality of the equivalence classes, as in definition 2.37. Then lemma 2.35 can also be stated for quasi-orders.

In the following we introduce an important class of well-founded (quasi-) orders.

**Definition 2.40.** A quasi-order $\succeq$ (respectively partial order $\succ$) over a set $S$ is a *well quasi-order* (respectively a *partial well-order*), abbreviated to *wqo* (respectively to *pwo*), if and only if every quasi-order (respectively partial order) extending it, including itself, is well-founded.

**Definition 2.41.** Given a quasi-order $\succeq$ (respectively partial order $\succ$) over a set $S$, a sequence $(s_i)_{i\geq 0}$ of elements of $S$ is *good* if there are indices $0 \leq i < j$ with $s_j \succeq s_i$. If no such indices exist, the sequence is named *bad*.

There are several equivalent characterizations of *wqo*'s and *pwo*'s. We also use the following (see Gallier [38], Middeldorp and Zantema [78]):

**Lemma 2.42.** *Given a quasi-order $\succeq$ (respectively partial order $\succ$) over a set $S$, the following assertions are equivalent:*

1. *$\succeq$ (respectively $\succ$) is a* wqo *(respectively* pwo*).*
2. *Every infinite sequence $(s_i)_{i\geq 0}$ of elements of $S$ contains an infinite subsequence $(s_{\phi(i)})_{i\geq 0}$ such that $s_{\phi(i+1)} \succeq s_{\phi(i)}$, for all $i \geq 0$.*
3. *Every infinite sequence is good.*

Note that in the case of a *pwo*, the equivalence part of $\succeq$ is just equality.

## 2.2 Term Rewriting Systems

We introduce some notions from the theory of first-order term rewriting systems.

**Definition 2.43.** A *signature* or *alphabet* $\mathcal{F}$ is a (non-empty) set of function symbols, each of which has associated an *arity* given by the function $\texttt{arity} : \mathcal{F} \to \mathbb{N}$. Elements of $\mathcal{F}$ with arity $0$ are also called *constants*; constants are denoted usually by $c$ instead of $c()$.

**Remark 2.44.** It is not essential to consider that each function symbol has an associated fixed arity. Instead $\texttt{arity}(f)$ can be any non-empty subset of the natural numbers, i. e., $\texttt{arity}(f) \in \mathcal{P}(\mathbb{N}) \setminus \emptyset$. If for at least one element $f \in \mathcal{F}$, $\texttt{arity}(f)$ contains more than one element, we speak of a *varyadic* signature. Otherwise we speak of a *fixed-arity* signature. We will use the function arity ambiguously meaning either a natural number, in the case of fixed-arity signatures, or a non-empty set of natural numbers, for varyadic signatures. Another way of expressing this is by defining $\mathcal{F} = \cup_{i\geq 0}\mathcal{F}_i$, where each $\mathcal{F}_i$, contains the function symbols of arity $i$. Then a varyadic signature corresponds to the case where the union is not disjoint. We will make clear when necessary what kind of signature we have in mind.

To define the set of terms we will also use variables. In the following $\mathcal{X}$ will represent a countable set of variables (whose elements we usually denote by letters $x, y, z, \dots$). The function $\texttt{arity}$ is extended to the elements of $\mathcal{X}$: they have arity $0$ or $\{0\}$, depending whether we are dealing with a fixed-arity or varyadic signature, respectively.

**Definition 2.45.** Let $\mathcal{F}$ be a signature and let $\mathcal{X}$ denote a countable set of variables with $\mathcal{F} \cap \mathcal{X} = \emptyset$. The set of terms over $\mathcal{F}$ and $\mathcal{X}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and the set of ground terms over $\mathcal{F}$ by $\mathcal{T}(\mathcal{F})$; they are defined inductively as follows:

- $\mathcal{X}, \mathcal{F}_0 \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, $\mathcal{F}_0 \subseteq \mathcal{T}(\mathcal{F})$; where $\mathcal{F}_0$ represents the set of constants,
- $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ (respectively $\mathcal{T}(\mathcal{F})$), if $f \in \mathcal{F}$ admits arity $n \geq 1$ and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ (respectively $\mathcal{T}(\mathcal{F})$) for any $1 \leq i \leq n$.

**Definition 2.46.** Given a term $t$, the set of variables occurring in $t$ is denoted by $var(t)$ and the multiset of variables occurring in $t$ is denoted by $mvar(t)$. By $\#_c(t)$ we denote the number of occurrences of the symbol or variable $c$ in $t$, and $|t|$ denotes the total number of function symbols and variables occurring in $t$ (obviously $|t| = \sum_{c \in \mathcal{F} \cup \mathcal{X}} \#_c(t)$).

**Definition 2.47.** A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be *linear* if each variable in $t$ occurs at most once.

We sometimes need to abstract from the actual form of the whole term and concentrate on parts of it. For that we use contexts. Intuitively a *context* is a term containing "holes" that can be filled with other terms. We formalize this concept.

**Definition 2.48.** Let $\mathcal{F}$ be a signature and $\Box$ a constant not occurring in $\mathcal{F}$. A *context* is a term over $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with at least one occurrence of $\Box$ (the trivial context). Given a context $C[\Box, \cdots, \Box]$ with $n$ occurrences of $\Box$, and $n$ terms $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $C[t_1, \ldots, t_n]$ denotes the term obtained by replacing each *hole* (occurrence of $\Box$) by a term $t_i$, $1 \leq i \leq n$, from left to right. More precisely: $\Box[t] = t$, for any term $t$, and

$$f(C_1, \ldots, C_n)[t_{1,1}, \ldots, t_{1,k_1}, \ldots, t_{n,1}, \ldots, t_{n,k_n}] \quad = \quad f(C_1[t_{1,1}, \ldots, t_{1,k_1}], \ldots, C_n[t_{n,1}, \ldots, t_{n,k_n}])$$

for any $f \in \mathcal{F}$, contexts $C_1, \ldots, C_n$ with respectively exactly $k_1, \ldots, k_n$ occurrences of $\Box$, and terms $t_{i,j}$, for any $1 \leq i \leq n$, $1 \leq j \leq k_i$.

**Notation 2.49.** We abbreviate contexts of the form $C[\Box, \cdots, \Box]$ to $C[\cdots]$. A context with only one occurrence of $\Box$ is denoted by $C[\Box]$ or $C[\ ]$ and is called a *linear* context.

We note that the concept of context and replacement of $\Box$ by a term can be stated using the notion of *position*. A position is a sequence of natural numbers indicating precise points within a term or context. Each occurrence of $\Box$ has a unique position within the term and the notation $s = t|_p u$ or $s = t[u]_p$ is used to indicate that term $s$ is obtained from term or context $t$, by replacing the term at position $p$ with the term $u$. We chose not to use this notation since we usually don't need to be too specific about the position in a term where replacement occurs.

**Remark 2.50.** We will often need to perform induction on the definition of linear contexts, i. e., if we want to prove some property for a term $C[t]$, for any linear context $C$ and (any) term $t$, we prove that the property holds for (all) $t$ and then assuming that the property holds for $D[t]$, where $D$ is a linear context, we prove the property holds for $f(\ldots, D[t], \ldots)$, for any

$f \in F$ with appropriate arity. It is not difficult to see that this is equivalent to proving the property for (all) $t$ and then prove that if the property holds for a term $s$ then it also holds for $f(\ldots, s, \ldots)$, for any $f \in F$, arity permitting. This fact will be used when performing induction on linear contexts.

**Definition 2.51.** We say that a term $t$ is a *subterm* of a term $s$ if we have $s = C[t]$, for some linear context $C$; $s$ is also called a *superterm* of $t$. If $C$ is not the trivial context then $t$ is a *proper* subterm of $s$. Furthermore if $s = f(t_1, \ldots, t_n)$, for some $n \geq 1$, the terms $t_i$, with $1 \leq i \leq n$ are called the *principal* subterms, or *arguments*, of $s$; the function symbol $f$ is the *root* symbol of $s$, usually denoted by $\texttt{root}(s)$.

**Definition 2.52.** A *substitution* $\sigma$ is a function from $\mathcal{X}$ to $\mathcal{T}(\mathcal{F}, \mathcal{X})$; such a function can be extended to an endomorphism over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows

- $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$, for any $f \in \mathcal{F}$ admitting arity $n \geq 0$, and terms $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

A *ground* substitution is a substitution whose image lies in $\mathcal{T}(\mathcal{F})$. A *renaming* is an injective substitution whose image lies in $\mathcal{X}$. We usually denote $\sigma(t)$ by $t\sigma$.

**Definition 2.53.** A *term rewriting system* (TRS) is a tuple $(\mathcal{F}, \mathcal{X}, R)$, where $R$ is a subset of $(\mathcal{T}(\mathcal{F}, \mathcal{X}) \setminus \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$. The elements of $R$ are called the rules of the TRS and are usually denoted by $l \to r$, with $l$ being the *lefthand-side* (lhs) of the rule, $r$ the *righthand-side* (rhs) and satisfying $var(r) \subseteq var(l)$.

Note that the definition does not exclude the bizarre case of $R = \emptyset$ and $\mathcal{F} = \mathcal{X} = \emptyset$. Obviously we are not interested in this case. In the following we identify a TRS with its set of rules $R$; $\mathcal{F}$ is defined implicitly: it is the set of function symbols occurring in $R$. We will only specify $\mathcal{F}$ or $\mathcal{X}$ if necessary. We will also sometimes associate labels with rules.

The rules of a TRS induce a relation on terms as follows.

**Definition 2.54.** A TRS $(\mathcal{F}, \mathcal{X}, R)$ induces a *reduction relation* on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, denoted by $\to_R$, as follows: $s \to_R t$ if and only if $s = C[l\sigma]$ and $t = C[r\sigma]$, for some linear context $C$, substitution $\sigma$ and rule $l \to r \in R$. We call $s \to_R t$ a *reduction* or *rewrite step* and say that $t$ is obtained from $s$ by *contracting* or *reducing* the *redex* $l\sigma$, i. e., replacing the redex $l\sigma$ by its *contractum* $r\sigma$. The transitive closure of $\to_R$ is denoted by $\to_R^+$ and its reflexive-transitive closure by $\to_R^*$.

**Remark 2.55.** It is not difficult to see that definition 2.54 is equivalent to the following inductive definition:

- $l\sigma \to_R r\sigma$,
- $f(\ldots, t_i, \ldots) \to_R f(\ldots, t_i', \ldots)$, if $f \in \mathcal{F}$ admits arity $n \geq 1$, $t_i, t_i' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and $t_i \to_R t_i'$.

A *rewrite sequence* is a sequence of reduction steps $t_0 \to_R t_1 \to_R \cdots$, and may be finite or infinite.

**Definition 2.56.** Let $R$ be a TRS. A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be a (or to be in) *normal form* (with respect to $R$) if for no term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ we have $s \to_R t$. The set of normal forms of $R$ is denoted by $\mathcal{N}_f(R)$.

Apart from the reduction relation we will also deal with some particular relations on terms we define next.

**Definition 2.57.** A binary relation $\Theta$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be *closed under contexts* (*monotonic* or satisfying the *replacement property*) if whenever $s\Theta t$ then for any linear context $C[\ ]$ also $C[s]\Theta C[t]$. Equivalently, $s\Theta t \Rightarrow f(\dots, s, \dots)\Theta f(\dots, t, \dots)$, for all non-constant $f \in \mathcal{F}$.

**Definition 2.58.** A binary relation $\Theta$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be *closed under substitutions* (*stable* or satisfying the *full invariance property*) if whenever $s\Theta t$ then for any substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ also $s\sigma\Theta t\sigma$.

**Definition 2.59.** A binary relation on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is said to be a *congruence* if it is an equivalence relation closed under contexts.

Some kinds of TRS's are important enough for us to name them apart. They usually are formed by rules having a certain shape, thus it is also useful to name those rules.

**Definition 2.60.** Let $(\mathcal{F}, \mathcal{X}, R)$ be a TRS and $l \to r$ a rule in $R$. We say that $l \to r$ is a

- *left-linear rule* (respectively *right-linear rule*) if $l$ (respectively $r$) is a linear term,
- *linear rule* if both $l$ and $r$ are linear terms,
- *collapsing rule* (c-rule) if $r \in \mathcal{X}$,
- *duplicating rule* (d-rule) if $mvar(l) \subsetneq mvar(r)$,
- *non-erasing rule* if $var(l) = var(r)$,
- *length-preserving rule* if $mvar(l) = mvar(r)$ and $|l| = |r|$ (equivalently $|l\sigma| = |r\sigma|$ for all ground substitutions $\sigma$, under the assumption that $\mathcal{F}$ contains at least one constant),
- *embedding rule* if $l \to r$ is of the form $f(x_1, \dots, x_n) \to x_i$, for some $1 \le i \le n$, where $f \in \mathcal{F}$ admits arity $n \ge 1$, and $x_1, \dots, x_n$ are pairwise different variables. We denote by $\mathcal{E}mb_{\mathcal{F}}$ all embedding rules for all function symbols in $\mathcal{F}$,
- *ground rule* if $l, r \in \mathcal{T}(\mathcal{F})$.

**Definition 2.61.** A TRS $(\mathcal{F}, \mathcal{X}, R)$ is said to be *left-linear* (respectively *right-linear*, *linear*, *duplicating*, *non-erasing*, *length-preserving*, *ground*) if all the rules in $R$ are left-linear (respectively right-linear, linear, duplicating, non-erasing, length-preserving, ground).

**Example 2.62.** Let $\mathcal{F} = \{f, g, h\}$ with $\mathtt{arity}(f) = 1, \mathtt{arity}(g) = 2, \mathtt{arity}(h) = 3$. Consider the rules

$$\begin{array}{rrcl} r_1: & g(x,x) & \to & h(x,x,x) \\ r_2: & h(x,y,z) & \to & z \\ r_3: & g(f(x),y) & \to & g(x,y) \\ r_4: & h(x,x,f(y)) & \to & g(x,g(x,y)) \end{array}$$

$r_1$ is a d-rule, $r_2$ is a (linear) c-rule and also an embedding rule, $r_3$ is a non-erasing (linear) rule, and $r_4$ is a length-preserving rule.

An important subclass of TRS's are the so-called *string rewriting systems* (SRS's) or *Semi-Thue systems* (see Jantzen [49] or Book [9]).

**Definition 2.63.** A TRS $(\mathcal{F}, \mathcal{X}, R)$ is said to be a *string rewriting system* (SRS) or *Semi-Thue system* if $\mathcal{F}$ contains only unary function symbols.

It is important and desirable to be able to infer properties of TRS's from the validity of the same properties in parts of the TRS (in general we call this concept *modularity*). There are different ways of defining what "parts" are. Here we take the simplest approach.

**Definition 2.64.** Given two TRS's $(\mathcal{F}_1, \mathcal{X}_1, R_1)$ and $(\mathcal{F}_2, \mathcal{X}_2, R_2)$ with $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, their disjoint union $R_1 \oplus R_2$ is the TRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{X}_1 \cup \mathcal{X}_2, R_1 \cup R_2)$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are not disjoint, we can rename one (or both) of the TRS's, forcing the disjointness condition to hold.

A thorough account of modular properties of TRS's can be found in Middeldorp [76].

### 2.2.1 Equational Rewriting

We introduce the notion of equation over terms and of equational rewriting.

**Definition 2.65.** Let $\mathcal{T}(\mathcal{F}, \mathcal{X})$ be a set of terms. An *equation* over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is a pair of terms $(s,t) \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$, sometimes denoted by $s = t$. An *equational system* over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is a set of equations $EQ$; if this set is finite we say that the equational system is *finitely presented*.

Given an equational system $EQ$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, we can define semantics for it in the style of the semantics for first-order logic (see Plaisted [89]). A "structure" $\mathcal{M}$ consists of a non-empty domain $D$, a function $f_D : D^n \to D$, for each function symbol $f \in \mathcal{F}$ (i. e., $D$ is an $\mathcal{F}$-algebra) and evaluations $\sigma : \mathcal{X} \to D$. The "meaning" of a term $t$ in the structure $\mathcal{M}$, for a given evaluation $\sigma$, is denoted by $[\![t, \sigma]\!]_{\mathcal{M}}$ and defined as $\sigma(x)$ if $t$ is a variable $x$, or

$f_D(\llbracket t_1, \sigma \rrbracket_{\mathcal{M}}, \dots, \llbracket t_n, \sigma \rrbracket_{\mathcal{M}})$ if $t = f(t_1, \dots, t_n)$. If $\mathcal{M}$ is a structure and $(s,t)$ an equation, we say that $\mathcal{M}$ satisfies the equation $(s,t)$ and write $\mathcal{M} \models s = t$ if $\llbracket s, \sigma \rrbracket_{\mathcal{M}} = \llbracket t, \sigma \rrbracket_{\mathcal{M}}$, for all evaluations $\sigma$. If $\mathcal{M}$ satisfies all the equations of $EQ$ we say that $\mathcal{M}$ is a *model* for $EQ$ and we write $\mathcal{M} \models EQ$. If $EQ_1$ and $EQ_2$ are two equational systems, we write $EQ_1 \models EQ_2$ if all models for $EQ_1$ are also models for $EQ_2$. In particular $EQ \models s = t$ if all models of $EQ$ satisfy $(s,t)$. We then say that $s = t$ is a logical consequence of $EQ$.

An equational system $EQ$ generates a congruence on the set of terms. We denote by $=_{EQ}$ the least congruence closed under substitutions containing $EQ$. Note that this congruence does exist since there is at least one congruence closed under substitutions and containing $EQ$, namely $\mathcal{T}(\mathcal{F},\mathcal{X}) \times \mathcal{T}(\mathcal{F},\mathcal{X})$, and the class of congruences closed under substitutions and containing $EQ$ is closed under intersection, so a least such congruence does exist, namely the intersection of all of them.

The following result connecting logical consequence of an equational system and congruence generated by it is due to Birkhoff [8].

**Theorem 2.66.** *If $EQ$ is a set of equations then $EQ \models s = t$ if and only if $s =_{EQ} t$.*

**Remark 2.67.** From now on we assume that any equational system contains its symmetric image, i. e., if $(s,t) \in EQ$ then also $(t,s) \in EQ$; however for the sake of simplicity, when expressing $EQ$ extensively we will omit the symmetric equations. With this assumption the equational theory generated by a set of equations becomes:

**Definition 2.68.** The equational theory generated by an equational system $EQ$ is denoted by $=_{EQ}$ and is the least congruence on $\mathcal{T}(\mathcal{F},\mathcal{X})$ containing $EQ$ and closed under substitutions, i. e., $s =_{EQ} t$ if and only if either

- $s \equiv t$,
- $s \equiv C[e_1\sigma]$ and $t \equiv C[e_2\sigma]$, for some equation $(e_1, e_2) \in EQ$, linear context $C$ and substitution $\sigma$,
- $s =_{EQ} u$ and $u =_{EQ} t$, for some term $u$.

where $\equiv$ stands for syntactical equality.

Some of the terminology applied to TRS's carries over to equational systems. Thus we have:

**Definition 2.69.** Let $(s,t)$ be an equation over $\mathcal{T}(\mathcal{F},\mathcal{X})$. We say that $(s,t)$ is

- *linear* if both $s$ and $t$ are linear terms,
- *variable-preserving* if $mvar(s) = mvar(t)$,
- *length-preserving* or *permutative* if $mvar(s) = mvar(t)$ and $|s| = |t|$ (equivalently $|s\sigma| = |t\sigma|$ for all ground substitutions $\sigma$, under the assumption that $\mathcal{F}$ contains at least one constant).

As for TRS's, an equational system is said to be *linear*, *non-erasing* or *length-preserving* if all its equations are linear, non-erasing or length-preserving.

An important classe of equational systems are the *permutative* or length-preserving theories (see Peterson and Stickel [86]). This class comprises the so-called AC-theories, where A stands for associativity and C for commutativity. AC-theories contain only

- associative axioms; these axioms have the form $f(x, f(y, z)) = f(f(x, y), z)$, for binary function symbols $f$,
- commutative axioms; these axioms have the form $f(x, y) = f(y, x)$, for binary function symbols $f$.

Finally, we define what equational rewriting is.

**Definition 2.70.** An equational rewrite system $R/EQ$ consists of a TRS $R$ and an equational system $EQ$, both defined over the same signature and set of variables. Its associated equational rewrite relation $\rightarrow_{R/EQ}$ is given by: $s \rightarrow_{R/EQ} t$ if and only if there are terms $u, v$ such that $s =_{EQ} u \rightarrow_R v =_{EQ} t$. We speak of *equational rewriting* or *rewriting modulo a set of equations*.

Another interesting more restricted notion of rewriting modulo a set of equations is the relation $\rightarrow_{R,EQ}$ defined by: $s \rightarrow_{R,EQ} t$ if and only if $s = C[u]$, $u =_{EQ} l\sigma$ and $t = C[r\sigma]$ (see Jouannaud and Muñoz [53]). In this case, given a term $s$, we are only allowed to rewrite subterms of $s$ modulo the equational system. For example, the term $(x + e) + y$ cannot be rewritten using the rule $e + x \rightarrow x$ and the associative axiom for the symbol "+". However we have $(x + e) + y \rightarrow_{R/EQ} x + y$. Note also that the two relations $\rightarrow_{R/EQ}$ and $\rightarrow_{R,EQ}$ can still be different even if we rewrite at the top of the term. For example if $a =_{EQ} b$, $b \rightarrow c$ and $c =_{EQ} d$, for some constants $a, b, c, d$, then $a \rightarrow_{R/EQ} d$ while $a$ only rewrites with $\rightarrow_{R,EQ}$ to $c$.

## 2.3 Orders on Terms

In the context of term rewriting, orders on terms are especially useful if they are compatible with the reductions, i. e., if $s \rightarrow_R t \Rightarrow s > t$, for some reduction relation $\rightarrow_R$ and order $>$. Checking if an order satisfies this property can be cumbersome and ideally we would like to be able to infer this property by performing a check on a finite special set of reductions steps. That is indeed possible if the order satisfies some properties (and the TRS has finitely many rules).

**Definition 2.71.** A partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is called a *rewrite order* if it is closed under contexts and substitutions. A quasi-order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is called a *rewrite quasi-order* if both its strict and equivalence parts are closed under contexts and substitutions.

We remark that in the literature a rewrite quasi-order is only required to be closed under contexts and substitutions. This means in particular that we can have $s > t$ and $C[s] \sim C[t]$, for a rewrite quasi-order $> \cup \sim$. We found it more convenient to define rewrite quasi-order as in definition 2.71.

**Definition 2.72.** A partial order (respectively quasi-order) over $\mathcal{T}(\mathcal{F},\mathcal{X})$ is called a *reduction order* (respectively *reduction quasi-order*) if it is a well-founded rewrite order (respectively a well-founded rewrite quasi-order) over $\mathcal{T}(\mathcal{F},\mathcal{X})$.

**Definition 2.73.** A partial order $>$ (respectively quasi-order $\geq$) is *compatible* with a TRS $R$ if $s > t$ whenever $s \to_R t$. If $>$ is a rewrite order, it is enough to require $l > r$ for every rule $l \to r$ in $R$.

Some orders play an important role on termination arguments. They are related to the beautiful Kruskal's theorem (theorem 2.78 below).

**Definition 2.74.** The *subterm ordering* is defined on $\mathcal{T}(\mathcal{F},\mathcal{X})$ as follows: $s \blacktriangleright t \iff s = C[t]$, for some non-trivial context $C$. We say that a partial order $>$ (respectively quasi-order $\geq$) satisfies or has the *subterm property* if $\blacktriangleright \subseteq >$ (respectively $\blacktriangleright \subseteq \geq$).

**Definition 2.75.** An order $>$ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ is called a *simplification order* if it is a rewrite order with the subterm property and satisfying the *deletion property*, i. e., $f(\ldots,t,\ldots) > f(\ldots\ldots)$ for all terms $t$ (provided the arities of $f$ allow it). A quasi-order $\succeq$ is called a *quasi-simplification order* if it satisfies the same properties, i. e., $f(\ldots t \ldots) \succeq t$ and $f(\ldots t \ldots) \succeq f(\ldots\ldots)$.

Note that the deletion property in the definition above is effective only in a varyadic setting. In a fixed-arity setting, it is trivially satisfied.

The interest of simplification orders stems from the fact that they are well-founded, provided that the signature $\mathcal{F}$ is finite (this result is a consequence of Kruskal's Theorem, presented later). For infinite signatures, the subterm property is not enough to guarantee well-foundedness. To see why, consider the signature $\mathcal{F} = \{a_i\}_{i\geq 0}$, where each symbol $a_i$ is a constant. Then the order $\succ$ defined by $a_i \succ a_j$, for all $i < j$, is a rewrite order having the subterm property and yet is not well-founded.

Since simplification orderings are related to Kruskal's Theorem, it seems reasonable to define them according to the requirements of the theorem itself. This is the approach taken in Middeldorp and Zantema [78] and that we adopt here (for a different approach see Ohlebush [85]).

First we introduce the homeomorphic embedding (see for example Gallier [38]).

**Definition 2.76.** Let $\geq$ be a quasi-order on $\mathcal{F}$. The quasi-order $\succeq_{emb}$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ is defined as follows:

1. $f(\ldots,s,\ldots) \succeq_{emb} s$ (*subterm property*)
2. $f(\ldots,s,\ldots) \succeq_{emb} f(\ldots)$ (*deletion property*) when the arities of $f$ allow it;
3. $f(s_1,\ldots,s_k) \succeq_{emb} g(t_1,\ldots,t_m)$ whenever $f \geq g$ and there are integers $j_1,\ldots,j_m$ such that $1 \leq j_1 < \ldots < j_m \leq k$, $s_{j_1} \succeq_{emb} t_1,\ldots,s_{j_m} \succeq_{emb} t_m$.

If $>$ is a partial order on $\mathcal{F}$, the partial order $\succ_{emb}$ is defined as above, replacing everywhere $\geq$ by $>$ and $\succeq_{emb}$ by $\succ_{emb}$.

We now define simplification ordering as in Middeldorp and Zantema [78]. Note that in the case that the signature is finite, definition 2.75 and definition 2.77 coincide, since any partial order over $\mathcal{F}$ and in particular the empty one, is a *pwo*.

**Definition 2.77.** A rewrite order $\gg$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is a *simplification ordering* with respect to a *pwo* $>$ on $\mathcal{F}$ if it satisfies $\succ_{emb} \subseteq \gg$.

The following result clarifies the relation between embedding, simplification orderings and termination.

**Theorem 2.78. Kruskal's Tree Theorem** *[64]*
*Let $\geq$ (respectively $>$) be a wqo (respectively pwo) on $\mathcal{F}$. Then $\succeq_{emb}$ (respectively $\succ_{emb}$) is a wqo (respectively pwo) on $\mathcal{T}(\mathcal{F})$.*

For an elegant proof of this result we suggest Nash-Williams [82] or Gallier [38]. A proof of the version for pwo's can be found in Middeldorp and Zantema [78]. We remark that a restricted version of this result for strings is due to Higman [45].

The following result is a corollary of Kruskal's Theorem. For finite signatures (and using the concept of simplification ordering as defined in 2.75), it appeared first in Dershowitz [19]; the general result is taken from Middeldorp and Zantema [78].

**Theorem 2.79.** *Let $\succ$ be any simplification ordering with respect to a pwo $>$ on $\mathcal{F}$. Then $\succ$ is a well-founded order.*

Note that if we consider fixed-arity signatures, condition $(2)$ in definition 2.76 can be left out. If furthermore the quasi-order $\geq$ on $\mathcal{F}$ is just the equality relation (so its strict part is empty), then condition $(3)$ is just closedness under contexts. As a consequence a rewrite order over a finite signature is well-founded if it contains the subterm relation (thus justifying the original definition 2.75).

## 2.4 Termination

An important notion in the theory of TRS's is the subject of this thesis, namely termination.

**Definition 2.80.** A TRS $R$ is said to be *terminating* (*strongly normalizing* or *noetherian*) if $\rightarrow_R$ is terminating.

The following is a well-known and simple observation going back to Manna and Ness [74].

**Theorem 2.81.** *A TRS $R$ over a set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is terminating if and only if there exists a well-founded order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $s \rightarrow_R t \Rightarrow s > t$.*

The conditions on the above theorem imply a possibly infinite test since possibly infinitely many rewrites have to be checked. A way of reducing the number of rewrites to be checked is given in the next (also well-known) result.

**Theorem 2.82.** *A TRS is terminating if and only if it is compatible with a reduction order.*

The previous result describes what is usually the task of proving termination of particular TRS's: finding a suitable well-founded order that can be turned into a reduction order. There are essentially two ways of accomplishing this: syntactically or semantically. In the syntactic approach a careful inspection of the structure of the terms is done (eventually using an auxiliary order on the set of function symbols) in order to define an order on terms. Most of the path orders are of this kind, being its best known representative the *recursive path order* due to Dershowitz [22]. In the semantic approach, terms are interpreted in an algebra $(A, >)$, where $>$ is a well-founded order. The interpretation is compositional and has to obey some monotonicity conditions (we will be more specific later). If all conditions are met, the order on the algebra and the interpretation induce a reduction order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

In the setting of rewriting modulo a set of equations, termination has a more general definition.

**Definition 2.83.** Let $EQ$ be an equational system and $R$ a TRS, both defined over $\mathcal{T}(\mathcal{F}, \mathcal{X})$. We say that $R$ is $E$-terminating (or that $R/EQ$ is terminating) if the relation $\rightarrow_{R/EQ}$ is terminating, i. e., if there are no infinite sequences of the form:

$$s_0 =_{EQ} s'_0 \rightarrow_R s_1 =_{EQ} s'_1 \rightarrow_R s_2 \ldots$$

Note that definition 2.80 is a particular case of definition 2.83 for which $EQ$ is just syntactic equality.

**Definition 2.84.** An equational rewrite system $R/EQ$ is compatible with a quasi-order $\succeq = \succ \cup \sim$ (on $\mathcal{T}(\mathcal{F}, \mathcal{X})$) if $=_{EQ} \subseteq \sim$ and $\succ$ is compatible with $R$.

We also have a result similar to theorem 2.82.

**Theorem 2.85.** *An equational rewrite system $R/EQ$ is terminating if and only if it is compatible with a reduction quasi-order.*

The following result can easily be proven using the definitions and properties of equational rewriting.

**Theorem 2.86.** *Let $R/EQ$ be an equational rewrite system over $\mathcal{T}(\mathcal{F}, \mathcal{X})$. We have that:*

1. $=_{EQ} \circ \rightarrow^+_{R/EQ} \circ =_{EQ} \ \subseteq \ \rightarrow^+_{R/EQ}$,
2. *If $R/EQ$ is terminating then $\rightarrow^+_{R/EQ}$ is a reduction order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Furthermore $\rightarrow^+_{R/EQ}$ is compatible with $=_{EQ}$ (in the sense of definition 2.1), so $\rightarrow^+_{R/EQ} \cup =_{EQ}$ is a well-founded reduction quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

Proving $E$-termination is in general much more complicated than proving termination. For a study of the problems involved see Jouannaud and Muñoz [53]. As far as this thesis is concerned, $E$-termination will only play a role in chapter 6.

### 2.4.1 Path Orders

The basic idea behind path orders is the construction of orders on terms starting from a well-founded order on the signature $\mathcal{F}$ (usually called a *precedence*). In general a term $s$ is greater than any term built from "smaller" terms connected together under a function symbol smaller, in the precedence, then the root of $s$ (in particular terms are bigger than proper subterms). Thus path orders compare the roots of the terms using the precedence and in the case of equal or equivalent roots, subterms are compared recursively in some manner. The different ways of doing this subterm's comparison give rise to different path orders.

Path orders originated with the work of Plaisted (*path of subterms orderings* [88, 87]) and Dershowitz (*recursive path order* [20, 22]) at the end of the seventies. Since then other orders have been proposed and the original ones improved. As examples of such orders we have (apart from the ones already mentioned): the *lexicographic path order* of Kamin and Lévy [54], the *recursive decomposition ordering* of Jouannaud, Lescanne and Reinig [52], the *path ordering* of Kapur, Narendran and Sivakumar [56].

Other orders have been proposed and a lot of work has been done on generalizing and improving existing ones (see for example Rusinowitch [96], Lescanne [70]). For an exhaustive account on path orders, their properties, history and connections between them, see Steinbach [100, 101].

An improvement which has been systematically added to path orders is the notion of *status*. Basically a status is associated to each function symbol with the purpose of indicating how arguments of the function symbol are to be taken for the purpose of comparison. The idea of status can be traced back to the work of Kamin and Lévy [54]; it was also presented informally in Lescanne [73] and formalized in Lescanne [69]. Here we present a definition of status based in the one given in Steinbach [99].

**Definition 2.87.** To each function symbol $f \in \mathcal{F}$ we associate a status $\tau(f)$. We consider two possible cases:

- $\tau(f) = mul$; indicates that, for the purpose of ordering, the arguments of $f$ are to be taken as a multiset,
- $\tau(f) = lex_\pi$, where $\pi$ is a permutation of the set $\{1, \ldots, \mathtt{arity}(f)\}$; indicates that, for the purpose of ordering, the arguments are to be taken as a lexicographic sequence whose order is given by $\pi$. Two common abbreviations are *left* and *right*. The first indicates the usual left-to-right lexicographic sequencing ($\pi = Id$) and the second the right-to-left lexicographic sequencing.

Probably the best known path order is the *recursive path order*. Using this notion of status we present the definition of *recursive path order* as it appeared in Steinbach [99]. First we introduce some notation.

**Definition 2.88.** Let $\mathcal{F}$ be a signature. A *precedence* is a partial or quasi-order on $\mathcal{F}$ denoted respectively by $\rhd$ or $\unrhd$. We sometimes use the term *quasi-precedence* to emphasize that we are dealing with a quasi-order on $\mathcal{F}$.

**Definition 2.89. (rpo with status)** Let $\rhd$ be a precedence (i. e., a partial order) on $\mathcal{F}$ and $\tau$ a status function. Given two terms $s, t$ we say that $s >_{rpo} t$ iff $s = f(s_1, \ldots, s_m)$ and either

1. $t = g(t_1, \ldots, t_n)$, $s >_{rpo} t_i$, for all $1 \leq i \leq n$, and
   (a) $f \rhd g$, or
   (b) $f = g$ and $(s_1, \ldots, s_m) >_{rpo,\tau} (t_1, \ldots, t_n)$; or
2. $\exists 1 \leq i \leq m : \ s_i >_{rpo} t$ or $s_i = t$.

where $>_{rpo,\tau}$ is the extension of $>_{rpo}$ associated with the status $\tau(f)$.

Note that in the case that we take the lexicographic status for all function symbols in $\mathcal{F}$, $>_{rpo}$ coincides with $>_{lpo}$, the *lexicographic path order* of Kamin and Lévy [54], and if the status is fixed to the multiset status, the order coincides with the original *recursive path order* of Dershowitz [20, 22].

It can be seen that $>_{rpo}$ is a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, closed under contexts and substitutions, well-founded if and only if the precedence is, satisfying the subterm property and monotone with respect to the precedence, i. e., if $\rhd$ and $\rhd'$ are two precedences satisfying $\rhd \subseteq \rhd'$ then $>_{rpo} \subseteq >_{rpo}'$, where $>_{rpo}$ and $>_{rpo}'$ are the recursive path orders associated with the precedences $\rhd$ and $\rhd'$ respectively. Furthermore $>_{lpo}$ is a total order on $\mathcal{T}(\mathcal{F})$ whenever $\rhd$ is total on $\mathcal{F}$ (we will see this in more detail in chapter 4).

The orders mentioned so far are simplification orders of an essentially syntactic nature. Bridging the gap between syntax and semantics we have orders such as the *Knuth-Bendix order (kbo)* of Knuth and Bendix [63], the *semantic path order (spo)* of Kamin and Lévy [54] and the *general path order (gpo)* of Dershowitz and Hoot [24]. It should be noted that while *kbo* is still a simplification order, *spo* and *gpo* are in general not so (we willl look at these orders in more detail in the next chapter).

*Knuth-Bendix order* is similar to a *polynomial interpretation* (see below). The idea behind *kbo* is to assign a natural number (a weight) to each function symbol and then extend the weight function to terms. Below we present a version of *kbo* which combines features from the definitions presented in Steinbach [99] and Dick, Kalmus and Martin [29].

Let $\phi_0 \in \mathbb{N}$ be a fixed natural greater than zero. Let $\phi : \mathcal{F} \cup \mathcal{X} \to \mathbb{N}$ be a function such that

$$\phi(f) \text{ is } \begin{cases} = \phi_0 & \text{if } f \in \mathcal{X} \\ \geq \phi_0 & \text{if } \texttt{arity}(f) = 0 \\ > 0 & \text{if } \texttt{arity}(f) = 1 \text{ and } \exists g \in \mathcal{F} : \ f \not\rhd g \end{cases}$$

where $\rhd$ is a precedence in $\mathcal{F}$. Note that the last condition means that we can allow a function symbol $f$ with $\texttt{arity}(f) = 1$ to have weight $0$ as long as that symbol is maximal in the precedence $\rhd$.

We extend $\phi$ to terms as follows: $\phi(f(s_1, \ldots, s_m)) = \phi(f) + \sum_{i=1}^{m} \phi(s_i)$.

Let $\#_x(t)$ denote the number of occurrences of variable $x$ in term $t$ (see definition 2.46). We define the Knuth-Bendix order with status as follows (Steinbach [99]; Dick, Kalmus and Martin [29]).

**Definition 2.90. (kbo with status)** Let $\rhd$ be a precedence (i. e., a partial order) on $\mathcal{F}$ and $\tau$ a status function. We say that $s >_{kbo} t$ iff $\forall x \in \mathcal{X} : \#_x(s) \geq \#_x(t)$ and

1. $\phi(s) > \phi(t)$, or
2. $\phi(s) = \phi(t)$, and
   (a) $t \in \mathcal{X}$ and $\exists k > 0 : \ s = f_0^k(t)$, where $f_0$ is the element of $\mathcal{F}$ having arity $1$ and weight $0$, and being maximal in the precedence,
   (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ and
   - $f \rhd g$ or
   - $f = g$ and $s_1, \ldots, s_m >_{kbo,\tau} t_1, \ldots, t_n$

where $>_{kbo,\tau}$ is the extension of $>_{kbo}$ associated with the status $\tau(f)$.

Knuth-Bendix order has properties similar to $>_{rpo}$ (see Steinbach [99]; Dick, Kalmus and Martin [29] and chapter 4), namely it is a simplification ordering, monotone with respect to precedences and total on ground terms if the precedence is total.

All the orders mentioned are meant for term rewriting; in the presence of an equational system they cannot be applied. Recently there has been quite some work done on defining or adapting path orders to equational rewriting, more especifically rewriting modulo AC-theories. One of the earliest examples of such orders is the *associative path ordering* of Plaisted [90] (see also Dershowitz, Hsiang, Josephson and Plaisted [25]) defined for ground terms. More recently Kapur, Sivakumar and Zhang [57] defined another associative ordering which can be applied also to non-ground terms, and Rubio and Nieuwenhuis [95] gave a modification of *rpo* which is compatible with AC-theories and total on non-AC-equivalent ground terms. For more information on this type of orders see Steinbach [100].

### 2.4.2 Monotone Algebras

The idea of using a well-founded $\mathcal{F}$-algebra and a monotonic morphism ("termination function" in the original terminology) to prove termination of TRS's goes back to Lankford [67], Manna and Ness [74]. We use the definitions and terminology of Zantema [109].

**Definition 2.91.** Let $\mathcal{F}$ be a signature. A *monotone $\mathcal{F}$-algebra* $(A, >)$ is a structure consisting of a non-empty set $A$ provided with a partial order $>$ and algebra operations $f_A : A^n \to A$, for each function symbol $f \in \mathcal{F}$ of arity $n \geq 0$; if $n = 0$, then $f_A$ is an arbitrary element of $A$. Furthermore each algebra operation is monotone[1] in all of its coordinates: for each function symbol $f \in \mathcal{F}$ of arity $n \geq 1$, and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ for which $a_i > b_i$ for some $i$, and $a_j = b_j$ for all $j \neq i$, we have $f_A(a_1, \ldots, a_n) > f_A(b_1, \ldots, b_n)$. A *well-founded* (respectively *total*) monotone $\mathcal{F}$-algebra $(A, >)$ is a monotone $\mathcal{F}$-algebra such that the order $>$ is well-founded (respectively total).

[1] By monotone we mean *strictly increasing*.

A monotone algebra is used to define interpretations for the terms as follows.

**Definition 2.92.** Let $(A, >)$ be a (well-founded) monotone $\mathcal{F}$-algebra. The interpretation function $[\![\ ]\!]_A : \mathcal{T}(\mathcal{F}, \mathcal{X}) \times A^{\mathcal{X}} \to A$ (where $A^{\mathcal{X}} = \{\rho : \mathcal{X} \to A\}$) is defined inductively by

$$\begin{aligned} [\![x, \rho]\!]_A &= \rho(x), \\ [\![c, \rho]\!]_A &= c_A, \text{ for any } c \in \mathcal{F}_0, \\ [\![f(t_1, \ldots, t_n), \rho]\!]_A &= f_A([\![t_1, \rho]\!]_A, \ldots, [\![t_n, \rho]\!]_A), \end{aligned}$$

for $x \in \mathcal{X}, \rho \in A^{\mathcal{X}}, f \in \mathcal{F}$ with arity $n$, $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $c_A \in A$ is fixed. This function induces a partial order $>_A$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows:

$$t >_A t' \iff (\forall \rho \in A^{\mathcal{X}} : [\![t, \rho]\!]_A > [\![t', \rho]\!]_A).$$

It is easy to see that $>_A$ is indeed a (well-founded) partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Intuitively $t >_A t'$ means that for each interpretation of the variables in $A$ the value of $t$ is greater than that of $t'$.

If we restrict ourselves to ground terms, it follows from the definition by an easy induction that the interpretation of the terms does not depend on the assignment of values to variables. In other words, if $t \in \mathcal{T}(\mathcal{F})$ then for any $\rho, \tau \in A^{\mathcal{X}}$, $[\![t, \rho]\!]_A = [\![t, \tau]\!]_A$. We denote this value by $[\![t]\!]_A$. In the following, we omit the subscript $A$ in $[\![]\!]_A$ when it is clear from context which algebra we refer to.

**Definition 2.93.** We say that a TRS $(\mathcal{F}, \mathcal{X}, R)$ and a non-empty (well-founded) monotone $\mathcal{F}$-algebra $(A, >)$ are *compatible* if $l >_A r$ for every rule $l \to r$ of $R$.

This terminology is motivated by the following proposition.

**Theorem 2.94.**

1. *If $(A, >)$ is a (well-founded) monotone algebra compatible with $R$ then $>_A$ is a rewrite (reduction) order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

2. *A TRS is terminating if and only if it admits a compatible non-empty well-founded monotone algebra.*

For the proof we refer to Zantema [108, 109].

Another way of proving termination of a TRS is now as follows: choose a well-founded poset $A$, define for each function symbol a corresponding operation that is strictly monotone in all of its coordinates, and for which the interpretation function satisfies $[\![l, \rho]\!]_A > [\![r, \rho]\!]_A$ for all rewrite rules $l \to r$ and all $\rho : \mathcal{X} \to A$. Then according to the above proposition the TRS is terminating.

**Example 2.95.** Consider the system

$$f(f(x,y),z) \;\rightarrow\; f(x,f(y,z)).$$

Choose $(A,>) = (\mathbb{N}_1,>)$, where $\mathbb{N}_1$ is defined to be the set of strictly positive integers, and choose $f_A(x,y) = 2x+y$. Clearly $f_A$ is strictly monotone in both coordinates, and

$$f_A(f_A(x,y),z) = 4x+2y+z > 2x+2y+z = f_A(x,f_A(y,z))$$

for all $x,y,z \in A$. Hence $f(f(x,y),z) >_A f(x,f(y,z))$, proving termination of the system.

Sometimes when trying to define algebras compatible with a TRS $R$ it is convenient to define algebras compatible with parts of $R$. If certain conditions are met, then the lexicographic product of those algebras will be an algebra compatible with the whole TRS. We now give those conditions.

**Definition 2.96.** Given a monotone $\mathcal{F}$-algebra $(A,>)$ the relation $=_A$ on terms is defined by

$$t =_A s \iff \forall \rho : \mathcal{X} \rightarrow A : \; [\![t,\rho]\!] = [\![s,\rho]\!]$$

The relation $=_A$ is an equational theory induced by the algebra and the interpretation.

**Lemma 2.97.** *In the conditions of definition 2.96, $=_A$ is a congruence, i. e., an equivalence relation closed under contexts, also closed under substitutions.*

**Proof** Checking that $=_A$ is an equivalence relation is quite straightforward so we will just show that $=_A$ is closed under contexts and substitutions. We first show that $=_A$ is closed under contexts, i. e., if $s =_A t$ then $C[s] =_A C[t]$, for any linear context $C$. We proceed by induction on the context. If $C$ is the trivial context, the result holds by hypothesis. Suppose now that $s =_A t$ and that $C$ is of the form $f(\ldots, D[\;], \ldots)$, for some $f \in \mathcal{F}$, and that the result holds for context $D$. Let $\rho : \mathcal{X} \rightarrow A$ be an arbitrary assignment. Then

$$\begin{array}{lll}
[\![C[s],\rho]\!] & = & \\
[\![f(\ldots, D[s], \ldots),\rho]\!] & = & \text{(by definition of interpretation)} \\
f_A(\ldots, [\![D[s],\rho]\!], \ldots) & = & \text{(by induction hypothesis)} \\
f_A(\ldots, [\![D[t],\rho]\!], \ldots) & = & \\
[\![f(\ldots, D[t], \ldots),\rho]\!] & = & \\
[\![C[t],\rho]\!] & &
\end{array}$$

Since $\rho$ was arbitrarily chosen, we can conclude that $C[s] =_A C[t]$.

To see that $=_A$ is closed under substitutions we need the following fact from Zantema [109]:

> Let $\sigma : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F},\mathcal{X})$ be any substitution and let $\rho : \mathcal{X} \rightarrow A$ be an assignment. Then $[\![t\sigma,\rho]\!] = [\![t,\tau]\!]$ where $\tau : \mathcal{X} \rightarrow A$ is given by $\tau(x) = [\![\sigma(x),\rho]\!]$.

Suppose now that $s =_A t$ and let $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ be any substitution. Let $\rho : \mathcal{X} \to A$ be any assignment. Then

$$\begin{array}{lll} [\![s\sigma, \rho]\!] & = & \text{(by fact above)} \\ [\![s, \tau]\!] & = & \text{(by definition of } =_A) \\ [\![t, \tau]\!] & = & \text{(by fact above)} \\ [\![t\sigma, \rho]\!] & & \end{array}$$

where $\tau$ is defined as in the fact above. Since $\rho$ was arbitrarily chosen we conclude that $s\sigma =_A t\sigma$. □

**Definition 2.98.** A *monotone quasi-model* for a TRS $(\mathcal{F}, \mathcal{X}, R)$ is a monotone $\mathcal{F}$-algebra $(A, >)$ such that $[\![l, \rho]\!] \geq [\![r, \rho]\!]$, for all assignments $\rho : \mathcal{X} \to A$ and all rules $l \to r$ in $R$. If $[\![l, \rho]\!] = [\![r, \rho]\!]$, for all evaluations $\rho$ and rules $l \to r$, we say that $(A, >)$ is a *model* for the TRS. A *well-founded* (respectively *total*) monotone (quasi-)model is just a monotone (quasi-)model where the order $>$ is well-founded (respectively total).

**Theorem 2.99.** *Let $(\mathcal{F}^i, \mathcal{X}^i, R^i)$, with $1 \leq i \leq k$ , for some fixed $k \geq 1$, be TRS's. Let $\mathcal{F} = \bigcup_{i=1}^k \mathcal{F}^i$. Suppose that for each $i$, $(A_i, >_i)$ is a non-empty (well-founded or total) monotone $\mathcal{F}$-algebra such that $(A_i, >_i)$ is compatible with $(\mathcal{F}^i, \mathcal{X}^i, R^i)$ and $(A_i, >_i)$ is a (well-founded or total) monotone quasi-model for $(\bigcup_{j=i+1}^k R^j) \setminus R^i$. Define $B = (A_1 \times \ldots \times A_k, >_{lex})$ where $>_{lex}$ is the lexicographic product of $>_1, \ldots, >_k$, and the interpretation $f_B : B^n \to B$ given by*

$$f_B((a_1^1, \ldots, a_k^1), \ldots, (a_1^n, \ldots, a_k^n)) = (f_{A_1}(a_1^1, \ldots, a_1^n), \ldots, f_{A_k}(a_k^1, \ldots, a_k^n))$$

*for all $f \in \mathcal{F}$ with arity $n \geq 0$. Then $(B, >_{lex})$ is a (well-founded or total) monotone $\mathcal{F}$-algebra compatible with $(\bigcup_{i=1}^k \mathcal{F}^i, \bigcup_{i=1}^k \mathcal{X}^i, \bigcup_{i=1}^k R^i)$.*

Before presenting the proof of the theorem, we give the following lemma (which appeared in Ferreira and Zantema [32]).

**Lemma 2.100.** *In the conditions of theorem 2.99, let $\rho : \mathcal{X} \to B$ be any evaluation (where $\mathcal{X} = \bigcup_{i=1}^k \mathcal{X}^i$). Then*

$$[\![t, \rho]\!]_B = ([\![t, \pi_1 \circ \rho]\!]_{A_1}, \ldots, [\![t, \pi_k \circ \rho]\!]_{A_k})$$

*for any term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and where the functions $\pi_i$, $1 \leq i \leq k$, are the projections in the $i$-th coordinate.*

**Proof** We proceed by induction over $t$. If $t$ is a variable $x$ then

$$[\![x, \rho]\!]_B = ((\pi_1 \circ \rho)(x), \ldots, (\pi_k \circ \rho)(x)) = ([\![x, \pi_1 \circ \rho]\!]_{A_1}, \ldots, [\![x, \pi_k \circ \rho]\!]_{A_k})$$

If $t = f(t_1, \ldots, t_n)$ then

$$\begin{array}{lll} [\![f(t_1, \ldots, t_n), \rho]\!]_B & = & \text{(by def. 2.92)} \\ f_B([\![t_1, \rho]\!]_B, \ldots, [\![t_n, \rho]\!]_B) & = & \text{(by IH)} \\ f_B((\ldots, [\![t_1, \pi_i \circ \rho]\!]_{A_i}, \ldots), \ldots, (\ldots, [\![t_n, \pi_i \circ \rho]\!]_{A_i}, \ldots)) & = & \text{(by def. of } f_B) \\ (f_{A_1}(\ldots, [\![t_i, \pi_1 \circ \rho]\!]_{A_1}, \ldots), \ldots, f_{A_k}(\ldots, [\![t_i, \pi_k \circ \rho]\!]_{A_k}, \ldots)) & = & \text{(by def. 2.92)} \\ ([\![f(t_1, \ldots, t_n), \pi_1 \circ \rho]\!]_{A_1}, \ldots, [\![f(t_1, \ldots, t_n), \pi_k \circ \rho]\!]_{A_k}) & & \end{array}$$

□

We present now the proof of theorem 2.99.

**Proof** Obviously $B$ is non-empty and by lemma 2.35, $>_{lex}$ is a (well-founded or total) order on $B$. We need to see that the interpretation functions $f_B$ are monotone on each argument. Take then $f_B : B^n \to B$ with

$$f_B((a_1^1, \dots, a_k^1), \dots, (a_1^n, \dots, a_k^n)) = (f_{A_1}(a_1^1, \dots, a_1^n), \dots, f_{A_k}(a_k^1, \dots, a_k^n))$$

Suppose we replace an element $(a_1^i, \dots, a_k^i)$ by a bigger one (with respect to the $>_{lex}$ order) $(a_1^i{\prime}, \dots, a_k^i{\prime})$. This means that

$$\exists 1 \leq j \leq k : \ (a_j^i{\prime} >_j a_j^i \text{ and } (\forall 1 \leq l < j : a_l^i{\prime} = a_l^i))$$

Consequently, $f_{A_l}(a_l^1, \dots, a_l^i{\prime}, \dots, a_l^n) = f_{A_l}(a_l^1, \dots, a_l^i, \dots, a_l^n)$, for all $1 \leq l < j$, and, due to the monotonicty of $f_{A_j}$, $f_{A_j}(a_j^1, \dots, a_j^i{\prime}, \dots, a_j^n) >_j f_{A_j}(a_j^1, \dots, a_j^i, \dots, a_j^n)$. By definition of $>_{lex}$, we conclude that

$$f_B(\dots, (a_1^i{\prime}, \dots, a_k^i{\prime}), \dots) >_{lex} f_B(\dots, (a_1^i, \dots, a_k^i), \dots)$$

as we wanted. Finally to see that $B$ is compatible with $R$, take $l \to r$ an arbitrary rule in $R$ and let $R^i$ be the TRS with smallest index $i$, $1 \leq i \leq k$, such that $l \to r \in R^i$. Let $\rho : \mathcal{X} \to B$ be any evaluation (with $\mathcal{X} = \bigcup_{p=1}^k \mathcal{X}^p$). The combination of $l \to r \notin R^l$, for $1 \leq l < i$ and the fact that $A_l$ is a monotone quasi-model for $(\bigcup_{j=l+1}^k R_j) \setminus R^l$, allows us to say that $[\![l, \pi_l \circ \rho]\!]_{A_l} \geq_l [\![r, \pi_l \circ \rho]\!]_{A_l}$, for $1 \leq l < i$ and since $A_i$ is compatible with $R^i$, also $[\![l, \pi_i \circ \rho]\!]_{A_i} >_i [\![r, \pi_i \circ \rho]\!]_{A_i}$. Then using lemma 2.100, we can write

$$\begin{aligned}
& [\![l, \rho]\!]_B = \\
& ([\![l, \pi_1 \circ \rho]\!]_{A_1}, \dots, [\![l, \pi_i \circ \rho]\!]_{A_i}, \dots, [\![l, \pi_k \circ \rho]\!]_{A_k}) \\
& >_{lex} \\
& ([\![r, \pi_1 \circ \rho]\!]_{A_1}, \dots, [\![r, \pi_i \circ \rho]\!]_{A_i}, \dots, [\![r, \pi_k \circ \rho]\!]_{A_k}) = \\
& [\![r, \rho]\!]_B
\end{aligned}$$

Arbitrariety of the rule chosen gives the result. □

Note that we don't require neither the signatures $\mathcal{F}^i$ nor the rewrite rules $R^i$ to be disjoint sets.

An interesting corollary of theorem 2.99 is the following.

**Corollary 2.101.** *Let $(\mathcal{F}, \mathcal{X}, R)$ be a TRS. Let $(A_1, >_1)$ be a non-empty (well-founded or total) monotone $\mathcal{F}$-algebra compatible with $(\mathcal{F}, \mathcal{X}, R)$ and suppose that $(A_i, >_i)$, with $2 \leq i \leq k$, for some $k \geq 2$, is a non-empty (well-founded or total) monotone $\mathcal{F}$-algebra for $(\mathcal{F}, \mathcal{X}, \emptyset)$. Define $(B, >_{lex})$ and $f_B$ as in theorem 2.99. Then $(B, >_{lex})$ is a non-empty (well-founded or total) monotone $\mathcal{F}$-algebra compatible with $(\mathcal{F}, \mathcal{X}, R)$.*

Theorem 2.99 is particularly useful when proving termination of $R \cup \mathcal{E}mb_{\mathcal{F}}$ since it is sometimes easier to provide a monotone algebra compatible with $\mathcal{E}mb_{\mathcal{F}}$ that is also a monotone model for $R$, and combine it with a monotone algebra compatible with $R$.

Theorem 2.94 tells us that each terminating TRS has associated with it a class of monotone algebras in which a termination proof can be given. This means that we can use intrinsic properties of the algebras to try to characterize differents aspects of terminating TRS's. Using this idea Zantema [109, 107] proposed a classification of different types of termination. He distinguished between *polynomial termination*, *$\omega$-termination*, *total termination* and *simple termination*. Below we introduce these concepts and give some examples.

### Simple Termination

The terminology *simple termination* was first used by Kurihara and Ohuchi [66]. They define a TRS to be *simply terminating* if the TRS is compatible with a simplification order in the sense of definition 2.75. As we have seen, compatibility with such a simplification order in the case that $\mathcal{F}$ is infinite, is not enough to guarantee termination of the TRS; for that we need to use the definition of simplification order of Middeldorp and Zantema [78], presented as definition 2.77. Since for finite signatures both definitions coincide, no harm is done. The following definition is taken from Middeldorp and Zantema [78].

**Definition 2.102.** A TRS $(\mathcal{F}, \mathcal{X}, R)$ is *simply terminating* if it is compatible with a simplification order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

**Example 2.103.** Let $\mathcal{F} = \{\mathit{fact}, s, \mathit{mul}, 0, 1\}$ where the function symbols $0, 1$ are constants, *fact* and $s$ have arity 1 and *mul* has arity 2. Let $R$ be given by the rules

$$\begin{array}{lcl} \mathit{fact}(s(x)) & \to & \mathit{mul}(s(x), \mathit{fact}(x)) \\ \mathit{fact}(0) & \to & 1 \end{array}$$

Consider the set $(A, >) = (\mathbb{N}, >)$, where $>$ is the usual total well-founded order on the natural numbers, and define the algebra operations:

$$\begin{array}{lcr} \mathit{fact}_A(x) & = & x^2 + 1 \\ s_A(x) & = & x + 2 \\ \mathit{mul}_A(x, y) & = & x + y + 1 \\ 0_A, 1_A & = & 0 \end{array}$$

It is not difficult to see that with these operations $(A, >)$ is indeed a monotone algebra for the TRS, compatible with the rewrite rules.

Since $\mathcal{F}$ is finite, we can take as *pwo* the empty relation and therefore the homeomorphic embedding relation associated to it is, according to definition 2.76, the least rewrite order satisfying the subterm property. Again it is not difficult to see that the interpretation defined indeed satisfies this property and so contains the embedding relation. In other words $>_A$, the induced order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, is a simplification order compatible with $R$ and so $R$ is simply terminating.

As a consequence of Kruskal's theorem, we have (see Middeldorp and Zantema [78]):

**Theorem 2.104.** *Every simply terminating TRS is terminating.*

The reverse is not true, i. e., not all terminating TRS's are simply terminating, as the following example shows.

**Example 2.105.** The TRS given by

$$R : f(f(x)) \to f(g(f(x)))$$

is terminating. To see that we define the following monotone algebra. Let $A = \{0,1\} \times \mathbb{N}$ and $\succ$ be defined on $A$ by

$$(a,n) \succ (b,m) \iff (a = b) \text{ and } n > m$$

where $>$ is the usual order on the natural numbers. It is not difficult to see that $\succ$ is a well-founded partial order over $A$. Define the algebra operations

$$\begin{array}{rcl} g(a,n) & = & (1,n) \text{ for all } n \in \mathbb{N},\ a \in \{0,1\} \\ f(a,n) & = & \left\{ \begin{array}{ll} (0,n+1) & \text{for all } n \in \mathbb{N}, \text{ if } a = 0 \\ (0,n) & \text{for all } n \in \mathbb{N}, \text{ if } a = 1 \end{array} \right. \end{array}$$

It can be seen that with these operations $(A, \succ)$ is a monotone algebra for $R$, compatible with the rewrite rule since, given any evaluation $\rho : \mathcal{X} \to A$,

$$[\![ f(f(x)), \rho ]\!] = \left\{ \begin{array}{ll} (0,n+2) & \text{if } \rho(x) = (0,n) \\ (0,n+1) & \text{if } \rho(x) = (1,n) \end{array} \right.$$

and

$$[\![ f(g(f(x))), \rho ]\!] = \left\{ \begin{array}{ll} (0,n+1) & \text{if } \rho(x) = (0,n) \\ (0,n) & \text{if } \rho(x) = (1,n) \end{array} \right.$$

and in any case $[\![ f(f(x)), \rho ]\!] \succ [\![ f(g(f(x))), \rho ]\!]$.

To see why $R$ is not simply terminating, remark that from the definition of homeomorphic embedding, any partial order containing it also satisfies the subterm property. Suppose $>$ is a simplification order and let $c$ be any constant. Then we must have

$$\begin{array}{rcl} f(f(c)) & > & \text{(compatibility of } > \text{ with the rewrite rule)} \\ f(g(f(c))) & > & \text{(subterm property and closedness under contexts)} \\ f(f(c)) & & \end{array}$$

which contradicts transitivity and irreflexivity of the order.

For finite fixed-arity signatures, the homeomorphic embedding is just the the least rewrite order having the subterm property and it is not difficult to see that this order can be "defined" by the rewrite system $\mathcal{E}mb_{\mathcal{F}}$, i. e., the homeomorphic embedding is the smallest rewrite order satisfying $f(\ldots, x_i, \ldots) > x_i$, for all embedding rules of $\mathcal{E}mb_{\mathcal{F}}$. Indeed in Zantema [109] the following result (implicit in many earlier works) can be found.

**Theorem 2.106.** *Let $R$ be a TRS over a finite fixed-arity signature $\mathcal{F}$. The following assertions are equivalent:*

1. *$R$ is simply terminating.*
2. *$R \cup \mathcal{E}mb_{\mathcal{F}}$ is simply terminating.*
3. *$R \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating.*

In order to have a similar statement for finite varyadic signatures, the TRS $\mathcal{E}mb_{\mathcal{F}}$ would have to contain the rewrite rules arising from the deletion property.

The following result is a simple observation but quite useful.

**Lemma 2.107.** *Let $R$ be a TRS over a finite fixed-arity signature $\mathcal{F}$. If $R$ is terminating and length-preserving then $R \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating.*

For the infinite case a characterization similar to theorem 2.106 also exists but now the embedding rules have to be defined with respect to a *pwo* on $\mathcal{F}$. From Middeldorp and Zantema [78] we recall the following definition and result.

**Definition 2.108.** Let $>$ be a partial order on a signature $\mathcal{F}$. The TRS $\mathcal{E}mb_{(\mathcal{F},>)}$ consists of all the rewrite rules of $\mathcal{E}mb_{\mathcal{F}}$ together with the rules

$$f(x_1, \ldots, x_m) \to g(x_{i_1}, \ldots, x_{i_n})$$

with $f \in \mathcal{F}$ admitting arity $m$, $g \in \mathcal{F}$ admitting arity $n$, $f > g$, $m \geq n$, $1 \leq i_1 < \ldots < i_n \leq m$, whenever $n \geq 1$, and where $x_1, \ldots, x_m$ are pairwise distinct variables.

Theorem 2.106 can now be re-phrased as:

**Theorem 2.109.** *$(\mathcal{F}, \mathcal{X}, R)$ is simply terminating if and only if $(\mathcal{F}, \mathcal{X}, R \cup \mathcal{E}mb_{(\mathcal{F},\succ)})$ is terminating, for some* pwo *$\succ$ over $\mathcal{F}$.*

**Total termination**

The concept of *total termination* was defined in Zantema [107, 109]. We present the definition.

**Definition 2.110.** A TRS $(\mathcal{F}, \mathcal{X}, R)$ is called *totally terminating* if it admits a compatible total well-founded monotone $\mathcal{F}$-algebra $(A, >)$. If $A = \mathbb{N}$ and $>$ is the usual order on the natural numbers, we speak about *$\omega$-termination*; if $A = \mathbb{N}$, $>$ is the usual order on the natural numbers, and the algebra operations are polynomials over $\mathbb{N}$, we speak about *polynomial termination*.

Note that totality of $>$ (in the algebra) does not imply totality of $>_A$, the induced relation on terms. For example if $\mathcal{X}$ contains at least two different elements $x$ and $y$, we cannot have $x >_A y$ nor $y >_A x$ since depending on the assignment $\rho \in A^{\mathcal{X}}$, we will sometimes have $[\![x, \rho]\!]_A > [\![y, \rho]\!]_A$ and sometimes $[\![y, \rho]\!]_A \geq [\![x, \rho]\!]_A$.

It is clear from the definition that any totally terminating TRS is also terminating. Furthermore we have

$$\text{polynomial termination} \;\Rightarrow\; \omega\text{-termination} \;\Rightarrow\; \text{total termination.}$$

The reverse of these implications do not hold. The following TRS

$$f(g(h(x))) \to g(f(h(g(x))))$$

is $\omega$-terminating and not polynomially terminating (see Zantema [109]), and the system

$$f(g(x)) \to g(f(f(x)))$$

is totally terminating and not $\omega$-terminating (see chapter 5).

The relation between total termination and simple termination varies depending on the cardinality of the signature. We have the following result from Zantema [109].

**Theorem 2.111.** *Let $(\mathcal{F}, \mathcal{X}, R)$ be a TRS. Then $R$ is totally terminating if and only if $R \cup \mathcal{E}mb_{\mathcal{F}}$ is totally terminating.*

**Proof** (Sketch). The "if" part is trivial. For the "only-if" part we remark that any function monotone on all coordinates and defined on a well-order $(A, >)$ has the property $f(a_1, \ldots, a_n) \geq a_i$, for any $1 \leq i \leq n$ (this result appeared in Zantema [108] and will be proved in chapter 5). Now since $R$ is totally terminating, it admits a compatible well-founded monotone $\mathcal{F}$-algebra $(A, >)$ for which the order $>$ is total on $A$. Furthermore, by the observation above, we have that $f(x_1, \ldots, x_n) \geq_A x_i$, for any embedding rule, i. e., $(A, >)$ is a monotone total well-founded model for $\mathcal{E}mb_{\mathcal{F}}$. We now define an interpretation in $(\mathbb{N}, \succ)$, where $\succ$ is the usual order on the naturals, as follows: $f_{\mathbb{N}}(x_1, \ldots, x_n) = x_1 + \ldots + x_n + 1$. Clearly $(\mathbb{N}, \succ)$ is a total well-founded monotone algebra for $\mathcal{F}$ and for each embedding rule $f(x_1, \ldots, x_n) \to x_i$, we have that $[\![f(x_1, \ldots, x_n), \rho]\!]_{\mathbb{N}} = \rho(x_1) + \ldots + \rho(x_n) + 1 \succ \rho(x_i)$, for any $\rho : \mathcal{X} \to \mathbb{N}$ and any $1 \leq i \leq n$. Consequently $(\mathbb{N}, \succ)$ is compatible with $\mathcal{E}mb_{\mathcal{F}}$. Due to theorem 2.99, the lexicographic product $B$ of the algebras $(A, >)$ and $(\mathbb{N}, \succ)$ with interpretations $f_B = (f_A, f_{\mathbb{N}})$ is a total well-founded monotone algebra compatible with $R \cup \mathcal{E}mb_{\mathcal{F}}$. □

For finite signatures, theorems 2.106 and 2.111 give the relation between total termination and simple termination, since

$$\begin{array}{c} R \text{ is totally terminating} \\ \Updownarrow \\ R \cup \mathcal{E}mb_{\mathcal{F}} \text{ is totally terminating} \\ \Downarrow \\ R \cup \mathcal{E}mb_{\mathcal{F}} \text{ is terminating} \\ \Updownarrow \\ R \text{ is simply terminating.} \end{array}$$

The reverse implication is not true. The system

$$\begin{array}{rcl} f(a) & \to & f(b) \\ g(b) & \to & g(a) \end{array}$$

is simply terminating. First note that $R$ is terminating since if $s \to_R t$ then $t$ is in normal form. Since $R$ is also length-preserving, we can apply lemma 2.107 to conclude that $R \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating, so $R$ is simply terminating.

To see that $R$ is not totally terminating, note that any possible interpretation of $a$ and $b$ combined with monotonicity and compatibility with the rewrite rules, forces incomparability of $a$ and $b$ while $a \neq b$, and that is impossible in any total algebra.

So for TRS's over finite signatures we have:

polynomial termination
$\Downarrow$
$\omega$-termination
$\Downarrow$
total termination
$\Downarrow$
simple termination.

For infinite signatures, things are not quite so as Middeldorp and Zantema showed in [78]. Indeed we have that polynomial termination no longer implies simple termination and consequently neither $\omega$-termination nor total termination imply simple termination. What happens is that theorem 2.111 no longer holds if we replace $\mathcal{E}mb_{\mathcal{F}}$ by $\mathcal{E}mb_{(\mathcal{F},>)}$ for an arbitrary *pwo* over $\mathcal{F}$. The next TRS, from Middeldorp and Zantema [78], provides a counterexample. Let $\mathcal{F} = \{f_i, g_i |\ i \geq 1\}$, where for each $i \geq 1$, $f_i$ and $g_i$ have arity 1. Let $R$ be given by the rules

$$f_i(g_j(x)) \to f_j(g_j(x)), \quad \text{for any } j > i$$

Take $(A, >)$ as $(\mathbb{N}_1, >)$, where $\mathbb{N}_1$ contains only strictly positive naturals, and define the following interpretation:

$$\begin{array}{rcll} f_{i_A}(x) & = & x^3 - ix^2 + i^2x & (i \geq 1) \\ g_{i_A}(x) & = & x + 2i & (i \geq 1) \end{array}$$

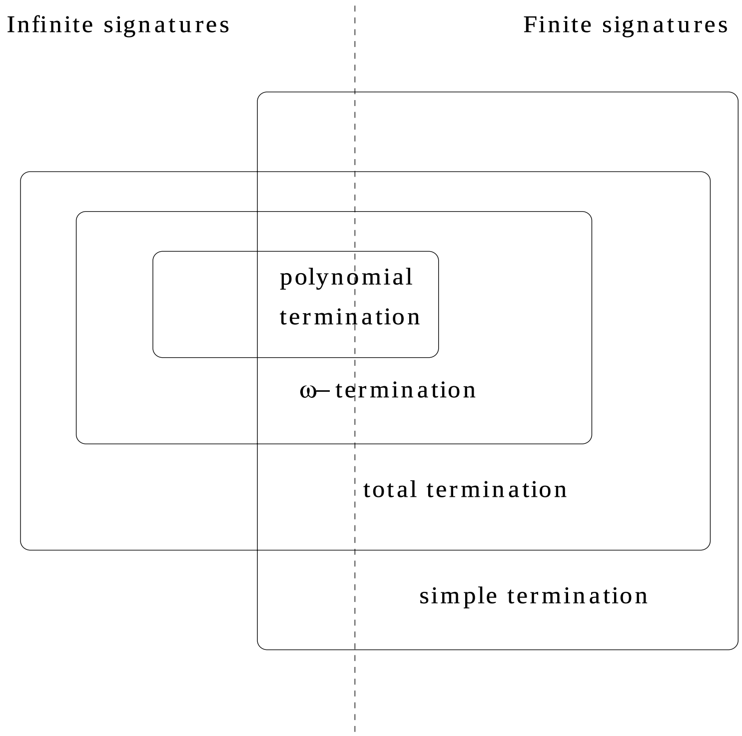
It can be seen that the interpretations of both $f_i$ and $g_i$, for all $i \geq 1$, are strictly monotone and that the induced order $>_A$ on terms satisfies $f_i(g_j(x)) >_A f_j(g_j(x))$. So $R$ is polynomially terminating.

However $R \cup \mathcal{E}mb_{(\mathcal{F},\succ)}$ is not terminating for any *pwo* $\succ$ on $\mathcal{F}$. Suppose $\succ$ is a *pwo* on $\mathcal{F}$. Since the sequence $(f_i)_{i\geq 1}$ must be good, we have two indices $j > l$ such that $f_j \succ f_l$. Let $\gg$ be any simplification order compatible with $R \cup \mathcal{E}mb_{(\mathcal{F},\succ)}$. Then we must have

$$\begin{array}{rcl} f_j(g_j(x)) & \gg & (\text{since } f_j \succ f_l) \\ f_l(g_j(x)) & \gg & (\text{compatibility of } \gg \text{ with } R) \\ f_j(g_j(x)) & & \end{array}$$

which contradicts well-foundedness of $\gg$.

The following picture from Middeldorp and Zantema [78] summarizes the relationship between the kinds of termination considered.




## Modularity of Termination

As mentioned before, we will consider the weakest form of modularity. It is well-known that termination is not a modular property of TRS's. The following famous counter-example is due to Toyama. Let $R_1$ and $R_2$ be given by:

$$R_1 : f(0,1,x) \quad \rightarrow \quad f(x,x,x) \qquad\qquad \begin{array}{rcl} R_2 : g(x,y) & \rightarrow & x \\ g(x,y) & \rightarrow & y \end{array}$$

Then both $R_1$ and $R_2$ are terminating while $R_1 \oplus R_2$ admits the cyclic derivation:

$$f(0,1,g(0,1)) \rightarrow f(g(0,1),g(0,1),g(0,1)) \rightarrow f(0,g(0,1),g(0,1)) \rightarrow f(0,1,g(0,1))$$

If the TRS's satisfy some conditions, termination of the union can be concluded from termination of the parts. The following result is proven in Middeldorp [76].

**Theorem 2.112.** *Let $(\mathcal{F}_1, \mathcal{X}_1, R_1)$ and $(\mathcal{F}_2, \mathcal{X}_2, R_2)$ be two disjoint and terminating TRS's such that one of them contains neither duplicating nor collapsing rules. Then $R_1 \oplus R_2$ is terminating.*

Simple termination is a modular property. For the case of finite signatures, the result is due to Kurihara and Ohuchi [66], and for infinite signatures (and for the notion of simple termination we use here) the result is due to Middeldorp and Zantema [79].

As far as total termination is concerned, modularity of termination holds under some conditions as we will prove in chapter 5. As a consequence of the results presented there, we have that both $\omega$-termination and polynomial termination are modular properties.

Modular properties of TRS's are treated extensively in Middeldorp [76]. Recently Rubio [94] has given different proofs of most known results on modularity, including modularity of total termination.

### 2.4.3 Undecidability Questions

In this subsection we restrict ourselves to TRS's for which both the signature $\mathcal{F}$ and the set of rewrite rules $R$ is finite.

Most interesting properties of TRS's are undecidable, i. e., it is not possible to devise a procedure that can decide whether or not a given TRS has that property. Confluence is an undecidable property though decidable for ground TRS's. For the property of termination, the situation is the same. In 1978, Huet and Lankford [47] proved that given a finite TRS $R$ and a term $t$, the problem of determing whether $t$ has an infinite rewrite sequence is undecidable. Furthermore determining whether $R$ is terminating is undecidable. For ground TRS they showed that these properties are decidable (termination is decidable even for right-ground TRS's; see Dershowitz [21]). In their proof Huet and Lankford used a translation from Turing machines to term rewriting systems such that the uniform halting problem for Turing machines (known to be undecidable) was equated with termination of the TRS. A characteristic of this proof is that only unary function symbols are used (so Huet and Lankford presented in fact a proof of undecidability of termination for string rewriting systems). In Klop [61] a similar simpler proof is given using also binary function symbols. We also present such a proof in Appendix B.

The original result of Huet and Lankford has been since refined. In 1987, Dershowitz [23] showed that termination is undecidable even for TRS's containing only two rewriting rules. Later on, in 1989, Dauchet [17] presented a complicated proof of the undecidabilty of termination for one-rule TRS's. Since the TRS used was *orthogonal*, Dauchet proved a stronger result, namely he showed undecidabilty of termination for one-rule *orthogonal* TRS's (i. e., TRS's which are left-linear and have no critical pairs - for a definition of critical pairs see any survey on TRS's, for example Klop [61]). The results mentioned so far have in common the fact that they all use a translation from Turing Machines to TRS's and equate the halting problem for Turing Machines with termination. Using a different approach, Lescanne [72] has shown the result proven in Dauchet [17] but in a simpler way. In his proof he uses a translation to another undecidable problem, namely the *Post Correspondence problem* from Post [91].

Since we have distinguished between different types of termination, we can ask ourselves whether these types of termination are or not decidable. Except for termination proofs using *recursive path order*, which are decidable neither simple termination nor total termination are decidable. Simple termination was proven undecidable even for one rule orthogonal systems by Middeldorp and Gramlich [77]. They used a translation to linear bounded automata, whose halting problem had been proven undecidable via the Post correspondence problem by Caron [11]. Undecidability of total termination has been shown recently by Zantema [110] via the Post correspondence problem, but no restriction has been put on the number of rules. It is still an open problem whether total termination of one-rule rewrite systems is decidable. As for polynomial and $\omega$-termination, their (un)decidability is still an open problem.

For the sake of completeness, we give here a proof of undecidability of termination for the general case. For this purpose, we will use a translation of Post correspondence problem to term rewriting.

A Post correspondence system is composed of an alphabet $\Sigma$ and a finite subset $P = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} \subseteq \Sigma^* \times \Sigma^*$ of ordered pairs. The *Post correspondence problem* (PCP for short) consists in, given such a system determine whether there are indexes $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $\alpha_{i_1} \cdots \alpha_{k_1} = \beta_{i_1} \cdots \beta_{k_1}$, (where "$\cdot$" represents string concatenation), i. e., determine whether the system has a solution. If we impose the restriction that $i_1 = 1$ , this problem becomes the *Modified Post Correspondence Problem* (MPCP for short).

It is well-known that (M)PCP is an undecidable problem even if $\Sigma$ contains only two elements (see Post [91], Rozenberg and Salomaa [93]). We will see that termination of TRS's is undecidable by translating the PCP to term rewriting in such a way that given a correspondence system $P$, $P$ has a solution if and only if $R_P$ (the associated TRS) is not terminating. As a consequence we obtain undecidability of termination for TRS's.

Suppose then that $P = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} \subseteq \Sigma^* \times \Sigma^*$, with $n > 0$. We are going to consider terms over the signature $\mathcal{F} = \Sigma \cup \{f, c\}$, where each symbol $a \in \Sigma$ has arity $1$, $f$ has arity $3$ and $c$ is a constant. For each word $\alpha \in \Sigma^*$ if $\alpha = a_1 \cdots a_k$ then the term $\alpha(x)$ is given by $a_1(\ldots a_k(x) \ldots)$. The TRS $R_P$ (based on ideas from Lescanne [72] and Zantema [110]) is defined over the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and contains the rules:

$$\begin{array}{lll} f(\alpha(x), \beta(y), z) & \rightarrow & f(x, y, z) \quad \text{for each pair } (\alpha, \beta) \in P \\ f(c, c, a(z)) & \rightarrow & f(a(z), a(z), a(z)) \quad \text{for each element } a \in \Sigma \end{array}$$

We have the following results connecting solutions of the PCP and reductions.

**Lemma 2.113.** *If the PCP for $P$ has a solution then $R_P$ is not terminating.*

**Proof** Let $\alpha_{i_1}. \ldots .\alpha_{i_k} = \beta_{i_1}. \ldots .\beta_{i_k}$ be a solution for $P$ then the term

$$f(\alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots))$$

admits the following cyclic rewrite sequence:

$$\begin{array}{ll} f(\alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots)) & \\ & \rightarrow^+ \\ f(c, c, \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots)) & \\ & \rightarrow \\ f(\alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots), \alpha_{i_1}(\ldots \alpha_{i_k}(c) \ldots)) & \end{array}$$

□

**Lemma 2.114.** *If the TRS $R_P$ is not terminating, then the PCP for $P$ has a solution.*

**Proof** The idea behind the proof is to show that if $R_P$ has infinite rewrite sequences, then there are terms $t$ admitting an infinite rewrite sequence and having the form $f(s_1, s_2, s_3)$, where the function symbol $f$ does not occur in any term $s_i$, $1 \leq i \leq 3$. This combined with the shape of the rules will give a solution for $P$. The easiest way of proving this assertion is to associate to $R_P$ a many-sorted TRS $R_\mathcal{S}$. We first introduce the idea of many-sorted rewriting (see Huet and Oppen [48]; Goguen and Meseguer [44]). We will follow the notation and conventions of Zantema [106, 109].

Let $\mathcal{S}$ be a set of sorts and $\mathcal{X}_\mathcal{S}$ an $\mathcal{S}$-sorted set of variables. Let $\mathcal{F}'$ be a set of function symbols such that with each symbol there is a sort and an arity associated, given respectively by the functions:

$$\begin{aligned} \mathtt{st} &: \mathcal{F}' \to \mathcal{S} \\ \mathtt{ar} &: \mathcal{F}' \to \mathcal{S}^* \end{aligned}$$

The $\mathcal{S}$-sorted set of terms, is $\mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S}) = \bigcup_{s \in \mathcal{S}} \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$, where $\mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$ is defined by:

- $\mathcal{X}_s \subseteq \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$, for any $s \in \mathcal{S}$ (we remark that all sets $\mathcal{X}_s$ are disjoint),
- $f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$, if $f \in \mathcal{F}'$, $\mathtt{ar}(f) = s_1 \ldots s_k$, $\mathtt{st}(f) = s$ and $t_i \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_{s_i}$, for $i = 1, \ldots, k$.

Note that substitutions now have to respect the sort of variables, i. e., a substitution $\sigma$ is a function $\sigma : \mathcal{X}_\mathcal{S} \to \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})$ such that $\sigma(\mathcal{X}_s) \subseteq \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$ for all sorts $s \in \mathcal{S}$.

An $\mathcal{S}$-sorted TRS has an $\mathcal{S}$-sorted set of rules $R_\mathcal{S} = \bigcup_{s \in \mathcal{S}} R_s$ such that, for any $s \in \mathcal{S}$, $R_s \subseteq \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s \times \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_s$. The reduction relation on a sorted TRS is a $\mathcal{S}$-sorted relation $\to_{R_\mathcal{S}} = \bigcup_{s \in \mathcal{S}} \to_{R_s}$ on $\mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})$, where $\to_{R_s}$ is given by:

- $l\sigma \to_{R_s} r\sigma$, for every $(l, r) \in R_s$ and every substitution $\sigma$,
- $f(t_1, \ldots, t_k, \ldots, t_n) \to_{R_s} f(t_1, \ldots, t'_k, \ldots, t_n)$, for every $f \in \mathcal{F}'$ with $\mathtt{ar}(f) = s_1 \ldots s_n$, $\mathtt{st}(f) = s$, $t_i \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_{s_i}$, for $i = 1, \ldots, n$, $t'_k \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_{s_k}$, and $t_k \to_{R_{s_k}} t'_k$.

We build our particular sorted TRS. In our case $\mathcal{S} = \{S, F\}$, $\mathcal{F}' = \{a' |\ a \in \Sigma\} \cup \{f', c'\}$, $\mathcal{X}_\mathcal{S}$ is the $\mathcal{S}$-sorted set of variables such that $\mathcal{X}_F = \emptyset$. Each function symbol $a'$ has arity $\mathtt{ar}(a') = S$ and sort $\mathtt{st}(a') = S$; $f'$ has arity $\mathtt{ar}(f') = S \cdot S \cdot S$ and sort $\mathtt{st}(f') = F$; $c'$ has arity $\epsilon$ (the empty word) and sort $S$.

We have only two kinds of terms, namely terms of sort $S$ and terms of sort $F$, such that

- $t \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_S \iff t = a'_1(\ldots(a'_k(\Delta)\ldots)$, where $\Delta \in \{c'\} \cup \mathcal{X}_S$,
- $t \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_F \iff t = f'(s_1, s_2, s_3)$, where $s_i \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_S$.

As for the reduction rules, our sorted system has two set of rules $R_S$ and $R_F$ with $R_S = \emptyset$ and $R_F$ given by:

$$\begin{array}{lll} f'(\alpha'(x), \beta'(y), z) & \rightarrow & f(x, y, z) & \text{for each pair } (\alpha, \beta) \in P \\ f(c', c', a'(z)) & \rightarrow & f(a'(z), a'(z), a'(z)) & \text{for each element } a \in \Sigma \end{array}$$

where for any word $\tau \in \Sigma^*$, if $\tau = a_1 \cdots a_k$, the term $\tau'(x)$ is given by $a'_1(\ldots a'_k(x)\ldots)$.

The $\mathcal{S}$-sorted TRS $R_\mathcal{S}$ is terminating if and only if for every $s \in \mathcal{S}$, the reduction relation $\rightarrow_{R_s}$ is terminating. In this case since $R_S = \emptyset$, $\rightarrow_{R_S}$ gives no reductions, so the system is terminating if and only if $\rightarrow_{R_F}$ is terminating.

Now associated with any sorted TRS $R_\mathcal{S}$, there is a one-sort TRS obtained from the sorted version by ignoring the sort information. This new TRS, that we denote by $\Theta(R_\mathcal{S})$, is given by:

- $\mathcal{F} = \{f | f' \in \mathcal{F}'\}$, $\mathcal{X} = \bigcup_{s\in\mathcal{S}} \mathcal{X}_s$,
- $(l, r) \in \Theta(R_\mathcal{S}) \iff (l', r') \in \bigcup_{s\in\mathcal{S}} R_s$, $l = \theta(l')$ and $r = \theta(r')$, where $\theta : \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S}) \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined inductively by:
  - $\theta(x) = x$, for all $x \in \mathcal{X}_s$, for all $s \in \mathcal{S}$ (this definition poses no problem since all $\mathcal{X}_s$ are disjoint),
  - $\theta(f'(t_1, \ldots, t_n)) = f(\theta(t_1), \ldots, \theta(t_n))$, for all $f' \in \mathcal{F}'$ and $t_i$ of the appropriate sort.

We remark that $p \rightarrow_{R_s} q \iff \theta(p) \rightarrow_{\Theta(R_\mathcal{S})} \theta(q)$.

From Zantema [106, 109] we know that if $R_\mathcal{S}$ is an $\mathcal{S}$-sorted TRS without both collapsing and duplicating rules then $R_\mathcal{S}$ is terminating if and only if $\Theta(R_\mathcal{S})$ is terminating.

Since by construction we have $R_P = \Theta(R_\mathcal{S})$ and $R_\mathcal{S}$ has duplicating but no collapsing rules, using the previous statement we can conclude that if $R_P$ does not terminate, neither does $R_\mathcal{S}$.

Let $R_1$ denote the system containing the rules:

$$f'(\alpha'(x), \beta'(y), z) \quad \rightarrow \quad f(x, y, z), \quad \text{for each pair } (\alpha, \beta) \in P$$

and $R_2$ denote the system containing the rules:

$$f(c', c', a'(z)) \quad \rightarrow \quad f(a'(z), a'(z), a'(z)), \quad \text{for each element } a \in \Sigma$$

It is not difficult to see that both $R_1$ and $R_2$ are terminating. Since $R_\mathcal{S}$ is not terminating, any infinite rewrite sequence must contain infinitely many applications of rules from both $R_1$ and $R_2$, such that there are no infinitely many consecutive steps of $R_1$ or $R_2$. This means that any infinite reduction will contain a reduction sequence of the shape:

$$s \rightarrow_{R_2} t \rightarrow^+_{R_1} u \rightarrow_{R_2} v$$

Since $s \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_F$ and due to the shape of the rules in $R_2$, we have that $s = f'(c', c', s_0)$ and $t = f'(s_0, s_0, s_0)$, for some term $s_0 \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_S$. For similar reasons, $u = f'(c', c', u_0)$ and $v = f'(u_0, u_0, u_0)$, for some term $u_0 \in \mathcal{T}(\mathcal{F}', \mathcal{X}_\mathcal{S})_S$. Now, due to the shape of the rules of $R_1$, we have that $s_0 = \alpha'_{i_1}(\dots(\alpha'_{i_k}(c')\dots) = \beta i_1{}'(\dots(\beta i_k{}'(c')\dots)$, and by replacing each symbol $a'$ by $a$ in the above expression, we get a solution for $P$. □

As a consequence of lemmas 2.113 and 2.114, we have:

**Theorem 2.115.** *Given a TRS $R$, the problem of determining whether $R$ is terminating is undecidable.*

The coding of the PCP $P$ in $R_P$ allows us to prove undecidability of termination but not of simple or total termination. It is however very easy to modify the system $R_P$ in order to obtain undecidability of simple termination. If we define $R_P^{st}$ to be

$$\begin{array}{llll} R_P^{st}: & f(\alpha(x), x', \beta(y), y') & \to & f(x, \overleftarrow{\alpha}(x'), y, \overleftarrow{\beta}(y')), \quad \text{for each pair } (\alpha, \beta) \in P \\ & f(x, a(z), x, a(z)) & \to & f(a(x), z, a(x), z), \quad \text{for each } a \in \Sigma \end{array}$$

where for any string $\alpha = a_1 \cdots a_k \in \Sigma^*$, $\overleftarrow{\alpha}(x)$ represents the term $a_k(\dots a_1(x)\dots)$; then in a similar way as done before we can prove that the PCP $P$ has a solution if and only if the TRS $R_P^{st}$ is terminating. But since $R_P^{st}$ is length-preserving, termination of $R_P^{st}$ is equivalent to simple termination (see lemma 2.107) and so we conclude that simple termination is undecidable. Furthermore if we introduce another unary function symbol $\overline{a}$ for each element $a \in \Sigma$ and define $R_P^{tt}$ to be:

$$\begin{array}{llll} R_P^{tt}: & f(\alpha(x), x', \beta(y), y') & \to & f(x, \overline{\alpha}(x'), y, \overline{\beta}(y')), \quad \text{for each pair } (\alpha, \beta) \in P \\ & f(x, \overline{a}(z), x, \overline{a}(z)) & \to & f(a(x), z, a(x), z), \quad \text{for each } a \in \Sigma \end{array}$$

where for any string $\alpha = a_1 \cdots a_k \in \Sigma^*$, the term $\overline{\alpha}(x)$ is now given by $\overline{a_k}(\dots\overline{a_1}(x)\dots)$, it can be proven that the PCP for $P$ has a solution if and only if $R_P^{tt}$ is terminating if and only if $R_P^{tt}$ is total terminating. The proof of this statement is however much more complicated (see Zantema [110]).

We have seen that termination is in general an undecidable property. This remains so if we are interested only in termination starting from a fixed term. For a proof of this statement see Huet and Lankford [47] or appendix B. Finally termination is decidable for ground rewriting.

## 2.5 Conclusions

In this chapter we tried to summarize the concepts needed for the subsequent chapters. Most of the results presented are not new, though some have a different presentation. Needless to say this chapter is quite incomplete. Many important notions and properties of term rewriting (including some strongly related to the topic of termination) have been left out; critical pairs, confluence, completion procedures, quasi-termination, relative termination, are some of them.

We tried both to provide the necessary information for reading this thesis and to be concise. As a consequence topics beyond the scope of this work had to be left out.

With respect to equational rewriting, we have treated this important topic very superficially due to the fact that most of the thesis makes no use of it and the part that does, needs little more than the definition of this concept. Nevertheless, we would like to point out that many concepts concerning termination of TRS's, like simple or total termination, can also be defined in an equational setting. We did not do this because the work presented here was done in the simple rewrite setting and we thought it to be wiser and more natural to keep the presentation restricted to the non-equational case.

# Chapter 3

# Well-foundedness of Term Orderings

Well-foundedness is the essential property of orderings for proving termination of term rewriting systems. In this chapter we introduce criteria on term orderings such that any term ordering possessing the subterm property and satisfying some of these criteria is well-founded. The usual path orders fulfil these criteria, yielding a much simpler proof of well-foundedness than the classical proof depending on Kruskal's theorem. Even more, our approach covers non-simplification orders like the *semantic path order (spo)* of Kamin and Lévy [55] and *general path order (gpo)* of Dershowitz and Hoot [24], that can not be dealt with by Kruskal's theorem.

For finite alphabets we present completeness results, i. e., a term rewriting system terminates if and only if it is compatible with an order satisfying the criterion. For infinite alphabets the same completeness results hold for a slightly different criterion.

## 3.1 Introduction

The usual way of proving termination of a term rewriting system is by finding a well-founded order such that every rewrite step causes a decrease according to this ordering. Proving well-foundedness is often difficult, in particular for recursively defined syntactic orderings. It is therefore desirable to have criteria that help decide whether a particular order is well-founded. A standard criterion of this type is implied by Kruskal's theorem: if a monotonic term ordering over a finite signature satisfies the subterm property then it is well-founded on ground terms; if additionally the order is closed under substitutions then it is well-founded on the set of all terms. However, this theorem does not apply to all terminating TRS's: there are terminating TRS's like

$$f(f(x)) \to f(g(f(x)))$$

that are not compatible with any monotonic term ordering satisfying the subterm property (for more details see example 2.105). Even *recursive path order (rpo)* with lexicographic status over a varyadic alphabet, is not covered directly by Kruskal's theorem as we shall see in chapter 4. This motivated us to look for other conditions ensuring well-foundedness. Our approach consists in removing the monotonicity condition and replacing it by some decomposability condition. For orderings satisfying the subterm property and this decomposability condition we prove well-foundedness in a way that is inspired by Nash-Williams' proof of Kruskal's theorem (Nash-Williams [82], as it appears in Gallier [38]), but which is much simpler. A similar technique, for a particular order, has already been used by Kamin and Lévy [55]. Standard orderings like *recursive path order* (Dershowitz [22], Steinbach [99]) and *semantic path order (spo)* (Kamin and Lévy [55], Dershowitz [23], Geser [42]) trivially satisfy our conditions, yielding a simple proof of well-foundedness for these orders. Moreover, our conditions cover all terminating TRS's: a TRS terminates if and only if it is compatible with an order satisfying our conditions.

We are concerned essentially with term rewriting systems over finite signatures. In the case of an infinite signature the same conditions yield well-foundedness if the signature is provided with a partial well-order satisfying some natural compatibility with the given term ordering.

The rest of the chapter is organized as follows. In section 3.2, we introduce the notion of lifting of an order, which plays an essential role in the theory presented. In section 3.3 we present our well-foundedness criterion for orders on terms built over a finite signature and give some surprising completeness results involving orders closed under substitutions and orders that are total.

In section 3.4, we present a well-foundedness criterion for orders on terms built over infinite signatures. First we follow an approach similar to the one used in section 3.3. For that we need the existence of *well quasi-orders* on the set of function symbols. This requirement is quite strong and to overcome it we introduce a different notion of lifting of orders on terms. Using this new notion we can present a very general and simple result on well-foundedness and show that in this case the completeness results of section 3.3 also hold. The criteria presented are used in section 3.5 to derive well-foundedness of *semantic path order* and *general path order*.

Finally we make some concluding remarks, including some comparison between our results and Kruskal's theorem.

In the sequel we will consider terms over different kinds of signature, for example finite or infinite signatures and finite or infinite sets of variables. We will make clear which restrictions apply at any point. We assume our signature may contain varyadic function symbols.

In the following we will also use "order" as an abbreviation of "partial order" (see definition 2.5).

## 3.2 Liftings and Status

As mentioned before, we replace monotonicity by another condition. This condition relates the comparison between terms $f(s_1, \ldots, s_m)$ and $f(t_1, \ldots, t_n)$ (i. e., terms having the same root) to the comparison of the sequences $\langle s_1, \ldots, s_m \rangle$ and $\langle t_1, \ldots, t_n \rangle$. Here we need to describe how an ordering on terms is lifted to an ordering on sequences of terms. To be able to conclude well-foundedness it is essential that this lifting preserves well-foundedness.

**Definition 3.1.** Let $(S, >)$ be a partial ordered set and $S^* = \cup_{n \in \mathbb{N}} S^n$. We define a *lifting* to be a partial order $>^\lambda$ on $S^*$ for which the following holds: for every $A \subseteq S$, if $>$ restricted to $A$ is well-founded, then $>^\lambda$ restricted to $A^*$ is also well-founded. We use the notation $\lambda(S, >)$ to denote all possible liftings of $>$ on $S^*$.

A typical example of a lifting is the *multiset extension* of an order, i. e., we say that $a_1 \cdots a_k >^\lambda b_1 \cdots b_m$ if $\{\{a_1, \ldots, a_k\}\} >_{mul} \{\{b_1, \ldots, b_m\}\}$. The usual *lexicographic extension* on unbounded sequences is not a lifting as example 2.33 shows, since well-foundedness is not always respected. If the lexicographic comparison is restricted to sequences whose size is bounded by some fixed natural $N$, then it is indeed a lifting.

Another type of lifting is a constant lifting, i. e., any fixed well-founded partial order on $S^*$. More precisely if $\gg$ is a fixed well-founded partial order on $S^*$, and if we define

$$a_1 \cdots a_k >^\lambda b_1 \cdots b_m \iff a_1 \cdots a_k \gg b_1 \cdots b_m.$$

then $>^\lambda$ is a lifting.

Clearly other liftings can be defined like combinations of the ones mentioned. In particular, combinations of multiset and lexicographic liftings can be very useful. For example, in a partial order $(S, >)$ where $a > b$ and $c$ is incomparable with $a$ and $b$, one cannot conclude $\langle a, c, c\rangle >^\lambda \langle c, b, a\rangle$, for the multiset lifting nor for any lexicographic lifting. If we define $>^\lambda$ by

$$\langle s_1, \ldots, s_m\rangle >^\lambda \langle t_1, \ldots, t_n\rangle \iff \begin{cases} (m = n = 3) & \text{and} \\ \langle s_1, s_2\rangle >^{mul} \langle t_1, t_2\rangle & \text{or} \\ \langle s_1, s_2\rangle =^{mul} \langle t_1, t_2\rangle & \text{and } s_3 > t_3 \end{cases}$$

it is not difficult to see that $>^\lambda$ satisfies the definition of lifting and also satisfies $\langle a, c, c\rangle >^\lambda \langle c, b, a\rangle$. This lifting will be used to obtain

$$f(s(x), y, y) >_{rpo} f(y, x, s(x))$$

Classical $>_{rpo}$ (see definition 2.89) cannot be used to compare these two terms.

Definition 3.1 is intended to be applied to terms over varyadic function symbols. If we consider signatures with fixed arity function symbols, we can simplify the notion of lifting: instead of taking liftings of any order we need only to take liftings of fixed order, i. e., the lifting is going to be a partial order over $S^n$, for a fixed natural number $n$. This is a special case of a lifting to $S^*$: $>^\lambda$ is defined on $S^*$ to be the order one has in mind for $S^n$ on sequences of length $n$, while all other pairs of sequences are defined to be incomparable with respect to $>^\lambda$. Note that for the case of infinite signatures with unbounded arities, we cannot find a natural $n$ where liftings for all the possible arities could be defined. This is not really a problem, since we will associate a lifting to each function symbol, and for each function symbol the arity is a fixed natural number.

Again typical examples of liftings are the *lexicographic extension* of $>$ on sequences and the *multiset extension* of $>$ restricted to multisets of a fixed size.

We are interested in orders on terms, so from now on we choose $S = \mathcal{T}(\mathcal{F}, \mathcal{X})$, with $\mathcal{F}$ containing varyadic function symbols, and we fix a partial order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

**Definition 3.2.** Given $(\mathcal{T}(\mathcal{F}, \mathcal{X}), >)$, a *status function* (with respect to $>$) is a function $\tau : \mathcal{F} \to \lambda(\mathcal{T}(\mathcal{F}, \mathcal{X}), >)$, mapping every $f \in \mathcal{F}$ to a lifting $>^{\tau(f)}$.

Again for the case of fixed-arity signatures, a status function will associate to each function symbol $f \in \mathcal{F}$ an order $n$ lifting $>^\lambda$ on $\mathcal{T}(\mathcal{F},\mathcal{X})^n$, where $n$ is the arity of $f$.

We note that the terminology "status" was not chosen arbitrarily. In fact our notion of status is a generalization of status as defined in definition 2.87.

**Example 3.3.** The following status will be used later in connection with the semantic path order. Let $>$ be a partial order and $\succeq$ a well-founded quasi-order both defined on $\mathcal{T}(\mathcal{F},\mathcal{X})$. Write $\succ$ for the strict part of $\succeq$ and $\sim$ for the equivalence relation induced by $\succeq$. For each $f \in \mathcal{F}$ the lifting $\tau(f)$ is given by

$$\langle s_1,\ldots,s_k\rangle >^{\tau(f)} \langle t_1,\ldots,t_m\rangle \iff \begin{cases} s \succ t, \text{ or} \\ s \sim t \text{ and } \langle s_1,\ldots,s_k\rangle >^{mul} \langle t_1,\ldots,t_m\rangle \end{cases}$$

for any $k, m \in \texttt{arity}(f)$ and where $>^{mul}$ is the multiset extension of $>$, $s = f(s_1,\ldots,s_k)$ and $t = f(t_1,\ldots,t_m)$. It is not difficult to see that $>^{\tau(f)}$ is indeed a partial order on $\mathcal{T}(\mathcal{F},\mathcal{X})^*$ and that $>^{\tau(f)}$ respects well-foundedness, being therefore a lifting.

We introduce some additional notation. In the following, $>$ stands for a strict partial order and $\geq$ stands for a quasi-order whose strict part is $>$. It is important to note that the equivalence part of $\geq$ does not need to be equality; what is essential is that $>$ is indeed the strict part of $\geq$ and therefore compatible with the equivalent part. This allows us to conclude that if $s > t \geq u$ or $s \geq t > u$, then also $s > u$, an argument often used in the sequel.

## 3.3 Finite Signatures

In this section we present one of the main results of this chapter. For the sake of simplicity we restrict ourselves to finite signatures (infinite signatures will be treated separately). Surprisingly we do not need to fix the arities of the function symbols.

### 3.3.1 Main Result

In the following we consider the set of terms $\mathcal{T}(\mathcal{F},\mathcal{X})$, over the set of varyadic function symbols $\mathcal{F}$ and such that $\mathcal{F} \cup \mathcal{X}$ is finite.

Recall that a term $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is *well-founded* (with respect to a certain order $>$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$) if there are no infinite descending chains starting with $t$; recall also that $|t|$ denotes the size of the term $t$ (see chapter 2, definition 2.46).

**Definition 3.4.** Let $>$ be a partial order over $\mathcal{T}(\mathcal{F},\mathcal{X})$ and $\tau$ a status function with respect to $>$. We say that $>$ is *decomposable* with respect to $\tau$ if $>$ satisfies

- if $f(s_1,\ldots,s_k) > f(t_1,\ldots,t_m)$ then either
  - $\exists 1 \leq i \leq k : s_i \geq f(t_1,\ldots,t_m)$, or

– $\langle s_1, \ldots, s_k\rangle >^{\tau(f)} \langle t_1, \ldots, t_m\rangle$.

for all $f \in \mathcal{F},\ k, m \in \texttt{arity}(f)$ and terms $s_1, \ldots, s_k, t_1, \ldots, t_m \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

We can now present the main result of this section.

**Theorem 3.5.** *Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\tau$ a status function with respect to $>$. Suppose $>$ has the subterm property and is decomposable with respect to $\tau$, then $>$ is well-founded.*

**Proof** Suppose that $>$ is not well-founded and take an infinite descending chain $t_0 > t_1 > \cdots > t_n > \cdots$, minimal in the following sense:

- $|t_0| \leq |s|$, for all non-well-founded terms $s$;
- $|t_{i+1}| \leq |s|$, for all non-well-founded terms $s$ such that $t_i > s$.

Note that from the first minimality condition follows that any principal subterm of $t_0$ is well-founded. Assume that $t_{i+1} = f(u_1, \ldots, u_k)$ and some $u_j$, with $1 \leq j \leq k$, is not well-founded. From the subterm property we obtain $t_i > t_{i+1} > u_j$, and transitivity gives $t_i > u_j$, hence the second minimality condition yields $|t_{i+1}| \leq |u_j|$ which is a contradiction. We conclude that all principal subterms of any term $t_i$, $i \geq 0$, are well-founded.

Since $\mathcal{F} \cup \mathcal{X}$ is finite and $>$ is irreflexive, the (infinite) sequence $(t_i)_{i\geq 0}$ must contain a subsequence $(t_{\phi(i)})_{i\geq 0}$ with $t_{\phi(i)} = f(u_{i,1}, \ldots u_{i,n_i})$, for a fixed $f \in \mathcal{F}$.[1] By hypothesis, for each $i \geq 0$, either

- $\exists 1 \leq j \leq n_i :\ u_{i,j} \geq t_{\phi(i+1)}$; or
- $\langle u_{i,1}, \ldots, u_{i,n_i}\rangle >^{\tau(f)} \langle u_{i+1,1}, \ldots, u_{i+1,n_{i+1}}\rangle$.

Since all terms $u_{i,j}$ are well-founded, the first case never occurs; note that whenever $u_{i,j}$ $\mathtt{eq}(\geq)$ $t_{\phi(i+1)} > t_{\phi(i+2)}$, then also $u_{i,j} > t_{\phi(i+2)}$, and therefore $u_{i,j}$ would be non-well-founded. Consequently we have an infinite descending chain

$$\langle u_{0,1}, \ldots, u_{0,n_0}\rangle >^{\tau(f)} \langle u_{1,1}, \ldots, u_{1,n_1}\rangle >^{\tau(f)} \langle u_{2,1}, \ldots, u_{2,n_2}\rangle >^{\tau(f)} \ldots$$

Since $>$ is well-founded over the set of terms $\bigcup_{i\geq 0}(\bigcup_{j=1}^{n_i}\{u_{i,j}\})$, this contradicts the assumption that $>^{\tau}(f)$ preserves well-foundedness. □

Theorem 3.5 provides a way of proving well-foundedness of orders on terms, including orders which are not closed under contexts nor closed under substitutions.

Consider the *recursive path order* with status (Dershowitz [22], Steinbach [99]; see also definition 2.89) extended to incorporate arbitrary liftings as status.

[1] Irreflexivity of the order in combination with finiteness of $\mathcal{F} \cup \mathcal{X}$ rules out an infinite number of occurrences of either variables or constants.

**Definition 3.6. (rpo with status)** Let $\triangleright$ be a partial order on $\mathcal{F}$ and $\tau$ a status function with respect to $>_{rpo}$. Given two terms $s, t$ we say that $s >_{rpo} t$ iff $s = f(s_1, \ldots, s_m)$ and either

1. $t = g(t_1, \ldots, t_n)$, $s >_{rpo} t_i$, for all $1 \leq i \leq n$, and
   (a) $f \triangleright g$, or
   (b) $f = g$, and $\langle s_1, \ldots, s_m \rangle >_{rpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle$; or
2. $\exists 1 \leq i \leq m : \; s_i >_{rpo} t$ or $s_i = t$.

Some care should be taken when using a status in recursive definitions such as definition 3.6 since for arbitrary status we may end up with relations that are not partial orders. In chapter 4 we make some considerations about this. Since here our aim is not to study the conditions in which rpo-like orders can be defined we assume that we have an order satisfying the conditions of definition 3.6 for some status. Well-foundedness of $>_{rpo}$, as defined in definition 3.6, follows from theorem 3.5: condition $2$ ensures that $>_{rpo}$ has the subterm property and conditions $1$-b and $2$ imply that $>_{rpo}$ is decomposable with respect to $\tau$. If we take the definition of $>_{rpo}$ over a precedence that is a quasi-order with the additional condition that each equivalence class of function symbols has one status associated (see chapter 4 for a precise definition), well-foundedness is still a direct consequence of theorem 3.5. We remark that by using our definition of lifting and status, definition 3.6 is a generalization of $>_{rpo}$ orders as found in the literature. With this definition we are able to prove termination of the following TRS (originally from Geerling [39]):

$$f(s(x), y, y) \;\; \rightarrow \;\; f(y, x, s(x))$$

For that we use a lifting given earlier, namely

$$\langle s_1, \ldots, s_m \rangle >^{\lambda} \langle t_1, \ldots, t_n \rangle \iff \begin{cases} (m = n = 3) & \text{and} \\ \langle s_1, s_2 \rangle >^{mul} \langle t_1, t_2 \rangle & \text{or} \\ (\langle s_1, s_2 \rangle =^{mul} \langle t_1, t_2 \rangle) & \text{and } s_3 > t_3 \end{cases}$$

and then take $>_{rpo}^{\tau(f)} = >_{rpo}^{\lambda}$. Termination of this system cannot be handled by earlier versions of $>_{rpo}$.

In section 3.5 we shall see that well-foundedness of both *semantic path order* and *general path order* also follows from theorem 3.5.

## 3.3.2 Completeness Results

The next result states that the type of term orderings described in theorem 3.5 covers all terminating TRS's.

**Theorem 3.7.** *A TRS $R$ is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function $\tau$ satisfying the following conditions:*

- *$>$ has the subterm property,*

- $>$ *is decomposable with respect to* $\tau$,
- *if* $s \to_R t$ *then* $s > t$, *i. e.,* $>$ *is compatible with* $R$.

**Proof** The "if" part follows from theorem 3.5: the order $>$ is well-founded and the assumption $\to_R \subseteq >$ implies that $R$ is terminating.

For the "only-if" part we define the relation $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ by:

$$s > t \iff s \neq t \text{ and } \exists C[\,] : s \to_R^* C[t]$$

By definition, the relation $>$ is irreflexive and has the subterm property.

For transitivity, suppose $s > t$ and $t > u$; then there are contexts $C[\,], D[\,]$ such that $s \to_R^* C[t]$, $t \to_R^* D[u]$, $s \neq t$ and $t \neq u$. Clearly $s \to_R^* C[D[u]]$ so to have $s > u$ we only need to check that $s \neq u$. If $s \to_R^+ C[t]$ or $t \to_R^+ D[u]$ then $s = u$ would give us $s \to_R^+ C[D[s]]$ contradicting termination of $R$. If both $s = C[t]$ and $t = D[u]$, since $s \neq t$ and $t \neq u$, we must have $C[\,], D[\,] \neq \Box$ and from $s = C[D[u]]$ we conclude $s \neq u$.

We check that $>$ is well-founded. Suppose it is not and let $s_0 > s_1 > \cdots$, be an infinite descending chain. By definition of $>$, for each $i \geq 0$, we have $s_i \to_R^* C_i[s_{i+1}]$, for some context $C_i[\,]$, with $s_i \neq s_{i+1}$. Thus we obtain the infinite chain

$$s_0 \to_R^* C_0[s_1] \to_R^* C_0[C_1[s_2]] \to_R^* \cdots$$

From termination of $R$, we conclude that there is an index $j \geq 0$ such that

$$s_j = C_j[s_{j+1}] = C_j[C_{j+1}[s_{j+2}]] = \ldots$$

Since the sequence is infinite and $C_k[\,] \neq \Box$ (since $s_k \neq s_{k+1}$), for all $k \geq j$, and all terms are finite, this is a contradiction.

For each function symbol $f \in \mathcal{F}$ we define $>^{\tau(f)}$ by:

$$\langle u_1, \ldots u_k \rangle >^{\tau(f)} \langle v_1, \ldots, v_m \rangle \iff f(u_1, \ldots, u_k) > f(v_1, \ldots, v_m)$$

for any $k, m \in \texttt{arity}(f)$. Since $>$ is well-founded, we see that $>^{\tau(f)}$ is indeed a lifting.

From the above reasoning follows that all the conditions of theorem 3.5 are satisfied. Finally if $s \to_R t$, we obviously have $s \to_R^* C[t]$, with $C$ the trivial context. Since $R$ is terminating we must have $s \neq t$ and consequently $s > t$. $\Box$

An alternative proof of theorem 3.7 can be given using the fact that a TRS $R$ is terminating if and only if it is compatible with some particular version of a semantic path order; in the proof of this fact the same order as above is used. Since *spo* fulfils the conditions of theorem 3.5, as we shall see in section 3.5, this provides an alternative proof for theorem 3.7.

The order defined in the proof of theorem 3.7 has the additional property of being closed under substitutions (but not under contexts). Consequently we also have the following stronger result.

**Theorem 3.8.** *A TRS $R$ is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ and a status function $\tau$ satisfying the following conditions:*

- *$>$ has the subterm property,*
- *$>$ is decomposable with respect to $\tau$,*
- *$>$ is closed under substitutions,*
- *if $s \rightarrow_R t$ then $s > t$.*

An interesting question raised by J.-P. Jouannaud is what can be said about totality of orders satisfying the conditions of theorem 3.5. It turns out that totality can very easily be achieved as we now show. However, totality is not compatible with closedness under substitutions. First we present a well-known lemma.

**Lemma 3.9.** *Any partial well-founded order $>$ on a set $A$ can be extended to a total well-founded order on $A$.*

For a proof of this result see Appendix A.

**Theorem 3.10.** *A TRS $R$ is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F},\mathcal{X})$, and a status function $\tau$ satisfying the following conditions:*

- *$>$ has the subterm property,*
- *$>$ is decomposable with respect to $\tau$,*
- *$>$ is total,*
- *if $s \rightarrow_R t$ then $s > t$.*

**Proof** Again the "if" part follows from theorem 3.5: the order $>$ is well-founded and the assumption $\rightarrow_R \subseteq >$ implies that $R$ is terminating.

For the "only-if" part we use theorems 3.5 and 3.7. Since $R$ is terminating, by theorem 3.7 there is an order $\gg$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and a status function $\tau$ satisfying the conditions of theorem 3.5 and such that $s \rightarrow_R t \Rightarrow s \gg t$. By theorem 3.5 the order $\gg$ is well-founded, but not necessarily total. By lemma 3.9, let $>$ be a total well-founded order extending $\gg$. Since $\gg$ has the subterm property, so does $>$. Furthermore $>$ is also compatible with $\rightarrow_R$, for if $s \rightarrow_R t$ then $s \gg t$ and so $s > t$. In order to apply theorem 3.5 we still have to define a status function $\tau$ for which $>$ is decomposable. For each function symbol $f \in \mathcal{F}$ we define: $\langle u_1, \ldots u_k \rangle >^{\tau(f)} \langle v_1, \ldots, v_m \rangle \iff f(u_1, \ldots, u_k) > f(v_1, \ldots, v_m)$, for any $k, m \in \texttt{arity}(f)$. Since $>$ is well-founded, $>^{\tau(f)}$ is indeed a lifting. Theorem 3.5 now gives the result. $\square$

The previous result may seem a bit strange since it tells us that we can achieve totality on all terms and not only ground terms. This is so because we do not impose any closure conditions on the order. Note that a total order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is never closed under substitutions as long as $\mathcal{X}$ contains more than one element. As for closure under contexts, this property is usually not maintained by naïve extensions of the order, it may even make the existence of certain extensions impossible. In our case the conditions imposed are subterm property and compatibility with the reduction relation; any extension will comply with those conditions whenever the original order does.

# 3.4 Infinite Signatures

In the previous section we presented some results which are applicable to orders and TRS's over finite signatures. Here we treat the infinite case, i. e., we consider the set of terms over an infinite alphabet $\mathcal{F}$, with varyadic function symbols, and an infinite set of variables $\mathcal{X}$. As usual we require that $\mathcal{F} \cap \mathcal{X} = \emptyset$.

We first discuss orders which are based on a precedence on the set of function symbols. Afterwards we present another simplified approach in which we can dispense with the precedence. This approach is based on a generalization of the notion of lifting.

## 3.4.1 Precedence-based Orders

Theorem 3.5 can also be extended to infinite signatures, but requires some extra conditions.

We introduce some more notation. Let $\trianglerighteq$ be a quasi-order on $\mathcal{F}$, called a *precedence* (see definition 2.88). As usual, we denote the strict partial order $\trianglerighteq \setminus \trianglelefteq$ by $\triangleright$ and the equivalence relation $\trianglerighteq \cap \trianglelefteq$ by $\sim$.

**Definition 3.11.** Given an order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a precedence $\trianglerighteq$ on $\mathcal{F}$, we say that $>$ is *compatible* with $\trianglerighteq$ if whenever $f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n)$ and $g \triangleright f$ then $s_i \geq g(t_1, \ldots, t_n)$, for some $1 \leq i \leq m$.

In theorem 3.5 we only needed to take into account comparisons between terms with the same head function symbol (root), but now we also need to consider the comparisons between terms whose head function symbols are equivalent under the precedence considered. As a consequence, we need to impose some constraint on the status associated with a function symbol.

**Definition 3.12.** Given a precedence $\trianglerighteq$ on $\mathcal{F}$, an order $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a status function $\tau$, with respect to $>$, we say that $\tau$ and $\trianglerighteq$ are *compatible* if whenever $f \sim g$ then $\tau(f) = \tau(g)$.

Recall from chapter 2, definition 2.40, that a *well quasi-order*, abbreviated to *wqo*, is a quasi-order $\succeq$ such that any extension of it is well-founded. We can now formulate theorem 3.5 for infinite signatures:

**Theorem 3.13.** *Let $\trianglerighteq$ be a precedence on $\mathcal{F}$, $>$ a partial order over $\mathcal{T}(\mathcal{F},\mathcal{X})$, and $\tau$ a status function with respect to $>$, such that that both $>$ and $\trianglerighteq$, and, $\tau$ and $\trianglerighteq$ are compatible. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f, g \in \mathcal{F},\ m \in \mathtt{arity}(f), n \in \mathtt{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$ : *if* $f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n)$ *with* $f \sim g$, *then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, *or*
  - $\langle s_1, \ldots, s_m \rangle >^{\tau(f)} \langle t_1, \ldots, t_n \rangle$.

*Suppose also that $\trianglerighteq$ is a wqo on $\mathcal{F} \setminus \mathcal{F}_0$ and $>$ is well-founded on $\mathcal{X} \cup \mathcal{F}_0$, where as usual $\mathcal{F}_0 = \{f \in \mathcal{F} : \mathtt{arity}(f) = \{0\}\}$. Then $>$ is well-founded on $\mathcal{T}(\mathcal{F},\mathcal{X})$.*

**Proof** We proceed, as in proof of theorem 3.5, by contradiction. First we remark that any infinite descending sequence $(t_i)_{i\geq 0}$ contains an infinite subsequence $(t_{\phi(i)})_{i\geq 0}$ such that $\mathtt{arity}(\mathtt{root}(t_{\phi(i)})) \neq \{0\}$, for all $i \geq 0$. If that would not be the case, the sequence would contain infinitely many variables or constants, contradicting the fact that $>$ is well-founded on $\mathcal{X} \cup \mathcal{F}_0$.

We take a minimal infinite descending sequence $(t_i)_{i\geq 0}$, in the same sense as in theorem 3.5. Again, as remarked in the proof of theorem 3.5, from the minimality of $(t_i)_{i\geq 0}$, the subterm property and transitivity of $>$, it follows that all (principal) subterms of any term $t_i$, $i \geq 0$, are well-founded.

Let $\mathtt{root}(t)$ be the head function symbol of the term $t$. Consider the infinite sequence $(\mathtt{root}(t_i))_{i\geq 0}$. From the first observation above, it follows that this sequence contains infinitely many elements of $\mathcal{F} \setminus \mathcal{F}_0$. Consequently and since $\trianglerighteq$ is a *wqo* on $\mathcal{F}\setminus\mathcal{F}_0$, we can conclude that the sequence $(\mathtt{root}(t_i))_{i\geq 0}$ contains an infinite subsequence $(\mathtt{root}(t_{\phi(i)}))_{i\geq 0}$ such that $\mathtt{root}(t_{\phi(i+1)}) \trianglerighteq \mathtt{root}(t_{\phi(i)})$ and $\mathtt{arity}(\mathtt{root}(t_{\phi(i)})) \neq \{0\}$, for all $i \geq 0$.

The infinite sequence $(\mathtt{root}(t_i))_{i\geq 0}$ contains no infinite subsequence $(\mathtt{root}(t_{\psi(i)}))_{i\geq 0}$ such that $\mathtt{root}(t_{\psi(i+1)}) \sim \mathtt{root}(t_{\psi(i)})$, for all $i \geq 0$. Suppose it is not so and let $(\mathtt{root}(t_{\psi(i)}))_{i\geq 0}$ be such a sequence. Since $t_{\psi(i)} > t_{\psi(i+1)}$, by hypothesis we must have

1. $s_{i,k} \geq t_{\psi(i+1)}$, with $s_{i,k}$ a principal subterm of $t_{\psi(i)}$, or
2. $\langle s_{i,1}, \ldots, s_{i,k_{\psi(i)}} \rangle >^{\lambda} \langle s_{i+1,1}, \ldots, s_{i+1,k_{\psi(i+1)}} \rangle$, where $>^{\lambda}$ is the lifting given by the status of $\mathtt{root}(t_{\psi(0)})$[2], and where $s_{i,1}, \ldots, s_{i,k_{\psi(i)}}$ and $s_{i+1,1}, \ldots, s_{i+1,k_{\psi(i+1)}}$ are the principal subterms of, respectively, $t_{\psi(i)}$ and $t_{\psi(i+1)}$, for all $i$.

Due to the minimality of $(t_i)_{i\geq 0}$ and the subterm property, case 1 above can never occur. Therefore we have an infinite descending sequence

$$\langle s_{0,1}, \ldots, s_{0,k_{\psi(0)}} \rangle >^{\lambda} \langle s_{1,1}, \ldots, s_{1,k_{\psi(1)}} \rangle >^{\lambda} \langle s_{2,1}, \ldots, s_{2,k_{\psi(2)}} \rangle >^{\lambda} \ldots$$

[2] Recall that for equivalent function symbols, their status coincides.

Since $>$ is well-founded on $\bigcup_{i\geq 0} \bigcup_{j=1}^{k_{\psi(i)}} \{s_{i,j}\}$, this contradicts the definition of lifting.

Therefore, and without loss of generality, we can state that the infinite subsequence $(\mathtt{root}(t_{\phi(i)})_{i\geq 0})$ has the additional property $\mathtt{root}(t_{\phi(i+1)}) \rhd \mathtt{root}(t_{\phi(i)})$, for all $i$.[3] Since $t_{\phi(i)} > t_{\phi(i+1)}$ and $>$ is compatible with $\unrhd$, we must have $u \geq t_{\phi(i+1)}$, for some principal subterm $u$ of $t_{\phi(i)}$, contradicting the minimality of $(t_i)_{i\geq 0}$. □

Some remarks are appropriate. It is not essential to require that $\unrhd$ should be a precedence on $\mathcal{F}$; requiring that $\unrhd$ is a precedence on $\mathcal{F} \setminus \mathcal{F}_0$ is enough. The condition • serves the same purpose as the decomposability condition (for $>$ with respect to $\tau$) in theorem 3.5.

The restriction $f \sim g \Rightarrow \tau(f) = \tau(g)$ can be relaxed if we allow the arguments of $f$ and $g$ to be permuted before being compared under $>^{\tau(f)}$. The result is still valid if (while keeping the restriction $f \sim g \Rightarrow \tau(f) = \tau(g)$) we associate permutations to function symbols indicating the order in which arguments are taken to form a sequence. Then $\langle s_1, \ldots, s_m\rangle >^{\tau(f)} \langle t_1, \ldots, t_n\rangle$ becomes $\pi_f(\langle s_1, \ldots, s_m\rangle) >^{\tau(f)} \pi_f(\langle t_1, \ldots, t_n\rangle)$, where $\pi_h$ is a permutation associated to function symbol $h$ (this may be a function that first checks the number or arguments and then permutes them, thus not a permutation in the strict sense), and where by $\pi_h(\langle u_1, \ldots, u_k\rangle)$ we mean the application of the permutation $\pi_h$ to the elements $u_1, \ldots, u_k$. This relaxation allows us to consider lexicographic status where sequences are first arranged before they are compared.

Since there are no substitutions involved, there is no essential difference between elements of $\mathcal{X}$ and $\mathcal{F}_0$. The condition stating that $>$ is well-founded on $\mathcal{X}$ is imposed to disallow the bizarre case where we can have an infinite descending sequence constituted solely by variables. Usually (e. g. in Kruskal's theorem) it is required that the precedence $\unrhd$ is a *wqo* over $\mathcal{F}$, we can however relax that condition to $\unrhd$ being a *wqo* over $\mathcal{F} \setminus \mathcal{F}_0$ provided $>$ is also well-founded on $\mathcal{F}_0$. This is weaker than requiring that $\unrhd$ be a *wqo* on $\mathcal{F}$. The *wqo* requirement cannot be weakened to well-foundedness as the following example shows. Consider $\mathcal{F} = \{f_i | i \geq 0\}$ with $\mathtt{arity}(f_i) = \{1\}$, for all $i \geq 0$. Let $>$ be an order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ with the subterm property and such that

$$f_0(x) > f_1(x) > f_2(x) > \cdots$$

Take $\unrhd$ to be equality. Obviously $\unrhd$ is well-founded and all the other conditions of theorem 3.13 are satisfied, however the order $>$ is not well-founded.

If we remove the condition "$>$ is well-founded on $\mathcal{X} \cup \mathcal{F}_0$", and strengthen the condition on $\unrhd$ to "$\unrhd$ is a *wqo* on $\mathcal{F} \cup \mathcal{X}$", then the same statement as above can be proved (and the proof is very similar). In this case and for finite signatures, theorem 3.5 is a direct consequence of theorem 3.13, since the discrete order is a *wqo* and the compatibility conditions are trivially fulfilled.

Theorem 3.13 holds in particular for precedences that are *partial well-orders (pwo's)*. In this case we only need to compare terms with the same root function symbol and the compatibility condition of definition 3.12 is trivially verified.

[3] Strictly speaking, an infinite subsequence of this sequence has that property.

As in the finite case, well-foundedness of orders as *rpo* over infinite signatures, is a consequence of theorem 3.13. For that we need to extend the well-founded precedence to a total well-founded one, keeping the equivalence part the same, which is then a *wqo*. Note that we also need to check that *rpo* is well-founded over $\mathcal{F}_0 \cup X$. All the other conditions also hold, so the theorem can be applied.

If it is the case that $\mathcal{F}$ is finite but we allow $\mathcal{X}$ to be an infinite set, the conditions imposed on the order on theorem 3.5 are not enough to guarantee that the order is well-founded: any non-well-founded order defined only in $\mathcal{X}$ is a counterexample. However, in the presence of an infinite set of variables, well-foundedness of $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is equivalent to well-foundedness of $>$ on $\mathcal{X}$, i. e., theorem 3.5 can be rewritten as:

**Theorem 3.14.** *Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\tau$ a status function with respect to $>$. Suppose $>$ has the subterm property and is decomposable with respect to $\tau$. Then $>$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ if and only if $>$ is well-founded on $\mathcal{X}$.*

One direction is trivial, the other is a consequence of theorem 3.13, by taking as precedence equality. Note that for $\mathcal{F}$ finite and $\mathcal{X}$ infinite, theorems 3.7, 3.8 and 3.10, hold under the additional assumption that the order considered is well-founded when restricted to $\mathcal{X}$.

If we relax the requirements on the precedence and strengthen the ones on the order, another interesting result arises.

**Theorem 3.15.** *Let $\trianglerighteq$ be a well-founded precedence on $\mathcal{F} \cup \mathcal{X}$ such that elements of $\mathcal{F}$ and $\mathcal{X}$ are incomparable under $\trianglerighteq$ and $\trianglerighteq$ restricted to $\mathcal{X}$ is equality. Let $>$ be a partial order over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and $\tau$ a status function with respect to $>$, such that that $\tau$ and $\trianglerighteq$ are compatible. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f, g \in \mathcal{F} \cup \mathcal{X},\ m \in \mathtt{arity}(f), n \in \mathtt{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X}):$ *if $f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n)$ then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, *or*
  - $f \triangleright g$, *or*
  - $f \sim g$ *and* $\langle s_1, \ldots, s_m \rangle >^{\tau(f)} \langle t_1, \ldots, t_n \rangle$.

*Then $>$ is well-founded on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

The proof is very similar to the proof of theorem 3.13, therefore we omit it. Note that condition • implies that $>$ and $\trianglerighteq$ are compatible. Furthermore we can require that $\trianglerighteq$ is only defined and well-founded on $\mathcal{F}$ provided $>$ is well-founded on $\mathcal{X}$. As for theorem 3.13, we can also generalize the result above by allowing arguments of $f$ and $g$ to be permuted before they are compared.

Note that well-foundedness of *rpo*, for an arbitrary well-founded precedence, is a direct consequence of this result. In the "classical" approach, first the precedence has to be extended via lemma 3.9 to a well-founded total precedence, maintaining the equivalence part, before Kruskal's theorem yields the desired result.

It would also be interesting to have a theorem similar to theorem 3.7 for the case of infinite signatures. However for infinite signatures the discrete relation (equality) is not a *wqo* any longer and it is not clear how to choose an appropriate *wqo*. A possibility is to take $\trianglerighteq$ defined by $f \sim g$ for any $f, g \in \mathcal{F}$, which is trivially a *wqo*, however this choice will not always work as the following example shows. Consider the infinite terminating TRS given by

$$a_i \quad \to \quad a_j \quad \text{for any } i > j \geq 0$$

and where each $a_i$ is a constant. Then any order compatible with $R$ will never satisfy a decomposability condition akin to • in theorem 3.13 or 3.15, for a precedence in which $a_i \sim a_j$, for all $i, j \geq 0$.

Another alternative is to take a total well-founded order on $\mathcal{F}$, again by definition a *wqo*, but then other compatibility problems arise. Just consider the rule

$$a \quad \to \quad f(0)$$

If we choose the precedence as an arbitrary total well-founded order on $\mathcal{F}$, we may have $f \rhd a$, and the conditions of theorem 3.13 will never hold.

## 3.4.2 Generalizing Liftings on Orders

The decomposability restriction $\langle s_1, \ldots, s_m \rangle >^{\tau(f)} \langle t_1, \ldots, t_n \rangle$ has the inconvenience of forgetting about the root symbols of the terms compared. In the case of finite signatures, that is irrelevant since we only need to compare terms with the same head symbol and the symbol can be encoded in the status $\tau$. For infinite signatures, however, that information is essential, since given an infinite sequence of terms we no longer have the guarantee that it contains an infinite subsequence of terms having the same root symbol. As a consequence we need to impose some strong conditions both on the set of function symbols and on the status and order used. A way of relaxing these conditions is by remembering the information lost with the decomposition and this can be achieved by changing the definition of lifting.

In this section we present another condition for well-foundedness on term orderings. Now we do not require the existence of an order or quasi-order on the set of function symbols $\mathcal{F}$. Instead we will use a different definition of lifting for orderings on terms.

**Definition 3.16.** Let $(\mathcal{T}(\mathcal{F}, \mathcal{X}), >)$ be a partially ordered set of terms. We define a *term lifting* to be a partial order $>^{\Lambda}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ for which the following holds: for every $A \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, if $>$ restricted to $A$ is well-founded, then $>^{\Lambda}$ restricted to $\bar{A}$ is also well-founded, where

$$\bar{A} = \{f(t_1, \ldots, t_n) : f \in \mathcal{F} \cup \mathcal{X},\ n \in \texttt{arity}(f),\ \text{and } t_i \in A,\ \text{for all } i, 0 \leq i \leq n\}$$

We use the notation $\Lambda(>)$ to denote all possible term liftings of $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Note that for any partial order $>$, the term lifting $>^{\Lambda}$ is well-founded on $\mathcal{X} \cup \mathcal{F}_0$. This is a consequence of the definition since $>$ is well-founded on $A = \emptyset$ and $\bar{A} = \mathcal{X} \cup \mathcal{F}_0$.

We remark that term liftings can make use of liftings and status functions since the well-foundedness requirement is preserved. Given an order $>$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$, every lifting in the sense of definition 3.1 induces a term lifting of the same order as follows:

$$f(s_1,\ldots,s_m) >^\Lambda g(t_1,\ldots,t_n) \stackrel{\text{def}}{\Longleftrightarrow} \langle s_1,\ldots,s_m\rangle >^\lambda \langle t_1,\ldots,t_n\rangle$$

It is not difficult to see that if $>^\lambda$ is a lifting in the sense of definition 3.1, then $>^\Lambda$ is irreflexive and transitive. Furthermore if $>$ is well-founded on a set of terms $A \subseteq \mathcal{T}(\mathcal{F},\mathcal{X})$ then $>^\lambda$ is well-founded on $A^*$ and so $>^\Lambda$ is well-founded on $\bar{A}$.

We present a new well-foundedness criterion.

**Theorem 3.17.** *Let $>$ be a partial order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and let $>^\Lambda$ be a term lifting of $>$. Suppose $>$ has the subterm property and satisfies the following condition:*

- $\forall f,g \in \mathcal{F}\cup\mathcal{X},\ m \in \texttt{arity}(f), n \in \texttt{arity}(g), s_1,\ldots,s_m,t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X}):$ *if* $s = f(s_1,\ldots,s_m) > g(t_1,\ldots,t_n) = t$ *then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1,\ldots,t_n)$, *or*
  - $s >^\Lambda t$

*Then $>$ is well-founded on $\mathcal{T}(\mathcal{F},\mathcal{X})$.*

**Proof** Suppose that $>$ is not well-founded and take an infinite descending chain

$$t_0 > t_1 > \cdots > t_n > \cdots$$

minimal in the same sense as in the proof of theorem 3.5, i. e.,

- $|t_0| \leq |s|$, for all non-well-founded terms $s$;
- $|t_{i+1}| \leq |s|$, for all non-well-founded terms $s$ such that $t_i > s$.

As remarked in the proof of theorem 3.5, from the minimality of $(t_i)_{i\geq 0}$, the subterm property and transitivity of $>$, it follows that all principal subterms of any term $t_i$, $i \geq 0$, are well-founded. Since $t_i > t_{i+1}$, for all $i \geq 0$, we must have either

1. $u_i \geq t_{i+1}$, for some principal subterm $u_i$ of $t_i$, or
2. $t_i >^\Lambda t_{i+1}$

Due to the minimality of the sequence, the first case above can never occur. Therefore we have an infinite descending chain

$$t_0 >^\Lambda t_1 >^\Lambda t_2 >^\Lambda \ldots$$

But due also to minimality, the order $>$ is well-founded over the set of terms

$$A = \{u : u \text{ is a principal subterm of } t_i, \text{ for some } i \geq 0\}.$$

By definition of term lifting we have that $>^\Lambda$ is well-founded over

$$\bar{A} = \{f(u_1,\ldots,u_k) : f \in \mathcal{F}\cup\mathcal{X}, k \in \texttt{arity}(f) \text{ and } u_i \in A, \text{ for all } 1 \leq i \leq k\}$$

and since $\{t_i |\ i \geq 0\} \subseteq \bar{A}$, we get a contradiction. □

It is interesting to remark that theorem 3.5 (for finite signatures) is a consequence of theorem 3.17. Let $>$ be a partial order over $\mathcal{T}(\mathcal{F},\mathcal{X})$ in the conditions of theorem 3.5. We define the following order $\gg$:

$$s \gg t \overset{\text{def}}{\Longleftrightarrow} \texttt{root}(s) = \texttt{root}(t) \text{ and } s > t$$

It is not difficult to see that $\gg$ is indeed a partial order, i. e., irreflexive and transitive.

Now we define the following term lifting

$$f(s_1,\ldots,s_m) \gg^{\Lambda} g(t_1,\ldots,t_n) \overset{\text{def}}{\Longleftrightarrow} f = g \text{ and } \langle s_1,\ldots,s_m\rangle >^{\tau(f)} \langle t_1,\ldots,t_n\rangle$$

where $>^{\tau(f)}$ is the lifting associated by the status function $\tau$ to the function symbol $f$. It is not difficult to see that since the lifting $>^{\tau(f)}$ respects well-foundedness of $>$, $\gg^{\Lambda}$ is a well-defined term lifting. Now theorem 3.17 gives well-foundedness of $\gg$. But since non-well-foundedness of $>$ would imply non-well-foundedness of $\gg$ (by an argument similar to the proof of theorem 3.5), we are done.

Furthermore when $\mathcal{F}\cup\mathcal{X}$ is finite, theorem 3.17 is also a consequence of theorem 3.5 (i. e., they are equivalent). Suppose that we have a partial order $>$ on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and a term lifting $>^{\Lambda}$ in the conditions of theorem 3.17. We define the status

$$\langle s_1,\ldots,s_m\rangle >^{\tau(f)} \langle t_1,\ldots,t_n\rangle \overset{\text{def}}{\Longleftrightarrow} f(s_1,\ldots,s_m) >^{\Lambda} f(t_1,\ldots,t_n)$$

It is now not difficult to check that $>^{\tau(f)}$ is a partial order and that if $>$ is well-founded over a set of terms $A \subseteq \mathcal{T}(\mathcal{F},\mathcal{X})$ then $>^{\tau(f)}$ is well-founded over $A^*$ (since otherwise $>^{\Lambda}$ would not be well-founded over $\bar{A}$). By theorem 3.5 we conclude that $>$ is well-founded.

Due to the required existence of a partial order on the set of function symbols, the relation of this theorem with theorems 3.13 and 3.15 is not yet clear.

An important consequence of the use of term liftings is that we manage to recover the completeness results stated in section 3.3.2 which we could not state for precedence-based orders.

**Theorem 3.18.** *Let $R$ be a TRS over an infinite varyadic signature. Then $R$ is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F},\mathcal{X})$ and a term lifting $>^{\Lambda}$ satisfying the following conditions:*

- *$>$ has the subterm property (and $>$ is closed under substitutions),*
- $\forall f,g \in \mathcal{F}\cup\mathcal{X},\ m \in \texttt{arity}(f), n \in \texttt{arity}(g), s_1,\ldots,s_m,t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{X}):$ *if $s = f(s_1,\ldots,s_m) > g(t_1,\ldots,t_n) = t$ then either*
    - $\exists 1 \le i \le m : s_i \ge g(t_1,\ldots,t_n)$*, or*
    - $s >^{\Lambda} t$
- *if $s \to_R t$ then $s > t$.*

**Proof** Sketch. The "if" part follows from theorem 3.17: the order $>$ is well-founded and the assumption $\to_R \subseteq >$ implies that $R$ is terminating.

For the "only-if" part the proof is similar to the proof of theorem 3.7. We define again the relation $>$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$: $s > t \iff s \neq t$ and $\exists C[\,] : s \to_R^* C[t]$. The only different part is the definition of term lifting. Since the order $>$ is well-founded we can use it as the term lifting itself. □

As for the finite case the completeness result concerning totality also holds and the proof is very similar, so we omit it.

**Theorem 3.19.** *Let $R$ be a TRS over an infinite varyadic signature. Then $R$ is terminating if and only if there is an order $>$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and a term lifting $>^\Lambda$ satisfying the following conditions:*

- *$>$ has the subterm property,*
- $\forall f, g \in \mathcal{F} \cup \mathcal{X},\ m \in \texttt{arity}(f), n \in \texttt{arity}(g), s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X}) :$ *if $s = f(s_1, \ldots, s_m) > g(t_1, \ldots, t_n) = t$ then either*
  - $\exists 1 \leq i \leq m : s_i \geq g(t_1, \ldots, t_n)$, *or*
  - $s >^\Lambda t$
- *$>$ is total,*
- *if $s \to_R t$ then $s > t$.*

## 3.5 Semantic Path Order and General Path Order

As an application of results presented earlier, we show here how well-foundedness of *semantic path order* (Kamin and Lévy [55]) and *general path order* (Dershowitz and Hoot [24]) can be derived using either theorem 3.5 or theorem 3.17.

**Definition 3.20.** (**semantic path order - spo**) Let $\geq$ be a well-founded quasi-order on $\mathcal{T}(\mathcal{F})$. The *semantic path order* $\succeq_{spo}$ is defined on $\mathcal{T}(\mathcal{F})$ as follows: $s = f(s_1, \ldots, s_m) \succeq_{spo} g(t_1, \ldots, t_n) = t$ if either

1. $s > t$ and $s \succ_{spo} t_i$, for all $1 \leq i \leq n$, or
2. $s \sim t$ and $s \succ_{spo} t_i$, for all $1 \leq i \leq n$, and $\langle s_1, \ldots, s_m \rangle \succeq_{spo}^{mul} \langle t_1, \ldots, t_n \rangle$, where $\succeq_{spo}^{mul}$ is the multiset extension of $\succeq_{spo}$, or
3. $\exists i \in \{1, \ldots, m\} : s_i \succeq_{spo} t$.

It can be seen that $\succeq_{spo}$ is a quasi-order with the subterm property and in general not monotonic (Geser [42]).

In the case the alphabet we consider is finite, define the following status. Let $\geq$ be the well-founded quasi-order used in the definition of $\succeq_{spo}$. For each $f \in \mathcal{F}$ the lifting $\tau(f)$ is given by

$$\langle s_1, \ldots, s_k \rangle \succ_{spo}^{\tau(f)} \langle t_1, \ldots, t_m \rangle \iff \begin{cases} s > t, \text{ or} \\ s \sim t \text{ and } \langle s_1, \ldots, s_k \rangle \; \mathtt{ord}(\succeq_{spo}^{mul}) \; \langle t_1, \ldots, t_m \rangle \end{cases}$$

for any $k, m \in \mathtt{arity}(f)$ and where $\mathtt{ord}(\succeq_{spo}^{mul})$ is the strict part of the multiset extension of $\succeq_{spo}$, $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$. It is not difficult to see that $\succ_{spo}^{\tau(f)}$ is indeed a partial order on $\mathcal{T}(\mathcal{F})^*$ and that $\succ_{spo}^{\tau(f)}$ preserves well-foundedness, being therefore a lifting. Since $\succ_{spo}$ has the subterm property and satisfies the other conditions of theorem 3.5, its well-foundedness follows from application of the theorem.

For the case we consider an infinite signature, we define the following term lifting: for $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$

$$s \succ_{spo}^{\Lambda} t \iff \begin{cases} (s > t) & \text{or} \\ (s \sim t) \quad \text{and} & \langle s_1, \ldots, s_m \rangle \; \mathtt{ord}(\succeq_{spo}^{mul}) \; \langle t_1, \ldots, t_n \rangle \end{cases}$$

where again $\geq$ is the well-founded quasi-order used in the definition of $\succeq_{spo}$. Since $>$ is well-founded and the multiset extension preserves well-foundedness, $\succ_{spo}^{\Lambda}$ is indeed a term lifting. Using this term lifting, we can apply theorem 3.17 to conclude that $\succ_{spo}$ is well-founded.

The *general path order*, that we denote by $\succeq_{gpo}$, was introduced by Dershowitz and Hoot [24]. We present the definition and show how well-foundedness of this order can be derived from theorem 3.5 or theorem 3.17.

**Definition 3.21.** A *termination function* $\theta$ is a function defined on the set of ground terms $\mathcal{T}(\mathcal{F})$ and is either

1. a homomorphism from terms to a set $S$ such that

$$\theta(f(s_1, \ldots, s_n)) = f_\theta(\theta(s_1), \ldots, \theta(s_n))$$

2. an extraction function that given a term associates to it a multiset of principal subterms, i. e.,

$$\theta(f(s_1, \ldots, s_n)) = \{\!\{ s_{i_1}, \ldots, s_{i_k} \}\!\}$$

where $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

**Definition 3.22.** A *component order* $\phi = \langle \theta, \geq \rangle$ consists of a termination function defined on the set $\mathcal{T}(\mathcal{F})$ of ground terms, along with an associated well-founded quasi-order $\geq$ (defined on the codomain of $\theta$).

**Definition 3.23.** **(general path order - gpo)** Let $\phi_i = \langle \theta_i, \geq_i \rangle$, with $0 \leq i \leq k$, be component orders, such that if $\theta_j$ is an extraction function then $\geq_j$ is the multiset extension of the general path order itself. The induced *general path order* $\succeq_{gpo}$ is defined on $\mathcal{T}(\mathcal{F})$ as follows: $s = f(s_1, \ldots, s_m) \succ_{gpo} g(t_1, \ldots, t_n) = t$ if either

1. $\exists i \in \{1, \ldots, m\} : s_i \succeq_{gpo} t$ or
2. $s \succ_{gpo} t_j$, for all $1 \leq j \leq n$, and $\Theta(s) >_{lex} \Theta(t)$, where $\Theta = \langle \theta_0, \ldots, \theta_k \rangle$ and $>_{lex}$ is the lexicographic combination of the component orderings $\theta_i$, with $0 \leq i \leq k$.

The equivalence part is defined as: $s = f(s_1, \ldots, s_m) \sim_{gpo} g(t_1, \ldots, t_n) = t$ if $s \succ_{gpo} t_j$, for all $1 \leq j \leq n$, and $t \succ_{gpo} s_j$, for all $1 \leq j \leq m$, and $\theta_i(s) \sim_i \theta_i(t)$, for all $0 \leq i \leq k$, where $\sim_i$ is the equivalence contained in $\geq_i$.

It is known (Dershowitz and Hoot [24]) that $\succeq_{gpo}$ is a quasi-order with the subterm property.

Well-foundedness of $\succeq_{gpo}$ is a consequence of the results previously presented. For the case of finite signatures we define the following status

$$\langle s_1, \ldots, s_m \rangle \succ_{gpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle \iff \Theta(f(s_1, \ldots, s_m)) >_{lex} \Theta(f(t_1, \ldots, t_k))$$

where as in definition 3.23, $\Theta(v) = \langle \theta_0(v), \ldots, \theta_k(v) \rangle$ and $>_{lex}$ is the lexicographic combination of the component orderings $\theta_i$, with $0 \leq i \leq k$. If $\theta_i$ is an homomorphism to a well-founded set, then $\theta_i$ is obviously a lifting, and if $\theta_i$ is a multiset extracting function, since the multiset construction preserves well-foundedness, we also have that $\theta_i$ is a lifting. Finally the finite lexicographic composition of liftings is still a lifting. As a consequence $\succ_{gpo}^{\tau(f)}$ is a well-defined status, and since $\succ_{gpo}$ has the subterm property and satisfies the other conditions of theorem 3.5, we can apply this result to conclude $\succ_{gpo}$ is well-founded.

For infinite signatures, well-foundedness of $\succ_{gpo}$ is a consequence of theorem 3.17. If we define the term lifting $\succ_{gpo}^{\Lambda}$ as $\Theta$, we see that $\succ_{gpo}^{\Lambda}$ is indeed well-defined. Since the other conditions of theorem 3.17 are satisfied, we can apply it to conclude well-foundedness of $\succ_{gpo}$. Finally it is interesting to remark that if we allow the termination function to be not only a multiset extraction function but an arbitrary lifting, we may be able to obtain a generalization of $\succ_{gpo}$ whose well-foundedness can still be derived from the results presented.[4]

## 3.6 Conclusions

We presented some criteria for proving well-foundedness of orders on terms. Our approach was inspired by Kruskal's theorem but is simpler. Kruskal's theorem (and extensions as the one in Puel [92]) is a stronger result in the sense that it establishes that a certain order is a *well quasi-order* (or *partial well-order*). Our result allows to conclude well-foundedness directly. However the essential difference is the domain of application: Kruskal's theorem implies well-foundedness of orders extending any monotonic order with the subterm property, hence only

[4] For generalizations of $\succ_{gpo}$ see Geser [41].

covers simplification orders and it is well-known that those orders do not cover all terminating TRS's. Our criteria do not require monotonicity and as a consequence, cover all terminating TRS's.

For infinite signatures we presented a criterion even simpler and the completeness results still hold.

# Chapter 4

# On Recursively Defined Term Orders

In the previous chapters we presented some recursively defined path orders and stated some properties these orders have. Here we look at some of these orders in more detail.

The chapter can be divided in two parts. In the first part we consider the problems surrounding the definition of recursive path orders. When giving a recursive definition of a path order, several problems are posed. One of them is well-definedness of the order, i. e., one should see that an object of the sort that is being defined exists. Another important aspect concerns the properties that make a relation a partial order, i. e., irreflexivity and transitivity. In general irreflexivity is not so difficult to prove but verification of transitivity is in general not a trivial task. We discuss these issues for the particular cases of *spo* and *rpo*. We will follow an approach similar to the one presented in Kamin and Lévy [55]. We will concentrate on the partial order case and will make some remarks on the quasi-order case.

In the second part we show that existing versions of *rpo* and *kbo* are related to total termination in the sense that any TRS proven terminating by such an order is totally terminating.

## 4.1 CPO's and Continuous Functions

The definitions we want to consider are recursive ones and recursive definitions are related to *fixed points*. Given a function $f : A \to A$, a fixed point of $f$ is an element $a \in A$ satisfying $f(a) = a$. Not all functions have fixed points, but it is possible to ensure the existence of fixed points if both the domain and the functions satisfy certain conditions. A possibility is to require $A$ to be a *CPO* and $f$ to be *continuous*. In this section we introduce these concepts. For more detailed information, see for example Davey and Priestley [18].

**Definition 4.1.** Let $(P, >)$ be a poset and let $S$ be a subset of $P$. An element $p \in P$ is said an *upper bound* for $S$ if it satisfies $p \geq s$, for all $s \in S$. The *supremum* of $S$, denoted by $\bigvee S$, when it exists, is the *least* upper bound of $S$, i. e.,

- $\bigvee S \geq s$, for all $s \in S$,
- if $p$ is an upper bound for $S$ then $p \geq \bigvee S$.

The supremum of $P$ (when it exists) is named the *greatest element* or *top*.

We note that the notions of *lower bound*, *greatest lower bound* or *minimum* and *least element* (or *bottom*) have a dual definition.

**Definition 4.2.** Let $D$ be a non-empty subset of a poset $(P, >)$. $D$ is said to be *directed* if for any finite subset $F$ of $D$ there is an element $d \in D$ which is an upper bound for $F$.

**Definition 4.3.** A poset $(P, >)$ is a *complete partial order*, abbreviated to *CPO*, if it satisfies the following conditions:

- $P$ has a least element,
- every directed subset of $P$ has a supremum.

**Example 4.4.** A very simple example of CPO that will be of use later, is the powerset of any set $P$, ordered by strict inclusion. The least element is the empty set and the supremum of any family of sets, and in particular a directed one, is the union of the elements in the family. This CPO also has a greatest element, namely $P$ itself.

**Definition 4.5.** Let $(P, >)$ and $(Q, \succ)$ be two CPO's. A function $f : P \to Q$ is said to be *continuous* if for every directed set $D$ of $P$ we have $f(\bigvee_{>} D) = \bigvee_{\succ} f(D)$.

Note that in the definition above we do not need the existence of a least element neither in $P$ nor in $Q$. In fact the definition of continuous function can be weakened to requiring that the condition for the supremums holds whenever they exist. Note also that if a function is continuous it is also order-preserving or weakly monotone (i. e., $x > y \Rightarrow f(x) \geq f(y)$) since for any pair of elements $x, y$ such that $x > y$, the set $\{x, y\}$ is directed and its supremum is $x$. Continuity now gives $f(\bigvee\{x, y\}) = f(x) \geq f(y)$, by definition of supremum.

We now present the fixed-point result that is needed (for a proof of these statements see Davey and Priestley [18]).

**Theorem 4.6.** *Let $(P, >)$ be a CPO with least element $\bot$. Let $f : P \to P$ be any function. We have:*

1. *if $f$ is order-preserving then $f$ has a least fixed point. Furthermore if $\bigvee_{n \geq 0} f^n(\bot)$ is a fixed point then it is the least fixed-point.*
2. *if $f$ is continuous then $f$ has a least fixed point given by $\bigvee_{n \geq 0} f^n(\bot)$.*

Note that the set $\{f^n(\bot) |\ n \geq 0\}$ is a directed set and so the supremum is well-defined.

## 4.2 Defining spo and rpo

We present a construction that allows us to define *spo* and *rpo* and easily prove that they are partial orders on terms.

First we introduce a new notion of status. Later it will become clear why we need to do so.

**Definition 4.7.** Let $S$ be a set. A status is a function $\Lambda : \mathcal{P}(S \times S) \to \mathcal{P}(S \times S)$ satisfying:

1. if $\theta$ is transitive over $S$ then $\Lambda(\theta)$ is transitive over $S$.
2. if $\theta$ is a partial order over $S$ then $\Lambda(\theta)$ is a partial order over $S$.
3. if we consider the CPO $(\mathcal{P}(S \times S), \supset)$ then $\Lambda$ is continuous.

Note that the last condition above implies that $\Lambda$ is order-preserving or weakly monotone.

We consider the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{X})$ over a set of variables $\mathcal{X}$ and a fixed arity signature $\mathcal{F}$. Since we do not restrict ourselves to finite signatures, the results presented apply also for the varyadic case since any varyadic signature can be simulated by a fixed-arity one with the function symbols labelled with their arities.

We now fix our CPO to be $\mathcal{P}(\mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X}))$ or abbreviatedly $\mathcal{PT}$, ordered by strict inclusion $\supset$. The least element is the empty set. Let $\succeq$ be a fixed quasi-order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and let $\Lambda$ be a status with domain $\mathcal{PT}$.

**Definition 4.8.** The function $\mathcal{G} : \mathcal{PT} \to \mathcal{PT}$ is defined as follows, $s = f(s_1, \dots, s_k)\ \mathcal{G}(\theta)\ t$, (for $\theta \in \mathcal{PT}$), with $f \in \mathcal{F}$ having arity $k \geq 0$, if one of the following conditions holds:

1. $t = g(t_1, \dots, t_m)$, for some $g \in \mathcal{F}$ having arity $m \geq 0$, and $s\ \mathcal{G}(\theta)\ t_j$, for all $1 \leq j \leq m$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s\ \Lambda(\theta)\ t$, or
2. $\exists 1 \leq i \leq k :\ s_i\ \mathcal{G}(\theta)\ t$ or $s_i = t$.

The first thing that needs to be checked is that the function $\mathcal{G}$ is well-defined, i. e., that $\mathcal{G}(\theta)$ is a relation on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, for any relation $\theta$ and that if $\theta = \theta'$ then $\mathcal{G}(\theta) = \mathcal{G}(\theta')$. That can be done without much work by induction over $|s| + |t|$.

The idea behind the definition of the function $\mathcal{G}$ is that we want to use it to obtain a definition of *spo* and *rpo*, i. e., we want those orders to be the least fixed point of this function. So now we have to see that the function $\mathcal{G}$ is continuous.

**Lemma 4.9.** *The function $\mathcal{G}$ as defined in definition 4.8 is continuous.*

**Proof** Let $D$ be a directed set in $\mathcal{P}(\mathcal{T}(\mathcal{F},\mathcal{X}) \times \mathcal{T}(\mathcal{F},\mathcal{X}))$. Recall that the supremum of $D$ in this CPO is just the union of the elements of $D$, that we denote by $\bigcup D$ to keep the notation simple. Thus what we need to see is that $\mathcal{G}(\bigcup D) = \bigcup \mathcal{G}(D)$, where again abusively $\bigcup \mathcal{G}(D)$ denotes $\bigcup_{A \in D} \mathcal{G}(A)$.

It is well-known that one inclusion always holds due to weak monotonicity and the definition of supremum. Indeed since $\bigvee D \supseteq A$, for any $A \in D$, if $\mathcal{G}$ is weakly monotone then $\mathcal{G}(\bigcup D) \supseteq \mathcal{G}(A)$ and consequently $\mathcal{G}(\bigcup D) \supseteq \bigcup_{A \in D} \mathcal{G}(A) = \bigcup \mathcal{G}(D)$.

So first we check weak monotonicity of $\mathcal{G}$. Let then $\theta \supset \theta'$. We need to see that if $s\ \mathcal{G}(\theta')\ t$ then also $s\ \mathcal{G}(\theta)\ t$ and we do it by induction on $|s| + |t|$. Let then $s, t$ be a pair of minimal terms in the sense that the property holds for terms $u, v$ with $|u|+|v| < |s|+|t|$. We have to do some case analysis. If $s\ \mathcal{G}(\theta')\ t$ by case 2 of definition 4.8, we have by induction hypothesis or definition of $\mathcal{G}$ that also $s\ \mathcal{G}(\theta)\ t$. For case 1, induction hypothesis gives us $s\ \mathcal{G}(\theta)\ t_j$, for all $1 \leq j \leq m$, where $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F}$ having arity $m \geq 0$. Now if we are in case 1a we are done since the same case allows us to conclude that $s\ \mathcal{G}(\theta)\ t$. If we are in case 1b, then $s\ \Lambda(\theta')\ t$ and since $\Lambda$ is monotone also $s\ \Lambda(\theta)\ t$. Again we can conclude that $s\ \mathcal{G}(\theta)\ t$.

We still need to see that the other inclusion holds, that is $\mathcal{G}(\bigcup D) \subseteq \bigcup \mathcal{G}(D)$. We see that for any $s, t \in \mathcal{T}(\mathcal{F},\mathcal{X})$, $s\ \mathcal{G}(\bigcup D)\ t \Rightarrow s\ \bigcup \mathcal{G}(D)\ t$, by induction on $|s| + |t|$. Let then $s, t$ be a pair of minimal terms for which the property is not yet known to hold. Again case analysis has to be done. If $s\ \mathcal{G}(\bigcup D)\ t$ by case 1 then $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F}$ having arity $m \geq 0$, and induction hypothesis gives us $s\ \bigcup \mathcal{G}(D)\ t_j$, for all $0 \leq j \leq m$. This means that for each $j$ there is an element of $D$, that we denote by $\theta_j$ such that $s\ \mathcal{G}(\theta_j)\ t_j$. Because $\{\theta_j \in D :\ 0 \leq j \leq m\}$ is a finite subset of the directed set $D$, we know that there is an element $\theta \in D$ such that $\theta \supseteq \theta_j$, for all $0 \leq j \leq m$, and weak monotonicity of $\mathcal{G}$ implies that $s\ \mathcal{G}(\theta)\ t_j$, for all $0 \leq j \leq m$. Now if case 1a is applicable, we conclude that $s\ \mathcal{G}(\theta)\ t$ and so that $s\ \bigcup \mathcal{G}(D)\ t$; if case 1b is applicable then continuity of the status gives us $s\ \bigcup \Lambda(D)\ t$, that is there is an element $\theta'$ in $D$ such that $s\ \Lambda(\theta')\ t$. Again the two elements $\theta, \theta'$ are majorated in $D$ by an element $\Theta$ and again monotonicity of $\mathcal{G}$ gives $s\ \mathcal{G}(\Theta)\ t_j$, for all $1 \leq j \leq m$ and monotonicity of $\Lambda$ gives $s\ \Lambda(\Theta)\ t$. By definition of $\mathcal{G}$, we conclude that $s\ \mathcal{G}(\Theta)\ t$ and thus that $s\ \bigcup \mathcal{G}(D)\ t$. Finally if case 2 is applicable, induction hypothesis (and definition of $\mathcal{G}$) gives the result. □

Since the function $\mathcal{G}$ is continuous, theorem 4.6 tell us that this function has a least fixed point. Furthermore this fixed point is given by $\bigcup_{n \geq 0} >^n$ where

$$>^0 = \emptyset \text{ and } >^{n+1} = \mathcal{G}(>^n), \text{ for all } n \geq 0.$$

We will now show that for each $n \geq 0$ the relation $>^n$ is transitive and irreflexive.

**Lemma 4.10.** *For each $n \geq 0$, the relation $>^n$ as defined above is transitive.*

**Proof** We have to see that

$$\forall n \geq 0\ \forall s, t, u \in \mathcal{T}(\mathcal{F}, \mathcal{X}):\ s >^n t \text{ and } t >^n u \Rightarrow s >^n u$$

We proceed by induction on the lexicographic product $(n, |s| + |t| + |u|)$. For $n = 0$ the result is trivially satisfied since $>^0$ is the empty relation. Suppose now that $(n+1, |s| + |t| + |u|)$ is a minimal tuple for which the property is not yet verified, i. e., $>^j$ is transitive for any $0 \leq j \leq n$ and if $|p| + |q| + |r| < |s| + |t| + |u|$ then $p >^{n+1} q$ and $q >^{n+1} r$ implies $p >^{n+1} r$. We have to do case analysis and we have to check the following nine cases:

| Case | $s >^{n+1} t$ | $t >^{n+1} u$ | $s >^{n+1} u$ |
|---|---|---|---|
| (1) | 1a | 1a | 1a |
| (2) | 1a | 1b | 1a |
| (3) | 1a | 2 | IH/2 |
| (4) | 1b | 1a | 1a |
| (5) | 1b | 1b | 1b |
| (6) | 1b | 2 | IH/2 |
| (7) | 2 | 1a | IH/2 |
| (8) | 2 | 1b | IH/2 |
| (9) | 2 | 2 | IH/2 |

The last column indicates the case of the definition of $\mathcal{G}$ used in the conclusion, apart from the induction hypothesis and other reasoning. For the cases when it is necessary we assume that $s, t, u$ are written, respectively as $f(s_1, \ldots, s_k)$, $g(t_1, \ldots, t_m)$ and $h(u_1, \ldots, u_o)$.

Cases (7), (8) and (9) follow directly from the induction hypothesis and the definition of $\mathcal{G}$ since $s_i >^{n+1} t$ and $t >^{n+1} u$ implies, by induction hypothesis, that $s_i >^{n+1} u$, and $s_i = t >^{n+1} u$ gives, using case 2 from definition 4.8, $s_i >^{n+1} u$. The same holds for cases (3) and (6). For case (1), since $s >^{n+1} t$ and $t >^{n+1} u_l$, for all $0 \leq l \leq o$ (where $u = h(u_1, \ldots, u_o)$), induction hypothesis gives $s >^{n+1} u_l$, for all $l$. Since $\succ$ is transitive the result follows. For case (2), similarly to (1), induction hypothesis gives us $s >^{n+1} u_l$, for all $0 \leq l \leq o$. Since $\succ$ and $\sim$ are compatible, we also have $s \succ u$ and the result follows. Case (4) is similar to case (3).

Case (5) is the most complicated. As in (1), induction hypothesis gives us $s >^{n+1} u_l$, for all $0 \leq l \leq o$. Transitivity of $\sim$ gives us $s \sim u$. By hypothesis we have $s\ \Lambda(>^n)\ t$ and $t\ \Lambda(>^n)\ u$. Recall that induction hypothesis tells us that $>^n$ is transitive and since the status $\Lambda$ preserves transitivity of relations, we conclude that $\Lambda(>^n)$ is also transitive and therefore that $s\ \Lambda(>^n)\ u$. Case 1b of the definition of $\mathcal{G}$, gives the result. □

**Lemma 4.11.** *For each $n \geq 0$, the relation $>^n$ as defined above is irreflexive.*

**Proof** We show that $\forall n \geq 0\ \forall s \in \mathcal{T}(\mathcal{F},\mathcal{X}) :\ \neg(s >^n s)$. We proceed by induction on the lexicographic tuple $(n, |s|)$ and will use transitivity of the relations $>^n$, $n \geq 0$ (proven in lemma 4.10).

For $n = 0$ again the result is trivially verified, since $>^0$ is the empty relation. Suppose then that $(n+1, |s|)$ is a minimal tuple for which the property has not yet been verified (i. e., the property holds for all tuples $(j, t)$ with $0 \leq j \leq n$ and $t$ arbitrary, and tuples $(n+1, t)$ with $|t| < |s|$). Suppose furthermore that $s >^{n+1} s$. We now proceed by case analysis. If case 1a would be applicable then we would have $s \succ s$, contradicting irreflexivity of order $\succ$. If case 1b would be applicable we would have $s\ \Lambda(>^n)\ s$. But by induction hypothesis $>^n$ is irreflexive and by lemma 4.10 it is transitive; therefore $>^n$ is a partial order and since the status $\Lambda$ preserves partial orders, we have that $\Lambda(>^n)$ is a partial order, thus arriving at a contradiction. Finally if case 2 would be applicable, we would get either

- $s_i >^{n+1} s$, for some $1 \leq i \leq k$, and since $s >^{n+1} s_i$, transitivity of $>^{n+1}$ (see lemma 4.10) gives $s_i >^{n+1} s_i$, contradicting the minimality of $s$,
- $s_i = s$; and this cannot occur.

□

In the following we shall denote the least fixed point of $\mathcal{G}$ by $\gg$. Of course we have:

**Proposition 4.12.** *$s = f(s_1, \ldots, s_k) \gg t$, with $k \geq 0$ and $f \in \mathcal{F}$ having arity $k$, if one of the following conditions holds:*

1. *$t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F}$ having arity $m \geq 0$, and $s \gg t_j$, for all $1 \leq j \leq m$, and either*
   *(a) $s \succ t$, or*
   *(b) $s \sim t$ and $s\ \Lambda(\gg)\ t$, or*
2. *$\exists 1 \leq i \leq k :\ s_i \gg t$ or $s_i = t$.*

The previous lemmas give us the result we were aiming at.

**Theorem 4.13.** *Let $\gg\ = \bigcup_{n\geq 0} >^n$. Then $\gg$ is irreflexive and transitive.*

**Proof** We first check irreflexivity. Suppose there is a term $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that $s \gg s$. Then we must have an index $i \geq 1$ such that $s >^i s$. But lemma 4.11 tells us that $>^i$ is irreflexive, so we reach a contradiction.

For transitivity, suppose that $s \gg t$ and $t \gg u$, for some terms $s, t, u \in \mathcal{T}(\mathcal{F},\mathcal{X})$. This means that there are indexes $i, j \geq 1$ such that $s >^i t$ and $t >^j u$. Now it is easy to see that the following fact holds: $\forall n \geq 0 :\ >^n \subseteq >^{n+1}$; indeed $>^0 = \emptyset \subseteq >^1$ and if

$>^k \subseteq >^{k+1}$, weak monotonicity of $\mathcal{G}$ gives $>^{k+1} \subseteq >^{k+2}$. As a consequence of this fact we have

$$>^0 \ \subseteq \ >^1 \ \subseteq \ \ldots >^n \ \ldots$$

Take now $k = max\{i, j\}$, then $>^i, >^j \ \subseteq \ >^k$, and since $>^k$ is transitive (by lemma 4.10), we conclude that $s >^k u$ and therefore that $s \gg u$. □

So now we know that the relation $\gg$ (satisfying 4.12) is a partial order on terms but what kind of order is it? If we consider the usual definition of *spo* (Kamin and Lévy [54]) we see that $\gg$ is a form of *spo*.

Also if we have a fixed precedence (quasi-order) $\trianglerighteq$ on $\mathcal{F}$ and associated to each function symbol $f \in \mathcal{F}$ we have:

- a permutation $\pi_f$ that indicates how the arguments of $f$ are to be taken to form a sequence (to simplify the notation, we denote a sequence $s_{\pi_f(1)} \cdots s_{\pi_f(k)}$ by $\pi_f(s_1, \ldots, s_k)$),
- a lifting $\lambda_f$ from relations on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ to relations on $\mathcal{T}(\mathcal{F}, \mathcal{X})^*$, preserving transitivity and partial orders (i. e., having the same properties as the status $\Lambda$) and continuous with respect to the CPO's $(\mathcal{P}(\mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})), \supset)$ and $(\mathcal{P}(\mathcal{T}(\mathcal{F}, \mathcal{X})^* \times \mathcal{T}(\mathcal{F}, \mathcal{X})^*), \supset)$.

If we additionally require that the liftings are compatible with the precedence, i. e., if $f \sim g$ then $\lambda_f = \lambda_g$, then by choosing the quasi-order $\succeq$ to be:

- $s \succ t \iff \texttt{root}(s) \triangleright \texttt{root}(t)$,
- $s \sim t \iff \texttt{root}(s)\texttt{eq}(\trianglerighteq)\texttt{root}(t)$, where $\texttt{eq}(\trianglerighteq)$ is the equivalence part of the fixed precedence $\trianglerighteq$,

and defining the status $\Lambda$ as

$$s = f(s_1, \ldots, s_k)\ \Lambda(\gg)\ g(t_1, \ldots, t_m) = t \text{ if } \begin{cases} \texttt{root}(s) \sim \texttt{root}(t) \text{ and} \\ \pi_f(s_1, \ldots, s_k)\ \lambda_f(\gg)\ \pi_g(t_1, \ldots, s_m) \end{cases}$$

(where $k, m \geq 0$), we see that definition 2.89 is a particular case of the definition of $\gg$. This is due to the fact (not difficult to prove) that multiset extension and lexicographic extensions of relations are continuous (essentially continuity is guaranteed because comparison of multisets or sequences envolves only a *finite* number of comparisons of individual elements). We denote the order $\gg$ obtained in these conditions by $\gg_{rpo}$.

The order $\gg$ (see 4.12) is a generalization of *spo* and *rpo*. To see that, consider the properties usually associated with *spo* and *rpo*, namely subterm property and closedness under substitutions for both orders, and additionally closedness under contexts for *rpo*.

Subterm property is not a problem, i. e., the order $\gg$ also enjoys this property, as is stated in the following lemma that can easily be proven by induction on the context and using case 2 from proposition 4.12.

**Lemma 4.14.** *The partial order $\gg$ satisfies $C[s] \gg s$, for any term $s$ and any non-trivial context $C$.*

However in general the order $\gg$ does not enjoy the other properties. The reason why stems from the use (and definition) of status used in the construction of the order. If the status function produces an order which is not closed under substitutions, $\gg$ will not be closed under substitutions. A similar observation applies to closedness under contexts for the particular case of *rpo*. For example, let $x \in \mathcal{X}$ be fixed. If we define the status $\Lambda$ by

$$s\ \Lambda(\theta)\ t \iff \begin{cases} s = f(s_1, \ldots, s_k) \text{ and } t = g(t_1, \ldots, t_m), k, m \geq 1, \text{ and} \\ s_1 = t_1 = x \text{ and } s_2 \cdots s_k\ \theta_{lex}\ t_2 \cdots t_m \end{cases}$$

then $\Lambda$ is a status in the sense of definition 4.7 but it is not closed under substitutions, so the relation $\gg$ associated with this status will not be closed under substitutions.

If we require the status function to preserve closedness under substitutions, i. e., to satisfy:

> *If $\theta$ is a relation closed under substitutions then $\Lambda(\theta)$ is closed under substitutions.*

then we are able to prove the following:

**Theorem 4.15.** *If the quasi-order $\succeq$ is closed under substitutions, i. e., both $\succ$ and $\sim$ are closed under substitutions, and $\Lambda$ preserves closedness under substitutions then $\gg$ is closed under substitutions.*

**Proof** Recall that $\gg\ = \bigcup_{n \geq 0} >^n$, where $>^0 = \emptyset$ and $>^{n+1} = \mathcal{G}(>^n)$. We will show that for each $n \geq 0$, the relation $>^n$ is closed under substitutions. We proceed by induction on $n$. For $n = 0$ the result is trivially satisfied since $>^0 = \emptyset$. Suppose the result is valid for $>^n$, for some $n \geq 0$. We see that for any terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and any substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$, $s >^{n+1} t \Rightarrow s\sigma >^{n+1} t\sigma$. We proceed by induction on $|s| + |t|$. Let then $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ be any substitution and let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be a pair of minimal terms for which the property is not yet verified (i. e., if $u, v$ are terms such that $|u| + |v| < |s| + |t|$ then $u >^{n+1} v \Rightarrow u\sigma >^{n+1} v\sigma$). Suppose $s >^{n+1} t$; we have to do some case analysis. If $s >^{n+1} t$ by case 1 then $s = f(s_1, \ldots, s_k)$ and $t = g(t_1, \ldots, t_m)$, for some function symbols $f, g \in \mathcal{F}$ having arities respectively $k, m \geq 0$; note that $t\sigma = g(t_1\sigma, \ldots, t_m\sigma)$. Furthermore $s >^{n+1} t_j$, for all $1 \leq j \leq m$, and the induction hypothesis gives $s\sigma >^{n+1} t_j\sigma$, for all $1 \leq j \leq m$. If case 1a is applicable then $s \succ t$ and so $s\sigma \succ t\sigma$. We then have that $s\sigma >^{n+1} t\sigma$ (using case 1a). If case 1b is applicable then $s \sim t$ and so also $s\sigma \sim t\sigma$, and $s\ \Lambda(>^n)\ t$. Since $>^n$ is closed under substitutions and $\Lambda$ respects this property, we conclude that $s\sigma\ \Lambda(>^n)\ t\sigma$. As a consequence we have that $s\sigma >^{n+1} t\sigma$. Finally if $s >^{n+1} t$ by case 2 then also $s = f(s_1, \ldots, s_k)$ for some $f \in \mathcal{F}$ having arity $k \geq 1$, and $s_i >^{n+1} t$ or $s_i = t$, for some $1 \leq i \leq k$. For the first case and by induction hypothesis we conclude that $s_i\sigma >^{n+1} t\sigma$, and the second case we have $s_i\sigma = t\sigma$ (equality is closed under substitutions) and since $s\sigma = f(s_1\sigma, \ldots, s_k\sigma)$, it follows (from 2) that $s\sigma >^{n+1} t\sigma$.

Now we see that $\gg\ = \bigcup_{n \geq 0} >^n$ is closed under substitutions. Let $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ be any substitution and let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be terms satisfying $s \gg t$. This means that there is an index $n \geq 1$ such that $s >^n t$ and since $>^n$ is closed under substitutions we conclude that $s\sigma >^n t\sigma$, and therefore that $s\sigma \gg t\sigma$, as we wanted. □

For closedness under contexts, we know that in general *spo* is not closed under contexts, but *rpo* is. As for closedness under substitutions, if we want to see that $\gg$ is closed under contexts for the particular case that $\gg$ is $\gg_{rpo}$, we have to require that the status $\Lambda$ satisfies some condition. In this case we additionally require that each lifting $\lambda_f$, $f \in \mathcal{F}$, respects or extends in some sense the relation lifted, i. e.:

> *If $\theta$ is a partial order then $\Lambda(\theta)$ respects $\theta$, i. e., if $s\theta t$ then $s_1 \cdots s \cdots s_n \; \lambda_f(\theta) \; s_1 \cdots t \cdots s_n$, for any terms $s, t, s_1, \ldots, s_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and function symbol $f \in \mathcal{F}$ having arity $n \geq 0$.*

Then we are able to prove the following result:

**Theorem 4.16.** *If $\Lambda(\gg_{rpo})$ respects $\gg_{rpo}$ then $\gg_{rpo}$ is closed under contexts.*

**Proof** We proceed by induction on the context $C$. If $C$ is the trivial context, the result holds so it suffices to show that

if $s \gg_{rpo} t$ then $p = h(u_1, \ldots, s, \ldots, u_n) \gg_{rpo} h(u_1, \ldots, t, \ldots, u_n) = q$,

for any $h \in \mathcal{F}$ having arity $k \geq 1$, terms $u_1, \ldots, u_{j-1}, u_{j+1}, \ldots u_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and where both $s$ and $t$ occur at position $1 \leq l \leq n$.

Suppose that $s \gg_{rpo} t$ by case 1. Then $s = f(s_1, \ldots, s_k)$, $t = g(t_1, \ldots, t_m)$ and $s \gg_{rpo} t_j$, for all $1 \leq j \leq m$. Since $\gg_{rpo}$ satisfies the subterm property, we have that $h(u_1, \ldots, s, \ldots, u_n) \gg_{rpo} u_j, s$, for all $1 \leq j \leq n$, $j \neq l$. But $s \gg_{rpo} t$ and so transitivity gives us $h(u_1, \ldots, s, \ldots, u_n) \gg_{rpo} t$. Since $\texttt{root}(p)\ \texttt{eq}(\trianglerighteq)\ \texttt{root}(q)$, we have to see that $u_1 \cdots s \cdots u_n \; \lambda_f(\gg_{rpo}) \; u_1 \cdots t \cdots u_n$, and this a consequence of the fact that $\Lambda(\gg_{rpo})$ respects $\gg_{rpo}$ and that $s \gg_{rpo} t$. Consequently, using case 1b we can conclude that $p \gg_{rpo} q$. Suppose now that $s \gg_{rpo} t$ by case 2; then $s = f(s_1, \ldots, s_k)$, and $s_i \gg_{rpo} t$ or $s_i = t$, for some $1 \leq i \leq k$. But $h(u_1, \ldots, s, \ldots, u_n) \gg_{rpo} s_i$ (due to the subterm property and transitivity) and also $h(u_1, \ldots, s, \ldots, u_n) \gg_{rpo} u_j$, for any $j \neq l$, so since $s \gg_{rpo} s_i \gg_{rpo} t$ or $s \gg_{rpo} s_i = t$, and $\Lambda(\gg_{rpo})$ respects $\gg_{rpo}$, we conclude that $u_1 \cdots s \cdots u_n \; \lambda_f(\gg_{rpo}) \; u_1 \cdots t \cdots u_n$, so by case 1b, the result holds. □

Another interesting property to look at is totality. Totality of an order closed under substitutions cannot be achieved in the set of terms. But if we restrict ourselves to ground terms, sometimes totality is possible. In general neither *spo* nor *rpo* will be total even on ground terms, but totality for *rpo* can be achieve at least in some cases.

In [70], Lescanne proved that if the precedence is a total quasi-order satisfying the *arity condition*:

*if* $f\ \texttt{eq}(\trianglerighteq)\ g$ *and* $f \neq g$ *then* $\texttt{arity}(f) \neq \texttt{arity}(g)$.

and the lifting $\lambda_f$ is the lexicographic extension for all $f \in \mathcal{F}$, then the order $\gg_{rpo}$ is total on ground terms.

Note that the arity condition is essential as the following example shows.

**Example 4.17.** Suppose that $a, b$ are constants and that $f$ admits arity $2$. Suppose that $\lambda_f(\gg_{rpo})$ is the left-to-right lexicographic extension of $\gg_{rpo}$, i. e., $\lambda_f(\gg)_{rpo} = \gg_{rpo,lex}$. Suppose also that $f \succ p$ for any $p \in \{a, b\}$, and that $\sim$ satisfies (apart from equality) $a \sim b$. Then $\succeq$ is total but we cannot conclude that $f(a, a) \gg_{rpo} f(b, b)$ nor the reverse. Since clearly these terms are different this example shows that we cannot weaken the arity condition.

It is also interesting to remark that there are "natural" examples of TRS's that require a quasi-order as precedence in the definition of *rpo* as we see in the following example.

**Example 4.18.** The following TRS (from Lescanne [70]), arising in the context of groups with left division, can only be oriented by a version of *rpo* where the function symbols $i$ and $\backslash$ are equivalent.

$$\begin{array}{lll} x \setminus e & \rightarrow & i(x) \\ i(x \backslash y) & \rightarrow & y \setminus x \\ (x \setminus y) \setminus z & \rightarrow & y \setminus (i(x) \setminus z) \end{array}$$

If we use *rpo* with left-to-right lexicographic status, and a quasi-precedence $\trianglerighteq$ satisfying $i \sim \backslash$, we can indeed orient the rules of the TRS.

### 4.2.1 Well-foundedness of spo, rpo

For orders on terms to be used in termination proofs, it is essential that the orders are well-founded. Here we discuss well-foundedness of the order $\gg$. From lemma 4.14 we know that the order $\gg$ has the subterm property.

We define the relation $\gg^{\bar{\Lambda}}$ as follows:

$$s \gg^{\bar{\Lambda}} t \iff \begin{cases} s \succ t \text{ or} \\ s \sim t \text{ and } s \; \Lambda(\gg) \; t \end{cases}$$

where $\succeq$ and $\Lambda$ are the quasi-order and status used in the definition of $\gg$.

It is not difficult to see that $\gg^{\bar{\Lambda}}$ is a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Also if $\Lambda(\gg)$ is a term lifting in the sense of definition 3.16 and $\succeq$ is well-founded, we also have that $\gg^{\bar{\Lambda}}$ is a term lifting. Furthermore if $s \gg t$ then either $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F}$ having arity $k \geq 1$, and $s_i \gg t$ or $s_i = t$, for some $1 \leq i \leq k$ (this is given by case 2), or $s \gg^{\bar{\Lambda}} t$ (consequence of cases 1a, 1b). Consequently theorem 3.17 gives us the following result.

**Theorem 4.19.** *If $\succeq$ is well-founded and $\Lambda(\gg)$ is a term lifting (in the sense of definition 3.16) then $\gg$ is well-founded.*

### 4.2.2 Some Remarks on Quasi-orders

The construction presented in section 4.2 only works for partial orders. Since the CPO we use is more general (it contains all relations) we can ask ourselves if such construction will also work for the quasi-order case. The answer is no. One of the problems is that in the case of partial orders inclusion of partial orders coincides with inclusion of sets but for quasi-orders we want inclusion of quasi-orders to respect the strict and equivalent parts and therefore inclusion of quasi-orders no longer coincides with inclusion of sets. So a different kind of CPO has to be defined and a good candidate seems to be the set of quasi-orders ordered by an appropriate partial order.

We proceed to define the CPO. Let $S$ be a set and define $\mathcal{QO}_S$ to be the set of all quasi-orders on $S$, i. e., $\mathcal{QO}_S = \{\theta \subseteq S \times S : \theta \text{ is a quasi-order}\}$. We now define a relation $\sqsupset$ in $\mathcal{QO}_S$ as follows:

$$\theta \sqsupset \theta' \iff \begin{cases} \theta \supset \theta', \text{ and} \\ \mathtt{ord}(\theta) \supseteq \mathtt{ord}(\theta'), \text{ and} \\ \mathtt{eq}(\theta) \supseteq \mathtt{eq}(\theta') \end{cases}$$

It is not difficult to see that $\sqsupset$ is indeed a partial order (irreflexivity follows from the first condition above and transitivity is a consequence of the fact that $\supset$ and $\supseteq$ are transitive). Furthermore we have:

**Lemma 4.20.** *The poset $(\mathcal{QO}_S, \sqsupset)$ is a CPO with bottom element given by equality, i. e., the relation $\{(s,s) | \; s \in S\}$, and with the supremum of directed sets given by the union of the elements in the set.*

**Proof** It is clear that the bottom element is equality since any quasi-order contains this relation. Suppose now that $D$ is a directed set of quasi-orders and take $\bigcup D$. We have to see that $\bigcup D$ is a quasi-order and that for any element $\theta \in D$, we have $\bigcup D \sqsupseteq \theta$. The relation $\bigcup D$ is indeed reflexive since it is the union of reflexive relations. As for transitivity, suppose we have elements $s, t, u \in S$ such that $s \; (\bigcup D) \; t$ and $t \; (\bigcup D) \; u$; then there are elements $\theta_1, \theta_2 \in D$ such that $s\theta_1 t$ and $t\theta_2 u$. Since $D$ is directed, there is an element $\theta_3 \in D$ such that $\theta_3 \sqsupseteq \theta_1, \theta_2$ and since $\theta_3$ is transitive, we conclude that $s \; \theta_3 \; u$ and so $s \; (\bigcup D) \; u$, as we wanted.

Now we see that $\bigcup D$ is an upper bound for each element in $D$. Let $\theta$ be an arbitrary element of $D$. It is obvious that $\bigcup D \supseteq \theta$, but we still have to see that $\mathtt{ord}(\bigcup D) \supseteq \mathtt{ord}(\theta)$ and $\mathtt{eq}(\bigcup D) \supseteq \mathtt{eq}(\theta)$. Suppose that $(s,t) \in \mathtt{eq}(\theta)$, then $s\theta t$ and $t\theta s$ and consequently $s \; (\bigcup D) \; t$ and $t \; (\bigcup D) \; s$, so $(s,t) \in \mathtt{eq}(\bigcup D)$. Suppose now that $(s,t) \in \mathtt{ord}(\theta)$, then $s\theta t$ and $\neg(t\theta s)$. We also have $s \; (\bigcup D) \; t$; suppose we have $t \; (\bigcup D) \; s$. Then an element $\theta' \in D$ has to exist such that $t \; \theta' \; s$. Since $D$ is directed, there is an element $\theta'' \in D$ such that $\theta'' \sqsupseteq \theta, \theta'$; thus we have $s \; \theta'' \; t$ and $t \; \theta'' \; s$, which means that $(s,t) \in \mathtt{eq}(\theta'')$. But this contradicts the fact that $\theta'' \sqsupseteq \theta$ and $(s,t) \in \mathtt{ord}(\theta)$. So we must have $\neg(t \; (\bigcup D) \; s)$ and $\mathtt{ord}(\bigcup D) \supseteq \mathtt{ord}(\theta)$. We have just seen that $\bigcup D$ is an upper bound for $\theta$, thus concluding the proof. □

We now define a new notion of status. In the previous sections the status was intended for application to partial orders but now we are dealing with quasi-orders, so the properties the status has to fulfil are different.

**Definition 4.21.** Let $S$ be a set. A *status* is a function $\Lambda : \mathcal{P}(S \times S) \to \mathcal{P}(S \times S)$ satisfying:

1. if $\theta$ is reflexive in $S$ then $\Lambda(\theta)$ is reflexive in $S$,
2. if $\theta$ is transitive in $S$ then $\Lambda(\theta)$ is transitive in $S$,
3. if we consider the CPO $(\mathcal{QO}_S, \sqsupset)$, then $\Lambda$ is weakly monotone, i. e., $\theta \sqsupset \theta' \Rightarrow \Lambda(\theta) \sqsupseteq \Lambda(\theta')$.

From now on we fix our CPO to be $(\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}, \sqsupset)$. Let $\succeq$ be a fixed quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ and let $\Lambda$ be a status, in the sense of 4.21, with domain $\mathcal{PT}$. We define the following function:

**Definition 4.22.** The function $\mathcal{H} : \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})} \to \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$ is defined as follows. For $\theta \in \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$, $s = f(s_1, \ldots, s_k)\ \mathcal{H}(\theta)\ t$, with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if one of the following conditions holds:

1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s\ \mathcal{H}(\theta)\ t_j$ and $\neg(t_j\ \mathcal{H}(\theta)\ s)$, and either
   (a) $s \succ t$, or
   (b) $s \sim t$ and $s\ \Lambda(\theta)\ t$, or
2. $\exists 1 \leq i \leq k :\ s_i\ \mathcal{H}(\theta)\ t$.

It is not difficult to see that the function $\mathcal{H}$ is well-defined. Below we give some other properties of $\mathcal{H}$.

**Lemma 4.23.** *If $\theta$ is reflexive then $\mathcal{H}(\theta)$ is reflexive.*

**Proof** (Sketch) We show that for any term $s$, we have $s\ \mathcal{H}(\theta)\ s$, by induction on the size of $s$ and using the fact that both $\sim$ and $\Lambda(\theta)$ are reflexive. □

**Lemma 4.24.** *If $\theta$ is transitive then $\mathcal{H}(\theta)$ is transitive.*

**Proof** (Sketch) We have to see that for any terms $s, t, u \in \mathcal{T}(\mathcal{F},\mathcal{X})$, if $s\ \mathcal{H}(\theta)\ t$ and $t\ \mathcal{H}(\theta)\ u$ then $s\ \mathcal{H}(\theta)\ u$. We proceed by induction on $|s| + |t| + |u|$. The proof is a case analysis similar to the one done in lemma 4.10. The only part that is essentially different is that a negation is involved. For example, for the combination of cases 1 versus 1 we have to see that $s\ \mathcal{H}(\theta)\ u_l$ and $\neg(u_l\ \mathcal{H}(\theta)\ s)$, for all $1 \leq l \leq m$ (where $u = h(u_1, \ldots, u_m)$). By induction hypothesis $s\ \mathcal{H}(\theta)\ t$ and $t\ \mathcal{H}(\theta)\ u_l$ gives $s\ \mathcal{H}(\theta)\ u_l$, for all $1 \leq l \leq m$. Suppose we have $u_l\ \mathcal{H}(\theta)\ s$, for some $1 \leq l \leq m$. Then, since $s\ \mathcal{H}(\theta)\ t$, the induction hypothesis gives us $u_l\ \mathcal{H}(\theta)\ t$, which is a contradiction (since we have $\neg(u_l\ \mathcal{H}(\theta)\ t)$, for all $1 \leq l \leq m$). Other combination of cases in which a negation occurs are solved similarly.

The rest of the case analysis is very similar to the one in 4.10, so we omit it. □

The following lemma will be quite useful.

**Lemma 4.25.** *For any $\theta \in \mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$, for any non-trivial context $C$ and any term $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, we have that $\neg(s\ \mathcal{H}(\theta)\ C[s])$.*

**Proof** Let $\theta$ be an arbitrary element of $\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}$. We proceed by induction on the lexicographic product $(|s|, C)$. For terms of size $1$, the result holds since for concluding that $s\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, for any $f \in \mathcal{F}$, with arity $\geq 1$, the only case of the definition of $\mathcal{H}$ applicable is case 1 and then we must have simultaneously $s\ \mathcal{H}(\theta)\ s$ and $\neg(s\ \mathcal{H}(\theta)\ s)$, which is impossible; and if $D$ is a context for which the result holds then $s\ \mathcal{H}(\theta)\ f(\ldots, D[s], \ldots)$ again would imply (case 1 is the only possibility) that $s\ \mathcal{H}(\theta)\ D[s]$, contradicting the induction hypothesis.

Take now a term $s$ with $|s| > k$, for a fixed $k \geq 1$, for which the result is not yet verified. i. e., the result holds for all terms $u$ and contexts $D$ if $|u| \leq k$. Take $f \in \mathcal{F}$ with arity $\geq 1$. If $s\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, case 1 of definition 4.22 is not applicable, since both $s\ \mathcal{H}(\theta)\ s$ and its negation would have to hold. So we must have $s = h(s_1, \ldots, s_m)$, for some $h \in \mathcal{F}$ with arity $m \geq 1$, and $s_i\ \mathcal{H}(\theta)\ f(\ldots, s, \ldots)$, for some $1 \leq i \leq m$. But $f(\ldots, s, \ldots)$ can be written as $D[s_i]$, for some non-trivial context $D$, so we have $s_i\ \mathcal{H}(\theta)\ D[s_i]$, contradicting the induction hypothesis. Suppose now that the result holds for the pair $s$ and some non-trivial context $C$. Suppose also that $s\ \mathcal{H}(\theta)\ f(\ldots, C[s], \ldots)$, for some $f \in \mathcal{F}$ with arity $\geq 1$. Again case 1 gives a contradiction (since we would have $s\ \mathcal{H}(\theta)\ C[s]$) and case 2 will also give a contradiction since then we conclude that $s_i\ \mathcal{H}(\theta)\ D[s]$, for some non-trivial context $D$ and proper subterm $s_i$ of $s$, contradicting the induction hypothesis. □

**Lemma 4.26.** *The function $\mathcal{H}$ is weakly monotone.*

**Proof** We have to see that if $\theta\ \sqsupset\ \theta'$ then $\mathcal{H}(\theta)\ \sqsupseteq\ \mathcal{H}(\theta')$ or equivalently that

- $s\ \mathcal{H}(\theta')\ t \Rightarrow s\ \mathcal{H}(\theta)\ t$, for all terms $s, t \in \mathcal{T}(\mathcal{F},\mathcal{X})$, and
- $\texttt{ord}(\mathcal{H}(\theta))\ \supseteq\ \texttt{ord}(\mathcal{H}(\theta'))$, and
- $\texttt{eq}(\mathcal{H}(\theta))\ \supseteq\ \texttt{eq}(\mathcal{H}(\theta'))$.

We prove, by induction on $|s| + |t|$, that if $s\ \mathcal{H}(\theta')\ t$ then $s\ \mathcal{H}(\theta)\ t$ and if additionally $\neg(t\ \mathcal{H}(\theta')\ s)$ then also $\neg(t\ \mathcal{H}(\theta)\ s)$. It is not difficult to see that the statement holds for terms $s, t$ with $|s| + |t| = 2$. Let $s, t$ be a minimal pair of terms such that $s\ \mathcal{H}(\theta')\ t$ and for which the property is not yet verified, i. e., if $u, v$ are terms such that $|u| + |v| < |s| + |t|$, then $u$ and $v$ satisfy the property. We have to do some case analysis. If $s\ \mathcal{H}(\theta')\ t$ by

1. case 1; then $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s\ \mathcal{H}(\theta')\ t_j$ and $\neg(t_j\ \mathcal{H}(\theta')\ s)$. By induction hypothesis we also have $s\ \mathcal{H}(\theta)\ t_j$ and $\neg(t_j\ \mathcal{H}(\theta)\ s)$, for all $1 \leq j \leq m$.
   (a) If case 1a is applicable then we have $s \succ t$ and consequently also $s\ \mathcal{H}(\theta)\ t$. Suppose additionally that $\neg(t\ \mathcal{H}(\theta')\ s)$. If we would have $t\ \mathcal{H}(\theta)\ s$, then cases 1a and 1b cannot be applied since we cannot have simultaneously $s \succ t$ and

$t \succ s$ or $t \sim s$; therefore we must have $t\ \mathcal{H}(\theta)\ s$ by case 2 and this means that $t_j\ \mathcal{H}(\theta)\ s$, for some $1 \leq j \leq m$, which gives a contradiction.

(b) If case 1b is applicable then we have $s \sim t$ and $s\ \Lambda(\theta')\ t$. Since $\Lambda$ is weakly monotone, we also have $s\ \Lambda(\theta)\ t$ and so also $s\ \mathcal{H}(\theta)\ t$. If additionally $\neg(t\ \mathcal{H}(\theta')\ s)$ and $t\ \mathcal{H}(\theta)\ s$, then we conclude that we must have $t\ \mathcal{H}(\theta)\ s$ by case 1b (case 1a is not applicable since we cannot have both $s \sim t$ and $t \succ s$; and case 2 leads to a contradiction as above). But in case 1b, we have both $s\ \Lambda(\theta)\ t$ and $t\ \Lambda(\theta)\ s$, which means that $(s,t) \in \mathtt{eq}(\Lambda(\theta))$. But $(s,t) \in \mathtt{ord}(\Lambda(\theta'))$ and due to weak-monotonicity of $\Lambda$, we also have $(s,t) \in \mathtt{ord}(\Lambda(\theta))$, giving a contradiction.

2. case 2; then $s = f(s_1, \ldots, s_n)$, for some $f \in \mathcal{F}$, having arity $n \geq 1$, and $s_i\ \mathcal{H}(\theta')\ t$, for some $1 \leq i \leq n$. By induction hypothesis we conclude that $s_i\ \mathcal{H}(\theta)\ t$ and so that $s\ \mathcal{H}(\theta)\ t$. Suppose additionally that $\neg(t\ \mathcal{H}(\theta')\ s)$. If we would have $t\ \mathcal{H}(\theta)\ s$ then transitivity of $\mathcal{H}(\theta)$ would give $s_i\ \mathcal{H}(\theta)\ s$, contradicting lemma 4.25.

Once we have established that $\mathcal{H}(\theta') \subseteq \mathcal{H}(\theta)$ and $\mathtt{ord}(\mathcal{H}(\theta')) \subseteq \mathtt{ord}(\mathcal{H}(\theta))$, we also have that $\mathtt{eq}(\mathcal{H}(\theta')) \subseteq \mathtt{eq}(\mathcal{H}(\theta))$, since $\mathcal{H}(\gamma) = \mathtt{ord}(\mathcal{H}(\gamma)) \cup \mathtt{eq}(\mathcal{H}(\gamma))$, for any quasi-order $\gamma$. Thus we establish that $\mathcal{H}$ is weakly monotone. □

Since the function $\mathcal{H}$ is weakly monotone (or order-preserving), theorem 4.6 tells us that $\mathcal{H}$ has a least fixed point. We denote this element by $\geqq$. Obviously, as a consequence of the definition of $\geqq$, we have that:

**Proposition 4.27.** *$\geqq$ is a quasi-order on $\mathcal{T}(\mathcal{F},\mathcal{X})$ satisfying $s = f(s_1, \ldots, s_k) \geqq t$, with $f \in \mathcal{F} \cup \mathcal{X}$, having arity $k \geq 0$, if and only if one of the following conditions holds:*

*1. $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, having arity $m \geq 0$, and for all $1 \leq j \leq m$, we have $s \geqq t_j$ and $\neg(t_j \geqq s)$, and either*

*(a) $s \succ t$, or*

*(b) $s \sim t$ and $s\ \Lambda(\geqq)\ t$, or*

*2. $\exists 1 \leq i \leq k :\ s_i \geqq t$.*

As for the partial order case we can discuss what kind of properties does the quasi-order $\geqq$ enjoy. Note that if we consider the usual definition of *spo* (Kamin and Lévy [54]) we see that $\geqq$ is a form of *spo*, in quasi-order version.

If we have a fixed precedence (quasi-order) $\trianglerighteq$ on $\mathcal{F} \cup \mathcal{X}$ such that elements of $\mathcal{F}$ and $\mathcal{X}$ are incomparable under the precedence and $\trianglerighteq$ restricted to $\mathcal{X}$ is equality, and associated to each function symbol $f \in \mathcal{F}$ we have:

- a permutation $\pi_f$ indicating how the arguments of $f$ are to be taken to form a sequence (again we denote a sequence $s_{\pi_f(1)} \cdots s_{\pi_f(1)}$ by $\pi_f(s_1, \ldots, s_k)$),

- a lifting $\lambda_f$ from relations on $\mathcal{T}(\mathcal{F},\mathcal{X})$ to relations on $\mathcal{T}(\mathcal{F},\mathcal{X})^*$, preserving reflexivity and transitivity (i. e., having the same properties as the status $\Lambda$) and weakly monotone with respect to the CPO's $(\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})}, \sqsupset)$ and $(\mathcal{QO}_{\mathcal{T}(\mathcal{F},\mathcal{X})^*}, \sqsupset)$.

If we additionally require that the liftings are compatible with the precedence, i. e., if $f \sim g$ then $\lambda_f = \lambda_g$, then by choosing the quasi-order $\succeq$ to be:

- $s \succ t \iff \texttt{root}(s) \rhd \texttt{root}(t)$,
- $s \sim t \iff \texttt{root}(s)\ \texttt{eq}(\unrhd)\ \texttt{root}(t)$,

and defining the status $\Lambda$ as

$$s = f(s_1, \ldots, s_k)\ \Lambda(\geqq)\ g(t_1, \ldots, t_m) = t \text{ if } \begin{cases} \texttt{root}(s) \sim \texttt{root}(t) \text{ and} \\ \pi_f(s_1, \ldots, s_k)\ \lambda_f(\geqq)\ \pi_g(t_1, \ldots, s_m) \end{cases}$$

(where $k, m \geq 0$), we obtain an order similar to *rpo* but in a quasi-order version.

We denote the order $\geqq$ obtained in these conditions by $\geqq_{rpo}$. Note that both multiset and lexicographic extensions of quasi-orders satisfy the properties required for the lifting $\lambda$, i. e., they preserve reflexivity and transitivity and they are weakly monotone.

The quasi-order $\geqq$ enjoys the subterm property; more precisely the strict part of $\geqq$ enjoys this property.

**Lemma 4.28.** *The partial order* $\texttt{ord}(\geqq)$ *satisfies* $C[s]\ \texttt{ord}(\geqq)\ s$*, for any term* $s$ *and any non-trivial context* $C$*.*

**Proof** (Sketch) Since reflexivity of $\geqq$ ensures that $s \geqq s$, case 2 of proposition 4.27 gives $f(\ldots, s, \ldots) \geqq s$, for any $f \in \mathcal{F}$ having arity $n \geq 1$. That the relation is strict, i. e., that $\neg(s \geqq f(\ldots, s, \ldots))$ is a consequence of lemma 4.25. $\square$

In general the order $\geqq$ does not enjoy the other usual properties of well-known recursively defined quasi-orders, like closedness under substitutions or contexts. Again the reason why stems from the use (and definition) of status appearing in the construction of the order. If the status function produces an order which is not closed under substitutions, $\geqq$ will not be closed under substitutions. A similar observation applies to closedness under contexts.

For closedness under contexts, we know that in general *spo* does not have this property, but *rpo* does. If we want to see that $\geqq$ is closed under contexts for the particular case that $\geqq$ is $\geqq_{rpo}$, we have to require that the status $\Lambda$ satisfies some condition. In this case we additionally require that each lifting $\lambda_f$, $f \in \mathcal{F}$, respects or extends in some sense the relation lifted, i. e.:

> *If* $\theta$ *is a quasi-order then* $\Lambda(\theta)$ *respects* $\theta$*, i. e., if* $s\theta t$ *then* $s_1 \cdots s \cdots s_n\ \lambda_f(\theta)\ s_1 \cdots t \cdots s_n$*, for any terms* $s, t, s_1, \ldots, s_k \in \mathcal{T}(\mathcal{F},\mathcal{X})$*, and function symbol* $f \in \mathcal{F}$ *having arity* $n \geq 0$ *(and lifting* $\lambda_f$*). Furthermore if additionally* $\neg(t\theta s)$ *then also* $\neg(s_1 \cdots t \cdots s_n\ \lambda_f(\theta)\ s_1 \cdots s \cdots s_n)$*.*

Then we are able to prove the following result:

**Theorem 4.29.** *If $\Lambda(\geqq_{rpo})$ respects $\geqq_{rpo}$ then $\geqq_{rpo}$ is closed under contexts. Furthermore both its strict and equivalent parts are closed under contexts.*

**Proof** (Sketch) The proof is by induction on the context $C$. If $C$ is the trivial context, the result holds. We then show that for any $h \in \mathcal{F}$ having arity $n \geq 1$, and terms $u_1, \ldots, u_{j-1}, u_{j+1}, \ldots u_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$,

$$\text{if } s \geqq_{rpo} t \text{ then } p = h(u_1, \ldots, s, \ldots, u_n) \geqq_{rpo} h(u_1, \ldots, t, \ldots, u_n) = q,$$

and if also $\neg(t \geqq_{rpo} s)$ then $\neg(h(u_1, \ldots, t, \ldots, u_n) \geqq_{rpo} h(u_1, \ldots, s, \ldots, u_n))$, as well. This is done by case analysis and is not so difficult (note that lemma 4.25 is also needed). As a consequence the equivalence part of $\geqq$ is also closed under contexts and the result holds. □

We consider now closedness under substitutions. In general the relation $\geqq$ will not be closed under substitutions, and the same holds for $\geqq_{rpo}$. In order to achieve closedness under substitutions it is essential that the quasi-order $\succeq$ used on the construction of $\geqq$ has the strict part $\succ$ as well as the equivalent part $\sim$ closed under substitutions. But this is not enough and the reason why stems again from the status. There are two ways to deal with the problem of closedness under substitutions. One is the "brute force" way, that is, when confronted with a particular order defined in the same way as $\geqq$, one tries to prove, for that particular choice of status and/or lifting, that the property holds; for example for multiset and lexicographic liftings, that is not so difficult. Another way is to try to provide general conditions on the status and/or liftings that will ensure that the property holds. This is of course a more elegant way of solving the problem.

We now propose such a solution. Our proposal requires that we know how the least fixed point of $\mathcal{H}$ looks like. If $\mathcal{H}$ is continuous we indeed know what the least fixed point is. In fact it is enough to show that the relation $\geq = \bigcup_{n \geq 0} \geq^n$, where $\geq^0$ is equality and $\geq^{n+1} = \mathcal{H}(\geq^n)$, is a fixed point of $\mathcal{H}$; then we also know that it is the least fixed point. In any case we have to require that the status $\Lambda$ is continuous. We can then prove that $\geq$ is a fixed point (and indeed that the function $\mathcal{H}$ is continuous).

**Lemma 4.30.** *If $\Lambda$ is continuous then $\geq$ is the least fixed point of $\mathcal{H}$.*

**Proof** (Sketch) First note that due to the (weak) monotonicity of $\mathcal{H}$ and the fact that $\geq^1 \sqsupseteq \geq^0$, the relations $\geq^n$, $n \geq 0$, satisfy $\geq^{n+1} \sqsupseteq \geq^n$, for all $n \geq 0$. So the relation $\geq$ is well-defined and is a quasi-order. Also due to weak monotonicity of $\mathcal{H}$, definition or the quasi-orders $\geq^n$, $n \geq 0$ and definition of supremum, we have that $\bigcup_{n\geq 0} \geq^n \sqsubseteq \mathcal{H}(\bigcup_{n\geq 0} \geq^n)$.

What we have to prove is that $\mathcal{H}(\bigcup_{n\geq 0} \geq^n) \sqsubseteq \bigcup_{n\geq 0} \geq^n$. This can be done by showing (by induction on $|s| + |t|$) that if $s \; \mathcal{H}(\bigcup_{n\geq 0} \geq^n) \; t$ then $s \bigcup_{n\geq 0} \geq^n \; t$ and if additionally

$\neg(t\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ s)$ then also $\neg(t \bigcup_{n\geq 0} \geq^n\ s)$. It is not difficult to see that this result holds for terms $s, t$ with $|s| + |t| = 2$. Take now minimal terms $s, t$ for which the property has still to be checked (i. e., if $u, v$ are terms with $|u| + |v| < |s| + |t|$, then the property holds for $u$ and $v$). Suppose then that $s\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ t$. We now proceed by case analysis on the case of the definition 4.22 used to establish this.

1. case 1; then $t = g(t_1, \dots, t_m)$, for some $g \in \mathcal{F} \cup \mathcal{X}$, with arity $m \geq 0$, and for all $1 \leq j \leq m$, both $s\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ t_j$ and $\neg(t_j\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ s)$. By induction hypothesis, we also have that $s \bigcup_{n\geq 0} \geq^n\ t_j$, for all $1 \leq j \leq m$. Suppose now that $t_j \bigcup_{n\geq 0} \geq^n\ s$, for some $1 \leq j \leq m$. Since $\bigcup_{n\geq 0} \geq^n \sqsubseteq\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)$, then also $t_j\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ s$, and that is not possible. So also $\neg(t_j \bigcup_{n\geq 0} \geq^n\ s)$, for all $1 \leq j \leq m$.

   Note that $s \bigcup_{n\geq 0} \geq^n\ t_j$, for all $1 \leq j \leq m$, implies that, for each $j$ there is a quasi-order $\geq^{n_j}$ such that $s\ \geq^{n_j}\ t_j$, and since the set $\{\geq^n \mid n \geq 0\}$ is directed, we conclude that there is an element $\geq^k$ in this set, such that $s\ \geq^k\ t_j$, for all $1 \leq j \leq m$. Note also that $\neg(t_j\ \geq^k\ s)$, for all $1 \leq j \leq m$. If $s \succ t$ (case 1a), we conclude from the definition of $\geq^k$ that $s \geq^k t$ and so also $s \bigcup_{n\geq 0} \geq^n\ t$. If $s \sim t$ (case 1b), then also $s\ \Lambda(\bigcup_{n\geq 0} \geq^n)\ t$ and since $\Lambda$ is continuous, we have that $s \bigcup_{n\geq 0} \Lambda(\geq^n)\ t$. This means that there is an element $\geq^K$ such that $s\ \Lambda(\geq^K)\ t$. Again due to the directenedness of $\{\geq^n \mid n \geq 0\}$, we conclude that there is an element $\geq^p$ such that $\geq^p\ \sqsupseteq\ \geq^k, \geq^K$ and so $s \geq^p t_j$ and $\neg(t_j \geq^p s)$, for all $1 \leq j \leq m$, and $s\ \Lambda(\geq^p)\ t$; consequently $s \geq^p t$ and so $s \bigcup_{n\geq 0} \geq^n t$.

2. case 2 tells us that $s = f(s_1, \dots, s_k)$, for some $f \in \mathcal{F}$, having arity $k \geq 1$, and $s_i\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ t$, for some $1 \leq i \leq k$. Then by induction hypothesis we also have that $s_i \bigcup_{n\geq 0} \geq^n t$. This means that $s_i \geq^q t$ for some $\geq^q \in \{\geq^n \mid n \geq 0\}$. From the definition of $\geq^q$ we conclude that $s \geq^q t$ and consequently that $s \bigcup_{n\geq 0} \geq^n t$.

Suppose additionally that $\neg(t\ \mathcal{H}(\bigcup_{n\geq 0} \geq^n)\ s$. If we would have $t \bigcup_{n\geq 0} \geq^n s$, by the fact that

$\bigcup\limits_{n\geq 0} \geq^n \ \sqsubseteq \ \mathcal{H}(\bigcup\limits_{n\geq 0} \geq^n)$, we would get a contradiction.

We have seen that $\mathcal{H}(\bigcup\limits_{n\geq 0} \geq^n) \ \subseteq \ \bigcup\limits_{n\geq 0} \geq^n$ and that $\texttt{ord}(\mathcal{H}(\bigcup\limits_{n\geq 0} \geq^n)) \ \subseteq \ \texttt{ord}(\bigcup\limits_{n\geq 0} \geq^n)$; as a consequence also $\texttt{eq}(\mathcal{H}(\bigcup\limits_{n\geq 0} \geq^n)) \ \subseteq \ \texttt{eq}(\bigcup\limits_{n\geq 0} \geq^n)$, so $\bigcup\limits_{n\geq 0} \geq^n$ is a fixed point of $\mathcal{H}$ and by theorem 4.6, it is the least fixed point. □

Now we can prove that the relation $\geqq$ is closed under substitutions. Apart from continuity of $\Lambda$, we also have to require that $\Lambda$ satisfies the following substitution condition:

> *If $\theta$ is a quasi-order with strict and equivalent parts closed under substitutions then $\Lambda(\theta)$ is closed under substitutions, i. e., both* $\texttt{ord}(\Lambda(\theta))$ *and* $\texttt{eq}(\Lambda(\theta))$ *are closed under substitutions.*

**Lemma 4.31.** *Suppose that $\succ$ and $\sim$ are closed under substitutions and that $\Lambda$ satisfies the substitution condition. Suppose also that $\geqq = \bigcup_{n\geq 0} \geq^n$. Then $\geqq$ is closed under substitutions, i. e.,* $\texttt{ord}(\geqq)$ *and* $\texttt{eq}(\geqq)$ *are closed under substitutions.*

**Proof** (Sketch) The proof is an induction on the lexicographic product $(n, |s| + |t|)$, i. e., we see that $\geq^0$ satisfies the property and then as induction hypothesis we have $>^i$ and $\sim^i$ are closed under substitutions, for any $i \leq k$, and $k \geq 0$ fixed, and if $u, v$ are terms such that $|u| + |v| < |s| + |t|$ then $u \geq^{k+1} v$ implies that $u\sigma \ \geq^{k+1} v\sigma$ and if additionally $\neg(v \geq^{k+1} u)$ then also $\neg(v\sigma \ \geq^{k+1} u\sigma)$, for any substitution $\sigma$. We have a case analysis similar to the ones done in previous results. Note that lemma 4.25 has to be used in case 2. Now if $\geq^n$, for all $n \geq 0$ satisfy the property, so does $\geqq$. □

### On Well-foundedness

Here we discuss well-foundedness of the quasi-order $\geqq$. From lemma 4.28 we know that the order $\geqq$ has the subterm property. Furthermore $\texttt{ord}(\geqq)$ has the subterm property.

We define the following relation $\geqq^{\bar{\Lambda}}$ as follows:

$$s \geqq^{\bar{\Lambda}} t \iff \begin{cases} s \succ t \text{ or} \\ s \sim t \text{ and } s \ \texttt{ord}(\Lambda(\geqq)) \ t \end{cases}$$

where $\succeq$ and $\Lambda$ are the quasi-order and status used in the definition of $\geqq$.

It is not difficult to see that $\geqq^{\bar{\Lambda}}$ is a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Also if $\texttt{ord}(\Lambda(\geqq))$ is a term lifting in the sense of definition 3.16 and $\succeq$ is well-founded, we also have that $\geqq^{\bar{\Lambda}}$ is a term lifting. Furthermore if $s \ \texttt{ord}(\geqq) \ t$ then either $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F}$ having arity $k \geq 1$, and $s_i \geqq t$, for some $1 \leq i \leq k$, (this is given by case 2), or $s \ \geqq^{\bar{\Lambda}} \ t$ (consequence of cases 1a, 1b). Consequently theorem 3.17 gives us the following result.

**Theorem 4.32.** *If $\succeq$ is well-founded and* $\texttt{ord}(\Lambda(\geqq))$ *is a term lifting (in the sense of definition 3.16) then* $\texttt{ord}(\geqq)$ *is well-founded.*

## 4.3 Revisiting rpo and kbo Orders

In chapter 2, subsection 2.4.1, we introduced the notion of path orders. Path orders are related to simplification orders in the sense that most path orders are also simplification orders. Actually those path orders form a restricted class of simplification orders: they are either total or extendable to total monotonic orders on ground terms modulo some congruence. In other words those orders when applied to proving termination of TRS's, do not prove only simply termination of TRS's, they prove total termination.

In practical applications it is very natural to require this totality: for example in Knuth-Bendix completion such a well-founded term ordering is required, and a highly desirable property is that all new critical pairs can be ordered by the ordering. Totality on non-ground terms can not be achieved since commutativity conflicts with well-foundedness; totality on ground terms is the strongest feasible requirement. The totality property is essential for the completeness of the unfailing completion strategies. In the case of ground AC-equational theories finitely presented, the existence of a reduction ordering AC-compatible and total on $\mathcal{T}(\mathcal{F})/=_{AC}$ ensures that such theories always admit a canonical rewrite system. For more information on AC-compatible total orders see for example Narendran and Rusinowitch [81], Rubio and Nieuwenhuis [95].

## 4.4 Making rpo and kbo Total

In [46], Hofbauer proved that for a finite TRS shown terminating by recursive path order with only multiset status, a proof of total termination can be given in the natural numbers with primitively recursive operations. Here we show that even if we take *rpo* or *kbo* in their most general form, these orders actually prove total termination, i. e., if a TRS $R$ is proven terminating by *rpo* (or *kbo*), then $R$ is totally terminating. We will restrict ourselves to fixed arity signatures and make some remarks about the varyadic case.

Recall from chapter 2 that for any quasi-order $\succeq$, $\succeq_{lex}$ and $\succeq_{mul}$ denote its lexicographic and multiset extensions, respectively. Also from chapter 2, section 2.4, recall the definition of status (definition 2.87). To each function $f \in \mathcal{F}$ we associate a status $\tau(f)$. Status indicates how the arguments of the function symbol are to be taken. We consider two possible cases namely the multiset status ($\tau(f) = mul$) and the (family of) lexicographic status ($\tau(f) = lex_\pi$), whose order is given by a permutation $\pi$. Given the set of function symbols $\mathcal{F}$, let $\trianglerighteq$ denote a quasi-precedence over $\mathcal{F}$ (see definition 2.88).

From now on we assume that a quasi-precedence over $\mathcal{F}$ is given as well as a status function $\tau$, under the following restriction: lexicographic and multiset status cannot be mixed, i. e.,

$$\text{if } f \sim g \text{ and } \tau(f) = mul \text{ then } \tau(g) = mul \tag{4.1}$$

Write $>_{rpo}^{=}$ for recursive path order with status as defined in definition 2.89 but with the equality $f = g$ in clause 1b of the definition replaced by $f \sim g$. This definition is not suitable to our purposes. We need to define a total well-founded monotone algebra $(A, >)$ and a good candidate is $(\mathcal{T}(\mathcal{F}), >_{rpo}^{=})$ but if $>_{rpo}^{=}$ is based on a quasi-precedence over $\mathcal{F}$, then $>_{rpo}^{=}$ is not necessarily total even if the quasi-precedence is so. The reason behind this is that the equivalence relation contained in the quasi-precedence gives rise to a kind of equivalence on terms more general than equality and the order $>_{rpo}^{=}$ does not take those equivalent terms into account. For example suppose $c \rhd a \sim b \rhd d$ and $f \rhd a, b, c, d$, where $a, b, c, d$ are constants and $f$ has arity 2. Suppose we associate to $f$ the multiset status, then neither $f(f(a,b),c) >_{rpo}^{=} f(f(b,a),d)$ nor vice-versa and since $c \rhd d$, the terms are not equivalent.

What we have to do is extend $>_{rpo}^{=}$ in order to be able to compare equivalence classes of terms. But $>_{rpo}^{=}$ is not amenable to such an extension: if we define the congruence $\simeq$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $s \simeq t$ iff $s = t$ or $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, $f \sim g$, $m = n$ and either

- $\tau(f) = \tau(g) =$ *mul* and there is a permutation $\pi$ of $\{1, \ldots m\}$ such that $s_i \simeq t_{\pi(i)}$, for any $1 \leq i \leq m$;
- $\tau(f) = lex_{\pi_f}$ and $\tau(g) = lex_{\pi_g}$ and $s_{\pi_f(i)} \simeq t_{\pi_g(i)}$ for all $1 \leq i \leq m$.

Then if for ground terms $s, t$, $s \simeq t$ and $s \neq t$, both $s \not>_{rpo}^{=} t$ and $t \not>_{rpo}^{=} s$. So $>_{rpo}^{=}$ is not always total on $\mathcal{T}(\mathcal{F})$ and due to the nature of the obstacle just mentioned, it seems reasonable to take $A = \mathcal{T}(\mathcal{F})/\simeq$. Unfortunately the natural extension of $>_{rpo}^{=}$ to the congruence classes of $\mathcal{T}(\mathcal{F}, \mathcal{X})/\simeq$ is not well-defined even for total precedences (condition (2.1) does not hold). This can be repaired by extending the definition of $>_{rpo}^{=}$, namely replace equality by $\simeq$. The following definition is just definition 2.89 for the quasi-order case. It can be found in Steinbach [100, 101] and in Dershowitz [23] (only for multiset status). To avoid problems with reflexivity we extend the quasi-precedence $\unrhd$ to the set of variables as follows: $x \sim x$, for any $x \in \mathcal{X}$. Variables remain incomparable, under $\unrhd$, with elements of $\mathcal{F}$.

**Definition 4.33. (rpo with status)** Given two terms $s, t$ we say that $s \geq_{rpo} t$ if $s = f(s_1, \ldots, s_m)$ for some $f \in \mathcal{X} \cup \mathcal{F}$ and $m \geq 0$, and either

1. $t = g(t_1, \ldots, t_n)$, for some $g \in \mathcal{X} \cup \mathcal{F}$ and $n \geq 0$, and for all $1 \leq i \leq n$ both $s \geq_{rpo} t_i$ and $\neg(t_i \geq_{rpo} s)$ and either
   (a) $f \rhd g$, or
   (b) $f \sim g$ and $((m = k = 0)$ or $s_1 \cdots s_m \;\geq_{rpo,\tau} t_1 \cdots t_n)$; or
2. $\exists 1 \leq i \leq m : \; s_i \geq_{rpo} t$.

From the results presented in section 4.2.2, we can state that $\geq_{rpo}$ has the following properties (note that the multiset and the lexicographic status have the properties required for the lifting/status in section 4.2.2):

- $\geq_{rpo}$ is a quasi-order; we denote its strict part by $>_{rpo}$ and its equivalence part by $\sim_{rpo}$,
- $>_{rpo}$ has the subterm property,
- $\geq_{rpo}$ and in particular $>_{rpo}$ and $\sim_{rpo}$, are closed under substitutions and contexts.

Note that if all function symbols have *lex* status, then $\geq_{rpo}$ coincides with Kamin and Lévy's [55] *lexicographic path order*, $\geq_{lpo}$. If $\rhd$ is total in $\mathcal{F}$ and $\sim$ is equality then, as a consequence of lemma 4.36 (to be presented later), we have that $>_{lpo}$ is total over $\mathcal{T}(\mathcal{F})$.

The following lemma is not difficult to prove by induction on the sum of the size of the terms, using the properties of the lexicographic and multiset extensions of quasi-orders and lemma 4.25.

**Lemma 4.34.** *Let $\geq_{rpo}$ be defined as in 4.33. Then*

- *$\sim_{rpo}$ and $\simeq$ coincide,*
- *$s >_{rpo} t$ if and only if $s = f(s_1, \ldots, s_k)$, with $f \in \mathcal{F}$ admitting arity $k \geq 0$, and either*
  1. *$t = g(t_1, \ldots, t_m)$, with $g \in \mathcal{F}$ admitting arity $m \geq 0$, and either*
     (a) *$f \rhd g$, or*
     (b) *$f \sim g$ and $s_1 \cdots s_m$ $\mathtt{ord}(\geq_{rpo,\tau})$ $t_1 \cdots, t_n$; or*
  2. *$\exists 1 \leq i \leq k :\ s_i \geq_{rpo} t$.*

The following lemma is not difficult to prove by induction on $|s| + |t|$, using the characterization of both $\sim_{rpo}$ and $>_{rpo}$ given in lemma 4.34, and using the monotonicity of both lexicographic and multiset extensions.

**Lemma 4.35.** *The relation $\geq_{rpo}$ is monotone with respect to quasi-precedences, i. e., if $\trianglerighteq, \trianglerighteq'$ are quasi-precedences such that $\trianglerighteq'$ extends $\trianglerighteq$, i. e., $\rhd \subseteq \rhd'$ and $\sim \subseteq \sim'$, then $\geq_{rpo}$ associated with $\trianglerighteq'$ extends $\geq_{rpo}$ associated with $\trianglerighteq$, i. e., $\sim_{rpo} \subseteq \sim'_{rpo}$ and $>_{rpo} \subseteq >'_{rpo}$, where $\geq_{rpo} = >_{rpo} \cup \sim_{rpo}$ and $\geq'_{rpo} = \sim'_{rpo} \cup >'_{rpo}$. (Consequently $\geq_{rpo}$ extends $>^{=}_{rpo}$, for any fixed quasi-precedence and status.)*

Note that we require that both $\trianglerighteq$ and $\trianglerighteq'$ are defined over $\mathcal{F} \cup \mathcal{X}$, coinciding with the identity in $\mathcal{X}$ and maintaining incomparable variables with elements of $\mathcal{F}$.

The following result is crucial for the rest of the section; though it is a well-known result we will present its proof.

**Lemma 4.36.** *Let $\trianglerighteq$ be quasi-precedence total on $\mathcal{F}$. Then the extension of $>_{rpo}$ to the equivalence classes of $\mathcal{T}(\mathcal{F})/{\sim_{rpo}}$ is a total order.*

**Proof** First note that since $>_{rpo}$ and $\sim_{rpo}$ are compatible, the extension of $>_{rpo}$ to the equivalence classes of $\mathcal{T}(\mathcal{F}, \mathcal{X})/{\sim_{rpo}}$ is well-defined and is a partial order. In order to keep the notation simple we will denote this extension also by $>_{rpo}$; it should be clear from context whether we mean the order on terms or on equivalence classes.

Consider the set of ground terms $\mathcal{T}(\mathcal{F})$. If $\mathcal{T}(\mathcal{F}) = \emptyset$, then the result is trivially satisfied. Suppose then that $\mathcal{T}(\mathcal{F}) \neq \emptyset$. It can be seen by induction on the sum of the size of the terms that if $u \sim_{rpo} v$ then $|u| = |v|$ and this allows us to prove the lemma by induction on $|s| + |u|$, i. e., we see that for any ground terms $s, t$, we have either $\langle s \rangle = \langle t \rangle$, or

$\langle s\rangle >_{rpo} \langle t\rangle$ or $\langle t\rangle >_{rpo} \langle s\rangle$, by induction on $|s|+|t|$. Since equivalent terms have the same size, this is possible.

Suppose then that $s, t \in \mathcal{T}(\mathcal{F})$ are two minimal terms for which the result has not yet been verified. Suppose we have $s = f(s_1, \ldots, s_k)$ and $t = g(t_1, \ldots, t_m)$, for some function symbols $f, g \in \mathcal{F}$, admitting arities respectively $k, m \geq 0$, and ground terms $s_1, \ldots, s_k, t_1, \ldots, t_m \in \mathcal{T}(\mathcal{F})$. By induction hypothesis, for each $s_i$, with $1 \leq i \leq k$ fixed, we have either:

- $\langle s_i\rangle = \langle t\rangle$; this means that $s_i \sim_{rpo} t$ and so $s >_{rpo} t$. Consequently $\langle s\rangle >_{rpo} \langle t\rangle$.
- $\langle s_i\rangle >_{rpo} \langle t\rangle$; in this case we have $s >_{rpo} s_i >_{rpo} t$ and consequently $s >_{rpo} t$, giving $\langle s\rangle >_{rpo} \langle t\rangle$.
- $\langle t\rangle >_{rpo} \langle s_i\rangle$.

So if there is a term $s_i$ satisfying one of the first two cases above, we are done. Suppose then that no such $s_i$ exists, i. e., suppose that $\langle t\rangle >_{rpo} \langle s_i\rangle$, for all $1 \leq i \leq k$.

Applying the same reasoning as above to the terms $s$ and $t_j$, for $1 \leq j \leq m$, the only case that remains to be analysed is $\langle s\rangle >_{rpo} \langle t_j\rangle$, for all $1 \leq j \leq m$. So we suppose we have both $\langle t\rangle >_{rpo} \langle s_i\rangle$, for all $1 \leq i \leq k$ and $\langle s\rangle >_{rpo} \langle t_j\rangle$, for all $1 \leq j \leq m$. Since $\trianglerighteq$ is total we have either

- $f \triangleright g$; in this case we conclude that $\langle s\rangle >_{rpo} \langle t\rangle$;
- $g \triangleright f$; in this case we conclude that $\langle t\rangle >_{rpo} \langle s\rangle$;
- $f \sim g$.

Suppose also that $f \sim g$. If the sequences $\langle s_{\pi_f(1)}\rangle \cdots \langle s_{\pi_f(k)}\rangle$ and $\langle t_{\pi_g(1)}\rangle \cdots \langle t_{\pi_g(m)}\rangle$ are equal, then we conclude that $\langle s\rangle = \langle t\rangle$, whether the status of both $f$ and $g$ is lexicographic or multiset. If the sequences are different and the status of both $f$ and $g$ is lexicographic, using the definition of lexicographic extension and the fact that each element $\langle s_i\rangle$ is comparable with each element $\langle t_j\rangle$, it is not difficult to see that either

$$\langle s_{\pi_f(1)}\rangle \cdots \langle s_{\pi_f(k)}\rangle \;>_{rpo,lex}\; \langle t_{\pi_g(1)}\rangle \cdots \langle t_{\pi_g(m)}\rangle$$

or vice-versa. In both cases we are able to establish a relation between $\langle s\rangle$ and $\langle t\rangle$.

Suppose the sequences are different and the status of both $f$ and $g$ is the multiset status. Note that we can write

$$S = (T \cap S) \cup S_1 \text{ with } S_1 = S \setminus T$$
$$T = (T \cap S) \cup T_1 \text{ with } T_1 = T \setminus S$$

where $S$ is the multiset containing the elements $\langle s_1\rangle, \ldots, \langle s_k\rangle$, and $T$ is the multiset containing the elements $\langle t_1\rangle, \ldots, \langle t_m\rangle$. Then, by induction hypothesis, and since both $S_1$ and $T_1$ cannot be simultaneously empty, either

- $\forall\langle u\rangle \in T_1\ \exists\langle v\rangle \in S_1 :\ \langle v\rangle >_{rpo} \langle u\rangle$, or
- $\exists\langle u\rangle \in T_1\ \forall\langle v\rangle \in S_1 :\ \langle u\rangle >_{rpo} \langle v\rangle$ or $\langle v\rangle = \langle u\rangle$.

In the first case we conclude that $S >_{rpo,mul} T$ and so that $\langle s\rangle >_{rpo} \langle t\rangle$. In the second case, we see that if $\langle v\rangle = \langle u\rangle$ then $\langle v\rangle \in T \cap S$; indeed $\langle v\rangle \in T \setminus S$ implies that the number of occurrences of $\langle v\rangle$ in $T$ is strictly greater than the number of occurrences of $\langle v\rangle$ in $S$, but also $\langle v\rangle \in S \setminus T$ implies that the number of occurrences of $\langle v\rangle$ in $S$ is strictly greater than the number of occurrences of $\langle v\rangle$ in $T$ and these facts contradict themselves, so we must have $\langle v\rangle \in S \cap T$ and the second case above can be rewritten as

$$\exists\langle u\rangle \in T_1\ \forall\langle v\rangle \in S_1 :\ \langle u\rangle >_{rpo} \langle v\rangle$$

and consequently $\langle t\rangle >_{rpo} \langle s\rangle$, concluding the proof. □

Note that if all function symbols have *lex* status then $>_{rpo}$ coincides with Kamin and Lévy's [55] *lexicographic path order*, $>_{lpo}$. If $\rhd$ is total and $\sim$ is syntactical equality then we have that $>_{lpo}$ is total over $\mathcal{T}(\mathcal{F})$.

**Example 4.37.** Let $\mathcal{F}$ consist of two constants $a \rhd b$ and function symbols $f_i$, $i \geq 1$, such that $f_i$ has arity $i$, $\tau(f_i) = lex_{Id}$ and $f_i \sim f_j$, for any $i, j$. Then we have the following infinite descending chain

$$f_1(a) >_{rpo} f_2(b,a) >_{rpo} f_3(b,b,a) >_{rpo} f_4(b,b,b,a) >_{rpo} \cdots$$

In order for $>_{rpo}$ to be useful for proving termination of term rewriting systems, the order has to be well-founded. Unfortunately, well-foundedness of $\unrhd$ alone is not sufficient to guarantee well-foundedness of $>_{rpo}$ as the above example showed. The problem stems from the fact that lexicographic sequences of unbounded size are not well-founded.[1] Kamin and Lévy [55] proved that $>_{lpo}$ is well-founded provided that equivalent function symbols have the same arity. In the following we prove that this restriction can be weakened. It is enough to require that for every equivalence class of function symbols with lexicographic status, there is a natural number bounding the arities of the function symbols in the class. That is

$$\forall f \in \mathcal{F} :\ \tau(f) = lex_\pi \Rightarrow (\exists n \geq 0 :\ \forall g \in \langle f\rangle :\ \mathtt{arity}(g) \leq n) \tag{4.2}$$

A traditional way of proving well-foundedness of $>_{rpo}$ is via Kruskal's theorem. Recall the definitions of well-quasi-order (*wqo*) and embedding relation $\succeq_{emb}$ from chapter 2. Kruskal's theorem (2.78) states that if $\unrhd$ is a *wqo* on $\mathcal{F}$ then $\geq_{emb}$ is also a *wqo* on $\mathcal{T}(\mathcal{F})$. Consequently any relation containing the embedding relation is well-founded. Previous versions of $>_{rpo}$ fall within this category. For definition 4.33 this does no longer hold: in example 4.37 we have $f_2(b,a) >_{emb} f_1(a)$, and $f_2(b,a) \not>_{rpo} f_1(a)$.

[1] Note that even if $\unrhd$ would be total or $\mathcal{F}$ finite, with a function symbol $f$ allowing different arities, the same problem would arise.

A way of dealing with orders for which Kruskal's theorem is not applicable was given in chapter 3. Well-foundedness of $>_{rpo}$ can be derived from results presented there (see also section 4.2.2). Nevertheless here we present a proof of well-foundedness of $>_{rpo}$ inspired by the proof of Kruskal's theorem itself as presented in Gallier [38], Nash-Williams [82], and closely following the results of chapter 3. The proof given does not rely on Kruskal's theorem and is therefore simpler if we consider the degree of difficulty involved in Kruskal's theorem itself.

Again we admit that $\trianglerighteq$ is a quasi-precedence defined on $\mathcal{F} \cup \mathcal{X}$ such that $\trianglerighteq$ restricted to $\mathcal{X}$ is equality and elements of $\mathcal{X}$ and $\mathcal{F}$ are incomparable under $\trianglerighteq$.

**Theorem 4.38.** *Let $\trianglerighteq$ be a quasi-precedence over $\mathcal{F} \cup \mathcal{X}$ and $\tau$ a status function such that conditions (4.1) and (4.2) are satisfied. Then $>_{rpo}$ is well-founded over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ iff $\trianglerighteq$ is well-founded over $\mathcal{F} \cup \mathcal{X}$.*

**Proof** For the if part, let $\trianglerighteq$ be a well-founded quasi-precedence over $\mathcal{F} \cup \mathcal{X}$ and $\tau$ a status function such that conditions (4.1) and (4.2) are satisfied. We first extend $\trianglerighteq$ to a total well-founded quasi-order $\trianglerighteq'$ on $\mathcal{F}$ such that $\sim' = \sim$. This is done in the "usual" way (see Appendix A). First we consider the extension $\triangleright$ to the equivalence classes, i. e., we take the extension of $\triangleright$ to $\mathcal{F}/\!\sim$. This relation is then extended, using Zorn's Lemma, to a total well-founded partial order $>'$ over $\mathcal{F}/\!\sim$ (see Appendix A for the details on how this can be done). Then $\triangleright'$ and $\sim$ are compatible and $\trianglerighteq'$ (with $\sim' = \sim$), is total and well-founded over $\mathcal{F}$, where as expected $\triangleright'$ is defined as $\forall f, g \in \mathcal{F} : f \triangleright' g \iff \langle f \rangle >' \langle g \rangle$. The reasons why we require that $\sim' = \sim$ are twofold: to ensure compatibility of $\triangleright'$ and $\sim'$ and to avoid problems with the status of equivalent symbols, i. e., to guarantee that conditions (4.1) and (4.2) still hold for the extended quasi-precedence.

The relation $\trianglerighteq'$ is total and well-founded on $\mathcal{F}$, hence $\trianglerighteq'$ is a *wqo* over $\mathcal{F}$. Suppose now that $>_{rpo}$ taken over $\trianglerighteq'$ is not well-founded. Take then an infinite descending chain

$$t_0 >_{rpo} t_1 >_{rpo} t_2 >_{rpo} \cdots$$

minimal in the following sense:

- $|t_0| \leq |s|$, for all non-well-founded terms $s$.
- $|t_{i+1}| \leq |s|$, for all non-well-founded terms $s$ such that $t_i >_{rpo} s$.

where $|t|$ represents the number of function symbols occurring in $t$.

We remark that no proper subterm of a term $t_i$, $i \geq 0$ in the above chain can be non-well-founded; for, suppose $u^i_j$ is such a non-well-founded subterm, then the chain

$$t_0 >_{rpo} \cdots >_{rpo} t_{i-1} >_{rpo} u^i_j >_{rpo} u_1 >_{rpo} \cdots$$

will be an infinite descending chain contradicting the minimality of $(t_i)_{i \geq 0}$ (since $|u^i_j| < |t_i|$). Note also that for all $i \geq 0$, $t_i$ cannot be a variable, since by the restrictions imposed on $\trianglerighteq$ and $\trianglerighteq'$, variables are never greater than any other term, so if some $t_i$ would be a variable, the sequence would be finite.

Let $\texttt{root}(t)$ be the head function symbol of the term $t$. We see that there is no infinite subsequence $(t_{\phi(i)})_{i\geq 0}$ of $(t_i)_{i\geq 0}$ such that $\texttt{root}(t_{\phi(i)}) \sim \texttt{root}(t_{\phi(j)})$, for all $i, j \geq 0$. Suppose it is not so and let $(t_{\phi(i)})_{i\geq 0}$ be such a subsequence. Due to condition (4.1), all root symbols in this sequence have the same status (either *mul* or *lex*). By definition of $>_{rpo}$, and since $t_{\phi(i)} >_{rpo} t_{\phi(i+1)}$, for all $i \geq 0$, we must have

$$\textit{args}(t_{\phi(0)}) >_{rpos,\tau} \textit{args}(t_{\phi(1)}) >_{rpos,\tau} \cdots$$

where $\textit{args}(t)$ are the proper subterms of $t$. From lemma 2.38 or 2.39, we conclude that $>_{rpo}$ is not well-founded over $\bigcup_{i\geq 0} \textit{Args}(t_{\phi(i)})$ (where $\textit{Args}(t)$ is the set of proper subterms of $t$), contradicting the minimality of $(t_i)_{i\geq 0}$.

Consider the sequence $(\texttt{root}(t_i))_{i\geq 0}$. This sequence is infinite and since $\trianglerighteq'$ is a *wqo* over $\mathcal{F}$, by lemma 2.42 this sequence contains an infinite subsequence $(\texttt{root}(t_{\phi(i)}))_{i\geq 0}$ such that $\texttt{root}(t_{\phi(i+1)}) \trianglerighteq' \texttt{root}(t_{\phi(i)})$, for all $i \geq 0$. But since every $\sim$-equivalence class appears only finitely many times in the sequence $(\texttt{root}(t_i))_{i\geq 0}$, we can say without loss of generality that the subsequence $(\texttt{root}(t_{\phi(i)}))_{i\geq 0}$ fulfils $\texttt{root}(t_{\phi(i+1)}) \triangleright' \texttt{root}(t_{\phi(i)})$, for all $i \geq 0$ (strictly speaking there is a subsequence of $(\texttt{root}(t_{\phi(i)}))_{i\geq 0}$ with this property). But if $t_{\phi(i)} >_{rpo} t_{\phi(i+1)}$ (for all $i \geq 0$), then, by definition of $>_{rpo}$, both $t_{\phi(i)}$ and $t_{\phi(i+1)}$ are not constants and we must have $u_{\phi(i)} >_{rpo} t_{\phi(i+1)}$ or $u_{\phi(i)} \sim_{rpo} t_{\phi(i+1)}$, for some $u_{\phi(i)} \in Args(t_{\phi(i)})$. In both cases a contradiction with the minimality of $(t_i)_{i\geq 0}$ arises.

Well-foundedness of $>_{rpo}$ over the original quasi-precedence $\trianglerighteq$ follows from the fact that $>_{rpo}$ is monotone with respect to precedences (since $\trianglerighteq'$ is an extension of $\trianglerighteq$).

For the only-if part, suppose that $>_{rpo}$ is well-founded over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and that $\trianglerighteq$ is not well-founded on $\mathcal{F} \cup \mathcal{X}$. Let $f_0 \triangleright f_1 \triangleright \cdots$ be an infinite descending sequence in $\mathcal{F}$. This sequence does not contain an infinite subsequence consisting only of constants, since if $(f_{\phi(i)})_{i\geq 0}$ would be such a sequence, we would have $f_{\phi(0)} >_{rpo} f_{\phi(1)} >_{rpo} \cdots$, contradicting well-foundedness of $>_{rpo}$; it also does not contain any variable since for any $p \in \mathcal{X} \cup \mathcal{F}$ and any variable $x$, $x \not\triangleright p$. Let then $(f_{\phi(i)})_{i\geq 0}$ be an infinite subsequence of $(f_i)_{i\geq 0}$ such that $f_{\phi(i)}$ admits arity $\geq$1, for all $i \geq 0$. Let $x$ be any variable. By definition of $>_{rpo}$, we conclude that

$$f_{\phi(0)}(x, \ldots, x) >_{rpo} f_{\phi(1)}(x, \ldots, x) >_{rpo} \cdots$$

contradicting well-foundedness of $>_{rpo}$. $\square$

Another approach to prove well-foundedness of our version of $>_{rpo}$ is the following. Every function symbol with status *lex* has its arity augmented to the maximal arity associated with its equivalence class. By this construction all function symbols in the same equivalence class are forced to have the same arity, hence the old version of $>_{rpo}$ is applicable, provided we change the status function consistently. Well-foundedness of our version of $>_{rpo}$ then follows from well-foundedness of previous $>_{rpo}$ versions. This classical proof of well-foundedness does make use of Kruskal's theorem. To conclude that $>_{rpo}$ is well-founded over the original signature we just have to note that an infinite descending sequence of terms over the old signature translates to an infinite descending sequence of terms over the new signature where new arguments introduced are filled with a dummy constant.

**Example 4.39.** The following TRS's

$$
\begin{array}{llll}
R_1: & f(1,x) & \to & g(0,x,x) \\
 & g(x,1,y) & \to & f(x,0)
\end{array}
\qquad
\begin{array}{llll}
R_2: & a & \to & g(c) \\
 & g(a) & \to & b \\
 & f(g(x),b) & \to & f(a,x)
\end{array}
$$

can be proven terminating with $>_{rpo}$. Just take quasi-precedences $\trianglerighteq$ and status function $\tau$ satisfying $1 \triangleright 0$, $f \sim g$, $\tau(f) = \tau(g) = \mathit{lex}_{Id}$, for $R_1$, and $a \triangleright g$, $a \triangleright c$, $a \sim b$ and $\tau(f) = \mathit{mul}$, for $R_2$. Earlier versions or $>_{rpo}$ fail to prove termination of these TRS's: for $R_1$ we cannot choose $f \triangleright g$ nor $g \triangleright f$ nor incomparability of $f$ and $g$, and if $f \sim g$, the status of these symbols cannot be the multiset status, and for $R_2$ we wouldn't be able to orient the second and third rules using $>_{rpo}^{=}$.

We now prove that $>_{rpo}$ as defined in definition 4.33 does indeed prove total termination. Again we extend the precedence $\trianglerighteq$ to $\mathcal{X}$ with $\trianglerighteq$ restricted to $\mathcal{X}$ being the equality, and maintaining symbols of $\mathcal{F}$ and $\mathcal{X}$ incomparable.

**Theorem 4.40.** *Given a TRS $R$, suppose $\trianglerighteq$ is a quasi-precedence well-founded over $\mathcal{F}$ and $\tau$ is a status function such that conditions (4.1) and (4.2) are satisfied. If $l >_{rpo} r$ for every rule $l \to r \in R$ then $R$ is totally terminating.*

**Proof** Suppose that for a TRS $R$, a termination proof using $>_{rpo}$ exists, i. e., we can define a well-founded quasi-precedence $\trianglerighteq$ over $\mathcal{F} \cup \mathcal{X}$ such that $\trianglerighteq$ restricted to $\mathcal{X}$ is equality, and $\trianglerighteq$ maintains symbols of $\mathcal{F}$ and $\mathcal{X}$ incomparable, and we can define a status function satisfying conditions (4.1) and (4.2) and such that $l >_{rpo} r$, for every rule $l \to r \in R$. In order to establish total termination of $R$ we need to define a total well-founded monotone algebra. For that we choose $\mathcal{T}(\mathcal{F})/{\sim_{rpo}}$, where $\sim_{rpo}$ is the congruence associated with $>_{rpo}$. If $\mathcal{F}$ does not contain any constant, we introduce one to force $\mathcal{T}(\mathcal{F})$ to be non-empty. With respect to the quasi-precedence $\trianglerighteq$, the relative order of this new element with respect to elements of $\mathcal{F}$ is irrelevant. A possibility is to consider this new element incomparable to all other function symbols, then $\trianglerighteq$ remains unchanged and the same holds for $>_{rpo}$. As in the proof of theorem 4.38, we extend $\trianglerighteq$ to a total well-founded quasi-precedence $\trianglerighteq^t$ on $\mathcal{F}$ such that the equivalence part remains the same (this is done using Zorn's lemma as described in Appendix A), and consider $>_{rpo}$ over this extended quasi-precedence. By theorem 4.38, we know that $>_{rpo}$ is well-founded, and combining this with lemma 4.36, we conclude that $>_{rpo}$ extended to $\mathcal{T}(\mathcal{F})/{\sim_{rpo}}$ is total and well-founded. In $\mathcal{A} = (\mathcal{T}(\mathcal{F})/{\sim_{rpo}}, >_{rpo})$ we interpret the function symbols of $\mathcal{F}$ by

$$f_A(\langle s_1 \rangle, \ldots, \langle s_n \rangle) = \langle f(s_1, \ldots, s_n) \rangle$$

where $n = \texttt{arity}(f)$, and $\langle\ \rangle$ denotes the $\sim_{rpo}$-equivalence classes. If we take $\langle s_i \rangle = \langle s'_i \rangle$, for some $1 \le i \le \texttt{arity}(f)$ then $s_i \sim_{rpo} s'_i$ and since $\sim_{rpo}$ is a congruence we also have

$$f(s_1, \ldots, s_i, \ldots, s_n) \sim_{rpo} f(s_1, \ldots, s'_i, \ldots, s_n),$$

i. e., $\langle f(s_1, \dots, s_i, \dots, s_n)\rangle = \langle f(s_1, \dots, s_i', \dots, s_n)\rangle$, so the function $f_A$ does not depend on the class representative and thus is well-defined. The interpretation function $[\![\ ]\!] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{A}^{\mathcal{X}} \to \mathcal{A}$ is given as usual (see definition 2.92).

Since $\mathcal{A}$ is total and well-founded, the only condition we need to check to establish total termination is compatibility with the rules of $R$. For that we need the following fact:

$$\forall t \in \mathcal{T}(\mathcal{F}, \mathcal{X})\ \forall \tau \in \mathcal{A}^{\mathcal{X}} :\ [\![t, \tau]\!] = \langle t\sigma\rangle$$

where $\sigma$ is any ground substitution satisfying $\sigma(x) \in \tau(x)$, for all $x \in \mathcal{X}$. First we see that the expression above makes sense, i. e., if $\sigma_1, \sigma_2$ are two different substitutions satisfying $\sigma_i(x) \in \tau(x)$, for $i = 1, 2$ and all $x \in \mathcal{X}$, then $\langle t\sigma_1\rangle = \langle t\sigma_2\rangle$. We proceed by induction on the structure of $t$. If $t$ is a variable then $\langle t\sigma_1\rangle = \langle \sigma_1(x)\rangle = \tau(x) = \langle \sigma_2(x)\rangle = \langle t\sigma_2\rangle$. Suppose now that $t = f(t_1, \dots, t_m)$, then

$$\langle t\sigma_1\rangle = \langle f(t_1, \dots, t_m)\sigma_1\rangle = \langle f(t_1\sigma_1, \dots, t_m\sigma_1)\rangle$$

By induction hypothesis we know that $\langle t_i\sigma_1\rangle = \langle t_i\sigma_2\rangle$, i. e., $t_i\sigma_1 \sim_{rpo} t_i\sigma_2$ for all $1 \leq i \leq m$. Since $\sim_{rpo}$ is a congruence, also $f(t_1\sigma_1, \dots, t_m\sigma_1) \sim_{rpo} f(t_1\sigma_2, \dots, t_m\sigma_2)$ so

$$\langle f(t_1, \dots, t_m)\sigma_1\rangle = \langle f(t_1\sigma_1, \dots, t_m\sigma_1) = \langle f(t_1\sigma_2, \dots, t_m\sigma_2) = \langle t\sigma_2\rangle$$

So indeed the class $\langle t\sigma\rangle$ does not depend on the choice of the representative for $\sigma(x)$ from $\tau(x)$.

We now prove the stated fact also by induction on the structure of $t$. If $t = x \in \mathcal{X}$ then, by definition of interpretation, $[\![t, \tau]\!] = \tau(x) = \langle \sigma(x)\rangle$, for any ground substitution $\sigma$ satisfying $\sigma(x) \in \tau(x)$. Suppose that $t = f(t_1, \dots, t_m)$. Then

$$\begin{array}{lll} [\![t, \tau]\!] & = & \\ [\![f(t_1, \dots, t_m), \tau]\!] & = & \text{(by definition of interpretation)} \\ f_A([\![t_1, \tau]\!], \dots, [\![t_m, \tau]\!]) & = & \text{(by induction hypothesis)} \\ f_A(\langle t_1\sigma_1\rangle, \dots, \langle t_m\sigma_m\rangle) & & \end{array}$$

where each ground substitution $\sigma_i$, $1 \leq i \leq m$, satisfies $\sigma_i(x) \in \tau(x)$. Note that we can have a variable $x$ occurring in $t_i$ and $t_j$, with $i \neq j$, and such that $\sigma_i(x) \neq \sigma_j(x)$, but from what we established before, the class $\langle s\alpha\rangle$ does not depend on the representative chosen from $\tau(x)$ for $\alpha(x)$, so if we fix some element in $\tau(x)$ and define a new ground substitution $\sigma$ such that $\sigma(x)$ equals that fixed element in $\tau(x)$, we have that, for all $x \in \mathcal{X}$ and all $1 \leq i \leq m$, $\langle t_i\sigma_i\rangle = \langle t_i\sigma\rangle$. Consequently we can write

$$\begin{array}{ll} [\![t, \tau]\!] & = \\ f_A(\langle t_1\sigma_1\rangle, \dots, \langle t_m\sigma_m\rangle) & = \\ f_A(\langle t_1\sigma\rangle, \dots, \langle t_m\sigma\rangle) & = \\ \langle f(t_1\sigma, \dots, t_m\sigma)\rangle & \end{array}$$

as we wanted.

Let $l \to r$ be a rule in $R$ and let $\tau : \mathcal{X} \to \mathcal{A}$ be an assignment. Let $\sigma$ be a ground substitution satisfying $\sigma(x) \in \tau(x)$ for all $x \in \mathcal{X}$. Since $>_{rpo}$ is monotone with respect to quasi-precedences and by hypothesis $l >_{rpo} r$, with $>_{rpo}$ taken over $\trianglerighteq$, we also have $l >_{rpo} r$, where now the order $>_{rpo}$ is based on the total quasi-precedence $\trianglerighteq^t$. Consequently $\langle l\sigma \rangle >_{rpo} \langle r\sigma \rangle$, thus $[\![l, \tau]\!] >_{rpo} [\![r, \tau]\!]$, and we conclude that $R$ is totally terminating, with $\mathcal{T}(\mathcal{F})/\sim_{rpo}$ as total well-founded monotone algebra. □

We make some remarks about varyadic signatures. First note that lexicographic status is not defined for function symbols with varyadic arity. We can extend the definition and consider *rpo* with status over such signatures (basicly the permutation chosen has to make sense for all possible arities of a function symbol). We need to impose condition 4.1. Furthermore, in order for *rpo* to be well-founded, we have to impose a new restriction (apart from condition 4.2), namely that a varyadic function symbol with lexicographic status has to have its arities bounded by a natural number (this is similar to restriction 4.2). Now we can easily code the varyadic signature in a fixed-arity one by labelling function symbols with its arity. The new precedence on this extended set of function symbols has to respect the old precedence, i. e., fixed-arity function symbols arising from the labelling of a particular varyadic function symbol, are made equivalent in this new precedence, and if $f$ was greater than (respectively equivalent to) symbol $g$, then all labelled versions of $f$ are greater (respectively equivalent) to the labelled versions of $g$. Note that conditions 4.1 and 4.2 remain valid. With some work it can be seen that *rpo* remains well-founded in this new setting, and that termination of such a varyadic system implies termination of the corresponding labelled fixed-arity system. In other words theorem 4.40 remains valid.

We turn now to the Knuth-Bendix order. Recall from chapter 2 the definition of *weight function*, $\phi : \mathcal{F} \cup \mathcal{X} \to \mathbb{N}$ satisfying

$$\phi(f) \text{ is } \begin{cases} = \phi_0 & \text{if } f \in \mathcal{X} \\ \geq \phi_0 & \text{if } \mathtt{arity}(f) = 0 \\ > 0 & \text{if } \mathtt{arity}(f) = 1 \text{ and } \exists g \in \mathcal{F} : \ f \not\trianglerighteq g \end{cases}$$

where $\triangleright$ is a quasi-precedence in $\mathcal{F}$ and $\phi_0 \in \mathbb{N}$ is a fixed natural greater than zero. Note that the last condition now means that if $f \in \mathcal{F}$, has arity $1$ and weight $0$, then it must satisfy $f \trianglerighteq g$ for all function symbols $g \in \mathcal{F}$.

We extend $\phi$ to terms as follows: $\phi(f(s_1, \ldots, s_m)) = \phi(f) + \sum_{i=1}^{m} \phi(s_i)$.

The following definition of Knuth-Bendix order with status is just definition 2.90 adapted for the case the precedence on function symbols is a quasi-order.

**Definition 4.41. (kbo with status)** We say that $s >_{kbo} t$ iff $\forall x \in \mathcal{X} : \#_x(s) \geq \#_x(t)$ and

1. $\phi(s) > \phi(t)$ or
2. $\phi(s) = \phi(t)$, and

(a) $t \in \mathcal{X}$ and $\exists k > 0 : \; s = f_0^k(t)$, where $f_0$ is the element of $\mathcal{F}$ having weight $0$, and being maximal in the precedence,

(b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ and
  - $f \rhd g$, or
  - $f \sim g$ and $s_1, \ldots, s_m >_{kbos,\tau} t_1, \ldots, t_n$

where $\#_x(t)$ denotes the number of occurrences of variable $x$ in term $t$.

Again this version of Knuth-Bendix order has properties similar to $>_{rpo}$ namely it is a partial order closed under substitutions and contexts, monotone with respect to quasi-precedences and having the subterm property.

We show some of these properties. First we introduce a lemma which was proven in Dick, Kalmus and Martin [29].

**Lemma 4.42.** *Let the relation $>_{kbo}$ be defined as in 4.41. We have:*

1. *Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If $\phi(s) = \phi(t)$ and $t$ is a proper subterm of $s$ then $s = f_0^k(t)$, for some $k \geq 1$, where $f_0 \in \mathcal{F}$ is a function symbol satisfying* `arity`$(f_0) = 1$ *and* $\phi(f_0) = 0$.

2. *Let $x \in \mathcal{X}$ then $C[x] >_{kbo} x$ and $x \not>_{kbo} C[x]$, for any non-trivial linear context $C$.*

Using this lemma and induction on the size of the terms (i. e., the number of symbol occurrences on the terms) we can prove the following.

**Lemma 4.43.** *The relation $>_{kbo}$ as defined in 4.41 is a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

We show now that $>_{kbo}$ has the subterm property.

**Lemma 4.44.** *For any term $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and any non-trivial linear context $C$, we have that $C[s] >_{kbo} s$.*

**Proof** Note that due to remark 2.50 we only need to see that $f(\ldots, s, \ldots) >_{kbo} s$, for all $f \in \mathcal{F}$. We will prove this property by induction on $s$. If $s = x$, for some $x \in \mathcal{X}$, the result follows from lemma 4.42. Suppose now that $s = g(s_1, \ldots, s_m)$. First we note that we obviously have that $\#_x(f(\ldots, s, \ldots)) \geq \#_x(s)$, for all variables $x \in \mathcal{X}$. Also, from the definition of weight function, we conclude that $\phi(f(\ldots, s, \ldots)) \geq \phi(s)$. If the inequality is strict, we are done. Suppose then that $\phi(f(\ldots, s, \ldots)) = \phi(t)$. From the definition of $\phi$ it is easy to derive that $\phi(t) \geq \phi_0 > 0$, for any term $t$. As a consequence $\phi(f(\ldots, s, \ldots)) \geq \phi(s) + \phi(f) + (n-1).\phi_0$, where $n$ is the arity of $f$ and $(n-1).\phi_0$ is a lower bound for the weight of the (not shown) arguments of $f$ other than $s$. In order to have $\phi(f(\ldots, s, \ldots)) = \phi(s)$, we must have $n = 1$ and $\phi(f) = 0$, and in this case definition of $\phi$ forces $f \unrhd p$, for all $p \in \mathcal{F}$. Recall that $s = g(s_1, \ldots, s_m)$. If $f \rhd g$ then we can conclude that $f(\ldots, s, \ldots) >_{kbo} s$. If $f \sim g$ then we need to see that $s >_{kbos,\tau} s_1, \ldots, s_m$. Applying the induction hypothesis to each $s_i$, we conclude that $s = g(s_1, \ldots, s_m) >_{kbo} s_i$, for all $1 \leq i \leq m$, and this implies that $s >_{kbos,\tau} s_1, \ldots, s_m$, no matter what the status of $f$ is (namely multiset or lexicographic). Consequently $f(\ldots, s, \ldots) >_{kbo} s$. $\square$

Finally we see that $>_{kbo}$ is closed under contexts and substitutions.

**Lemma 4.45.** *Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $s >_{kbo} t$. Then:*

1. *$C[s] >_{kbo} C[t]$, for any context $C$,*
2. *$s\sigma >_{kbo} t\sigma$, for any substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$.*

**Proof** For $1$ we proceed by induction on the context. For the trivial context the result holds by hypothesis. Suppose now that $s >_{kbo} t$. According to remark 2.50 we just need to see that $f(u_1, \ldots, s, \ldots, u_k) >_{kbo} f(u_1, \ldots, t, \ldots, u_k)$, for any $f \in \mathcal{F}$ with arity $k \geq 1$, and terms $u_1, \ldots, u_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Since $s >_{kbo} t \Rightarrow \phi(s) \geq \phi(t)$ and due to the definition of $\phi$, we also have $\phi(f(u_1, \ldots, s, \ldots, u_k)) \geq \phi(f(u_1, \ldots, t, \ldots, u_k))$, for any $f \in \mathcal{F}$ with arity $k \geq 1$, and terms $u_1, \ldots, u_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If the inequality is strict, we are done, otherwise in order to conclude that $f(u_1, \ldots, s, \ldots, u_k) >_{kbo} f(u_1, \ldots, t, \ldots, u_k)$ we need to have $u_1, \ldots, s, \ldots, u_k >_{kbos,\tau} u_1, \ldots, t, \ldots, u_k$ and this holds for any status since $s >_{kbo} t$.

For $2$, we proceed by induction on $t$. If $t$ is a variable then we must have that $s$ is not a variable (since $s >_{kbo} t$) and so $t$ is a proper subterm of $s$. Consequently $t\sigma$ is a proper subterm of $s\sigma$ and by lemma 4.44 we conclude that $s\sigma >_{kbo} t\sigma$. Suppose that $t = g(t_1, \ldots, t_m)$, for some $g \in \mathcal{F}$ with arity $m \geq 0$, and terms $t_1, \ldots, t_m \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Given that $s >_{kbo} t$, the following facts are not difficult to establish:

- $\#_x(s\sigma) \geq \#_x(t\sigma)$, for any $x \in \mathcal{X}$ and any substitution $\sigma$,
- $\phi(s\sigma) \geq \phi(t\sigma)$ and $\phi(s\sigma) = \phi(t\sigma) \Rightarrow \phi(s) = \phi(t)$, for any substitution $\sigma$.

If $\phi(s\sigma) > \phi(t\sigma)$ then due to the first part of the fact above, we are done. If $\phi(s\sigma) = \phi(t\sigma)$, again by the fact above, we also have $\phi(s) = \phi(t)$. Since $t = g(t_1, \ldots, t_m)$ and due to the fact that $s >_{kbo} t$, we must have $s = f(s_1, \ldots, s_k)$, for some $f \in \mathcal{F}$, with arity $k \geq 0$, and terms $s_1, \ldots, s_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If $f \rhd g$ then again using the first part of the above fact, we conclude that $s\sigma >_{kbo} t\sigma$. Otherwise we must have $f \sim g$, $k, m \geq 1$, and $s_1, \ldots, s_k >_{kbos,\tau} t_1, \ldots, t_m$ and using the induction hypothesis (applied to each $t_i$) is not difficult to see with a little case analysis for the status, that this implies that $s_1\sigma, \ldots, s_k\sigma >_{kbos,\tau} t_1\sigma, \ldots, t_m\sigma$. Since $s\sigma = f(s_1\sigma, \ldots, s_k\sigma)$, $t\sigma = g(t_1\sigma, \ldots, t_m\sigma)$, we can conclude that $s\sigma >_{kbo} t\sigma$. $\square$

We can define a congruence $\sim_{kbo}$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows: $s \sim_{kbo} t$ iff $s = t$ or $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, $f \sim g$, $m = n$, $\phi(s) = \phi(t)$ and either

- $\tau(f) = mult$ and $s_i \sim_{kbo} t_{\pi(i)}$, for any $1 \leq i \leq m$, where $\pi$ is a permutation of $\{1, \ldots m\}$;
- $\tau(f) = lex_{\pi_f}$, $\tau(g) = lex_{\pi_g}$ and $s_{\pi_f(i)} \sim_{kbo} t_{\pi_g(i)}$ for all $1 \leq i \leq m$.

It can be seen that $\sim_{kbo}$ is indeed a congruence i. e., a reflexive, symmetric and transitive relation, closed under contexts. Further $\sim_{kbo}$ is also closed under substitutions and it is not difficult to see that $>_{kbo}$ and $\sim_{kbo}$ are compatible, so we can extend $>_{kbo}$ to $\mathcal{T}(\mathcal{F},\mathcal{X})/\sim_{kbo}$ in the usual way. As with $>_{rpo}$, given a total quasi-precedence over $\mathcal{F}$, $>_{kbo}$ is total over $\mathcal{T}(\mathcal{F})/\sim_{kbo}$. As for well-foundedness we have

**Theorem 4.46.** *Let $\trianglerighteq$ be a well-founded quasi-precedence over $\mathcal{F}$ and $\tau$ a status function such that condition (4.1) is satisfied. Then $>_{kbo}$ is well-founded over $\mathcal{T}(\mathcal{F},\mathcal{X})$.*

The proof of this theorem is very similar to the proof of theorem 4.38 so we omit it. Note however that condition (4.2) is not necessary since the use of the weight function ensures that the lexicographic extension is well-founded.

We now present the result stating that $>_{kbo}$ also proves total termination.

**Theorem 4.47.** *Given a TRS $R$, suppose $\trianglerighteq$ is a well-founded quasi-precedence over $\mathcal{F}$ and $\tau$ is a status function such that condition (4.1) is satisfied. Let $\phi$ be a weight function. If $l >_{kbo} r$ for every rule $l \to r \in R$ then $R$ is totally terminating.*

**Proof (Sketch)** We proceed in a manner similar as for $>_{rpo}$. If $\mathcal{F}$ is empty, we add a dummy constant to it and assign weight $\phi_0$ to that constant. We then extend the well-founded quasi-precedence $\trianglerighteq$ to a total one whose underlying equivalence is the same, and take $>_{kbo}$ over this total well-founded quasi-precedence. Note that if $f$ is a function symbol maximal with respect to $\trianglerighteq$, then it remains maximal with respect to the total quasi-precedence.

As total well-founded monotone algebra we choose $\mathcal{T}(\mathcal{F})/\sim_{kbo}$ and interpret the function symbols of $\mathcal{F}$ in the same way. It is not difficult to see that all requirements of total termination are met. □

We have shown that orders as $>_{rpo}$ and $>_{kbo}$ do prove total termination. The reverse in not true, i. e., total termination is a more general notion since not all totally terminating TRS's can be proven terminating using $>_{rpo}$ or $>_{kbo}$, as the following example shows.

**Example 4.48.** Let $R$ be given by:

$$f(s(x),y) \quad \to \quad h(s(f(h(y),x)))$$

This system cannot be proven terminating by either *rpo* or *kbo*:

- for *kbo*, note that if the weight of the function symbol $h$ is non-negative, we will always have that the weight of the rhs is bigger that the weight of the lhs of the rule, no matter what values we choose for the weight of the function symbols; and if $h$ is given weight $0$ , we must have $h \trianglerighteq p$, for all $p \in \mathcal{F}$, and even if $f \sim h$, we cannot have $f(s(x),y) >_{kbo} h(s(f(h(y),x)))$, for any possible status.

- for *rpo*, note that in order to have $>_{rpo}$ orient the rule we need to have $f(s(x), y) >_{rpo} f(h(y), x)$ and that is impossible: multiset status (for $f$) is ruled out because $y \not>_{rpo} h(y)$ and any lexicographic status implies the comparison of terms having different variables, something not possible under *rpo*.

We now see that $R$ is polynomially terminating. Choose $(A, >)$ to be $\mathbb{N}_2$, the set of naturals greater or equal to $2$, with the usual well-order $>$. Define the operations:

$$\begin{array}{lcl} h_A(x) & = & x \\ s_A(x) & = & x+1 \\ f_A(x,y) & = & xy \end{array}$$

These operations are monotone and if $\tau : \mathcal{X} \to \mathbb{N}_2$ is any assignment, we have

$$[\![f(s(x), y), \tau]\!] = \tau(x)\tau(y) + \tau(y) > \tau(x)\tau(y) + 1 = [\![h(s(f(h(y), x))), \tau]\!]$$

since $\tau(y) \geq 2$. Since $\tau$ is arbitrary, the result follows.

Another interesting example is the system $f(g(x)) \to g(f(f(x)))$. This system cannot be proven terminating with $>_{rpo}$. It can be proven terminating with $>_{kbo}$ if we choose $\phi(f) = 0$, $\phi(g) > 0$ and $f \rhd g$. As we shall see in chapter 5, this TRS is totally terminating but not $\omega$-terminating.

## 4.5 Conclusions

In the first part of this chapter we tried to provide a characterization of recursively defined path orders. Though many such orders are known, proofs of their well-definedness are, as far as we know, not to be found in the literature. With the characterizations given, we can also prove the "difficult" property of transitivity of both partial and quasi-orders defined in this recursive way. An interesting aspect of our approach is that it abstracts from the form of the status/liftings and enables us to concentrate on the properties of those status/liftings needed for the relations obtained to fulfil different requirements.

In the second part of the chapter we concentrated on $\geq_{rpo}$ and $>_{kbo}$. As we mentioned before, many path orders are either total or extendable to total monotonic orders on ground terms (eventually modulo some congruence). For status functions and quasi-precedences satisfying certain conditions, Lescanne [70] showed that *rpo* is contained in the *recursive decomposition ordering (rdo)* and that both orders can be extended to total orders on ground terms (modulo some congruence). In Rusinowitch [97], relations between different path orders for the case of total precedences, are given.[2] Continuing on this line of work, Steinbach [100] provides also a comparison of the behaviour of different path orders on ground terms (being the orders taken over total precedences). He states that in these conditions the *path of subterms ordering* of Plaisted [87] and the *path ordering with status* of Kapur, Narendran and Sivakumar [56], amongst others, are total on ground terms modulo some congruence. Furthermore the effect of

[2] As far as we know Rusinowitch was the first to provide this kind of comparisons.

those orders on ground terms is the same, i. e., given a total precedence and a status function these orders are equivalent between themselves and equivalent to *rpo*, in the sense that ground terms are comparable with respect to an order $\Theta$ if and only if they are comparable with respect to *rpo*.

In Steinbach [100], *kbo* is also identified as being an order different in nature from the other orders. Even when such strong conditions as total precedence and ground terms are imposed, *kbo* remains incomparable with the other orders mentioned. This is essentially the reason why we decided to deal in this chapter with *rpo* and *kbo*: they are representatives of the two essentially distinct classes of path orders amenable to totalization.

Finally we should remark that not all path orders are amenable to totalization. The example that comes to mind is *spo* of Kamin and Lévy [54]. It is well-known that any terminating TRS can be proven terminating via some *spo*. Since not all terminating TRS's are totally terminating, *spo* is not amenable to totalization in general.

# Chapter 5

# Total Termination of Term Rewriting

In this chapter we investigate proving termination of term rewriting systems by a compositional interpretation of terms in a total well-founded order. As introduced in chapter 2, this kind of termination is called *total termination*. Equivalently total termination can be characterized by the existence of an order on ground terms which is total, well-founded and closed under contexts, as we shall see.

When trying to prove termination of TRS's by interpretation, a major problem is how to choose a suitable well-founded ordered set. The variation among well-founded ordered sets is so unwieldy that some restriction is reasonable. A natural one is the restriction to total orders: then the ordered sets correspond to ordinal numbers, having a very elegant structure that has been studied extensively in the past. For finite signatures, total termination turns out to be a slightly stronger restriction than simple termination in the sense that every totally terminating TRS is also simply terminating but not vice-versa. Simple termination has been extensively studied while total termination is a relatively new notion. However, it turns out that most of the existing methods for proving termination of TRS's prove in fact total termination. By definition the methods of *polynomial interpretations* Lankford [68], Ben-Cherifa and Lescanne [6], and *elementary interpretations* Lescanne [71] are nothing else than our approach in which the algebra is chosen to be the naturals and the operations have a particular shape. Hence a termination proof by these interpretations implies total termination. The same can be said for recursive path order and Knuth-Bendix order.

In this chapter we investigate several aspects of total termination, in particular which totally ordered sets are useful. One of the main conclusions is that apart from some minor exceptions only ordinals of the shape $\omega^\alpha$ are of interest. The basic observation leading to this result is the following. The existence of a binary operation in a total well-founded order that is strictly monotonic in both coordinates implies that the order type is $\omega^\alpha$. Stated without ordinals this means that the order is isomorphic to the finite multisets over another order. Below the ordinal $\epsilon_0$ this implies that all totally ordered sets of interest can be constructed from the natural numbers in finitely many steps using only the constructions of lexicographic product and finite multisets. We show that these constructions are essential by presenting examples of TRS's for which a termination proof can be given (by an interpretation) in $\omega^\eta$, for any fixed $\eta \leq \omega$, but not in a totally ordered set of a smaller order type.

Another main topic of this chapter is the modularity of total termination. Surprisingly the tree structure of mixed terms that is essential in other modularity questions (see Middeldorp [76]) does not play a role here. The essential problem is how to lift an interpretation in an ordinal to an interpretation in a greater ordinal without affecting monotonicity and compatibility. We did not succeed in proving modularity of total termination in full generality (which is still an open problem) but found some interesting partial results.

The chapter is organized as follows. Since most of the techniques used are based on ordinal arithmetic, we present some needed ordinal theory in section 5.1. In section 5.2 the important multiset construction is introduced and given an ordinal characterization. In the same section we prove that existence of monotone functions, of arity greater than one, in a certain monotone algebra forces the algebra to have a multiset type. In section 5.3 we present some closure properties for the class of ordinals associated with a totally terminating TRS. An important consequence of those closure properties is the modularity (under certain conditions) of total termination. Section 5.4 discusses the particular case of string rewriting systems and in section 5.5 we make some considerations about the minimal ordinal associated with a totally terminating TRS. In section 5.6 we try to give a more syntactically oriented characterization of the notion of total termination. We show that total termination can indeed be equated with some total orders on ground terms. We also look at the problem of proving non-total termination: as a consequence some partial characterizations of total termination in terms of relations on terms, arise. Section 5.7 concludes the chapter.

Except when explicitly noted, all the results presented apply to both finite and infinite signatures and TRS's.

## 5.1 Tools from Ordinal Theory

A main topic of this chapter is the investigation of useful total orders for total termination. The main tool is the arithmetic of ordinals, i. e., of total well-founded orders modulo order-isomorphism. We say that a proof of total termination is in an ordinal $\alpha$ if the underlying order of the monotone algebra has order type $\alpha$. Since in this algebra we allow all possible monotone functions this does not mean that the proof can be given in $\alpha$ in the proof-theoretical sense. For example, the term rewriting system describing the Ackermann function can be proven terminating by a monotone algebra whose underlying order corresponds to the natural numbers, so in our sense its termination proof is in $\omega$. Another approach connecting termination orders and ordinals is given, for example, in Martin and Scott [75].

In this section we summarize notions and results from ordinal arithmetic we need. For many of the proofs we refer to Kuratowski and Mostowski [65]. A few of these results are, as far as we know, new (usually results about properties of monotone functions on ordinals) so their proofs are also presented. Note however that not all proofs presented are from new results.

**Definition 5.1.** A *well-order* is a total well-founded order.

In a well-order every non-empty subset has an unique minimal element (a minimum). A simple but useful lemma is the following.

**Lemma 5.2.** *Let $\mathcal{A} = (A, >)$ be well-ordered and let $f : A \to A$ be any monotone function. Then $f(x) \geq x$ for every $x \in A$. Furthermore, for any $x, y \in A$, $f(x) > f(y) \iff x > y$.*

**Proof** Suppose there is $x \in A$ such that $x > f(x)$. Monotonicity of $f$ leads to an infinite decreasing sequence: $x > f(x) > f(f(x)) > f(f(f(x))) > \cdots$, contradicting well-foundedness.

Suppose now that $f(x) > f(y)$, for some $x, y \in A$. Since $A$ is well-ordered we must have $x > y$, $y > x$ or $x = y$. The last two cases contradict the monotonicity of $f$ so we must have $x > y$. Conversely, if $x > y$, definition of monotonicity gives $f(x) > f(y)$. □

**Corollary 5.3.** *Let $\mathcal{A} = (A, >)$ be well-ordered and let $f : A \times \ldots \times A \to A$ be any function with $n \geq 1$ arguments and (weakly) monotone in all arguments. Then $f(x_1, \ldots, x_n) \geq x_i$ for all $1 \leq i \leq n$ and $x_i \in A$.*

**Proof** Fix $i$ with $1 \leq i \leq n$. By fixing all arguments of $f$ except the $i^{th}$ argument, we obtain a monotone function from $A$ to $A$. Now lemma 5.2 gives the result. □

**Definition 5.4.** Two ordered sets are called *similar* if they are *order-isomorphic*, i. e., there is a monotone bijection between them.

Since, in a total order, monotonicity implies injectivity we have:

**Lemma 5.5.** *Let $\mathcal{A}$ and $\mathcal{B}$ be totally ordered sets and $f : \mathcal{A} \to \mathcal{B}$. Then $f$ is monotone and surjective $\iff$ $f$ is an order-isomorphism.*

**Proof** The "if" part is always satisfied by definition of order-isomorphism. For the "only-if" part we just need to see that $f$ is injective. Let then $x, y \in \mathcal{A}$ and suppose $f(x) = f(y)$. Since $\mathcal{A}$ is well-ordered, we must have $x = y$ or $x >_{\mathcal{A}} y$ or $y >_{\mathcal{A}} x$. The last two cases would imply $f(x) >_{\mathcal{B}} f(y)$ or vice-versa (since $f$ is monotone), so we must have $x = y$. □

Similarity classes of well-orders are called *ordinal numbers* (or for short *ordinals*). For finite well-ordered sets their ordinals coincide with their cardinality and are denoted by natural numbers. The ordinal corresponding to a well-ordered set is called its *order type* or *type*.

**Definition 5.6.** A proper subset $X$ of a well-order $\mathcal{A} = (A, >)$ is called an *initial segment* of $\mathcal{A}$ if $\forall x \in X \; \forall y \in A \; (y < x \Rightarrow y \in X)$. Equivalently $X$ is an initial segment of $\mathcal{A}$ if and only if $X = \{y \mid y < x\}$ for some $x \in \mathcal{A}$.

The following result is essential for the theory of ordinals.

**Theorem 5.7.** *Let $\mathcal{A}$ and $\mathcal{B}$ be well-ordered sets. Then either $\mathcal{A}$ is similar to $\mathcal{B}$ or $\mathcal{A}$ is similar to an initial segment of $\mathcal{B}$ or $\mathcal{B}$ is similar to an initial segment of $\mathcal{A}$.*

Let $Ord$ denote the class of ordinal numbers and define a relation $<$ on $Ord$ by:
$\alpha < \beta \iff$ any set of type $\alpha$ is similar to an initial segment of a set of type $\beta$.
From theorem 5.7 follows that $<$ totally orders $Ord$.

An ordinal is an equivalence class and it is convenient to describe it by a canonical representative of this class. If $\mathcal{A} = (A, \succ)$ has type $\alpha$, it can be seen that $\mathcal{A}$ is similar to the set $\{\beta \in Ord \mid \beta < \alpha\}$.[1] We choose this set to be the canonical representative for the ordinal $\alpha$. As a consequence we have: $\beta < \alpha \iff \beta \in \alpha \iff \beta \subset \alpha$. We shall freely switch between the class and the canonical representative. Sometimes we will also use the notation $>_\alpha$ to emphasize that we are comparing elements of $\alpha$.

Below some basic properties of $Ord$ are listed.

**I.** $<$ well-orders the class $Ord$, that is:

- $<$ is a total order in $Ord$,
- Every non-empty class $B \subseteq Ord$ has a minimal element in $B$,
- For every $\alpha \in Ord$, $\{\xi \in Ord | \xi < \alpha\}$ is a set.

**II.** For every set of ordinals $U$ there is an ordinal $\alpha$ such that $\alpha = sup(U) = \bigvee U$ (If $U = \{f(\xi) | \; p(\xi)\}$, for any predicate $p$, we sometimes use the notation $\bigvee_{p(\xi)} f(\xi)$.)

**III.** $W(\alpha) = \{\xi \mid \xi < \alpha\}$ is well-ordered and has type $\alpha$.

Note that the existence of the ordinal $sup(U)$ in **II** above, is given by the fact that $sup(U) = min\{\beta \in Ord | \; \forall\gamma \in U : \gamma \leq \beta\}$, since this class is not empty. The second condition in **I** above is equivalent to the principle of transfinite induction that we will use in some proofs.

**Theorem 5.8.** **(Principle of Transfinite Induction - PTI)** *Let $\mathcal{A}$ be a class well-ordered by $>$ and let $F$ be a propositional function such that*

$$\forall x \in \mathcal{A} : (\forall y < x : F(y)) \Rightarrow F(x)$$

*Then* $\forall x \in \mathcal{A} : F(x)$.

**Proof** Suppose the result is not true and define $B = \{x \in \mathcal{A} | \; \neg F(x)\}$. By hypothesis $B$ is not empty and using the second condition of **I.** above we conclude that $B$ has a minimal element $x_m$. Then, by definition of $B$, $y < x_m \Rightarrow F(y)$, for any $y < x_m$, and this implies (by hypothesis) $F(x_m)$, contradicting the definition of $x_m$. □

To see that the Principle of Transfinite Induction implies the second condition of **I** above, i. e., that every non-empty class $B \subseteq A$ contains a minimal element, we consider the equivalent formulation of **PTI**:

$$\exists x \in A : \neg F(x) \Rightarrow \exists x \in A : (\forall y < x : F(y)) \wedge \neg F(x)$$

[1] Note that here the symbol $\in$ is used to denote a relation between an element and a class; the intuitive meaning is the same as for sets.

This is valid for all propositional functions $F$. By instantiating $F$ to $F(x) = x \notin B$, we obtain exactly the minimality requirement.

The ordinal $0$ is defined to be the minimal element of $Ord$; it is the type of the empty set. For every ordinal $\xi$, its *successor* $\xi'$ is defined by $\xi' = min\{\alpha \mid \xi < \alpha\}$. We use the notation $0' = 1$, $1' = 2$, and so forth. We will sometimes denote the successor ordinal by $\xi + 1$. Clearly $\xi < \xi'$ and there is no ordinal $\alpha$ such that $\xi < \alpha < \xi'$.

An ordinal $\xi$ is defined to be a limit ordinal if

$$(\exists \alpha < \xi) \ \wedge \ (\forall \alpha < \xi \ \exists \eta < \xi : \ \alpha < \eta)$$

The first condition states that a limit ordinal is non-empty, and the second condition says that it has no maximal element. An ordinal $\xi \neq 0$ is a limit ordinal if and only if $(\alpha < \xi) \Rightarrow (\alpha' < \xi)$; if and only if $\xi = \bigvee_{\alpha < \xi} \alpha$. The class of limit ordinals is denoted by $Lim$. The ordinal $\omega$ is defined to be the minimum of $Lim$; it is the type of the natural numbers.

Every ordinal is either $0$, a *successor* ordinal or a *limit* ordinal. These three kinds often appear in inductive proofs and definitions.

The operations of addition, multiplication and exponentiation are inductively defined in $Ord$ as follows:

| | $\alpha + \beta$ | $\alpha.\beta$ | $\alpha^{\beta}$ |
|---|---|---|---|
| $\beta = 0$ | $\alpha$ | $0$ | $1$ |
| $\beta = \beta_0'$ | $(\alpha + \beta_0)'$ | $\alpha.\beta_0 + \alpha$ | $\alpha^{\beta_0}.\alpha$ |
| $\beta \in Lim$ | $\bigvee_{\xi < \beta} (\alpha + \xi)$ | $\bigvee_{\xi < \beta} (\alpha.\xi)$ | $\bigvee_{0 < \xi < \beta} (\alpha^{\xi})$ |

We list some properties of these operations.

**Addition**: it is associative and non-commutative; commutativity does hold for ordinals smaller than $\omega$. Addition is weakly monotone in the left argument, i. e., $\alpha < \beta \Rightarrow \alpha + \delta \leq \beta + \delta$, and strictly monotone in the right argument; consequently there is a left-cancellation law, i. e., $\alpha + \beta < \alpha + \delta \Rightarrow \beta < \delta$. For any $\alpha$, if $\beta \in Lim$, then $\alpha + \beta \in Lim$.

**Multiplication**: it is associative and non-commutative; commutativity does hold for ordinals smaller than $\omega$. Multiplication is weakly monotone in the left argument provided right arguments are $> 0$; for left arguments $> 0$, it is strictly monotone in the right argument and left-cancellation holds. Also $\alpha.\beta = 0 \iff \alpha = 0 \vee \beta = 0$; if $\alpha \neq 0$ and $\beta \in Lim$, then $\alpha.\beta, \ \beta.\alpha \in Lim$. We also have that multiplication left-distributes over addition, i. e., $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.

**Exponentiation**: for a fixed base greater than $1$, exponentiation is strictly monotone in the exponent; also $(\alpha^{\beta})^{\gamma} = \alpha^{\beta.\gamma}$, for any $\alpha, \ \beta, \ \gamma$.

**Lemma 5.9.** $\lambda \in Lim \iff \lambda = \omega.\beta$, *for some* $\beta \neq 0$.

Some ordinals are closed under the operations of addition and/or multiplication; they are crucial in this chapter.

**Definition 5.10.** An ordinal $\alpha \neq 0$ is named *additive principal* if it satisfies $\xi, \eta < \alpha \Rightarrow \xi+\eta < \alpha$. An ordinal $\alpha > 1$ is said to be *multiplicative principal* if it satisfies $\xi, \eta < \alpha \Rightarrow \xi.\eta < \alpha$.

**Lemma 5.11.** *Let $\alpha \in Ord$. Then the following conditions are equivalent:*

- *$\alpha$ is additive (respectively multiplicative) principal;*
- *$\alpha = \omega^\eta$ (respectively $\alpha = \omega^{\omega^\eta}$ or $\alpha = 2$), for some $\eta \geq 0$;*
- *$\forall \beta < \alpha : \beta + \alpha = \alpha$ (respectively $\beta.\alpha = \alpha$).*

**Lemma 5.12.** *If $\alpha \leq \beta$ then there is a unique ordinal $\delta$ such that $\beta = \alpha + \delta$.*

The unique ordinal $\delta$ of the previous lemma is usually written $\beta - \alpha$ and we speak about subtraction of ordinals. It is not difficult to see that subtraction is weakly anti-monotone in the right argument, i. e., if $\beta < \delta \leq \alpha$ then $(\alpha - \delta) \leq (\alpha - \beta)$.

**Lemma 5.13.** *If $\tau \leq \alpha$ then $(\alpha + \delta) - \tau = (\alpha - \tau) + \delta$.*

**Proof** First we remark that the difference $\alpha - \tau$ is well-defined given that $\tau \leq \alpha$. By definition of difference, we have $\tau + ((\alpha + \delta) - \tau) = \alpha + \delta = (\tau + (\alpha - \tau)) + \delta$. Associativity of addition and left-cancellation give the result. □

As we would expect, additive principal ordinals are closed under subtraction.

**Lemma 5.14.** *If $\alpha < \beta$ and $\beta = \omega^\eta$, for some ordinal $\eta$, then $\beta - \alpha = \beta$.*

**Proof** By lemma 5.11 and lemma 5.12 we have $\alpha + (\beta - \alpha) = \beta = \alpha + \beta$. The result follows from left-cancellation. □

**Lemma 5.15.** *Let $f : \alpha \to \alpha$ be a monotone function. Then for any ordinals $a, b$ such that $a + b < \alpha$ we have $f(a + b) \geq f(a) + b$.*

**Proof** Fix $a \in \alpha$. Define $g(x) = f(a + x) - f(a)$, for any $x \in (\alpha - a)$. Since $f$ is monotone, $g$ is well-defined and is a function from $\alpha - a$ to $\alpha - f(a)$. Furthermore $g$ is also monotone for if $x > y$ we have by the properties of addition that $a + x > a + y$ and monotonicity of $f$ implies $f(a + x) > f(a + y)$. But then $f(a) + (f(a + x) - f(a)) = f(a + x) > f(a + y) = f(a) + (f(a + y) - f(a))$ and by left-cancellation we get $f(a + x) - f(a) > f(a + y) - f(a)$, proving that $g$ is monotone. From lemma 5.2 we conclude that $f(a+x) - f(a) = g(x) \geq x$, hence $f(a+x) \geq f(a) + x$, for all $x < \alpha - a$. Since $a + b < \alpha$ implies that $b < \alpha - a$, the result follows. □

**Lemma 5.16.** *Let $f : \alpha \to \beta$ be monotone. Then $\alpha \leq \beta$.*

**Proof** For any $\xi \in \alpha$, $f(\xi) \in \beta$. But since f is monotone, we have $\xi \leq f(\xi)$ (by lemma 5.2), hence $\xi \in \beta$. Therefore $\alpha \subseteq \beta$ and $\alpha \leq \beta$. □

We conclude this section with some useful standard results.

**Lemma 5.17.**

1. $\forall \alpha, \beta \in Ord\ \exists! \gamma, \delta \in Ord:\ \beta = \alpha.\gamma + \delta,\ \delta < \alpha.$

2. $\forall \beta \geq 1\ \forall \alpha \geq 2\ \exists! \eta:\ \alpha^\eta \leq \beta < \alpha^{\eta+1}.$

3. *If* $\alpha < \beta.\gamma$ *then* $\exists! \beta_1, \gamma_1 : \beta_1 < \beta\ \wedge\ \gamma_1 < \gamma\ \wedge\ \alpha = \beta.\gamma_1 + \beta_1.$

**Theorem 5.18. (Cantor Normal Form)** *For every ordinal* $\alpha > 0$ *there is a natural number* $k \geq 1$ *and uniquely determined ordinals* $\eta_1 \geq \cdots \geq \eta_k$ *such that* $\alpha = \omega^{\eta_1} + \cdots + \omega^{\eta_k}$.

Besides Cantor normal form, there are other ways of representing ordinals. One such representation that we will make use of later is the so-called *normal form*. Given an ordinal $\alpha$ its unique normal form $\overline{\alpha}$ is the expansion of $\alpha$ with base $\omega$, that is $\overline{\alpha} = \omega^{\eta_1}.p_1 + \cdots + \omega^{\eta_k}.p_k$, with $\eta_1 > \eta_2 > \cdots > \eta_k$, $0 \leq p_i < \omega$, for $1 \leq i \leq k$, and $\omega > k \geq 1$. Using this normal form we can define *natural addition*, denoted by $\oplus$. Given ordinals $x, y$, $\overline{x} \oplus \overline{y}$ is performed by adding the expansions of both $x$ and $y$ as polynomials in $\omega$ (well-defined since ordinal addition is commutative for ordinals smaller than $\omega$). Natural addition is commutative, associative and strictly increasing in each argument. Furthermore ordinals of the form $\omega^\gamma$, for $\gamma \geq 0$, are principal ordinals for addition, and therefore closed for natural addition. Similarly we can define natural multiplication (for details see Kuratowski and Mostowski [65]).

**Example 5.19.** Let $\alpha = \omega^{\omega.\omega} + \omega^2.(\omega^3 + 1) + 1$ and $\beta = \omega^2 + \omega + \omega + \omega$. Then $\overline{\alpha} = \omega^{\omega^2} + \omega^5 + \omega^2 + 1$ and $\overline{\beta} = \omega^2 + \omega.3$. We have $\overline{\alpha} \oplus \overline{\beta} = \overline{\beta} \oplus \overline{\alpha} = \alpha \oplus \beta = \beta \oplus \alpha = \omega^{\omega^2} + \omega^5 + \omega^2.2 + \omega.3 + 1$ while $\alpha + \beta = \omega^{\omega^2} + \omega^5 + \omega^2.2 + \omega.3$ and $\beta + \alpha = \omega^{\omega^2} + \omega^5 + \omega^2 + 1 = \alpha$.

We end this section by presenting a constructive characterization of ordinal multiplication. Later we will freely switch between the different characterizations of multiplication.

**Lemma 5.20.** *Let* $\alpha, \beta \in Ord$. *Then* $\alpha.\beta$ *is the order type of the lexicographic product* $(\beta \times \alpha, >_{lex})$, *where* $>_{lex}$ *is the lexicographic product of the orders* $>_\beta$ *by* $>_\alpha$ *as defined in 2.34.*

**Proof** First note that if either $\alpha$ or $\beta$ are $0$, the result holds since $\alpha.\beta$ is $0$ and the lexicographic product is empty. Suppose now that both $\alpha, \beta > 0$. By lemma 2.35, $>_{lex}$ is total and well-founded over $\beta \times \alpha$, so $\beta \times \alpha$ is also an ordinal and we see that it coincides with $\alpha.\beta$. Let $x \in \alpha.\beta$, by lemma 5.17 there are unique ordinals $\alpha_x < \alpha, \beta_x < \beta$ such that $x = \alpha.\beta_x + \alpha_x$. Let $f : \alpha.\beta \to \beta \times \alpha$ be defined by $f(x) = (\beta_x, \alpha_x)$. Due to the uniqueness of the decomposition mentioned, function $f$ is well-defined. According to lemma 5.5 in order to see that both ordinals are the same it is enough to show that $f$ is monotone and surjective. Suppose then that $x, y \in \alpha.\beta$ are uniquely decomposed in $x = \alpha.\beta_x + \alpha_x$, $y = \alpha.\beta_y + \alpha_y$ and that $x > y$. If $\beta_x < \beta_y$ we can write $\beta_y = \beta_x + \delta$, for some $\delta > 0$. Consequently we have

$$x = \alpha.\beta_x + \alpha_x > \alpha.\beta_y + \alpha_y = \alpha.\beta_x + \alpha\delta + \alpha_y = y$$

Left-cancellation gives $\alpha_x > \alpha\delta + \alpha_y \geq \alpha$, which is a contradiction. So $\beta_x \geq \beta_y$. If $\beta_x > \beta_y$, we have that $f(x) = (\beta_x, \alpha_x) >_{lex} (\beta_y, \alpha_y) = f(y)$, so we are done. If $\beta_x = \beta_y$, suppose that $\alpha_y > \alpha_x$. Again we can write $\alpha_y = \alpha_x + \delta$, for some $\delta > 0$, and $x > y$ becomes $\alpha.\beta_x + \alpha_x > \alpha.\beta_x + \alpha_y = \alpha.\beta_x + \alpha_x + \delta$. Left-cancellation gives $\delta < 0$ which is a contradiction. So if $\beta_x = \beta_y$ then $\alpha_x > \alpha_y$ and again $f(x) = (\beta_x, \alpha_x) >_{lex} (\beta_y, \alpha_y) = f(y)$, so $f$ is monotone. Let $(b, a) \in \beta \times \alpha$, then $b < \beta$ and $a < \alpha$, so $\alpha.b + a < \alpha.\beta$, since $b < \beta \Rightarrow (b+1) \leq \beta$ and $\alpha.b + a < \alpha.b + \alpha = \alpha.(b+1) \leq \alpha.\beta$. Trivially the image of $\alpha.b + a$ under $f$ is $(b, a)$, so $f$ is surjective. □

## 5.2 Multisets and Binary Functions

We give a constructive description of ordinal exponentiation. Let

$$Exp(\alpha, \eta) = \{\sigma : \eta \to \alpha \mid \{y \in \eta \mid \sigma(y) \neq 0\} \text{ is finite}\},$$

for any $\alpha, \eta \in Ord$. Note that if $\eta$ is zero, then $Exp(\alpha, \eta)$ contains only one element, namely the empty function. If $\alpha = 0$ and $\eta \neq 0$ then $Exp(\alpha, \eta)$ contains no elements. In $Exp(\alpha, \eta)$ we define the relation $\succ$ by

$$\sigma \succ \sigma' \iff \exists x \in \eta : \ (\sigma(x) >_\alpha \sigma'(x)) \ \wedge \ (\forall y \in \eta : y >_\eta x \Rightarrow \sigma(y) = \sigma'(y))$$

for any $\sigma, \sigma' \in Exp(\alpha, \eta)$. One easily verifies that $\succ$ is a total order (note that it is a kind of lexicographic order).

**Theorem 5.21.** *Let $\alpha, \eta \in Ord$. Then $(Exp(\alpha, \eta), \succ)$ is order-isomorphic to ordinal exponentiation $\alpha^\eta$.*

We present only a sketch of the proof. If $\alpha = 0$ or $\eta = 0$, the result can easily be derived from the definitions of $Exp(\alpha, \eta)$ and ordinal exponentiation. Suppose then that $\alpha, \eta \geq 1$. Any $x \in \alpha^\eta$ admits a unique finite decomposition in base $\alpha$ (see Kuratowski and Mostowski [65]), i. e., we can write $x = \alpha^{\eta_1}.\gamma_1 + \cdots + \alpha^{\eta_k}.\gamma_k$, with $1 \leq k < \omega$, $\eta > \eta_1 > \cdots > \eta_k$ and $\alpha > \gamma_i$, for $1 \leq i \leq k$. Further it is not difficult to see that given two elements $x, y$ and their respective decompositions $x = \alpha^{\eta_1}.\gamma_1 + \cdots + \alpha^{\eta_k}.\gamma_k$, $y = \alpha^{\tau_1}.\delta_1 + \cdots + \alpha^{\tau_m}.\delta_m$, with $1 \leq k, m < \omega$, $\eta > \eta_1 > \cdots > \eta_k$, $\eta > \tau_1 > \cdots > \tau_m$ and $\alpha > \gamma_i, \delta_j$, for $1 \leq i \leq k$ and $1 \leq j \leq m$, then the following fact holds

$$x > y \iff \begin{cases} \exists 1 \leq i \leq min\{k, m\} : & (\forall 1 \leq j < i : \alpha^{\eta_j}.\gamma_j = \alpha^{\tau_j}.\delta_j) \text{ and} \\ & (\eta_i > \tau_i \text{ or } (\eta_i = \tau_i \text{ and } \gamma_i > \delta_i)) \end{cases}$$

Given such decompositions, the function $\phi : \alpha^\eta \to Exp(\alpha, \eta)$ is defined as

$$\forall x \in \alpha^\eta \ \forall \xi \in \eta : \ \phi(x)(\xi) \ = \ \begin{cases} \gamma_i & \text{if } \xi = \eta_i, \text{ for some } 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to see that $\phi$ is an order-isomorphism. Due to the uniqueness and finiteness of the decomposition, the function $\phi$ is well-defined. Using the fact stated above and the definition of $\succ$ in $Exp(\alpha, \eta)$, it is not difficult to derive that $\phi$ is monotonic. For its surjectivity, let $\sigma \in Exp(\alpha, \eta)$ and define $D = \{x \in \eta \mid \sigma(x) \neq 0\}$. $D$ is finite and enumerating its elements in decreasing order we obtain a sequence $\eta_1 > \cdots > \eta_k$. Let $x = \alpha^{\eta_1}.\sigma(\eta_1) + \cdots + \alpha^{\eta_k}.\sigma(\eta_k)$. Since $\eta > \eta_1 > \cdots > \eta_k$ and $\alpha > \sigma(\eta_i)$, for all $1 \leq i \leq k$, $x \in \alpha^\eta$, and by definition of $\phi$, $\phi(x) = \sigma$, proving its surjectivity. The result now follows from lemma 5.5.

**Remark 5.22.** If $\alpha = \omega$ the definition of $Exp(\omega, \eta)$ is the set $\mathcal{M}(\eta)$ of finite multisets over $\eta$, together with its multiset order as described in Dershowitz and Manna [27]. So the order type of $\mathcal{M}(\eta)$ is $\omega^\eta$. In the sequel we shall freely switch between $\mathcal{M}(\eta)$ and $\omega^\eta$. Considering multisets in $\mathcal{M}(\eta)$ as functions from $\eta$ to $\omega$, multiset union is pointwise addition; it is known that if $\eta$ is totally ordered then $\mathcal{M}(\eta)$ is ordered by lexicographic order on sorted list of the elements of the multiset (see Jouannaud and Lescanne [51]). This corresponds exactly to natural addition of ordinals below $\omega^\eta$. Furthermore given $X, Y \in \omega^\eta$, $X >_{mul} Y \iff X \succ Y$, where $>_{mul}$ is the multiset extension of $>_\eta$ as defined in 2.25 and $\succ$ is the order on elements of $Exp(\omega, \eta)$ defined above.

We shall prove that the existence of a monotone function of arity greater than one in some ordinal implies that the ordinal has the form $\omega^\eta$. As a consequence, for a TRS containing function symbols of arity $> 1$ the only monotone algebras of interest are those whose underlying order is a multiset order. First we introduce some notation and some lemmas. Let $\times$ denote cartesian product. We have:

**Lemma 5.23.** *Let $\lambda > 0$ be an ordinal for which functions from $\lambda \times \cdots \times \lambda$ to $\lambda$, with more than one argument and monotone in all arguments, do exist. Then $\lambda$ satisfies: $\forall \alpha < \lambda : \alpha < \lambda - \alpha$.*

**Proof** Suppose the conclusion does not hold, i. e., there is an ordinal $\alpha < \lambda$ such that $\lambda - \alpha \leq \alpha$. Let $f : \lambda \times \cdots \lambda \to \lambda$ be a function with more than one argument and monotone in all arguments. Without loss of generality we can suppose that $f$ has two arguments (if not by fixing all arguments except two, such a function would be obtained). Define $\varphi : \lambda \to \lambda$ by $\varphi(x) = f(x, \alpha) - \alpha$. We see that $\varphi$ is well-defined. If we fix the first argument of $f$ to $0$, the minimal element of $\lambda$, we have, since $f(0, x)$ is strictly monotone and by lemma 5.2, that $f(0, \alpha) \geq \alpha$. So $f(x, \alpha) \geq \alpha$ for any $x$, hence $\varphi$ is well-defined. Actually $\varphi$ is a function from $\lambda$ to $\lambda - \alpha$. If $x > y$ then $\alpha + \varphi(x) = f(x, \alpha) > f(y, \alpha) = \alpha + \varphi(y)$. Due to the left-cancellation law, we conclude that $\varphi$ is strictly monotone. By lemma 5.16 we conclude that $\lambda \leq \lambda - \alpha$. Since $\alpha < \lambda \leq \lambda - \alpha \leq \alpha$, we get a contradiction. □

**Lemma 5.24.** *For $\lambda \neq 0$, $\lambda = \omega^\gamma$ for some $\gamma \iff \forall \alpha < \lambda : \lambda - \alpha > \alpha$.*

**Proof** We prove that $(\forall \alpha, \beta < \lambda : \alpha + \beta < \lambda)$ if and only if $(\forall \alpha < \lambda : \lambda - \alpha > \alpha)$; then the result follows from lemma 5.11.

For the only-if part, let $\alpha < \lambda$. We always have $\lambda - \alpha \leq \lambda$. If $\lambda - \alpha < \lambda$, by hypothesis we get $\alpha + (\lambda - \alpha) < \lambda$, a contradiction. Therefore $\lambda - \alpha = \lambda$, so $\lambda - \alpha > \alpha$.

For the if part, take $\alpha, \beta < \lambda$. The hypothesis implies $\alpha < \lambda - \alpha$ and $\beta < \lambda - \beta$. We may suppose without loss of generality, that $\beta \leq \alpha$. Then

$$\alpha + \beta \leq \alpha + \alpha < \alpha + (\lambda - \alpha) = \lambda$$

□

**Theorem 5.25.** *Let $\mathcal{A} = (A, >)$ be a well-ordered set such that $A \neq \emptyset$. Then $\mathcal{A}$ is order-isomorphic to $\mathcal{M}(\mathcal{B})$, for some well-ordered set $\mathcal{B}$, if and only if there is a function from $A \times \cdots \times A$ to $A$ with more than one argument, monotone in all arguments.*

**Proof** Assume $\mathcal{A}$ is order-isomorphic to $\mathcal{M}(\mathcal{B})$, for some well-ordered set $\mathcal{B}$. The multiset union from $\mathcal{M}(\mathcal{B}) \times \cdots \times \mathcal{M}(\mathcal{B})$ to $\mathcal{M}(\mathcal{B})$ is monotone in all arguments. The order-isomorphism gives us a similar function in $\mathcal{A}$.

On the other hand assume there is a function that is monotone in several arguments. According to lemmas 5.23 and 5.24 the order type of $\mathcal{A}$ is $\omega^\gamma$, so $\mathcal{A}$ is order-isomorphic to $\mathcal{M}(\gamma)$, for some $\gamma \geq 0$. □

Recall that a TRS is said to be totally terminating if it admits a compatible non-empty well-founded monotone algebra in which the underlying order is total. Thus, stated in different words, the previous result says that if we have a TRS $R$ containing at least a function symbol of arity $n \geq 2$ and totally terminating in an algebra $\mathcal{A}$, then $\mathcal{A}$ has type $\omega^\gamma$, for some $\gamma \geq 0$.

## 5.3 Extension to Higher Ordinals and Modularity

In this section we look at modularity of total termination (for finite and infinite signatures). If two TRS's are totally terminating, what can be said about their disjoint union? From Kurihara and Ohuchi [66] it follows that the disjoint union is simply terminating in the case of finite signatures, but is it also totally terminating? This is not clear if the proofs of total termination are given in distinct ordinals. That leads to the question of whether a total termination proof in some ordinal can be lifted to a similar proof in another ordinal.

**Definition 5.26.** For a TRS $R$ we define $U(R)$ to be the class of ordinals in which a proof of total termination of $R$ can be given. The minimum of $U(R)$ is denoted by $u_R$.

By definition $U(R)$ is non-empty for every totally terminating TRS $R$, since any algebra used in the termination proof of a non-empty TRS has to be non-empty; also if $R$ or $\mathcal{F}$ are non-empty then $0 \notin U(R)$. For example, if $R$ consists of one rule involving two different constants then $U(R)$ is the class of all ordinals $> 1$. The following lemma characterizes the class of ordinals associated with a disjoint union of TRS's.

**Lemma 5.27.** *Let $R_1 \oplus R_2$ be the disjoint union of TRS's $(\mathcal{F}_1, \mathcal{X}_1, R_1)$ and $(\mathcal{F}_2, \mathcal{X}_2, R_2)$. Then $U(R_1 \oplus R_2) = U(R_1) \cap U(R_2)$. Consequently $R_1 \oplus R_2$ is totally terminating if and only if $U(R_1) \cap U(R_2) \neq \emptyset$.*

**Proof** It is clear that $U(R_1 \oplus R_2) \subseteq U(R_1) \cap U(R_2)$, since any well-founded monotone (total) algebra compatible with $R_1 \oplus R_2$ is also compatible with both $R_1$ and $R_2$. Take now $\alpha \in U(R_1) \cap U(R_2)$. This means that a proof by interpretation of termination of both $R_1$ and $R_2$ can be given in $\alpha$. By using the same interpretations for each function symbol in $\mathcal{F}_1$ and $\mathcal{F}_2$ (possible since these sets are disjoint so there can be no clashes in the interpretations), we obtain an interpretation compatible with $R_1 \oplus R_2$. □

The following theorems state some basic properties of $U(R)$.

**Theorem 5.28.** *Let $\alpha \in U(R)$ and let $\beta$ be an arbitrary non-zero ordinal. Suppose that either all function symbols in $R$ have arity $\leq 1$ or that $\beta = \omega^\gamma$ for some ordinal $\gamma$. Then $\beta.\alpha \in U(R)$.*

**Proof** Interpret $\beta.\alpha$ as the lexicographic product with weight on $\alpha$ (see lemma 5.20). Its elements will be denoted by pairs $(b, a)$, with $a \in \alpha$ and $b \in \beta$. Since $\alpha \in U(R)$, we have an interpretation $f_\alpha$ of every function symbol $f$ of $R$ in $\alpha$, strictly monotone in each argument, such that for every rule $l \to r$ in $R$ and every assignment $\tau : \mathcal{X} \to \alpha$, we have $[\![l, \tau]\!]_\alpha >_\alpha [\![r, \tau]\!]_\alpha$. For every function symbol $f$ we introduce an interpretation $f_\beta$ in $\beta$: for constants $c$ we choose $c_\beta = 0$ and for unary $f$ we choose $f_\beta$ to be the identity on $\beta$. If there are symbols of arity $> 1$ we assumed $\beta$ to be the finite multisets over $\gamma$, in this case we define $f_\beta$ to be the multiset union of all of its arguments. For every $f$ define

$$f_{\beta.\alpha}((b_1, a_1), \ldots, (b_n, a_n)) = (f_\beta(b_1, \ldots, b_n), f_\alpha(a_1, \ldots, a_n))$$

By applying corollary 2.101 with $k = 2$, $(A_1, >_1) = \alpha$ and $(A_2, >_2) = \beta$, we conclude that $f_{\beta.\alpha}$ is monotone in all arguments and that with this interpretation, $\beta.\alpha$ is a well-founded total monotone algebra compatible with $R$, thus $\beta.\alpha \in U(R)$. □

Note that if $R$ contains function symbols of arity $\geq 2$, then $\beta$ has to have the form $\omega^\gamma$, for some $\gamma \geq 0$. Consider the (empty) TRS $(\{f\}, \mathcal{X}, \emptyset)$ with the function symbol $f$ having arity 2. Then $R$ is totally terminating and $u_R = \omega^0 = 1 = \alpha$. Take $\beta = \omega + 1$. Then $\beta.\alpha = \beta$ and $\beta \notin U(R)$ since by lemmas 5.23 and 5.24 it is not possible to define a function $f_\beta : \beta \times \beta \to \beta$ monotone on both arguments.

**Theorem 5.29.** *If $\alpha \in U(R)$ then $\omega^\alpha \in U(R)$.*

**Proof** Again $f_\alpha$ will denote the interpretation of the function symbols $f$ of $R$ in $\alpha$. In this proof we identify $\omega^\alpha$ with the finite non-empty multisets over $\alpha$ instead of all finite multisets. In terms of ordinals this does not make any difference since $\omega^\alpha - 1 = \omega^\alpha$, for $\alpha \geq 1$. Write $\{\!\{a\}\!\}$ for the multiset containing only one element $a$ and $\bigsqcup$ for multiset union.

For constants $c$ and function symbols $f$ of arity $n \geq 1$, we define (see 2.23):

- $c_{\omega^\alpha} = \{\!\{c_\alpha\}\!\}$.
- $f_{\omega^\alpha}(X_1, \ldots, X_n) = \bigsqcup_{x_1 \in X_1} \cdots \bigsqcup_{x_n \in X_n} \{\!\{f_\alpha(x_1, \ldots, x_n)\}\!\} = \overline{f_\alpha}(X_1, \ldots, X_n)$

That the functions $f_{\omega^\alpha}$, for all function symbols $f$, are strictly monotone in each argument is a consequence of lemma 2.29 (note that it is essential that for arities greater than one we restrict ourselves to non-empty multisets).

Let $l \to r$ be an arbitrary rewrite rule and let $\tau : \mathcal{X} \to \omega^\alpha$. We still have to prove that $[\![l, \tau]\!]_{\omega^\alpha} >_{\omega^\alpha} [\![r, \tau]\!]_{\omega^\alpha}$. For any such $\tau$, we define an assignment $\sigma_{max} : \mathcal{X} \to \alpha$ by $\sigma_{max}(x) = \max(\tau(x))$, the maximal element of $\tau(x)$ (recall that for any $x \in \mathcal{X}$, $\tau(x) \neq \emptyset$). Using the definition of $f_{\omega^\alpha}$, it can be easily proven by induction that, for any term $t$, $\max([\![t, \tau]\!]_{\omega^\alpha}) = [\![t, \sigma_{max}]\!]_\alpha$. For all $a \in [\![r, \tau]\!]_{\omega^\alpha}$ we have

$$a \leq \max([\![r, \tau]\!]_{\omega^\alpha}) = [\![r, \sigma_{max}]\!]_\alpha < [\![l, \sigma_{max}]\!]_\alpha \in [\![l, \tau]\!]_{\omega^\alpha}$$

Consequently we obtain $[\![l, \tau]\!]_{\omega^\alpha} >_{\omega^\alpha} [\![r, \tau]\!]_{\omega^\alpha}$. We have proven that $R$ is totally terminating in $\omega^\alpha$, so $\omega^\alpha \in U(R)$. □

Now we are ready to prove modularity of total termination under certain conditions.

**Theorem 5.30.** *Let $(\mathcal{F}_1, \mathcal{X}_1, R_1)$ and $(\mathcal{F}_2, \mathcal{X}_2, R_2)$ be totally terminating TRS's, at least one of which does not containing duplicating rules. Then $R_1 \oplus R_2$ is totally terminating.*

**Proof** Let $\alpha$ and $\beta$ be ordinals in which the proofs of total termination of $R_1$ and $R_2$ can respectively be given. By theorem 5.29 we may, and shall, assume that $\alpha = \omega^\gamma$ and $\beta = \omega^\eta$, for some $\gamma, \eta \geq 1$. Suppose that $R_1$ has no duplicating rules (the other case is symmetric). Identify $\beta = \omega^\eta$ with finite multisets over $\eta$ (including the empty multiset) and define interpretations in $\beta$ for the functions symbols of $R_1$ in the following way:

- $c_\beta = \{\!\{\ \}\!\}$, for any constant $c$, where $\{\!\{\ \}\!\} = 0_\beta$ represents the empty multiset.
- $f_\beta(x_1, \ldots, x_n) = \bigsqcup_{i=1}^{n} x_i$, where $\bigsqcup$ represents multiset union.

For a term $t$ let $X_t$ be the multiset of variables occurring in $t$. For any $\tau : \mathcal{X} \to \beta$ we obtain $[\![t, \tau]\!]_\beta = \bigsqcup_{x \in X_t} \tau(x)$; here the multiset union over an empty index is defined to be the empty multiset. Since there are no duplicating rules the multiset $X_r$ is contained in $X_l$ for all rewrite rules $l \to r \in R_1$. Consequently,

$$[\![l, \tau]\!]_\beta = \bigsqcup_{x \in X_l} \tau(x) \geq \bigsqcup_{x \in X_r} \tau(x) = [\![r, \tau]\!]_\beta$$

Note that the inequality is not strict in general. We have just turned $\beta$ into a well-founded total quasi-model for $R_1$.

Now in $\alpha.\beta$ (the lexicographic product with weight on $\beta$) we define for any $n$-ary function symbol $f$ of $R_1$, $n \geq 0$:

$$f_{\alpha.\beta}((a_1, b_1), \dots, (a_n, b_n)) = (f_\alpha(a_1, \dots, a_n), f_\beta(b_1, \dots, b_n)),$$

where $f_\alpha$ comes from the total termination proof of $R_1$ in $\alpha$. From lemma 5.20 we can identify $\alpha.\beta$ with $(\beta\times\alpha, >_\beta \times >_\alpha)$. Furthermore we can apply theorem 2.99, (with $k = 2$, $(\mathcal{F}^1, \mathcal{X}^1, R^1) = (\mathcal{F}_2, \mathcal{X}_2, R_2)$ and $(\mathcal{F}^2, \mathcal{X}^2, R^2) = (\mathcal{F}_1, \mathcal{X}_1, R_1)$) since $\beta$ is compatible with $R_2$ and a quasi-model for $R_1$ and $\alpha$ is compatible with $R_1$. Consequently $f_{\alpha.\beta}$ is well-defined and strictly monotone in all coordinates (for any $f \in \mathcal{F}_1$) and $(\beta \times \alpha, >_\beta \times >_\alpha)$ is compatible with $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{X}_1 \cup \mathcal{X}_2, R_1 \cup R_2)$, i. e., with $R_1 \oplus R_2$. Using again the identification between $\alpha.\beta$ and $(\beta \times \alpha, >_\beta \times >_\alpha)$, we conclude that $\alpha.\beta \in U(R_1 \oplus R_2)$, proving total termination of $U(R_1 \oplus R_2)$. $\square$

A trivial consequence of theorem 5.30 is the modularity of total termination for string rewriting systems, since by definition they cannot have duplicating rules.

Note that if both $R_1$ and $R_2$ contain duplicating rules, there are particular cases in which we can prove the union is totally terminating; the obvious case is when the proof of termination is given in the same ordinal for both TRS's. For example, let $R_1$ and $R_2$ be totally terminating in $\alpha, \beta$, respectively, and assume there are ordinals $\gamma,\ \delta$ such that $\gamma + \omega^{\cdot^{\cdot^{\omega^\alpha}}} = \delta + \omega^{\cdot^{\cdot^{\omega^\beta}}} = A$, for finite exponentiations on both right summands. Then it easily follows from theorem 5.28 and theorem 5.29 that $\omega^A \in U(R_1 \oplus R_2)$, so $R_1 \oplus R_2$ is totally terminating. Also if there are ordinals $\zeta, \xi$ such that $\zeta.\alpha = \xi.\beta$, total termination of $R_1 \oplus R_2$ can be proven. This last case is more interesting from a practical point of view. For example, since ordinal addition is commutative for ordinals smaller than $\omega$, we have that total termination is modular for TRS's for which a termination proof can be given in ordinals smaller than $\omega^\omega$.

However, not all $\alpha, \beta$ satisfy these properties; for example $\alpha = \omega^2$ and $\beta = \omega^\omega$. The problem boils down to extending functions (of any arity) defined on a certain ordinal, to a given higher one, in such a way that the requirements of total termination are met. That is, in the new ordinal the functions are strictly monotone in all coordinates and for every rule the interpretation of the lefthand-side is greater than that of the righthand-side.

Recently Rubio [94] has provided a new way of proving modularity results. The idea behind his technique is as follows: since each TRS is terminating, there are well-founded rewrite orders compatible with them. By translating terms over the extended signature $\mathcal{F}_1 \cup \mathcal{F}_2$ to multisets of terms over $\mathcal{F}_1$ and $\mathcal{F}_2$ and using some lexicographic product of extensions of the orders compatible with $R_1$ and $R_2$, a new well-founded order for $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{X})$ is built that (in some conditions) is compatible with $R_1 \oplus R_2$. In this way, Rubio presented another proof of theorem 5.30. Interesting enough his technique suffers from the same problem as ours in the presence of duplicating rules in both systems. It is still an open problem whether the condition of non-existence of duplicating rules in one of the systems can be dispensed with.

## 5.4 String Rewriting Systems

In the previous sections we saw that when trying to prove total termination of TRS's containing at least a function symbol of arity $n \geq 2$, only ordinals of the form $\omega^\eta$ were relevant. In this section, we discuss whether the same holds for string rewriting systems (SRS's), i. e., rewriting systems containing only unary function symbols. First we need a lemma.

**Lemma 5.31.** *Let $\alpha \neq 0$ and $f : \alpha \to \alpha$ be (strictly) monotone. Then there is a unique ordinal $\eta$ such that $\omega^\eta \leq \alpha < \omega^{\eta+1}$ and $f(\omega^\eta) \subseteq \omega^\eta$.*

**Proof** Existence and uniqueness of $\eta$ satisfying $\omega^\eta \leq \alpha < \omega^{\eta+1} = \omega^\eta.\omega$ is guaranteed by lemma 5.17. If $\alpha = \omega^\eta$, we are done. Suppose now that $\alpha > \omega^\eta$. Again by lemma 5.17 we can write $\alpha = \omega^\eta.\gamma + \delta$, where the uniquely determined ordinals $\gamma, \delta$ satisfy $1 \leq \gamma < \omega$, $0 \leq \delta < \omega^\eta$.

We suppose $f(\omega^\eta) \not\subseteq \omega^\eta$ and will derive a contradiction. That means there is an element $b \in \omega^\eta$ such that $f(b) \geq \omega^\eta$. Since $\omega^\eta$ is principal additive and $b < \omega^\eta$, we can conclude that $b + \omega^\eta = \omega^\eta$ and therefore $b + \omega^\eta.n = \omega^\eta.n$, for any ordinal $1 \leq n < \omega$.

Recall that $\alpha = \omega^\eta.\gamma + \delta$. We consider two cases, namely $\delta = 0$ and $\delta > 0$. If $\delta = 0$ then $\alpha = \omega^\eta.\gamma$ and $1 < \gamma < \omega$. Since $b + \omega^\eta.(\gamma - 1) = \omega^\eta.(\gamma - 1) < \alpha$ we can apply $f$ to $b + \omega^\eta.(\gamma - 1)$. We have

$$\begin{array}{rcl}
\alpha & > & \\
f(b + \omega^\eta.(\gamma - 1)) & \geq & \text{(by lemma 5.15)} \\
f(b) + \omega^\eta.(\gamma - 1) & \geq & \text{(by hypothesis)} \\
\omega^\eta + \omega^\eta.(\gamma - 1) & = & \\
\omega^\eta.(1 + (\gamma - 1)) & = & \text{(definition of difference)} \\
\omega^\eta.\gamma & = & \\
\alpha & &
\end{array}$$

which is a contradiction.

Suppose now that $\delta > 0$. Similarly we have $b + \omega^\eta.\gamma = \omega^\eta.\gamma < \alpha$ (recall that $1 \leq \gamma < \omega$). Again we can apply $f$ to $b + \omega^\eta.\gamma$ and conclude[2]

$$\alpha > f(b + \omega^\eta.\gamma) \geq f(b) + \omega^\eta.\gamma \geq \omega^\eta + \omega^\eta.\gamma = \omega^\eta.(1 + \gamma) = \omega^\eta.(\gamma + 1)$$

Then $\alpha = \omega^\eta.\gamma + \delta > \omega^\eta.\gamma + \omega^\eta$, and by left-cancellation we get $\delta > \omega^\eta$, which is a contradiction. □

Remember that for a totally terminating TRS $R$ the ordinal $u_R$ is defined to be the minimal ordinal in which the total termination proof can be given.

**Theorem 5.32.** *Let $R$ be a totally terminating SRS. Then $u_R = \omega^\eta$ for some $\eta \geq 1$.*

[2] Note that addition is commutative for ordinals smaller than $\omega$.

**Proof** From lemma 5.31 we obtain a unique ordinal $\eta$ such that $\omega^\eta \leq u_R < \omega^{\eta+1}$ and $f_\alpha(\omega^\eta) \subseteq \omega^\eta$ for all function symbols $f$. We now prove, by induction on the structure of terms, that $[\![t, \tau]\!]_{\omega^\eta} \in \omega^\eta$, for any assignment $\tau : \mathcal{X} \to \omega^\eta$ and where $f_{\omega^\eta}$ is taken to be $f_\alpha$ (since $f_\alpha(\omega^\eta) \subseteq \omega^\eta$, this is possible). If $t$ is a variable then $[\![t, \tau]\!]_{\omega^\eta} = \tau(t) \in \omega^\eta$ and the result holds. Suppose now that $t = f(s)$, for some $f \in \mathcal{F}$, and the result is valid for term $s$. Then $[\![t, \tau]\!]_{\omega^\eta} = f_{\omega^\eta}([\![s, \tau]\!]_{\omega^\eta})$. By induction hypothesis $[\![s, \tau]\!]_{\omega^\eta} \in \omega^\eta$ and so $f_{\omega^\eta}([\![s, \tau]\!]_{\omega^\eta}) \in \omega^\eta$. This means that by restricting $f : u_R \to u_R$ to $\omega^\eta$ for all function symbols $f$, we also have a proof of total termination of $R$ in $\omega^\eta$, so $\omega^\eta \in U(R)$. Since $u_R$ is the minimum of $U(R)$ and $\omega^\eta \leq u_R$ we obtain $u_R = \omega^\eta$. □

Note that this result is essentially weaker than theorem 5.25 for the case of arity $> 1$. The fact that $u_R = \omega^\eta$ does not imply that every ordinal in $U(R)$ is of that shape. For example, every proof of total termination of a SRS in $\omega$ is easily extended to a similar proof in $\omega + \omega$, which is not of the required shape. We can nevertheless impose some restrictions on $U(R)$ as we see below.

**Lemma 5.33.** *Let $R$ be a totally terminating (non-empty) SRS. Then $U(R) \subseteq Lim$.*

**Proof** Since we only use non-empty monotone algebras, we have that $0 \notin U(R)$. Let $\alpha + 1$ be a successor ordinal and suppose that $\alpha + 1 \in U(R)$. It can be seen, by induction on the context and using lemma 5.2, that $[\![C[x], \tau]\!]_\gamma \geq \tau(x)$, for any $\tau : \mathcal{X} \to \gamma$, where $\gamma$ is an arbitrary ordinal in $U(R)$ (non-empty by hypothesis). Since $R$ is a SRS, rules have the form $C[x] \to D[x]$, where $C, D$ are contexts with $D$ possibly trivial. Let $\tau : \mathcal{X} \to \alpha + 1$ be such that $\tau(x) = \alpha$. Then by total termination requirements, $[\![C[x], \tau]\!]_{\alpha+1} > [\![D[x], \tau]\!]_{\alpha+1} \geq \tau(x) = \alpha$. That gives $[\![C[x], \tau]\!]_{\alpha+1} \geq \alpha + 1$, which is a contradiction with the fact that $[\![C[x], \tau]\!]_{\alpha+1} < \alpha + 1$. So if $\gamma \in U(R)$, $\gamma$ is not zero and it is not a successor ordinal; therefore it must be a limit ordinal. □

A natural operation on SRS's is reversal: all lefthand-sides and righthand-sides are reversed, considered as strings. For example, the reverse of $f(f(g(x)))$ is $g(f(f(x)))$. Clearly there is a bijective correspondence between reductions in the original system and reductions in the reversed system. As a consequence, a SRS is terminating if and only if the reversed system is terminating. However, a similar observation does not hold for total termination. For example, the system

$$\begin{array}{rcl} f(f(x)) & \to & f(g(x)) \\ g(g(x)) & \to & g(f(x)) \end{array}$$

is not totally terminating since $f(a)$ and $g(a)$ are incomparable for any $a$ in any corresponding monotone algebra. On the other hand, the reversed system

$$\begin{array}{rcl} f(f(x)) & \to & g(f(x)) \\ g(g(x)) & \to & f(g(x)) \end{array}$$

is totally terminating in the natural numbers: just take the interpretation $f(x) = 4x+2, g(x) = 4x+1$, for $x$ even, and $f(x) = 4x, g(x) = 4x+3$, for $x$ odd. Further, if for a totally terminating system the reversed system is totally terminating too, the corresponding ordinal may change. An example is $f(g(x)) \to g(f(f(x)))$; in the next section we shall see that the minimal ordinal of this totally terminating system is $\omega^2$, while termination of the reversed system $g(f(x)) \to f(f(g(x)))$ can be proved in the natural numbers by choosing $f(x) = x + 1, g(x) = 3x$.

We conclude this section with some remarks about TRS's that also contain constants, and no function symbols of arity $> 1$. In all other cases we know, from theorems 5.25 and 5.32, that total termination implies that $u_R = \omega^\eta$ for some $\eta \geq 0$. However, if there are constants then the proof of theorem 5.32 no longer holds since the interpretation of the constants may be too big. The simplest example is the TRS $R$ consisting of the rule $a \to b$, where $a$ and $b$ are constants. It is totally terminating and $u_R = 2$. If we allow infinitely many constants and rewrite rules then for any ordinal $\alpha$ a TRS $R$ can be given with $u_R = \alpha$.

The infinite TRS $R$ consisting of the rules

$$\begin{array}{lcl} c_{j+1} & \to & f^i(c_j), \text{ with } 0 \leq j < k, \text{ and for each } i < \omega, \\ f(x) & \to & x \end{array}$$

(where $k \geq 1$ is a fixed natural) satisfies $u_R = \omega.(k+1)$. Note that by lemma 5.2 and the definition of successor ordinal, the interpretation $f(x) = x + 1$ is the "least" possible one for $f$ compatible with the rule $f(x) \to x$. If we interpret $c_0$ as $0$ (again the smallest possible choice) then the interpretation of $c_1$ has to be at least $\omega$ and similarly if we interpret $c_j$ by $\omega.j$ then we must have $c_{j+1} \geq \bigvee_{x<\omega} \omega.j + x = \omega.(j+1)$. Finally $f(c_k) > c_k$ implies that $u_R$ has to be the least limit ordinal greater than $\omega.k$ , i. e., $\omega.(k+1)$. It seems that in a similar way, for every ordinal $\alpha < \omega^\omega$ an infinite TRS $R$ with finitely many unary symbols and constants can be constructed satisfying $u_R = \alpha$.

We conjecture that for any finite totally terminating TRS $R$ without function symbols of arity $> 1$ and containing at least one rule of the form $C[x] \to D[x]$, for some contexts $C, D$, the ordinal $u_R$ is of the form $\omega^\eta$.

However, even if $u_R$ is not of the form $\omega^\eta$, by theorem 5.29 we need only consider those ordinals for proving total termination.

## 5.5 Minimal Ordinals

As we have seen previously, when trying to establish total termination of (finite or infinite) SRS's or TRS's containing symbols of arity $> 1$, we only need to consider algebras with type $\omega^\eta$ for some $\eta \geq 0$. Furthermore the minimal ordinal $u_R$ associated with any totally terminating SRS or TRS with function symbols of arity $\geq 2$ is always of the form $\omega^\alpha$, for some ordinal $\alpha$. This has interesting consequences if the ordinals considered are below $\epsilon_0$. As usual $\epsilon_0$ is defined to be the minimal $\epsilon$-ordinal, i. e., the minimal ordinal $\alpha$ satisfying $\alpha = \omega^\alpha$. It can also be defined as $\lim_{n<\omega} \gamma_n$ where $\gamma_0 = 1$ and $\gamma_{n+1} = \omega^{\gamma_n}$; finally it is the only ordinal $\lambda$ satisfying $\alpha < \lambda \Rightarrow \alpha < \omega^\alpha < \lambda$. Therefore if $u_R = \omega^\alpha < \epsilon_0$, then $u_R$ can be constructed in finitely many steps using the ordinal $\omega$, ordinal (lexicographic) product and ordinal exponentiation (multiset construction). This follows by induction on $\alpha$. If $\alpha = 0$, it is true. For $\alpha \geq 1$, if $u_R = \omega^\alpha < \epsilon_0$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_k}$ $(\omega > k \geq 1)$ is the Cantor normal form of $\alpha$, we have

$\epsilon_0 > \omega^\alpha > \alpha \geq \omega^{\alpha_i} > \alpha_i$, for all $i$, $1 \leq i \leq k$. By induction hypothesis each $\alpha_i$ is finitely constructed and since $u_R = \omega^\alpha = \omega^{\omega^{\alpha_1}+\cdots+\omega^{\alpha_k}} = \omega^{\omega^{\alpha_1}} \cdot \cdots \cdot \omega^{\omega^{\alpha_k}}$, so is $u_R$.

Is it the case that all ordinals of the form $\omega^\alpha$ are important or can we restrict the class even further? Partially answering this question, we have the following result.

**Theorem 5.34.** *For any ordinal $\eta$ with $1 \leq \eta \leq \omega$, there is a SRS $R$ such that $u_R = \omega^\eta$.*

**Proof** For $\eta = 1$, the string rewriting system $f(x) \to x$ satisfies the requirements by interpreting $f$ as the successor function in $\omega$.

For $1 < \eta < \omega$, let $R_\eta$ consist of the $\eta - 1$ rules

$$f_i(f_{i+1}(x)) \to f_{i+1}(f_i(f_i(x)))$$

for $i = 1, \ldots, \eta - 1$. We will show that $u_{R_\eta} = \omega^\eta$ for any $\eta$; for $\eta = 2$ this was already shown in Zantema [108] (report version).

For the TRS $R$ defined by

$$\begin{array}{lcl} f(g(x)) & \to & g(f(f(x)) \\ f(h(x)) & \to & h(g(x)) \end{array}$$

we shall prove that $u_R = \omega^\omega$.

According to theorem 5.32 the only ordinals of interest are of the shape $\omega^\alpha$, for some ordinal $\alpha$. In the following we will prove that $u_{R_\eta} \geq \omega^\eta$, for $1 < \eta \leq \omega$, and finally we establish $u_{R_\eta} \leq \omega^\eta$ by giving an interpretation in $\omega^\eta$ that satisfies all the requirements of total termination. Proving that $u_{R_\eta} \geq \omega^\eta$ requires some work; we first have to introduce some auxiliary lemmas and definitions.

To simplify the treatment we will use the same representation for a function symbol in a TRS and for the corresponding interpretation function in an ordinal.

**Lemma 5.35.** *Let $R$ be a TRS totally terminating in an ordinal $\alpha$ and containing a rule of the form*

$$F(G(x)) \to G(F(F(x)))$$

*Then* $\forall\ \omega > k \geq 1\ \forall a \in \alpha :\ G(a) > F^k(a)$.

**Proof** By induction on $k$. If $k = 1$, for any $a \in \alpha$, we have

$$\begin{array}{rcl} F(G(a)) & > & \text{(by total termination)} \\ G(F(F(a))) & \geq & \text{(by lemma 5.2)} \\ F(F(a)) & & \end{array}$$

From $F(G(a)) > F(F(a))$ using lemma 5.2 we conclude that $G(a) > F(a)$, as we wanted.

Assume that $G(a) > F^l(a)$ for some $\omega > l \geq 1$ and for all $a \in \alpha$. Then

$$\begin{array}{rcl} F(G(a)) & > & \text{(by total termination)} \\ G(F(F(a))) & > & \text{(by induction hypothesis)} \\ F^l(F(F(a))) & = & \\ F(F^{l+1}(a)) & & \end{array}$$

Again by lemma 5.2, we conclude that $G(a) > F^{l+1}(a)$. Since $a$ was arbitrary, the result holds. □

From the above lemma we conclude that $u_{R_\eta} \geq \omega^2$, since the property only holds in $\omega$ if $F$ is the identity function $Id$ (but the interpretation of the rule rules out that possibility) and, from theorem 5.32, $u_{R_\eta} = \omega^\alpha$, for some ordinal $\alpha$ (this is in fact another proof that for $\eta = 2$, $u_{R_\eta} \geq \omega^2$).

Given a function $F : \omega^m \to \omega^m$, we define

$$\mathcal{O}(F) = min\{k < \omega \mid 0 \leq k \leq m \wedge \forall a \in \omega^m : a + \omega^k > F(a)\}$$

The meaning of $\mathcal{O}(F)$ is as follows: if we consider the elements of $\omega^m$ as vectors of size $m$ (possible since $\omega^m$ is the lexicographic product $\omega \times \cdots \times \omega$), then $\mathcal{O}(F)$ denotes the highest-order coordinate $k$ which may be changed by function $F$. We note that $0 \leq \mathcal{O}(F) \leq m$; $\mathcal{O}(F) = 0 \iff F$ is the identity function. Next some necessary properties of $\mathcal{O}$ are introduced.

**Lemma 5.36.** *Let $F, G$ be monotone functions from $\omega^m$ to $\omega^m$, for some $m \geq 1$. Then $(\forall x \in \omega^m : G(x) \geq F(x)) \Rightarrow \mathcal{O}(G) \geq \mathcal{O}(F)$.*

**Proof** Suppose $0 \leq j = \mathcal{O}(G) < \mathcal{O}(F) = k \leq m$. Then $\exists a \in \omega^m$ such that $G(a) < a + \omega^j \leq a + \omega^{k-1} \leq F(a)$, contradicting the hypothesis. □

**Lemma 5.37.** *Let $F, G$ be monotone functions from $\omega^m$ to $\omega^m$, for some $m \geq 1$. Then $\mathcal{O}(F \circ G) = \max(\mathcal{O}(F), \mathcal{O}(G))$.*

**Proof** Let $k = \max(\mathcal{O}(F), \mathcal{O}(G))$. If $k = 0$ then $F \circ G = Id$, so $F = G = Id$ and the result holds. Suppose that $k \geq 1$. For any $0 \leq j < k$, $\exists a \in \omega^m$ such that either $F(a) \geq a + \omega^j$ or $G(a) \geq a + \omega^j$. In both cases, using monotonicity and lemma 5.2, we conclude $F(G(a)) \geq a + \omega^j$, hence $\mathcal{O}(F \circ G) \geq k$.

Note that since $a \leq G(a)$, we can write $G(a) = a + (G(a) - a)$, for any $a$. Since $\mathcal{O}(G) \leq k$ we conclude that $a + \omega^k > G(a) = a + (G(a) - a)$, for any $a$. By strict monotonicity of addition in the right argument, we conclude that $\omega^k > G(a) - a$, for any $a$. This combined with $\mathcal{O}(F) \leq k$ and lemma 5.11, gives us

$$a + \omega^k = a + (G(a) - a) + \omega^k = G(a) + \omega^k > F(G(a))$$

Hence $\mathcal{O}(F \circ G) \leq k$. □

**Lemma 5.38.** *Let $R$ be a TRS containing a rule of the form*

$$F(G(x)) \to G(F(F(x)))$$

*and totally terminating in an ordinal $\omega^m$ for some $m < \omega$. Then $\mathcal{O}(G) > \mathcal{O}(F)$.*

**Proof** By the assumption of total termination of $R$, both $F$ and $G$ are not the identity, so $\mathcal{O}(F), \mathcal{O}(G) > 0$. Let $\mathcal{O}(F) = k$, for some $1 \leq k \leq m$. We have $\forall x \in \omega^m$ : $x + \omega^k > F(x)$ and $\exists a \in \omega^m : a + \omega^{k-1} \leq F(a)$. Fix this element $a \in \omega^m$. Next we prove by induction on $j$ that $F^j(a) \geq a + \omega^{k-1}.j$, for any $1 \leq j < \omega$.[3] For $j = 1$ this holds by hypothesis. Suppose the property holds for any $i \leq j$. Then

$$\begin{array}{rcl} F^{j+1}(a) & = & \\ F(F^j(a)) & \geq & \text{(by monotonicity of } F \text{ and induction hypothesis)} \\ F(a + \omega^{k-1}.j) & \geq & \text{(by lemma 5.15)} \\ F(a) + \omega^{k-1}.j & \geq & \text{(by induction hypothesis)} \\ a + \omega^{k-1} + \omega^{k-1}.j & = & \text{(associativity of addition)} \\ a + \omega^{k-1}(1 + j) & = & \text{(commutativity of "+" below } \omega) \\ a + \omega^{k-1}.(j + 1) & & \end{array}$$

But by lemma 5.35, $G(a) \geq F^j(a) \geq a + \omega^{k-1}.j$, for any $j < \omega$. Applying this lemma we also conclude that $G \geq F$ (in the usual pointwise sense) and therefore $\mathcal{O}(G) \geq \mathcal{O}(F)$ (by lemma 5.36). Let $\mathcal{O}(F) = k$, then

$$a + \omega^k > F^j(a) \geq a + \omega^{k-1}.j$$

for any $j < \omega$. But then

$$G(a) \geq \bigvee_{j<\omega} (a + \omega^{k-1}.j) = a + \omega^{k-1}.(\bigvee_{j<\omega} j) = a + \omega^{k-1}.\omega = a + \omega^k$$

and therefore $\mathcal{O}(G) > k = \mathcal{O}(F)$. □

Now going back to our original system $R_\eta$ and applying the previous results to every rule, we get $0 < \mathcal{O}(f_1) < \cdots < \mathcal{O}(f_\eta) \leq m$. So $m \geq \eta$, hence, $u_{R_\eta} \geq \omega^\eta$. In order to show that $u_{R_\eta}$ is indeed $\omega^\eta$ we still have to give an interpretation in $\omega^\eta$. Identify $\omega$ with strictly positive integers and define in $\omega^\eta$:

$$f_i(x_1, \ldots, x_\eta) = (x_1, \ldots, x_{i-1}, x_i + 2^{x_{i+1}}, x_{i+1}, \ldots, x_\eta)$$

for $i = 1, \ldots, \eta$, where $x_{\eta+1}$ is defined to be $1$ and where $(x_1, \ldots, x_\eta)$ is an element of the right-to-left lexicographic product $\omega^\eta$.[4] It is not difficult to see that the functions $f_i$,

[3] Recall that "." denotes ordinal multiplication.

[4] The element $(x_1, \ldots x_\eta)$ can also be represented as $\omega^{\eta-1}.x_\eta + \cdots + \omega^0.x_1$, its normal formal form as introduced after theorem 5.18.

$i = 1, \ldots, \eta$, are monotone. We check that this interpretation is compatible with the rewrite rules. Given $\tau : \mathcal{X} \to \omega^\eta$, for a rule $f_i(f_{i+1}(x)) \to f_{i+1}(f_i(f_i(x)))$ we have

$$\begin{array}{rcl} [\![ f_i(f_{i+1}(x)), \tau ]\!]_{\omega^\eta} & = & f_i(f_{i+1}(x_1, \ldots, x_\eta)) \\ & = & f_i(x_1, \ldots, x_i, x_{i+1} + 2^{x_{i+2}}, x_{i+2}, \ldots, x_\eta) \\ & = & (x_1, \ldots, x_i + 2^{x_{i+1}+2^{x_{i+2}}}, x_{i+1} + 2^{x_{i+2}}, \ldots, x_\eta) \end{array}$$

and

$$\begin{array}{rcl} [\![ f_{i+1}(f_i(f_i(x))), \tau ]\!]_{\omega^\eta} & = & f_{i+1}(f_i(f_i(x_1, \ldots, x_\eta))) \\ & = & f_{i+1}(f_i(x_1, \ldots, x_{i-1}, x_i + 2^{x_{i+1}}, \ldots, x_\eta) \\ & = & f_{i+1}(x_1, \ldots, x_{i-1}, x_i + 2^{x_{i+1}+1}, \ldots, x_\eta) \\ & = & (x_1, \ldots, x_i + 2^{x_{i+1}+1}, x_{i+1} + 2^{x_{i+2}}, \ldots, x_\eta) \end{array}$$

where $\tau(x) = (x_1, \ldots, x_\eta)$. Since $x_{i+2} \geq 1$ we have that $x_i + 2^{x_{i+1}+2^{x_{i+2}}} >_{\mathbb{N}} x_i + 2^{x_{i+1}+1}$ so the interpretation of the lhs is always greater than that of the rhs. Since all the requirements of total termination are fulfilled the system is totally terminating in $\omega^\eta$.

For the ordinal $\omega^\omega$ we consider the SRS $R$

$$\begin{array}{rcl} f(g(x)) & \to & g(f(f(x))) \\ f(h(x)) & \to & h(g(x)) \end{array}$$

We shall prove $u_R = \omega^\omega$; first we show that we cannot prove total termination of $R$ in $\omega^n$, for any $1 < n < \omega$. Suppose we can, then there are strictly monotone functions $f, g, h : \omega^n \to \omega^n$ satisfying, for all $x \in \omega^n$,

$$f(g(x)) > g(f(f(x))) \quad \text{and} \quad f(h(x)) > h(g(x))$$

Let $\mathcal{O}$ be defined as before. By lemma 5.38, $\mathcal{O}(f) < \mathcal{O}(g)$. Let $j = \mathcal{O}(f)$ and thus $0 < j < \mathcal{O}(g) \leq n$; then $\forall a \in \omega^n : \ a + \omega^j > f(a)$ and $\exists a' \in \omega^n : a' + \omega^j \leq g(a')$. Using lemma 5.15, we derive $h(g(a')) \geq h(a' + \omega^j) \geq h(a') + \omega^j > f(h(a'))$, contradicting the requirements of total termination.

To prove $u_R = \omega^\omega$ we still need to present an interpretation in $\omega^\omega$. Identify $\omega$ with natural numbers, including $0$. Recall from Theorem 5.21 that we can identify an element $X \in \omega^\omega$ with a certain function $X : \omega \to \omega$; we therefore denote such an element by the sequence $(p_0, \ldots, p_k)$ where:

- $X(i) = p_i$, if $0 \leq i \leq k$.
- $X(k) \neq 0$ and $X(i) = 0$ for $i > k$.

Let $\mathcal{A}$ be $\omega^\omega$ restricted to the part for which $k \geq 1$ in this notation. This means that we skip the first $\omega$ elements of $\omega^\omega$; since $\omega^\omega - \omega = \omega^\omega$ this does not affect the ordinal. We now define $f, g, h : \mathcal{A} \to \mathcal{A}$ by:

$$\begin{array}{rcl} f(p_0, \ldots, p_{k-1}, p_k) & = & (p_0 + p_k, \ldots, p_{k-1} + p_k, p_k) \\ g(p_0, \ldots, p_{k-1}, p_k) & = & (p_0, \ldots, p_{k-1}, 2.p_k + 1) \\ h(p_0, \ldots, p_{k-1}, p_k) & = & (p_0, \ldots, p_{k-1}, p_k, 0, 1) \end{array}$$

Given two multisets $(p_0, \ldots, p_k)$ and $(q_0, \ldots, q_m)$, if $(p_0, \ldots, p_k) >_{\omega^\omega} (q_0, \ldots, q_m)$ then either $k >_\omega m$ or $k = m$ and $\exists 0 \le i \le k : \; p_i >_\omega q_i$ and for $i < j \le k : \; p_j = q_j$. Using this fact and with some easy calculations, it can be shown that the functions are indeed strictly monotone. We check that the interpretation of the lhs of the rules is always greater than that of the rhs. Let $\tau : \mathcal{X} \to \omega^\omega$ be an assignment such that $\tau(x) = (p_0, \ldots, p_k)$. For the rule $f(g(x)) \to g(f(f(x)))$ we have

$$\begin{aligned} [\![f(g(x)), \tau]\!]_{\omega^\omega} &= f(g(p_0, \ldots, p_k)) \\ &= f(p_0, \ldots, 2p_k + 1) \\ &= (p_0 + 2p_k + 1, \ldots, p_{k-1} + 2p_k + 1, 2p_k + 1) \end{aligned}$$

and

$$\begin{aligned} [\![g(f(f(x))), \tau]\!]_{\omega^\omega} &= g(f(f(p_0, \ldots, p_k))) \\ &= g(f(p_0 + p_k, \ldots, p_{k-1} + p_k, p_k)) \\ &= g(p_0 + 2p_k, \ldots, p_{k-1} + 2p_k, p_k) \\ &= (p_0 + 2p_k, \ldots, p_{k-1} + 2p_k, 2p_k + 1) \end{aligned}$$

and indeed $[\![f(g(x)), \tau]\!]_{\omega^\omega} >_{\omega^\omega} [\![g(f(f(x))), \tau]\!]_{\omega^\omega}$, since $p_{k-1} + 2p_k + 1 >_\omega p_{k-1} + 2p_k$. For the second rule $f(h(x)) \to h(g(x))$, we have

$$\begin{aligned} [\![f(h(x)), \tau]\!]_{\omega^\omega} &= f(h(p_0, \ldots, p_k)) \\ &= f(p_0, \ldots, p_{k-1}, p_k, 0, 1) \\ &= (p_0 + 1, \ldots, p_{k-1} + 1, p_k + 1, 1, 1) \end{aligned}$$

while

$$\begin{aligned} [\![h(g(x)), \tau]\!]_{\omega^\omega} &= h(g(p_0, \ldots, p_k)) \\ &= h(p_0, \ldots, p_{k-1}, 2p_k + 1) \\ &= (p_0, \ldots, p_{k-1}, 2p_k + 1, 0, 1). \end{aligned}$$

Because the coordinate of order $k + 2$ is greater for the lhs than for the rhs ($1$ and $0$ respectively) and the coordinate of order $k + 3$ is the same for both lhs and rhs, we conclude that $[\![f(h(x)), \tau]\!]_{\omega^\omega} >_{\omega^\omega} [\![h(g(x)), \tau]\!]_{\omega^\omega}$. This concludes the proof of theorem 5.34. □

We end this section with an example based on the battle of Hercules and the Hydra (see Kirby and Paris [59]; another version of this game appears in Dershowitz and Jouannaud [26]). For this system we conjecture $u_R = \epsilon_0$.

The Hydra is represented as a finite tree. We code the tree using a binary symbol $c$: a tree consisting of a root and descendants $t_1, \ldots, t_k$ is represented as $c(t_1, c(t_2, \ldots, c(t_{k-1}, t_k) \ldots))$, that is $c(D, S)$ represents a node whose descendants are coded in the subtree $D$ and whose siblings are coded in subtree $S$. Leaves are represented by the constant $nil$.

On each stage, a leaf node is selected and deleted. Afterwards, $k \ge 0$ copies of the subtree containing the now missing leaf, are added to the second ancestor of the selected leaf. The number of copies $k$ is chosen randomly. The game can be represented as the infinite TRS $H$:

$$\begin{array}{rcll} c(nil,x) & \to & x & \\ c(c(nil,x),y) & \to & copy(n,x,y) & \\ copy(s(k),x,y) & \to & copy(k,x,c(x,y)) & \\ copy(0,x,y) & \to & y & \\ n & \to & s^i(0) & \text{for each } i \geq 0 \end{array}$$

Termination of the system above cannot be proven by recursive path order with status. This remains true even if we allow the precedence to be a quasi-order. To see why note that in the third rule (counting from top to bottom) in order to have $copy(s(k),x,y) >_{rpo} copy(k,x,c(x,y))$, the status of $copy$ has to be lexicographic with permutation $\pi$ which satisfies $\pi(1)=1$ or satisfies $\pi(1)=2$ and $\pi(2)=1$, and the precedence $\triangleright$ has to satisfy $copy \triangleright c$. But then, the version of $>_{rpo}$ thus obtained is incompatible with the second rule.

The Knuth-Bendix order will not yield a termination proof either since it can only be applied to non-duplicating systems (i. e., systems where for every rule, the number of occurrences of a variable on the righthand-side is never bigger than the number of occurrences of the same variable in the lefthand-side) and the third rule is duplicating. The system is however totally terminating in $\epsilon_0$ with the following interpretation:

$$\begin{array}{lcl} 0 & = & 0 \\ n & = & \omega \\ nil & = & 2 \\ s(x) & = & x+1 \\ c(x,y) & = & \omega^x \oplus y \\ copy(k,x,y) & = & \omega^{k\oplus x\oplus 1} \oplus x \oplus y. \end{array}$$

Here elements of $\epsilon_0$ are identified with ordinals $< \epsilon_0$; the operation $\oplus$ represents natural addition (see remarks after theorem 5.18 and example 5.19). Well-definedness of these functions follows from standard properties of $\epsilon_0$. Since natural addition in associative, commutative and strictly monotone in both coordinates, it is not difficult to see that the functions above are strictly monotone in each coordinate. Further it is easy to check that the interpretations of all lefthand-sides are strictly greater than the interpretations of the corresponding righthand-sides. We show it here only for the second rule. For any assignment $\tau : \mathcal{X} \to \epsilon_0$, we have

$$\begin{array}{lll} [\![ l_2,\tau ]\!]_{\epsilon_0} & = [\![ c(c(nil,x),y),\tau ]\!]_{\epsilon_0} & = \omega^{\omega^2\oplus\tau(x)} \oplus \tau(y) \\ [\![ r_2,\tau ]\!]_{\epsilon_0} & = [\![ copy(n,x,y),\tau ]\!]_{\epsilon_0} & = \omega^{\omega\oplus\tau(x)\oplus 1} \oplus \tau(x) \oplus \tau(y) \end{array}$$

Since $\omega \oplus 1 < \omega^2$ (because $\omega^2$ is additive principal and $\omega, 1 < \omega^2$), we get

$$\omega \oplus \tau(x) \oplus 1 < \omega^2 \oplus \tau(x) \;\Rightarrow\; \omega^{\omega\oplus\tau(x)\oplus 1} < \omega^{\omega^2\oplus\tau(x)}$$

But also, $\tau(x) < \epsilon_0 \;\Rightarrow\; \tau(x) < \omega^{\tau(x)} < \omega^{\omega^2\oplus\tau(x)}$. Because $\omega^{\omega^2\oplus\tau(x)}$ is additive principal, we get

$$\omega^{\omega\oplus\tau(x)\oplus 1} \oplus \tau(x) < \omega^{\omega^2\oplus\tau(x)} \;\Rightarrow\; \omega^{\omega\oplus\tau(x)\oplus 1} \oplus \tau(x) \oplus \tau(y) < \omega^{\omega^2\oplus\tau(x)} \oplus \tau(y)$$

and thus $[\![l_2, \tau]\!]_{\epsilon_0} >_{\epsilon_0} [\![r_2, \tau]\!]_{\epsilon_0}$.

So $u_H \leq \epsilon_0$. It can be proven that if $f : \alpha \times \alpha \to \alpha$ is strictly increasing in each argument then $f(x, y) \geq max\{x + y, y + x\}$, for any $x, y \in \alpha$. Using this fact and rules 2, 3 and 4, it can be seen that for any substitution $\tau$, the lhs of rule 2 ($l_2$) has to fullfil $[\![l_2, \tau]\!]_{u_H} > \tau(x) + \cdots + \tau(x)$, where $\tau(x)$ can appear any finite number of times. Consequently $[\![l_2, \tau]\!]_{u_H} \geq \tau(x). \bigvee_{i<\omega} i = \tau(x).\omega$. So $[\![l_2, \tau]\!]_{u_H} \geq \omega^{\eta_1+1}$, where $\tau(x)$ has as Cantor normal form $\omega^{\eta_1} p_1 + \cdots + \omega^{\eta_0} p_0$. With this last inequality it is not difficult to derive $u_H \geq \omega^\omega$. Consequently $\omega^\omega \leq u_H \leq \epsilon_0$ and we conjecture that $u_H = \epsilon_0$.

## 5.6 Characterizing Total Termination

The concept of total termination as defined in chapter 2, definition 2.110, relies on the concept of monotone algebra. Here we give an alternative definition for total termination that does not refer to monotone algebras. First we need a very weak form of modularity of total termination.

**Lemma 5.39.** *Let $\mathcal{F}$ and $\mathcal{F}'$ be two disjoint signatures. Then $(\mathcal{F}, \mathcal{X}, R)$ is totally terminating if and only if $(\mathcal{F} \cup \mathcal{F}', \mathcal{X}, R)$ is totally terminating.*

**Proof** For the if part, since $(\mathcal{F} \cup \mathcal{F}', \mathcal{X}, R)$ is totally terminating there is a total monotone algebra compatible with $(\mathcal{F} \cup \mathcal{F}', \mathcal{X}, R)$. The same algebra is obviously compatible with $(\mathcal{F}, \mathcal{X}, R)$.

For the only-if part, we take $R_1$ to be $(\mathcal{F}', \emptyset, \emptyset)$ and $R_2$ to be $(\mathcal{F}, \mathcal{X}, R)$. Then all conditions of theorem 5.30 are satisfied so we can conclude that $R_1 \oplus R_2$ is totally terminating (note that $(\mathcal{F}', \emptyset, \emptyset)$ is totally terminating in, for example, $\omega$, since in $\omega$ we can define monotone functions with an arbitrary number of arguments). Since $R_1 \oplus R_2 = (\mathcal{F} \cup \mathcal{F}', \mathcal{X}, R)$, the result holds. □

**Theorem 5.40.** *Let $\mathcal{F}'$ be $\mathcal{F}$ extended with a new constant if $\mathcal{F}$ does not contain any. Then $R$ is totally terminating if and only if there is a strict partial order $>$ on $\mathcal{T}(\mathcal{F}')$, such that*

- *$>$ is a well-order (i. e., total and well-founded);*
- *$>$ is closed under ground contexts, i. e., if $C[\ ]$ is a linear ground context, and $t$ and $s$ are ground terms with $s > t$ then $C[s] > C[t]$;*
- *$l\sigma > r\sigma$ for every rule $l \to r$ in $R$ and every ground substitution $\sigma$.*

**Proof** First, consider the if part. Since $>$ is total and well-founded on $\mathcal{T}(\mathcal{F}')$, we can make $(\mathcal{T}(\mathcal{F}'), >)$ a well-founded total monotone algebra over $\mathcal{F}$ by interpreting each function symbol in $\mathcal{F}$ by itself. From the properties of $>$ follows that $R$ is compatible with this interpretation, yielding total termination of $R$.

For the only-if part, first note that total termination of $(\mathcal{F}, \mathcal{X}, R)$ implies total termination of $(\mathcal{F}', \mathcal{X}, R)$ (see lemma 5.39), so we consider total monotone algebras over $\mathcal{F}'$.

The essential step in this part is the existence of a total order on the set of ground terms, well-founded and closed under contexts. To construct such an order, consider the set of function symbols $\mathcal{F}'$. By Zermelo's Theorem (see for example Kuratowski and Mostowski [65]) there is a total, well-founded order on $\mathcal{F}'$. Let $\succ$ be such a total precedence. Consider the order $>_{lpo}$ associated with this precedence and taking lexicographic sequences from left to right. In chapter 2, section 2.4, we stated that this order has all the required properties.

Since $(\mathcal{F}', \mathcal{X}, R)$ is totally terminating, we know that $R$ is compatible with a (non-empty) monotone $\mathcal{F}'$-algebra $(A, >)$, with $>$ total and well-founded. Again let $[\![t]\!]$ be the interpretation in $A$ of a ground term $t$.

In $\mathcal{T}(\mathcal{F}')$ we define the order $\sqsupset$ by

$$s \sqsupset t \iff ([\![s]\!] > [\![t]\!]) \text{ or } ([\![s]\!] = [\![t]\!] \text{ and } s >_{lpo} t)$$

Irreflexivity and transitivity of $\sqsupset$ follows from irreflexivity and transitivity of both $>$ and $>_{lpo}$. Given any two ground terms $s, t$ then either $[\![s]\!] > [\![t]\!]$ or $[\![t]\!] > [\![s]\!]$ or $[\![t]\!] = [\![s]\!]$, since $>$ is total. In the first two cases we conclude $s \sqsupset t$ or $t \sqsupset s$, respectively. In the last case, since $>_{lpo}$ is total we know that either $s >_{lpo} t$ or $t >_{lpo} s$ or $s = t$, hence the order $\sqsupset$ is total. For well-foundedness of $\sqsupset$, suppose there is an infinite descending chain

$$t_0 \sqsupset t_1 \sqsupset t_2 \sqsupset \cdots$$

Consider the sequence $([\![t_i]\!])_{i \in \mathbb{N}}$. By definition of $\sqsupset$, we have

$$[\![t_0]\!] \geq [\![t_1]\!] \geq [\![t_2]\!] \geq \cdots$$

Since $>$ is well-founded, there must be an index $j$ such that $[\![t_j]\!] = [\![t_{j+1}]\!] = \cdots$, and consequently $t_j >_{lpo} t_{j+1} >_{lpo} \cdots$, contradicting the well-foundedness of $>_{lpo}$.

Note that another way of proving these properties is by remarking that the relation $\sqsupset$ can be injected in a subset of the lexicographic product of $>$ by $>_{lpo}$ (over $A \times \mathcal{T}(\mathcal{F}')$). Since both $>$ and $>_{lpo}$ are total well-founded partial orders on, respectively $A$ and $\mathcal{T}(\mathcal{F}')$, by lemma 2.35 we conclude that their lexicographic product is also a total well-founded partial order on $A \times \mathcal{T}(\mathcal{F}')$, and so on any subset of it; as a consequnece $\sqsupset$ is a total well-founded partial order on $\mathcal{T}(\mathcal{F}')$.

For closedness under ground contexts, note that $>_A$ and $=_A$, respectively the partial order and congruence induced on $\mathcal{T}(\mathcal{F}', \mathcal{X})$ by $(A, >)$ and the interpretation function, are both closed under contexts, and in particular ground contexts (see theorem 2.94 and lemma 2.97). Furthermore, since we are dealing with ground terms then $[\![s]\!] > [\![t]\!] \iff s >_A t$ and $[\![s]\!] = [\![t]\!] \iff s =_A t$. This combined with the fact that $>_{lpo}$ is closed under (ground) contexts gives that $\sqsupset$ is closed under ground contexts.

If $\sigma$ is any ground substitution and $l \to r$ is a rule in $R$, then $[\![l\sigma]\!] > [\![r\sigma]\!]$, since $(A, >)$ is compatible with $R$, and therefore $l\sigma \sqsupset r\sigma$, concluding the proof. $\square$

It is also of interest to consider under what conditions is a TRS *not* totally terminating. We define the truncation closure $TC(R)$ of a TRS $R$ to be the set

$$TC(R) = \{(t,t')|\ \exists C[] :\ C[t] \to_R^+ C[t']\}.$$

In general $TC(R)$ is not a TRS, however if $R$ is totally terminating, then $TC(R)$ is indeed a TRS. To see this note that $C[t] \to_R^+ C[t'] \Rightarrow C[t] >_A C[t']$, where $(A, >)$ is a total well-founded monotone algebra compatible with $R$ and $>_A$ is the order on terms induced by the algebra and the interpretation. By lemma 5.45, we have that $t >_A t'$. If $t$ would be a variable, this would be impossible (any assignment $\tau : \mathcal{X} \to A$ such that $\tau(t)$ is the minimal element of $A$ will give $[\![t', \tau]\!] > [\![t, \tau]\!]$). If $t'$ contains some variable not occurring in $t$, say $x$, then we can write $t' = D[x]$, with $D$ possibly trivial. Since $x$ does not occur in $t$, we can define an assignment $\tau : \mathcal{X} \to A$ satisfying $\tau(x) > [\![t, \tau]\!]$. Consequently (and using lemma 5.45), we get $[\![t', \tau]\!] \geq [\![x, \tau]\!] > [\![t, \tau]\!]$, contradicting $t >_A t'$. So if $R$ is totally terminating and $(t,t') \in TC(R)$ then $t \notin \mathcal{X}$ and $var(t') \subseteq var(t)$, so $TC(R)$ defines a TRS over $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

It is not difficult to see that if $R$ is totally terminating then $TC(R \cup \mathcal{E}mb_\mathcal{F})$ is also totally terminating. Indeed, from lemma 2.111, we conclude that $R \cup \mathcal{E}mb_\mathcal{F}$ is totally terminating, so $TC(R \cup \mathcal{E}mb_\mathcal{F})$ is a well-defined TRS. Suppose that $(A, >)$ is a total well-founded monotone algebra compatible with $R \cup \mathcal{E}mb_\mathcal{F}$, and take $t \to t'$, a rule in $TC(R \cup \mathcal{E}mb_\mathcal{F})$. Then there is a context $C$ such that $C[t] \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ C[t']$. This means that $C[t] >_A C[t']$, and by lemma 5.45 we have that $t >_A t'$, thus $(A, >)$ is compatible with $TC(R \cup \mathcal{E}mb_\mathcal{F})$, proving its total termination.

A trivial consequence of this remark is that if $R$ is totally terminating then $TC(R \cup \mathcal{E}mb_\mathcal{F})$ is a well-defined terminating TRS. Therefore if $TC(R \cup \mathcal{E}mb_\mathcal{F})$ is not a TRS or is a TRS allowing an infinite rewrite sequence, $R$ is not totally terminating. This is a useful tool for proving that a TRS is not totally terminating. Unfortunately this characterization is not complete, as we had conjectured in Ferreira and Zantema [32]. As Uwe Waldmann pointed out to us, termination of $TC(R \cup \mathcal{E}mb_\mathcal{F})$ does not imply total termination of $R$. The following TRS $R$ is a counter-example.

$$\begin{array}{lcl} f(a,a) & \to & f(b,b) \\ g(b,b) & \to & g(a,a) \end{array}$$

Clearly the elements $a$ and $b$ have to be incomparable, so the system cannot be totally terminating. The system $R \cup \mathcal{E}mb_\mathcal{F}$ is terminating; $R$ is terminating (each rewrite step eliminates a redex and no new redexes are created) and length-preserving (see lemma 2.107).

Furthermore $\to_{TC(R\cup\mathcal{E}mb_\mathcal{F})}^+ \subseteq \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+$. It can be seen by induction on the context that

$$C[s] \quad \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ \quad C[t] \quad \Rightarrow \quad s \quad \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ \quad t.$$

We sketch how this can be done. Suppose $C[]$ is of the form $f(u, \Box)$, for some term $u$. Suppose that $f(u,s) \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ f(u,t)$. If this reduction does not contain root reductions then either $u \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ u$ or $s \to_{R\cup\mathcal{E}mb_\mathcal{F}}^+ t$. The first case gives a contradiction (with termination of $R \cup \mathcal{E}mb_\mathcal{F}$) so the second case must hold and the result is satisfied. Suppose there is at least one root reduction, i. e., we can write

- $f(u,s) \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u',s') \to_{R\cup\mathcal{E}mb_\mathcal{F}} u' \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u,t)$, or
- $f(u,s) \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u',s') \to_{R\cup\mathcal{E}mb_\mathcal{F}} s' \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u,t)$, or
- $f(u,s) \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u',s') \to_{R\cup\mathcal{E}mb_\mathcal{F}} f(u'',s'') \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u,t)$,

with $u \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} u'$ and $s \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} s'$, and where the first root reduction is the reduction showed. It is not difficult to see that the first case gives a contradiction. The second case gives $s \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} s' \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} f(u,t) \to_{R\cup\mathcal{E}mb_\mathcal{F}} t$, so the result holds. In the third case the reduction is a $R$-step and this implies that $u' = a = s'$, $s'' = b = u''$. Furthermore we must have $u = b$, $t = b$, so $f(u,s) = f(b,s)$ and $b \to^*_{R\cup\mathcal{E}mb_\mathcal{F}} a$ which is not possible. For the other contexts, the proof is similar.

Consequently $t \to^+_{R\cup\mathcal{E}mb_\mathcal{F}} t'$, for any rule $t \to t' \in TC(R \cup \mathcal{E}mb_\mathcal{F})$; therefore

$$\to^+_{TC(R\cup\mathcal{E}mb_\mathcal{F})} \subseteq \to^+_{R\cup\mathcal{E}mb_\mathcal{F}}$$

so we conclude that $TC(R \cup \mathcal{E}mb_\mathcal{F})$ is terminating.

A next step in the characterization of total termination is context removal: if $C[t] \to^+_R C[u]$ then $R$ is totally terminating if and only if $R \cup \{t \to u\}$ is totally terminating. Note that in general $t \to u$ may be an incorrect rule; however if $R$ is totally terminating then $R \cup \{t \to u\}$ is a well-defined TRS.

A first rough attempt to characterize total termination resulted in the following definition.

**Definition 5.41.** Given a TRS $R$ we define the relation $\rhd \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ as follows: $s \rhd t$ iff $s \neq t$ and $(R \cup \mathcal{E}mb_\mathcal{F} \cup \{t \to s\})$ is not terminating.

It is not difficult to see that $\rhd$ has the following properties:

- if $C[s] \rhd C[t]$, for any ground context $C[\,]$, then $s \rhd t$,
- $\to^+_{R\cup\mathcal{E}mb_\mathcal{F}}$ restricted to $\mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$, is contained in $\rhd$,
- $\rhd$ is in general not transitive.

Given a binary relation $\theta$ over a set $A$, not necessarily transitive, we say that $\theta$ is well-founded if there is no infinite sequence $(a_i)_{i\in\mathbb{N}}$ such that $a_i \theta a_{i+1}$, for all $i \in \mathbb{N}$. The connection between this relation and total termination is given below.

**Theorem 5.42.** *If $R$ is totally terminating then $\rhd$ is well-founded.*

**Proof** Suppose $R$ is a totally terminating TRS. By theorem 2.111, $R \cup \mathcal{E}mb_\mathcal{F}$ is also totally terminating. Without loss of generality, we can assume that $\mathcal{T}(\mathcal{F}) \neq \emptyset$, since by lemma 5.39 and theorem 2.111, adding a constant to $\mathcal{F}$ does not change the behaviour of neither $R$ nor $R \cup \mathcal{E}mb_\mathcal{F}$ with respect to total termination. By theorem 5.40, there is a strict total and well-founded partial order $>$ over $\mathcal{T}(\mathcal{F})$, such that:

- $l\sigma > r\sigma$, for any rule $l \to r \in R \cup \mathcal{E}mb_\mathcal{F}$, and any ground substitution $\sigma$.

- $>$ is closed under ground contexts.

We will prove that $\rhd \subseteq >$.Then well-foundedness of the later relation will yield well-foundedness of the former relation. Suppose then that $s \rhd t$, with $s, t \in \mathcal{T}(\mathcal{F})$ and $s \neq t$. Since $>$ is total on $\mathcal{T}(\mathcal{F})$, we have either $s > t$ or $t > s$ . If $t > s$, we will see that $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{t \to s\}$ is terminating (in fact that it is totally terminating), contradicting $s \rhd t$. We remark that $>$ has the property $t\sigma > s\sigma$, for any (ground) substitution $\sigma$, since if $s$ and $t$ are ground terms we have that $t\sigma = t$ and $s\sigma = s$. Consequently we can apply theorem 5.40 in the opposite direction to conclude that $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{t \to s\}$ is totally terminating, so we must have $s > t$ as we wanted. □

The relation $\rhd$ can be used to prove that a system is not totally terminating, as the next example shows. Consider the TRS

$$\begin{array}{lcl} f(g(x)) & \to & f(f(x)) \\ g(f(x)) & \to & g(g(x)) \end{array}$$

The first rule combined with $f(c) \to g(c)$, where $c$ is an arbitrary constant, gives a non-terminating system, hence $g(c) \rhd f(c)$. Similarly the second rule combined with $g(c) \to f(c)$ results in a non-terminating system, hence $f(c) \rhd g(c)$. Consequently $\rhd$ is not well-founded and the system cannot be totally terminating. Note that the previous TRS:

$$\begin{array}{lcl} f(a,a) & \to & f(b,b) \\ g(b,b) & \to & g(a,a) \end{array}$$

can also be proven not totally terminating using the relation $\rhd$. In this case we have $a \rhd b \rhd a$ and so $\rhd$ is not well-founded.

The converse of theorem 5.42 does not hold, even if only constant and unary function symbols are allowed. Let $R$ be:

$$\begin{array}{lcl} f(a) & \to & f(b) \\ g(g(b)) & \to & g(c) \\ f(c) & \to & f(g(a)) \end{array}$$

Suppose $R$ is totally terminating and let $(A, >)$ be a total well-founded monotone algebra compatible with $R$. The first rule tells us that $[\![a]\!] > [\![b]\!]$. Then monotonicity of the algebra operations and compatibility with the rules give us $[\![g(b)]\!] > [\![c]\!] > [\![g(a)]\!] > [\![g(b)]\!]$, which is a contradiction.

We now give a sketch of the proof of well-foundedness of $\rhd$. Define the following weight function $\rho : \mathcal{T}(\mathcal{F}) \to \mathbb{N}$ by

- $\rho(a) = \rho(b) = 1$; $\rho(c) = 2$
- $\rho(p(t)) = 1 + \rho(t)$, for any $t \in \mathcal{T}(\mathcal{F})$, $p \in \{f, g\}$.

It is easy to see that for any ground substitution $\sigma$ and any rule $l \to r$, we have

- $\rho(l\sigma) = \rho(r\sigma)$, if $l \to r \in R$.
- $\rho(l\sigma) > \rho(r\sigma)$, if $l \to r \in \mathcal{E}mb_{\mathcal{F}}$.

Furthermore $\rho$ is closed under ground contexts.

The following fact is also not difficult to prove:

$$s \triangleright t \Rightarrow \rho(s) \geq \rho(t)$$

As a consequence $\triangleright \setminus =_\rho$ is well-founded, where $=_\rho$ is the equivalence relation generated by $\rho$, i. e., for any $t, s \in \mathcal{T}(\mathcal{F})$, $t =_\rho s \iff \rho(t) = \rho(s)$.

It is well known that given a ground TRS, if the system is not terminating then it contains a rule $l \to r$ such that $r$ admits an infinite rewrite sequence. Using this fact we can derive that $(a, b)$ is the only pair in $\triangleright$ of size one and that $(g(b), c), (c, g(a))$ are the only pairs in $\triangleright$ of size two involving $c$. Also $g(a) \not\triangleright g(b)$.

We see now that $g(u) \not\triangleright f(v)$, for any ground terms $u, v$ such that $\rho(u) = \rho(v)$. Suppose that is not so, i. e., there are terms $u, v \in \mathcal{T}(\mathcal{F})$ with $\rho(u) = \rho(v)$ and $g(u) \triangleright f(v)$. This means that the TRS $R \cup \mathcal{E}mb_{\mathcal{F}} \cup \{f(v) \to g(u)\}$ is not terminating. Since for any rule in this TRS and any ground substitution $\sigma$ we have $\rho(l\sigma) \geq \rho(r\sigma)$, $\rho$ is closed under contexts and $(\mathbb{N}, >)$ is well-founded, we can conclude that if this TRS admits an infinite rewrite sequence then so does $R_1 = R \cup \{f(v) \to g(u)\}$, and since $R_1$ is a ground system, at least one rhs of a rewriting rule admits an infinite rewrite sequence. With a bit of case analysis it is possible to see that no reduction rule has a rhs leading to an infinite rewrite sequence, giving a contradiction.

Suppose then that $\triangleright \cap =_\rho$ is not well-founded and take an infinite chain $t_0 \triangleright t_1 \triangleright \ldots$, such that the size of the chain, given by $n = \rho(t_i) = \rho(t_j)$, for any $i, j$, is minimal. Since $(a, b)$ is the only pair in $\triangleright$ of size one, it must be $n \geq 2$. If $n = 2$ and $c$ occurs in the chain, its occurrence has to follow the pattern $g(b) \triangleright c \triangleright g(a)$ or $c \triangleright g(a)$. But from what we have seen $g(a) \not\triangleright t$, for any $t \in \{c, g(b), f(a), f(b)\}$, which are all the possible terms of size two. Therefore the chain stops at $g(a)$ and cannot be infinite. Consequently any infinite chain of size $n \geq 2$ cannot contain $c$. So the head symbol of $t_0$ has to be either $f$ or $g$. If the head symbol never changes then the chain is of the form

$$p(t'_0) \triangleright p(t'_1) \triangleright \ldots \triangleright p(t'_i) \triangleright \ldots$$

where $p \in \{f, g\}$. By eliminating the head symbol, we get an infinite chain $(t'_i)_{i \in \mathbb{N}}$ with a strictly smaller size, contradicting the minimality of $(t_i)_{i \in \mathbb{N}}$. So the head symbol has to change infinitely many often and that contradicts the fact that $g(u) \not\triangleright f(v)$, for any terms $u, v \in \mathcal{T}(\mathcal{F})$ with the same weight. As a result $\triangleright \cap =_\rho$ is well-founded and so is $\triangleright$.

Furthermore the characterization of total termination via $\triangleright$ is not complete even for string rewriting systems. If we modify slightly the TRS above we can get a string rewriting system $R$ that is not totally terminating and such that $R \cup \mathcal{E}mb_{\mathcal{F}}$ terminates and $\triangleright$ is well-founded. In fact the following system

$$\begin{array}{lcl} f(h(x)) & \to & f(k(x)) \\ g(g(k(x))) & \to & g(i(x)) \\ f(i(x)) & \to & f(g(h(x))) \end{array}$$

is a string rewriting system in those conditions. For proving termination of $R \cup \mathcal{E}mb_{\mathcal{F}}$ we choose as monotone algebra $A = \mathbb{N} \times (\{0,1\} \times \mathbb{N})$ with the order $\succ$ defined by

$$(a,(x,n)) \succ (b,(y,m)) \iff (a > b \text{ or } (a = b \text{ and } x = y \text{ and } n > m))$$

where $>$ is the usual order in the natural numbers, and the interpretations

$$\begin{array}{lcll} k_A((a,(x,n))) & = & (a+1,(1,n)) & \text{for } x \in \{0,1\} \\ h_A((a,(x,n))) & = & (a+1,(0,n)) & \text{for } x \in \{0,1\} \\ i_A((a,(x,n))) & = & (a+2,(0,n)) & \text{for } x \in \{0,1\} \\ f_A((a,(x,n))) & = & \begin{cases} (a+1,(0,3n+1)) & \text{if } x = 0 \\ (a+1,(0,n)) & \text{otherwise} \end{cases} & \\ g_A((a,(x,n))) & = & \begin{cases} (a+1,(1,n)) & \text{if } x = 0 \\ (a+1,(1,2n+1)) & \text{otherwise} \end{cases} & \end{array}$$

It is not difficult to see that these functions are strictly monotone and that for every $\alpha : \mathcal{X} \to A$ and every rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$, $[\![l,\alpha]\!] \succ [\![r,\alpha]\!]$. The system cannot be totally terminating since for any possible total interpretation we would have $i_A(a) > g_A(h_A(a)) > g_A(k_A(a)) > i_A(a)$, for any algebra element $a$. For the well-foundedness of $\triangleright$ we proceed as in the previous example (with substantially more case analysis).

The next step is based on the following observation: if $C_0[t] \to_R^+ C_1[u]$ and $C_1[t] \to_R^+ C_0[u]$ then adding $t \to u$ to $R$ still does not affect total termination. These ideas were combined in the following definition.

**Definition 5.43.** Given a TRS $R$ we define the relation $\gtrdot \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ as follows: $s \gtrdot t$ if one of the following conditions holds

- $s \to_R t$ or $s \to_{\mathcal{E}mb_{\mathcal{F}}} t$
- $s = C[a]$ and $t = C[b]$ and $a \gtrdot b$
- for some $n > 0$, there are contexts $C_0[\;], \ldots, C_n[\;]$ such that $C_0[\;] = C_n[\;]$ and $C_i[s] \gtrdot C_{i+1}[t]$, for each $0 \leq i < n$,
- $\exists u \in \mathcal{T}(\mathcal{F}) :\; s \gtrdot u$ and $u \gtrdot t$

The relation $\gtrdot$ is a bit more elaborate than $\triangleright$ but a similar result as theorem 5.42 holds for $\gtrdot$. We need some auxiliary results.

**Lemma 5.44.** *Let $(A,>)$ be a (total well-founded) monotone $\mathcal{F}$-algebra. If $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and $[\![s,\sigma]\!] \geq [\![t,\sigma]\!]$, for some $\sigma \in A^{\mathcal{X}}$, then $[\![C[s],\sigma]\!] \geq [\![C[t],\sigma]\!]$, for any context $C[\;]$.*

**Proof** We proceed by induction. The assertion holds for the trivial context by hypothesis. Suppose it also holds for a context $C'[\;]$. Then

$$\begin{array}{lcl} [\![f(\ldots,C'[s],\ldots),\sigma]\!] & = & \text{(by definition of } [\![\;]\!]) \\ f_A(\ldots,[\![C'[s],\sigma]\!],\ldots) & \geq & \text{(by IH and monotonicity of } f_A) \\ f_A(\ldots,[\![C'[t],\sigma]\!],\ldots) & \geq & \text{(by definition of } [\![\;]\!]) \\ [\![f(\ldots,C'[t],\ldots),\sigma]\!] & & \end{array}$$

□

**Lemma 5.45.** *Let $(A, >)$ be any total (well-founded) monotone algebra compatible with $R$. Then $C[s] >_A C[t] \Rightarrow s >_A t$, for any terms $s, t$ and context $C[\;]$, where $>_A$ is the order over terms induced by $(A, >)$. Furthermore $[\![C[s], \tau]\!] \geq [\![s, \tau]\!]$, for any context $C$, term $s$ and assignment $\tau \in A^{\mathcal{X}}$, and if $(A, >)$ is also compatible with $\mathcal{E}mb_{\mathcal{F}}$, then $C[s] >_A s$, for any non-trivial context $C[\;]$ and term $s$.*

**Proof** Let then $C[s] >_A C[t]$. We have to see $\forall \sigma \in A^{\mathcal{X}} : [\![s, \sigma]\!] > [\![t, \sigma]\!]$. Suppose $\exists \tau \in A^{\mathcal{X}} : [\![s, \tau]\!] \not> [\![t, \tau]\!]$. Due to the totality of $>$, this means that $[\![s, \tau]\!] \leq [\![t, \tau]\!]$. By lemma 5.44 we have $[\![C[s], \tau]\!] \leq [\![C[t], \tau]\!]$, contradicting $C[s] >_A C[t]$. So $s >_A t$.

We prove that $[\![C[s], \tau]\!] \geq [\![s, \tau]\!]$, for any context $C$, term $s$ and assignment $\tau \in A^{\mathcal{X}}$. If $C$ is the trivial context, the result holds since $C[s] = s$. Suppose that $C[s] = f(\ldots, s, \ldots)$, for some $f \in \mathcal{X}$ admitting arity $n \geq 1$, and where $s$ occurs at position $i$, $1 \leq i \leq n$. Let $\tau \in A^{\mathcal{X}}$ be an arbitrary assignment. Then $[\![C[s], \tau]\!] = f_A(\ldots, [\![s, \tau]\!], \ldots)$. Since $f_A$ is monotone in all arguments, corollary 5.3 gives $f_A(\ldots, [\![s, \tau]\!], \ldots) \geq [\![s, \tau]\!]$, as we wanted.

Suppose now that $A$ is compatible with $\mathcal{E}mb_{\mathcal{F}}$. Let $C[s] = f(t_1, \ldots, s, \ldots, t_n)$, with $s$ occurring in position $i$, $1 \leq i \leq n$. Since $f(\ldots, x_i, \ldots) \to x_i$ is a rule in $\mathcal{E}mb_{\mathcal{F}}$, compatibility ensures that $f(\ldots, x_i, \ldots) >_A x_i$. We define the substitution $\sigma$ by

$$\sigma(x) = \begin{cases} t_j & \text{if } x = x_j \text{, for some } j \neq i \\ s & \text{if } x = x_j \\ x & \text{otherwise} \end{cases}$$

Since $>_A$ is closed under substitutions, we have $C[s] = f(x_1, \ldots, x_n)\sigma >_A \sigma(x_i) = s$.

Suppose $C'[s] >_A s$ for some context $C'[\;]$. Since $>_A$ is closed under contexts, we get $f(t_1, \ldots, C'[s], \ldots, t_n) >_A f(t_1, \ldots, s, \ldots, t_n)$. But $f(t_1, \ldots, s, \ldots, t_n) >_A s$, so transitivity of $>_A$ yields the result. □

**Theorem 5.46.** *If $R$ is totally terminating then $\gtrdot$ is well-founded.*

**Proof** Due to lemma 5.39 we can assume without loss of generality that $\mathcal{F}$ contains at least one constant, so $\mathcal{T}(\mathcal{F})$ is not empty. Since $R$ is totally terminating, from theorem 2.111 we know that $R \cup \mathcal{E}mb_{\mathcal{F}}$ is also totally terminating. By theorem 5.40 we know there is a total well-founded order $>$ over $\mathcal{T}(\mathcal{F})$ such that:

- $l\sigma > r\sigma$, for any rule $l \to r \in R \cup \mathcal{E}mb_{\mathcal{F}}$ and any ground substitution $\sigma$.
- $>$ is closed under ground contexts.

We will see, by induction on the definition of $\gtrdot$, that $s \gtrdot t \Rightarrow s > t$. Well-foundedness of $>$ will then yield the result. Suppose that $s \gtrdot t$, for some terms $s, t$.

- If $s \to_R t$ or $s \to_{\mathcal{E}mb_{\mathcal{F}}} t$, since $>$ is compatible with $R \cup \mathcal{E}mb_{\mathcal{F}}$ we have $\to^+_{R \cup \mathcal{E}mb_{\mathcal{F}}} \subseteq$ $>$ and therefore $s > t$.

- If $s = C[a]$, $t = C[b]$ with $a \gtrdot b$ and $a > b$ (by induction hypothesis) then $s > t$, since $>$ is closed under ground contexts.
- If $s \gtrdot t$ because for some $n > 0$, there are contexts $C_0[\ ], \ldots, C_n[\ ]$, such that $C_0[\ ] = C_n[\ ]$ and for each $0 \le i < n$, $C_i[s] \gtrdot C_{i+1}[t]$, then by induction hypothesis we have $C_0[s] > C_1[t], C_1[s] > C_2[t]$, etc. Since $>$ is total either $s > t$ or $t \ge s$. Suppose that $t \ge s$. Using the induction hypothesis, the fact that $>$ is closed under ground contexts and its transitivity, we get

$$C_0[s] > C_1[t] \ge C_1[s] > C_2[t] > \ldots > C_n[t] \ge C_n[s] = C_0[s]$$

  contradicting well-foundedness of $>$; therefore $s > t$.
- Finally if $\exists u \in \mathcal{T}(\mathcal{F}) :\ s \gtrdot u$ and $u \gtrdot t$, then also by induction hypothesis $s > u$ and $u > t$ and transitivity of $>$ gives the result.

□

The previous result can be used to show that a TRS is not totally terminating and in particular that it cannot be proven terminating by $>_{rpo}$ (or $>_{kbo}$). For example let $R$ be:

$$\begin{array}{ll} p(f(f(x))) \to q(f(g(x))) & p(g(g(x))) \to q(g(f(x))) \\ q(f(f(x))) \to p(f(g(x))) & q(g(g(x))) \to p(g(f(x))) \end{array}$$

This system (actually $R \cup \mathcal{E}mb_{\mathcal{F}}$) is terminating (in each step the number of redexes decreases) but not totally terminating. Let $c$ be a constant, then from the leftmost rules we get $p(f(f(c))) \gtrdot q(f(g(c)))$ and $q(f(f(c))) \gtrdot p(f(g(c)))$ and consequently $f(c) \gtrdot g(c)$ (with $C_0 = p(f(\square)) = C_2$ and $C_1 = q(f(\square))$). Similarly using the rightmost rules we get $g(c) \gtrdot f(c)$; therefore $\gtrdot$ is not well-founded and so $R$ cannot be totally terminating.

One can wonder whether the reverse of theorem 5.46 holds. This is not the case. For example one can prove that the system (due to Uwe Waldmann)

$$\begin{array}{ll} f(0,a) \to f(1,b) & h(1,a) \to h(0,b) \\ g(0,b) \to g(1,a) & k(1,b) \to k(0,a) \end{array}$$

is not totally terminating while $\gtrdot$ is well-founded. To see this note that $\gtrdot$ coincides with $\to_{R \cup \mathcal{E}mb_{\mathcal{F}}}$ and $R \cup \mathcal{E}mb_{\mathcal{F}}$ is terminating since in each $R$-rewriting step the number of redexes decreases and $R$ is length-preserving. It is easy to see that the interpretations of $a$ and $b$ (or $0$ and $1$) have to be incomparable and so the system is not totally terminating.

It is also interesting to remark that we can prove that the TRS's presented in connection with the relation $\triangleright$ can be proven not totally terminating using $\gtrdot$. For example for the TRS

$$\begin{array}{lcl} f(h(x)) & \to & f(k(x)) \\ g(g(k(x))) & \to & g(i(x)) \\ f(i(x)) & \to & f(g(h(x))) \end{array}$$

given an arbitrary constant $c$, from the definition and properties of $\gtrdot$ we can derive $g(k(c)) \gtrdot i(c) \gtrdot g(h(c)) \Rightarrow k(c) \gtrdot h(c)$. From the first rule we get $h(c) \gtrdot k(c)$, so $\gtrdot$ is not well-founded.

It is not clear whether the reverse of theorem 5.46 holds for string rewriting systems.

## 5.7 Conclusions

Proving termination of term rewriting systems by interpretation is not easy. We focussed on interpretations in monotone algebras in which the underlying order is total. This gives rise to the concept of total termination. Total termination covers many techniques used in practice to prove termination, including recursive path order.

We have shown that the existence of a function symbol of arity greater than one implies that the underlying order has type $\omega^\eta$, i. e., is equivalent to finite multisets over some well-order. Furthermore, for any TRS $R$, the class of total orders in which $R$ can be shown to be totally terminating, is closed under multiset construction and lexicographic product. Note that in this case (multiset and lexicographic extensions of well-orders) there is no essential difference between the multiset and lexicographic construction in the sense that multisets can be simulated by lexicographic sequences (see Jouannaud and Lescanne [51]).

It is not clear how to extend a total termination proof in a particular well-order to well-orders that cannot be finitely obtained from the original one by these constructions. This problem is closely connected to modularity of total termination, on which we obtained some interesting partial results.

We found examples of TRS's showing that proofs of total termination cannot always be given in well-orders of type smaller than $\omega^\omega$. Most of our techniques are based upon ordinal arithmetic which appears to be a strong and useful tool for proving termination of TRS's.

We also showed that the notion of total termination can be given independently from the concept of monotone algebra. Total termination can be identified with the existence of a total well-founded monotone order on ground terms compatible with (ground instances of) the rewrite rules. This was a first step on trying to find a syntactical characterization of total termination of the following shape: if a TRS is totally terminating then some syntactically defined relation is well-founded. This kind of characterization leads to a method of proving non-total termination: if the constructed relation admits an infinite descending chain then the TRS is not totally terminating. We defined some relations obeying this principle, yet for all the relations given the converse does not hold: we presented TRS's for which the constructed relations are well-founded while the TRS's are not totally terminating.

Ideally we would like to have a characterization of total termination in the same lines as the characterization of simple termination presented in chapter 2, theorem 2.106. This, however, remains a puzzle yet to be solved.

# Chapter 6

# Termination by Transformation

It should be clear by now that proving termination of term rewriting systems is a difficult task. In this chapter we investigate a method for transforming TRS's whose goal is to simplify that task. The method consists of a family of transformations that can be applied to any equational rewrite system (provided some conditions are satisfied). The transformations are all of the same type: function symbols considered "useless" are eliminated therefore simplifying the rewrite rules. If the eliminated function symbols do not interfere with the equational theory then the transformations are sound with respect to termination, i. e., termination of the original system (modulo the equational theory) can be inferred from termination of the transformed one (modulo the equational theory). Since we eliminate function symbols, something has to be done to its arguments. The different ways of dealing with those arguments characterize the different transformation techniques.

## 6.1 Motivation

In general, we are interested in simplifying the process of proving termination of term rewriting systems. A possible approach to this goal is to devise sound transformations on TRS's such that the transformed systems are somehow easier to deal with, with respect to termination proofs, than the original ones. As examples of such transformations we have *transformation orderings* from Bellegarde and Lescanne [5] (see also Geser [40]), *semantic labelling* Zantema [111] and *distribution elimination* Zantema [108] [1]. In fact it was the technique of *distribution elimination* plus the observation that "created" function symbols (i. e., function symbols occurring only in the righthand-side of rewriting rules) seemed to be irrelevant for reductions, which motivated this work.

In [107, 108], Zantema devised a transformation on terms and TRS's that consists in eliminating functions symbols whose occurrences in the rules of the TRS were restricted to righthand-sides or "distribution rules" (hence the name of the technique). If the transformed system satisfies the right-linearity condition, the transformation is sound with respect to termi-

[1] For an example of application of some of these techniques, including one described in this chapter, see Zantema and Geser [112].

nation. Right-linearity was shown to be essential in the presence of distribution rules and it was conjectured that if no distribution rules were present, right-linearity was no longer necessary. While trying to solve this conjecture, we arrived at a different transformation both simpler and seemingly more powerful, *dummy elimination* (see Ferreira and Zantema [35]). Later, combining the ideas of *distribution elimination* and *dummy elimination*, a whole family of transformations was devised. As remarked by Middeldorp and Ohsaki (personal communication) who presented another proof of the soundness of *dummy elimination* based on the technique of semantic labelling of Zantema [111], *dummy elimination* admits an extension in which the symbol to be eliminated is also allowed in the lefthand-sides of rewrite rules. In the same spirit, the family of transformations could be extended in order to lift the restriction that the eliminated function symbol could only occur in the righthand-side of rewrite rules. Furthermore, we found that the transformations could also be applied to rewriting modulo a set of equations, provided the equations have a certain shape. Here we present the transformations in its most complete form, but for didactical as well as historical reasons we start by explaining *dummy elimination*.

First we fix some conventions. We will consider fixed-arity signatures and elimination of only one function symbol. The theory can also be presented for varyadic signatures and/or simultaneous elimination of several function symbols. But, not only the presentation does not become clearer, but also the results obtained are actually weaker, as we will show later. We will denote by $a$ the function symbol we want to eliminate. We also fix its arity to be some natural number $N \geq 1$.

Suppose we want to prove termination of the following system

$$f(g(x)) \rightarrow f(a(g(g(f(x)))))$$

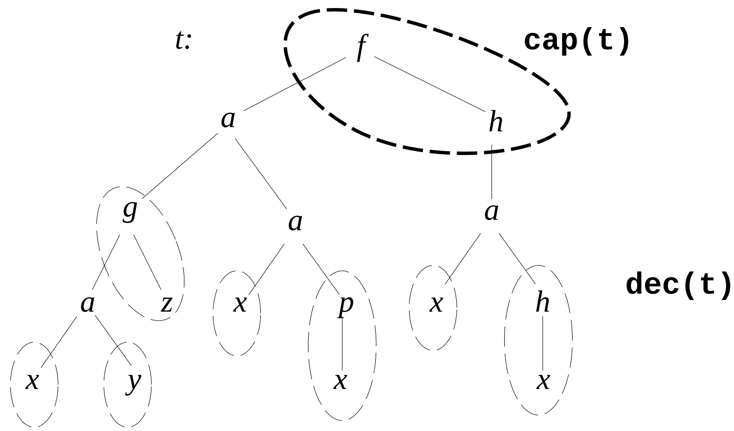
Intuitively, the function symbol $a$ is created but seems not to have any influence on the reductions. Taking that into account, we can eliminate it and transform the given rule into

$$\begin{array}{lcl} f(g(x)) & \rightarrow & f(\diamond) \\ f(g(x)) & \rightarrow & g(g(f(x))) \end{array}$$

where $\diamond$ is a fresh constant. Termination of the first system is not easy to prove, since the system is self-embedding orders like *recursive path order (rpo)* cannot be used (see chapter 2), while termination of the second system is trivially proved with *rpo* by choosing the precedence $\triangleright$ satisfying $f \triangleright g \triangleright \diamond$. If this transformation is sound, i. e., termination of the original system can be inferred from termination of the transformed one, our task is done. Proving that the transformation is sound constitutes the main result of this chapter.

The above example gives an idea of how the transformation associated with *dummy elimination* works: *alien* tems (i. e., terms whose root symbol is the one to be eliminated) are replaced by a fresh constant $\diamond$ and its subterms (themselves recursively rid of alien terms) are treated as separate entities. So a term $t$ is decomposed in blocks that do not contain the function symbol to be eliminated. One of these blocks, namely the one above all occurrences of alien terms, is denoted by $\mathtt{cap}(t)$ and treated especially. The other blocks are collected in a set denoted by $\mathtt{dec}(t)$. We give an example.

**Example 6.1.** The following term $t$




has as cap the term $f(\diamond, h(\diamond)))$ and its decomposition is given by

$$\mathtt{dec}(t) = \{g(\diamond, z), x, y, \diamond, p(x), h(x)\}$$

Both $\mathtt{cap}(t)$ and $\mathtt{dec}(t)$ have very simple inductive definitions, namely:

| $t$ | $\mathtt{cap}(t)$ | $\mathtt{dec}(t)$ |
|---|---|---|
| $x$ | $x$ | $\emptyset$ |
| $f(t_1, \ldots, t_m)$ | $f(\mathtt{cap}(t_1), \ldots, \mathtt{cap}(t_m))$ | $\bigcup_{i=1}^{m} \mathtt{dec}(t_i)$ |
| $a(t_1, \ldots, t_N)$ | $\diamond$ | $\bigcup_{i=1}^{N} (\{\mathtt{cap}(t_i)\} \cup \mathtt{dec}(t_i))$ |

The transformation on TRS's is now what we should expect. If we denote the transformed system by $\mathcal{E}(R)$ then

$$\mathcal{E}(R) = \{\mathtt{cap}(l) \to u \mid l \to r \ \in R \text{ and } u = \mathtt{cap}(r) \text{ or } u \in \mathtt{dec}(r)\}$$

Note that we allow the function symbol $a$ to occur on the lefthand-side of rewrite rules. We can state the soundness result:

If $\mathcal{E}(R)$ is terminating then $R$ is terminating.

## 6.2 General Dummy Elimination

If we compare the transformations associated with *distribution elimination* and *dummy elimination*, we see that the essential difference is in the way that subterms of alien terms are treated with respect to the whole term. In the case of *distribution elimination*, all (recursively treated)

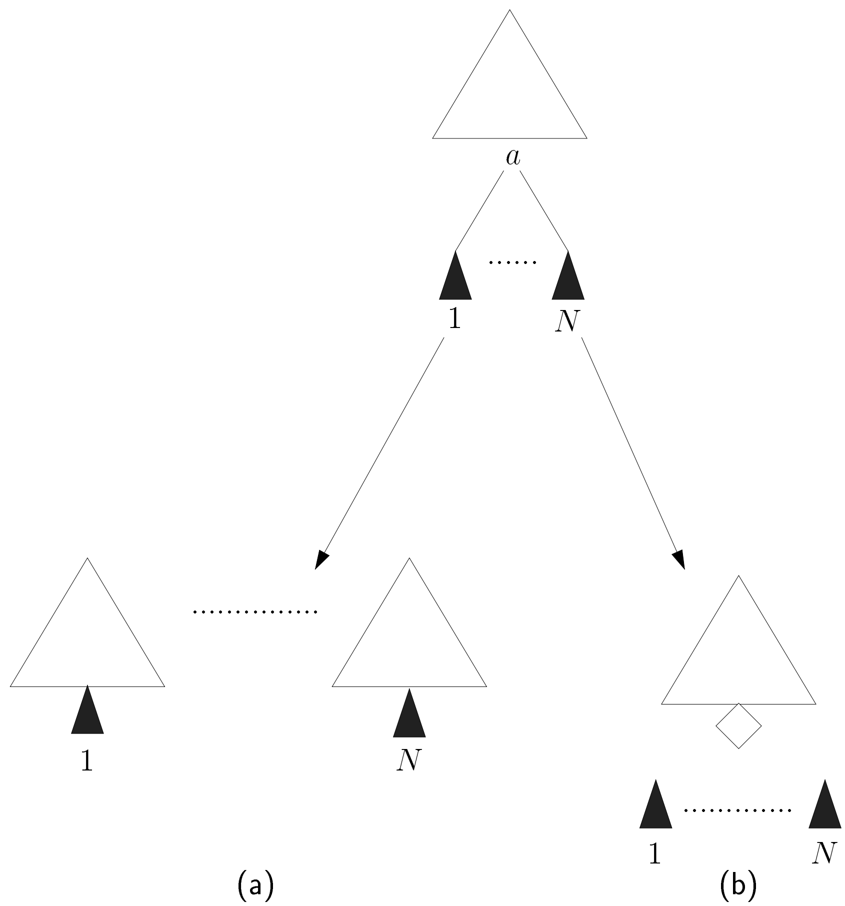



Figure 6.1: Transforming a term via (a): *distribution* or (b): *dummy* elimination.

subterms of an alien term are connected on the point where the alien term was hanging, and no subterms are treated as separate entities, while in *dummy elimination* we have precisely the opposite situation: no subterm is connected, thus all of them are treated as separate entities, and a constant has to be inserted at the point where the alien term was hanging. Figure 6.1 illustrates this situation.

But there is no reason to choose only from these two extreme cases. The essential question is "for each argument of the function symbol to be eliminated, how are we going to treat it?". Different answers to this question give us different transformations on terms and so on TRS's. Fig 6.2 shows all possible decompositions for the elimination of a function symbol of arity $2$.

The choice of arguments has to be incorporated into the cap and the decomposition of a term. Their meaning is now:

- $\texttt{cap}(t)$ is the part of the term starting at the root and where alien subterms are replaced by a fixed chosen argument (if no argument is chosen a fresh constant is used).

- $\texttt{dec}(t)$ contains the arguments (recursively decomposed) of the alien terms in $t$ that are

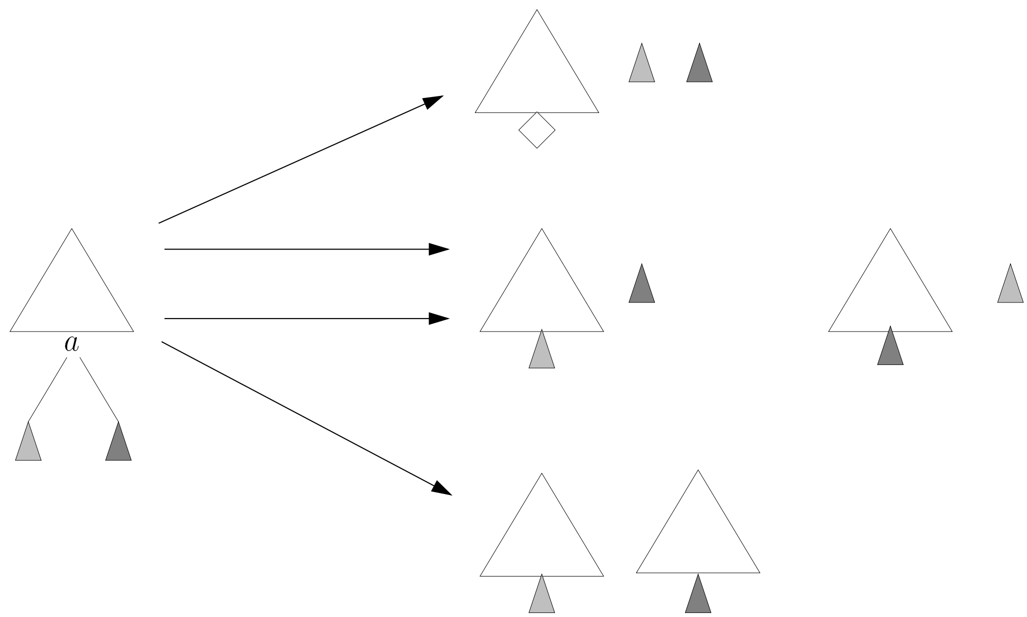



Figure 6.2: All possible eliminations for a function symbol of arity $2$.

to be separated from the superterm.

The motivation and intuition of our transformations is already set, but before we proceed with their definition and their soundness, we will introduce the main technical tool needed. Because the technique doesn't depend on term rewriting, we choose to present it in a general framework of quasi-ordered sets. Later we specialize to the case that the quasi-ordered set is the set ot terms and the order is a variant of the rewrite relation.

### 6.2.1 Ordering Trees

Given a non-empty set $S$, we consider non-empty trees over $S$, defined by the following data type: $Tr(S) \cong S \times \mathcal{M}(Tr(S))$, i. e., if $f$ is the function from sets to sets given by $f(X) = S \times \mathcal{M}(X)$, then $Tr(S)$ is the least fixed point of $f$. Therefore a tree is either a root (or leaf), represented by $(s, \emptyset)$, with $s \in S$ and $\emptyset$ being the empty multiset, or a tree with root $s \in S$ and subtrees $t_1, \ldots, t_n$, represented by $(s, \{\!\{t_1, \ldots, t_n\}\!\})$. Since we are not interested in the order of the subtrees, we choose the multiset representation for the subtrees instead of a sequence representation.

The *depth* of a tree is given by the function `depth` and is defined inductively as usual, i. e.,

- $\texttt{depth}(s, \emptyset) = 1.$
- $\texttt{depth}(s, \{\!\{t_1, \ldots, t_n\}\!\}) = 1 + \; max_{1 \leq i \leq n}(\texttt{depth}(t_i))$

We now describe how to lift a quasi-order on a set $S$ to a quasi-order on $Tr(S)$ in such a way that well-foundedness is preserved. This lifting will be used later on in the context of term rewriting.

**Definition 6.2.** Let $(S, \geq)$ be a quasi-ordered set, with $\geq = > \cup \sim$. Consider $Tr(S)$, the finite non-empty trees over $S$. In $Tr(S)$ we define the relation $\succeq$ by $t = (a, M) \succeq (b, M')$ if and only if $\forall u \in M' : (t \succ u)$ or $(\exists v \in M : v \succeq u)$, and either

- $a > b$, or
- $a \sim b$ and $M \succeq_{mul} M'$.

where $\succeq_{mul}$ is the multiset extension of $\succeq$ and $\succ = \succeq \setminus \preceq$. We call the relation $\succeq$ the tree lifting of $> \cup \sim$.

The following result justifies the previous definition.

**Lemma 6.3.** *The relation $\succeq$ is well-defined and is a quasi-order on $Tr(S)$.*

**Proof** The proof follows closely chapter 4, section 4.2.2. We consider the CPO $\mathcal{QO}_{Tr(S)}$ ordered by $\sqsupset$ (see lemma 4.20), and define a function $\mathcal{R} : \mathcal{QO}_{Tr(S)} \to \mathcal{QO}_{Tr(S)}$ as follows: for any $\theta \in \mathcal{QO}_{Tr(S)}$, $s = (a, M_s)\ \mathcal{R}(\theta)\ (b, M_t) = t$ if

$$\forall u \in M_t :\ (s\ \mathcal{R}(\theta)\ u \text{ and } \neg(u\ \mathcal{R}(\theta)\ s) \text{ or } (\exists v \in M_s :\ v\ \mathcal{R}(\theta)\ u)),$$

and either

- $a > b$, or
- $a \sim b$, and $M_s\ \theta_{mul}\ M_t$.

With proofs similar to the ones presented for the function $\mathcal{H}$ (see definition 4.22), we can see that:

- if $\theta$ is reflexive then $\mathcal{R}(\theta)$ is reflexive,
- if $\theta$ is transitive then $\mathcal{R}(\theta)$ is transitive,
- if $\theta \sqsupset \gamma$ then $\mathcal{R}(\theta) \sqsupseteq \mathcal{R}(\gamma)$, i. e., $\mathcal{R}$ is weakly monotone.

As a consequence $\mathcal{R}$ is well-defined and has a least fixed point, which is the element we use in definition 6.2 □

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Note that if $M_s \succeq_{mul} M_t$, by definition of multiset extension of quasi-orders, then

$$\forall u \in M_t\ \exists v \in M_s :\ v \succeq u.$$

As a consequence we can rewrite the order $\succeq$ as:

**Definition 6.4.** Let $(S, \geq)$ be a quasi-ordered set, with $\geq = > \cup \sim$. Consider $Tr(S)$, the finite non-empty trees over $S$. In $Tr(S)$ we define the tree lifting $\succeq$ of $\geq$ by $t = (a, M) \succeq (b, M')$ if and only if either

- $a > b$, and $\forall u \in M' : (t \succ u)$ or $(\exists v \in M : v \succeq u)$; or
- $a \sim b$ and $M \succeq_{mul} M'$.

where $\succeq_{mul}$ is the multiset extension of $\succeq$ and $\succ = \succeq \setminus \preceq$.

From now on we use definition 6.4 instead of definition 6.2.

As we saw $\succeq$ defines a quasi-order on $Tr(S)$. For practical purposes we want to distinguish its strict and equivalence components denoted, as usual, by $\succ$ and $\approx$. It is not difficult, though cumbersome, to see that the following lemmas provide a different characterization of both $\succ$ and $\approx$.

**Lemma 6.5.** *For any $A, B \in Tr(S)$, we have that $A \approx B$ if and only if one of the following conditions holds*

- $A = (a, \emptyset)$ *and* $B = (b, \emptyset)$ *and* $a \sim b$*; or*
- $A = (a, \{\{u_1, \ldots, u_k\}\})$, $B = (b, \{\{v_1, \ldots, v_m\}\})$ *and* $a \sim b$, $k = m$, *and there is a permutation* $\pi$ *of* $\{1, \ldots, k\}$ *such that* $u_i \approx v_{\pi(i)}$, *for all* $1 \leq i \leq k$.

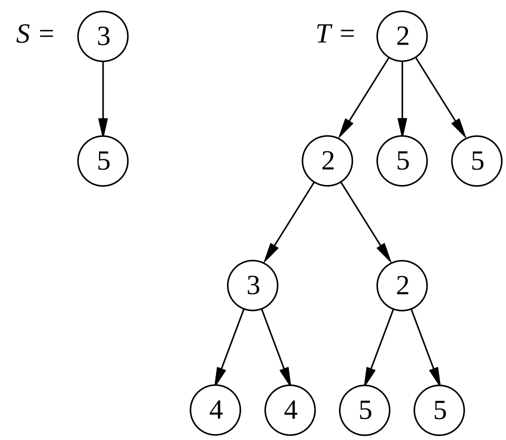
Note that if $A \approx B$ then $\texttt{depth}(A) = \texttt{depth}(B)$.

**Notation 6.6.** We will use the notation $U \approx_{mul} V$, for finite multisets $U, V \in \mathcal{M}(Tr(S))$, as a shortening for the second condition in the above lemma, i. e., $U = \{\{u_1, \ldots, u_k\}\}$, $V = \{\{v_1, \ldots, v_m\}\}$, for some $k, m \geq 0$, $k = m$ and $u_i \approx v_{\pi(i)}$, for some permutation $\pi$ of the set $\{1, \ldots, k\}$.

**Lemma 6.7.** *For any $A, B \in Tr(S)$ we have that $A \succ B$ iff $A = (a, M)$, $B = (b, M')$ and either*

- $a > b$ *and* $\forall u \in M' : (t \succ u)$ *or* $(\exists v \in M : v \succeq u)$*; or*
- $a \sim b$ *and* $M \; \texttt{ord}(\succeq_{mul}) \; M'$.

**Example 6.8.** We present an example of this construction. Let $(A, >)$ be the natural numbers with the usual order. Let




From definition 6.4 it follows that $S \succ T$. Note that even though $S \succ T$, the depth of $T$ is greater than the depth of $S$.

It is interesting to remark that $\geq_{rpo}$, the multiset recursive path order on trees, is a proper extension of $\succeq$, in the sense of definition 2.12, i. e., $\approx \subseteq \sim_{rpo}$ and $\succ \subseteq >_{rpo}$, as we see below. First recall the definition of $\geq_{rpo}$, with only multiset status, modified to be applied on trees. Suppose then that $\geq$ is a quasi-order on a set $S$ and consider the trees over $S$.

**Definition 6.9.** For any trees $s, t \in Tr(S)$, $s = (a, M) \geq_{rpo} t$ if and only if either

- $t = (b, M')$ and $\forall u \in M' : \; s >_{rpo} u$, and either
  - $a > b$, or
  - $a \sim b$ and $M \;\; \geq_{rpo,mul} \;\; M'$; or
- $\exists v \in M_s : \; v \geq_{rpo} t$.

In a similar way as we did for $\succeq$, we can characterize the strict and equivalent parts of $\geq_{rpo}$.

**Lemma 6.10.** *For any trees $A, B \in Tr(S)$, we have that:*

1. $A \sim_{rpo} B$ *if and only if one of the following conditions holds*
   - $A = (a, \emptyset)$ *and* $B = (b, \emptyset)$ *and* $a \sim b$*; or*
   - $A = (a, \{\{u_1, \ldots, u_k\}\})$, $B = (b, \{\{v_1, \ldots, v_m\}\})$ *and* $a \sim b$, $k = m$, *and there is a permutation* $\pi$ *of* $\{1, \ldots, k\}$ *such that* $u_i \sim_{rpo} v_{\pi(i)}$, *for all* $1 \leq i \leq k$.
2. $A >_{rpo} B$ *if and only if* $A = (a, M)$, $B = (b, M')$ *and either*
   - $a > b$ *and* $\forall u \in M' : t >_{rpo} u$*; or*
   - $a \sim b$ *and* $M$ $\texttt{ord(}\geq_{rpo,mul}\texttt{)}$ $M'$*; or*
   - $\exists U \in M : \; U >_{rpo} B$ *or* $U \sim_{rpo} B$.

*where $\geq_{rpo,mul}$ is the multiset extension of $\geq_{rpo}$.*

We have the following lemma.

**Lemma 6.11.** *Let $\geq$ be a quasi-order on a set $S$ and consider $Tr(S)$, the set of non-empty trees over $S$. Let $\succeq$ denote the tree lifting of $\geq$ and $\geq_{rpo}$ the multiset recursive path order on $Tr(S)$ associated with $\geq$. Then:*

1. $\approx = \sim_{rpo}$,
2. $\succ \subseteq >_{rpo}$.

**Proof** Note that $(1)$ is a direct consequence of lemmas 6.10 and 6.5. For $(2)$, by induction on $k = \mathtt{depth}(U) + \mathtt{depth}(V)$ and using $(1)$, we prove that $U \succ V \Rightarrow U >_{rpo} V$, for any trees $U, V \in Tr(S)$. For $k = 2$ we must have $U = (a, \emptyset)$ and $V = (b, \emptyset)$ and by lemma 6.7, $a > b$. By lemma 6.10 we also have $U >_{rpo} V$. Suppose the result holds for trees $U, V$ with $\mathtt{depth}(U) + \mathtt{depth}(V) \leq n$ and let $U' = (a, M_a)$, $V' = (b, M_b)$, such that $\mathtt{depth}(U') + \mathtt{depth}(V') = n + 1$ and $U' \succ V'$. By lemma 6.7 we must have either

- $a > b$ and $\forall B \in M_b :\ (U' \succ B)$ or $(\exists A \in M_a :\ (A \succ B$ or $A \approx B))$. We then also have that $\forall B \in M_b :$ either $U' >_{rpo} B$ (by induction hypothesis), or $A >_{rpo} B$ (also by induction hypothesis), or $A \sim_{rpo} B$ (by $(1)$), for some tree $A \in M_a$. In these two last cases since $U' >_{rpo} A$, for all $A \in M_a$ (the tree equivalent of the subterm property; see chapter 2), $>_{rpo}$ is transitive and compatible with $\sim_{rpo}$ and using $(1)$, we conclude that $U' >_{rpo} B$. Since $a > b$, lemma 6.10 gives $U' >_{rpo} V'$.
- $a \sim b$ and $M_a$ $\mathtt{ord}(\succeq_{mul})$ $M_b$, where $\succeq_{mul}$ is the multiset extension of $\succeq$. This means that $M_a = \{\!\{U_1, \ldots U_k\}\!\}$, for some $k \geq 1$ and $M_b = \{\!\{V_1, \ldots, V_m\}\!\}$, for some $m \geq 0$, and we have

$$\{\!\{\langle V_1\rangle, \ldots, \langle V_m\rangle\}\!\} = (\{\!\{\langle U_1\rangle, \ldots, \langle U_k\rangle\}\!\} \setminus X) \sqcup Y$$

for some finite multisets $\emptyset \neq X \sqsubseteq \{\!\{\langle U_1\rangle, \ldots, \langle U_k\rangle\}\!\}$ and $Y$ satisfying

$$\forall \langle y\rangle \in Y\ \exists \langle x\rangle \in X :\ \langle x\rangle \succ \langle y\rangle$$

Since $\langle x\rangle \succ \langle y\rangle \Rightarrow x \succ y$ and since $\approx$-equivalent trees have the same depth, we can apply the induction hypothesis to $x$ and $y$ to conclude that $x >_{rpo} y$. Since this conclusion holds for any representatives chosen, $>_{rpo}$ and $\sim_{rpo}$ are compatible and since $\approx = \sim_{rpo}$, we have that $\langle x\rangle = \langle x\rangle_{\sim_{rpo}}$, $\langle y\rangle = \langle y\rangle_{\sim_{rpo}}$ and $\langle x\rangle_{\sim_{rpo}} >_{rpo} \langle y\rangle_{\sim_{rpo}}$. We can thus conclude that $\{\!\{\langle U_1\rangle, \ldots, \langle U_k\rangle\}\!\} >_{rpo,mul} \{\!\{\langle V_1\rangle, \ldots, \langle V_m\rangle\}\!\}$, which is the same as $M_a$ $\mathtt{ord}(\geq_{rpo,mul})$ $M_b$. By lemma 6.10 we conclude that $U' >_{rpo} V'$, as we wanted.

□

The tree lifting of a quasi-order has other interesting properties, as we show below. Namely it is monotonic with respect to the (quasi-)order lifted, preserves well-foundedness and is a proper generalization of the multiset construction. We start with the last property.

**Lemma 6.12.** *Let $(S, \geq)$ be a quasi-ordered set. Then there is an order-preserving injection from $(\mathcal{M}(S), \geq_{mul})$ to $(Tr(S), \succeq)$, where $(\mathcal{M}(S), \geq_{mul})$ is the multiset extension of $(S, \geq)$.*

**Proof** Fix $r \in S$, arbitrarily chosen. Let the function $\phi_r : \mathcal{M}(S) \to Tr(S)$ be given by:

- $\phi_r(\emptyset) = (r, \emptyset)$
- $\phi_r(\{\{s_1, \ldots, s_k\}\}) = (r, \{\{(s_1, \emptyset), \ldots, (s_k, \emptyset)\}\})$

It is not difficult to see that $\phi_r$ is well-defined, injective and order-preserving. □

**Lemma 6.13.** *Let $S$ be a set and $\geq^1, \geq^2$ two quasi-orders in $S$ such that $>^1 \subseteq >^2$ and $\sim^1 \subseteq \sim^2$. Consider $Tr(S)$ with the quasi-orders $\succ^1 \cup \approx^1$ and $\succ^2 \cup \approx^2$, the tree liftings of respectively $\geq^1$ and $\geq^2$. Then $\succ^1 \subseteq \succ^2$ and $\approx^1 \subseteq \approx^2$.*

**Proof** (Sketch) We need to see that given two trees $A, B \in Tr(S)$ if $A \succ^1 B$ then $A \succ^2 B$ and if $A \approx^1 B$ then also $A \approx^2 B$.

Using lemma 6.5 and induction on $\texttt{depth}(A) + \texttt{depth}(B)$, it is very easy to see that $A \approx^1 B \Rightarrow A \approx^2 B$, thus $\approx^1 \subseteq \approx^2$. Using this result, we now see, also by induction on on $k = \texttt{depth}(A) + \texttt{depth}(B)$, that $A \succ^1 B \Rightarrow A \succ^2 B$. If $k = 2$ then $A = (a, \emptyset)$ and $B = (b, \emptyset)$ and we must have $a >^1 b$. By hypothesis we have $a >^2 b$, and therefore $A \succ^2 B$.

Suppose the result holds for trees $U$, $V$ with $\texttt{depth}(U) + \texttt{depth}(V) \leq n$. Let $A = (a, M_a)$ and $B = (b, M_b)$ be trees with $\texttt{depth}(A) + \texttt{depth}(B) = n + 1$. If $A \succ^1 B$ we must have either

- $a >^1 b$ and for all $u \in M_b$ either $A \succ^1 u$ or there is a tree $v \in M_a$ such that $v \succeq^1 u$. In this case also $a >^2 b$ and by induction hypothesis either $A \succ^2 u$ or $v \succeq^2 u$ (for the equivalence part we also have to use the fact that $\approx^1 \subseteq \approx^2$), so $A \succ^2 B$.
- $a \sim^1 b$ and $M_a$ $\texttt{ord}(\succeq^1_{mul})$ $M_b$. Let $\langle\ \rangle_i$ denote equivalence classes with respect to $\approx^i$, with $i = 1, 2$. We can write $M_a = \{\{u_1, \ldots, u_k\}\}$, with $k \geq 1$, and $M_b = \{\{v_1, \ldots, v_m\}\}$, with $m \geq 0$. Furthermore

$$\{\{\langle v_1\rangle_1, \ldots, \langle v_m\rangle_1\}\} = (\{\{\langle u_1\rangle_1, \ldots, \langle u_k\rangle_1\}\} \setminus X) \sqcup Y$$

for some finite multisets $\emptyset \neq X \sqsubseteq \{\{\langle u_1\rangle_1, \ldots, \langle u_k\rangle_1\}\}$ and $Y$ satisfying

$$\forall \langle y\rangle_1 \in Y\ \exists \langle x\rangle_1 \in X : \ \langle x\rangle_1 \succ^1 \langle y\rangle_1$$

Now $\langle x\rangle_1 \succ_1 \langle y\rangle_1 \Rightarrow x \succ_1 y \Rightarrow$ and by hypothesis we get $x \succ^2 y$ and consequently $\langle x\rangle_2 \succ^2 \langle y\rangle_2$. Note that this holds independently of the representatives of the classes. Since for any trees $p, q$ we have $\langle p\rangle_1 = \langle q\rangle_1 \Rightarrow \langle p\rangle_2 = \langle q\rangle_2$ (this is a simple consequence of $\approx^1 \subseteq \approx^2$), we can write:

$$\{\{\langle v_1\rangle_2, \ldots, \langle v_m\rangle_2\}\} = (\{\{\langle u_1\rangle_2, \ldots, \langle u_k\rangle_2\}\} \setminus X_2) \sqcup Y_2$$

where $\emptyset \neq X_2 \sqsubseteq \{\{\langle u_1\rangle_2, \ldots, \langle u_k\rangle_2\}\}$ and $Y_2$ are just the multisets $X$ and $Y$ with the equivalence classes taken with respect to $\approx^2$, and thus satisfying

$$\forall\langle y\rangle_2 \in Y_2\ \exists\langle x\rangle_2 \in X_2 :\ \langle x\rangle_2 \succ^2 \langle y\rangle_2$$

and so $M_a$ $\mathtt{ord}(\succeq^2_{mul})$ $M_b$. Since it also holds that $a \sim^2 b$, we have that $A \succ^2 B$.

□

Essential for our purposes is the preservation of well-foundedness, stated in the next result.

**Theorem 6.14.** *Let $(S, \geq)$ be a quasi-ordered set. Then $\geq$ is well-founded on $S$ if and only if $\succeq$ is well-founded on $Tr(S)$.*

**Proof** For the "if" part, suppose that $\geq$ is not well-founded in $S$. Then there is an infinite descending chain $a_0 > a_1 > \cdots$. According to the definition of $\succeq$ and lemma 6.7, then $(a_0, \emptyset) \succ (a_1, \emptyset) \succ \ldots$ is an infinite descending chain on $Tr(S)$, contradicting well-foundedness of $\succeq$.

For the "only-if" part consider $\geq_{rpo}$, the multiset recursive path order associated with $\geq$, as given in lemma 6.10. As was seen in chapter 4, well-foundedness of $\geq$ implies well-foundedness of $\geq_{rpo}$. Since lemma 6.11 gives that $\succeq \subseteq \geq_{rpo}$, we may also conclude well-foundedness of $\succeq$. □

A property not preserved by the tree lifting is totality. Again take $(S, >)$ to be the natural numbers with the usual order. Let


Then according to the definition of $\succeq$ neither $U \succ V$ nor $V \succ U$. Since $U \neq V$, the order $\succ$ is obviously not total. Note that $V \succ_{rpo} U$.

## 6.3 Transforming the TRS

We introduce now the transformations sketched in the beginning of this section. First we establish some terminology and the setting of our problem. Let $\mathcal{F}$ be a set of fixed arity function symbols and $\mathcal{X}$ a set of variables with $\mathcal{F} \cap \mathcal{X} = \emptyset$. Let $a$ be a function symbol with arity $N > 0$, and not occurring in $\mathcal{F}$; $a$ is the function symbol to be eliminated. Let $\diamond$ be a constant also not occurring in $\mathcal{F}$. We denote by $\mathcal{F}_a$ and $\mathcal{F}_\diamond$ respectively the sets $\mathcal{F} \cup \{a\}$ and $\mathcal{F} \cup \{\diamond\}$. We name *alien terms* those terms of $T(\mathcal{F}_a, X)$ whose root symbol is the symbol $a$.

We consider TRS's and equational systems over $T(\mathcal{F}_a, X)$ such that the function symbol $a$ may occur in the TRS but **not** in the equations. Since we are concerned with termination modulo a set of equations, we want to exclude equations that will force non-termination (independently of the form of the TRS). In [53], Jouannaud and Muñoz identified two essential restrictions equations have to satisfy, namely:

1. $var(e_1) = var(e_2)$, for any equation $(e_1, e_2)$;
2. $(e_1, e_2)$ can not be of the form $e_1 = x$ while $e_2$ contains more than one occurrence of the variable $x$.

We see why this is so. If $(e_1, e_2) \in E$ is an equation violating the first restriction, and supposing $e_1 = D[x, \ldots, x]$ where $x \notin var(e_2)$, we can write

$$D[r\sigma, \ldots, r\sigma] =_E e_2 =_E D[l\sigma, \ldots, l\sigma] \to D[r\sigma, \ldots, r\sigma],$$

where $l \to r$ is any rewrite rule and $\sigma$ a renaming to avoid conflict between variables of $l \to r$ and the equation $(e_1, e_2)$. If $(e_1, e_2) \in E$ is an equation violating the second restriction, for example $e_1 = x$ and $e_2 = D[x, \ldots, x]$ then

$$D[l\sigma, \ldots, l\sigma] \to D[l\sigma, \ldots, r\sigma] =_E D[D[l\sigma, \ldots, l\sigma], \ldots, r\sigma] \to \ldots$$

Condition (2) above can easily be generalized. Indeed equations of the form $(D[C[x]], C[x])$ where $x$ is a variable also occurring in the context $D$, prevent termination as we can see in a similar way as for equations $(C[x], x)$, where $x$ occurs in context $C$.

This last condition indicates that it is not always a good idea to mix equations with subterms. Due to a technicality in our proof, we need indeed to exclude such equations, but furthermore we need also to ensure that the equational theory is *length-limited*, i. e., for all (ground) equivalence classes $\langle s \rangle$, there is a natural number $n_{\langle s \rangle}$ such that $u \in \langle s \rangle \Rightarrow |u| \leq n_{\langle s \rangle}$; note that for finite signatures, length-limited is equivalent to finiteness of equivalence classes, for infinite signatures that is not so. We also need to ensure that the equations are variable-preserving, i. e., that $mvar(e_1) = mvar(e_2)$ for any equation $(e_1, e_2)$.

We also use *length-preserving* equations, i. e., equations satisfying $mvar(e_1) = mvar(e_2)$ and $|e_1| = |e_2|$. Provided that $T(\mathcal{F}_a) \neq \emptyset$, this is equivalent to the following characterization: for all *ground* substitutions $\sigma : X \to T(\mathcal{F}_a)$, $|e_1\sigma| = |e_2\sigma|$.

We are interested in rewriting modulo a set of equations, i. e., the relation $\to_{R/EQ}$, where $R$ is a TRS over $T(\mathcal{F}_a, \mathcal{X})$ and $EQ$ is a set of (length-limited variable preserving) equations over $T(\mathcal{F}_a, \mathcal{X})$, not containing the function symbol $a$. More specifically we are interested in termination of $\to_{R/EQ}$. Proving termination of this relation involves, amongst other things, proving termination of $\to_R$, so by simplifying this step, we simplify the whole process. We define a set of transformations on terms that induce transformations on the TRS's. Then we show that termination of $\to_{R/EQ}$ can be inferred from termination of $\to_{\mathcal{E}(R)/EQ}$, where $\mathcal{E}$ is the transformation used.

To the function symbol to be eliminated we associate a *e-status* (where "e" stands for elimination) whose role is to indicate how the subterms of alien terms are going to be treated.

**Definition 6.15.** A *e-status* is a partial function $\tau : \mathcal{F}_a \to \mathcal{P}(\mathbb{N}) \times \mathbb{N}$, satisfying the following condition: if $f$ has arity $n > 0$ and $\tau(f) = (X, i)$ then either

- $X = \emptyset$ and $i = 0$; or
- $X \neq \emptyset$ and $X \subseteq \{1, \ldots, n\}$ and $i \in X$.

Since we will only eliminate one function symbol at a time, the e-status needs only to be defined for that particular function symbol.

Note that the transformation associated with the e-status $\tau(a) = (\emptyset, 0)$ is *dummy elimination* and the transformation associated with the e-status $\tau(a) = (\{1, \ldots, n\}, i)$, is a generalization of *distribution elimination* (actually a set of generalizations indexed by $i$).

Recall that we want to eliminate the function symbol $a$, with arity $N \geq 1$. We present the new versions of the functions cap and dec, plus some needed auxiliary definitions.

**Definition 6.16.** For any $0 \leq i \leq N$, and term $t \in T(\mathcal{F}_a, \mathcal{X})$, the cap of $t$ of order $i$, denoted by $\mathsf{cap}_i(t)$, is a term over $T(\mathcal{F}_\diamond, \mathcal{X})$ given by the function $\mathsf{cap}_i$: $T(\mathcal{F} \cup \{a, \diamond\}, \mathcal{X}) \to T(\mathcal{F}_\diamond, \mathcal{X})$, defined inductively as follows:

- $\mathsf{cap}_i(x) = x$, for any $x \in \mathcal{X}$,
- $\mathsf{cap}_i(f(t_1, \ldots, t_m)) = f(\mathsf{cap}_i(t_1), \ldots, \mathsf{cap}_i(t_m))$, if $f \in \mathcal{F}_\diamond$ has arity $m \geq 0$,
- $\mathsf{cap}_i(a(t_1, \ldots, t_N)) = \begin{cases} \mathsf{cap}_i(t_i) & \text{if } i \neq 0 \\ \diamond & \text{if } i = 0 \end{cases}$

Note that strictly speaking the domain of $\mathsf{cap}_i$ need only be $T(\mathcal{F}_a, \mathcal{X})$, however to simplify the treatment later (basicly avoid defining an extension of cap only to include $\diamond$ in its domain), we use the extended signature $\mathcal{F}_a \cup \{\diamond\}$. This same observation applies to other definitions.

We can extend the function cap to substitutions as follows.

**Definition 6.17.** For any arbitrary substitution $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ and for any $0 \leq i \leq N$, the substitution $\mathsf{cap}_i(\sigma) : \mathcal{X} \to T(\mathcal{F}_\diamond, \mathcal{X})$ is defined by $\mathsf{cap}_i(\sigma)(x) = \mathsf{cap}_i(\sigma(x))$, for all $x \in \mathcal{X}$.

Next lemma states that cap distributes over substitution application.

**Lemma 6.18.** *Let $t \in T(\mathcal{F}_a, \mathcal{X})$ and let $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ be an arbitrary substitution. Then for any $0 \leq i \leq N$, we have $\mathsf{cap}_i(t\sigma) = \mathsf{cap}_i(t)\mathsf{cap}_i(\sigma)$.*

**Proof** Fix $0 \leq i \leq N$ arbitrarily. We proceed by induction on $t$. If $t = x \in \mathcal{X}$ then

$$\mathsf{cap}_i(t\sigma) = \mathsf{cap}_i(\sigma(x)) = \mathsf{cap}_i(\sigma)(x) = t\ \mathsf{cap}_i(\sigma) = \mathsf{cap}_i(t)\mathsf{cap}_i(\sigma),$$

by definition 6.17. If $t = f(t_1, \dots, t_m)$ (with $f \neq a$) then

$$\begin{array}{rcl}
\mathsf{cap}_i(f(t_1, \dots, t_m)\sigma) & = & \\
\mathsf{cap}_i(f(t_1\sigma, \dots, t_m\sigma)) & = & \text{(by definition 6.16)} \\
f(\mathsf{cap}_i(t_1\sigma), \dots, \mathsf{cap}_i(t_m\sigma)) & = & \text{(by induction hypothesis)} \\
f(\mathsf{cap}_i(t_1)\mathsf{cap}_i(\sigma), \dots, \mathsf{cap}_i(t_m)\mathsf{cap}_i(\sigma)) & = & \\
f(\mathsf{cap}_i(t_1), \dots, \mathsf{cap}_i(t_m))\mathsf{cap}_i(\sigma) & = & \text{(by definition 6.16)} \\
\mathsf{cap}_i(f(t_1, \dots, t_m))\mathsf{cap}_i(\sigma) & &
\end{array}$$

If $t = a(t_1, \dots, t_N)$ we distinguish two cases:

- if $i = 0$ then by definition 6.16, $\mathsf{cap}_i(t\sigma) = \diamond = \diamond\ \mathsf{cap}_i(\sigma) = \mathsf{cap}_i(t)\mathsf{cap}_i(\sigma)$ and the result holds.
- if $i \neq 0$ then

$$\begin{array}{rcl}
\mathsf{cap}_i(a(t_1, \dots, t_N)\sigma) & = & \\
\mathsf{cap}_i(a(t_1\sigma, \dots, t_N\sigma)) & = & \text{(by definition 6.16)} \\
\mathsf{cap}_i(t_i\sigma) & = & \text{(by induction hypothesis)} \\
\mathsf{cap}_i(t_i)\mathsf{cap}_i(\sigma) & = & \text{(by definition 6.16)} \\
\mathsf{cap}_i(a(t_1, \dots, t_N))\mathsf{cap}_i(\sigma) & &
\end{array}$$

□

The function cap is also idempotent as the next lemma shows.

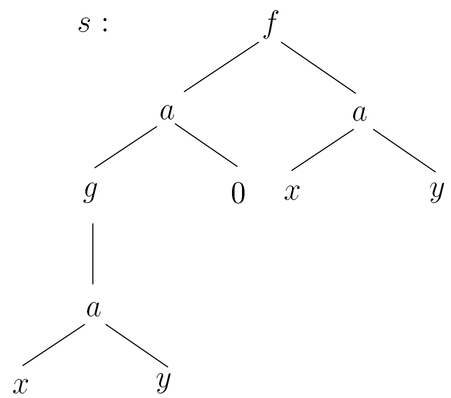
**Lemma 6.19.** *Let $t \in T(\mathcal{F}_a, \mathcal{X})$ be any term. Then $\mathsf{cap}_i(\mathsf{cap}_i(t)) = \mathsf{cap}_i(t)$.*

**Proof** This lemma can easily be proved by induction on terms; however note that for any $i$, $1 \leq i \leq N$, and for any term $t \in T(\mathcal{F} \cup \{a, \diamond\}, \mathcal{X})$, $\mathsf{cap}_i(t)$ is an element of $T(\mathcal{F}_\diamond, \mathcal{X})$, and if $s$ is an arbitrary term in $T(\mathcal{F}_\diamond, \mathcal{X})$ then $\mathsf{cap}_i(s) = s$. □

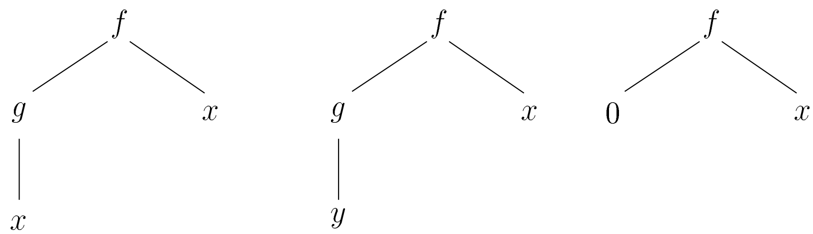
Since we are interested also in the subterms below occurrences of the symbol $a$, we need other operations for collecting the parts of those subterms that are relevant. In the case of *dummy elimination* the only operation needed was performed by the function dec. However, due to the fact that some arguments of alien terms need to be connected to the superterm where the alien term occurs, the decomposition of a term becomes a little more complicated. Also we want to minimize the combinations of these arguments, in the case of terms with nested $a$'s. We give an example to make clear what we mean.

**Example 6.20.** Consider the following term $s$:

$s$ :




Suppose that $a$ has e-status $\tau(a) = (\{1,2\},1)$. We want to have the following partitioning of $s$:




Note that we don't consider terms like $f(0,x)$. This is what we name a "crossed term" since it is obtained by choosing different arguments for aliens terms occurring at the same level of nesting (of alien terms). In other words the partition of the term we are interested in should always choose the same argument for alien terms occurring at the same level of nesting. We note that this is an essential difference with respect to *distribution elimination* since this transformation considers all possible combinations of subterms of alien terms.

The operations that partition a term will be defined in such a way that the combinations of subterms of alien terms will be done by always picking the same branch and thus minimizing the number of "crossed" terms obtained.

**Definition 6.21.** Given a term $t \in T(\mathcal{F}_a, \mathcal{X})$, its *residue of order* $i$, with $1 \leq i \leq N$, and its *residue* (with respect to $\tau$), are denoted respectively by $E_i(t)$ and by $E(t)$, where $E_i, E : T(\mathcal{F}_a, \mathcal{X}) \to \mathcal{P}(T(\mathcal{F}_\diamond, \mathcal{X}))$ are defined inductively as follows:

- $E_i(x) = \{x\}$; $E(x) = \{x\}$,
- if $f \in \mathcal{F}$ has arity $m \geq 0$ ($f \neq a$), then

$$E_i(f(t_1,\ldots,t_m)) = \{f(u_1,\ldots,u_m), \text{ with } u_j \in E_i(t_j), \text{ for all } 1 \leq j \leq m\}$$

$$E(f(t_1,\ldots,t_m)) = \begin{cases} \displaystyle\bigcup_{j \in X} E_j(f(t_1,\ldots,t_m)) & \text{if } X \neq \emptyset \\ \{\mathsf{cap}_0(f(t_1,\ldots,t_m))\} & \text{if } X = \emptyset \end{cases}$$

- $E_i(a(t_1, \dots, t_N)) = E(t_i)$ and

$$E(a(t_1, \dots, t_N)) = \begin{cases} \bigcup_{j \in X} E(t_j) & \text{if } X \neq \emptyset \\ \{\mathsf{cap}_0(a(t_1, \dots, t_N))\} & \text{if } X = \emptyset \end{cases}$$

**Definition 6.22.** Let the e-status of $a$ be $\tau(a) = (X, i)$. For any term $t \in T(\mathcal{F}_a, \mathcal{X})$, its *decomposition* is denoted by $\mathtt{dec}(t)$, where $\mathtt{dec} : T(\mathcal{F}_a, \mathcal{X}) \to \mathcal{P}(T(\mathcal{F}_\diamond, \mathcal{X}))$ is defined inductively as follows:

- $\mathtt{dec}(x) = \emptyset$
- $\mathtt{dec}(f(t_1, \dots, t_m)) = \bigcup_{i=1}^{m} \mathtt{dec}(t_i)$, if $f \in \mathcal{F}$ has arity $m \geq 0$ ($f \neq a$)
- $\mathtt{dec}(a(t_1, \dots, t_N)) = \bigcup_{i=1}^{N} \mathtt{dec}(t_i) \bigcup \begin{cases} \bigcup_{j \in \{1, \dots, N\} \setminus X} E(t_j) & \text{if } X \neq \emptyset \\ \bigcup_{j=1}^{N} \{\mathsf{cap}_0(t_j)\} & \text{if } X = \emptyset \end{cases}$

**Definition 6.23.** Given a term $t \in T(\mathcal{F}_a, \mathcal{X})$, its *leading residue* with respect to the e-status $\tau(a) = (X, i)$ is the term $\mathsf{cap}_i(t)$.

The following lemma relates the leading residue and the residue of a term, and will be of use later.

**Lemma 6.24.** *Let the e-status of $a$ be $\tau(a) = (X, i)$. Let $t \in T(\mathcal{F}_a, \mathcal{X})$ be an arbitrary term. Then $\mathsf{cap}_i(t) \in E(t)$, i. e., the residue of a term always contains the leading residue of the same term. Furthermore if $i \neq 0$ then $\mathsf{cap}_i(t) \in E_i(t)$.*

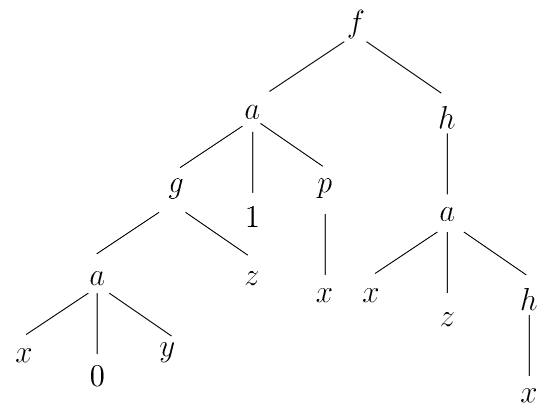
**Proof** If $i = 0$ then the result follows immediately from the definitions of cap and $E$. Suppose then that $i \in X$. We will prove that $\mathsf{cap}_i(t) \in E_i(t), E(t)$ by induction on $t$.

If $t = x$ then $\mathsf{cap}_i(x) = x$ and $E_i(t) = E(t) = \{x\}$, so the result holds.

If $t = f(t_1, \dots, t_m)$ then $\mathsf{cap}_i(t) = f(\mathsf{cap}_i(t_1), \dots, \mathsf{cap}_i(t_m))$. By induction hypothesis $\mathsf{cap}_i(t_j) \in E_i(t_j)$, for any $1 \leq j \leq m$. By definition of $E_i$, we have then that $f(\mathsf{cap}_i(t_1), \dots, \mathsf{cap}_i(t_m)) \in E_i(t)$. Since $E_i(t) \subseteq E(t)$ (note that $i \in X$), the result also holds.

If $t = a(t_1, \dots, t_N)$ then $\mathsf{cap}_i(t) = \mathsf{cap}_i(t_i)$. By induction hypothesis $\mathsf{cap}_i(t_i) \in E_i(t_i), E(t_i)$. Since $E_i(t) = E(t_i)$ and $E_i(t) \subseteq E(t)$, we have the result. □

**Example 6.25.** Suppose we want to eliminate the function symbol $a$, with arity $3$. Consider the following term:




Consider the following e-status for $a$: $\tau_0 = (\emptyset, 0)$, $\tau_1 = (\{1,3\}, 1)$ and $\tau_2 = (\{2\}, 2)$. With respect to the different e-status, the leading residue, residue and the decomposition of $t$ are given respectively by:

- for $\tau_0$:

$$\begin{array}{rcl} \mathtt{cap}_0(t) & = & f(\diamond, h(\diamond)) \\ E(t) & = & \{f(\diamond, h(\diamond))\} \\ \mathtt{dec}(t) & = & \{g(\diamond, z), x, 0, y, 1, p(x), z, h(x)\} \end{array}$$

- for $\tau_1$:

$$\begin{array}{rcl} \mathtt{cap}_1(t) & = & f(g(x,z), h(x)) \\ E(t) & = & \{f(g(x,z), h(x)), f(g(y,z), h(x)), f(p(x), h(h(x)))\} \\ \mathtt{dec}(t) & = & \{0, 1, z\} \end{array}$$

- for $\tau_2$:

$$\begin{array}{rcl} \mathtt{cap}_2(t) & = & f(1, h(z)) \\ E(t) & = & \{f(1, h(z))\} \\ \mathtt{dec}(t) & = & \{g(0,z), x, y, p(x), h(x)\} \end{array}$$

We can now define the transformation on TRS's. As can be expected we will transform the left and righthand-sides of the rules in $R$ and create new rules using this transformation. Since the transformation of a term gives, in general, a set containing more than one term, we have to decide which element we choose for the lefthand-side of the new rules. The choice of that element is given by the e-status of the symbol to be eliminated. Recall that $\tau(a) = (X, i)$; we use the index $i$ to determine that choice. In other words:

**Definition 6.26.** Given an equational rewrite system $R/EQ$ over $T(\mathcal{F}_a, \mathcal{X})$ such that the function symbol $a$ does not occur in the equations of $EQ$, and given a well-defined e-status $\tau(a) = (X, i)$, $\mathcal{E}(R)/EQ$ is an equational rewrite system over $T(\mathcal{F}_\diamond, \mathcal{X})$ where $\mathcal{E}(R)$ is given by

$$\mathcal{E}(R) = \{\mathtt{cap}_i(l) \to u \mid l \to r \in R \text{ and } u \in E(r) \cup \mathtt{dec}(r)\}$$

We make some remarks about the transformation defined.

- The transformation $\mathcal{E}$ depends on the e-status of the symbol to be eliminated, and therefore by changing the e-status, we obtain, associated to each function symbol to be eliminated, a family of transformations. Note also that in some cases $\mathcal{E}(R)$ may not be a TRS in the usual sense, since $\mathtt{cap}_i(l)$ may either be a variable or eliminate variables needed in the righthand-sides of the transformed rules. We are interested in the cases where $\mathcal{E}(R)$ is a well-defined TRS.

- Allowing the function symbol to be eliminated to occur in the lefthand-sides of rules of the TRS is a generalization with respect to previous work and its possibility was first remarked by Middeldorp and Ohsaki (personal communication).

- Note that we only need to extend the signature $\mathcal{F}$ to $\mathcal{F}_\diamond$ if we intend to eliminate function symbols with e-status $(\emptyset, 0)$. Otherwise, as can be seen from the definitions 6.16, 6.21, 6.22, the constant $\diamond$ is never introduced by the transformation, and so we can restrict ourselves to the signature $\mathcal{F}$.

- When the function symbol $a$ does not occur in a term $t$, we have that $\mathtt{dec}(t) = \emptyset$ and $E(t) = \{t\}$. Since the equations on $EQ$ satisfy this restriction, it is not necessary to apply the transformation to them.

**Example 6.27.** The following example was taken from Fokkink and Zantema [36]. Let $\mathcal{F} = \{+, *, ., c, d\}$, with $+, ., *$ having arity $2$ and $c, d$ being constants. Let $EQ$ consist of the associative and commutative equations for the function symbol "$+$", i. e.,

$$EQ = \{(x + (y + z), (x + y) + z), ((x + y) + z, x + (y + z)), (x + y, y + x)\}.$$

Let $R$ be given by the rules (in infix notation):

$$\begin{array}{lcl} (c * y) + z & \rightarrow & (c.(c * y)) + (y + z) \\ c * (d * z) & \rightarrow & c * ((d.(d * z)) + z) \end{array}$$

By eliminating "." with e-status $(\emptyset, 0)$, we get the TRS

$$\begin{array}{lcl} (c * y) + z & \rightarrow & \diamond + (y + z) \\ (c * y) + z & \rightarrow & c \\ (c * y) + z & \rightarrow & c * y \\ c * (d * z) & \rightarrow & c * (\diamond + z) \\ c * (d * z) & \rightarrow & d \\ c * (d * z) & \rightarrow & d * z \end{array}$$

From the definition of $\mathcal{E}$, we can see that in general the TRS $\mathcal{E}(R)$ has more rules but is syntactically simpler than the original one, so the transformation can be quite useful if we are able to infer termination of $R/EQ$ from termination of $\mathcal{E}(R)/EQ$. Termination however is not preserved, i. e., if $R/EQ$ is terminating, $\mathcal{E}(R)/EQ$ is not necessarily terminating, as the following example shows.

**Example 6.28.** Let $EQ = \emptyset$ and let $R$ be the terminating TRS given by:

$$f(x,x) \;\rightarrow\; f(a(x),x)$$

Suppose $a$ has e-status $\tau(a) = (\emptyset, 0)$. The transformed TRS $\mathcal{E}(R)$ is given by:

$$\begin{array}{lcl} f(x,x) & \rightarrow & f(\Diamond, x) \\ f(x,x) & \rightarrow & x \end{array}$$

and is obviously not terminating. Note that the only other possible e-status for $a$, namely $(\{1\}, 1)$, would also result in a non-terminating TRS:

$$f(x,x) \;\rightarrow\; f(x,x)$$

In general, however, different e-status can lead to TRS's with different termination properties (these issues are further discussed in section 6.5). Note also that it is not essential that $EQ = \emptyset$. Any equational system such that $R/EQ$ terminates, would give a similar example.

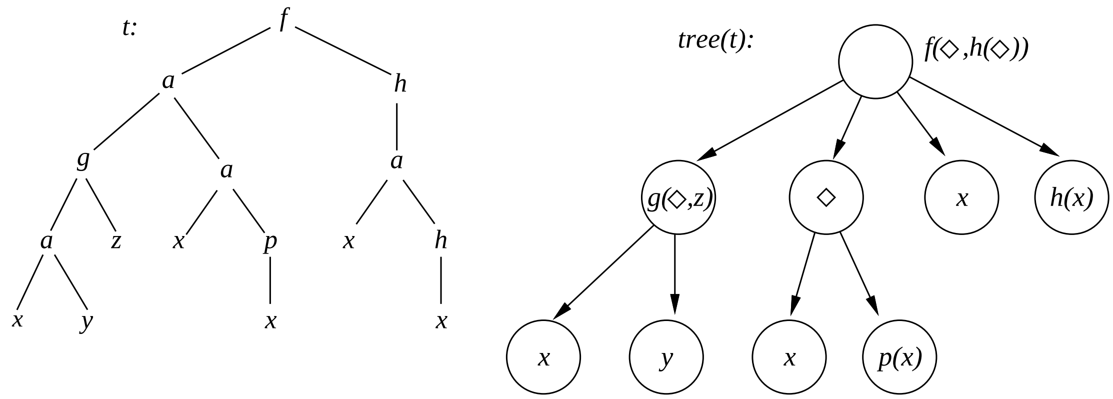
In the following we will show that, for any well-defined e-status $\tau$, the transformation associated with it is sound with respect to termination, i. e., termination of $\mathcal{E}(R)/EQ$ implies termination of $R/EQ$. Before going into the technical details we give a general idea of the proof. If $\mathcal{E}(R)/EQ$ is terminating, there is a well-founded quasi-order $\geq$ on $T(\mathcal{F}_\Diamond, \mathcal{X})$ compatible with $\mathcal{E}(R)/EQ$. If we consider the quasi-ordered set $(Tr(T(\mathcal{F}_\Diamond, \mathcal{X})), \succeq)$ (where $\succeq$ is the tree lifting of $\geq$ as defined in 6.4) then $\succeq$ is also well-founded. We now use the trees over $T(\mathcal{F}_\Diamond, \mathcal{X})$ to interpret the terms of $T(\mathcal{F}_a, \mathcal{X})$ in such a way that rewrite chains in $R/EQ$ translate to descending chains of trees. Preservation of well-foundedness by the tree lifting gives then termination of $\rightarrow_{R/EQ}$.

We introduce some definitions and auxiliary results.

**Definition 6.29.** Let the e-status of $a$ be $\tau(a) = (X, i)$. A term $t \in T(\mathcal{F}_a, \mathcal{X})$ is mapped to a tree $\texttt{tree}(t) \in Tr(T(\mathcal{F}_\Diamond, \mathcal{X}))$, by the function $\texttt{tree} : T(\mathcal{F} \cup \{a, \Diamond\}, \mathcal{X}) \rightarrow Tr(T(\mathcal{F}_\Diamond, \mathcal{X}))$, defined inductively as follows:

- $\texttt{tree}(x) = (x, \emptyset)$, for any $x \in \mathcal{X}$
- $\texttt{tree}(f(s_1, \ldots, s_m)) = \left(\mathsf{cap}_i(f(s_1, \ldots, s_m)), \bigsqcup_{j=1}^{m} M_j\right)$,
  where $\texttt{tree}(s_j) = (\mathsf{cap}_i(s_j), M_j)$.
- $\texttt{tree}(a(s_1, \ldots, s_N)) = \left(\mathsf{cap}_i(a(s_1, \ldots, s_N)), \left(\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\texttt{tree}(s_j)\}\!\}\right) \sqcup M_i\right)$,
  where, $M_i = \emptyset$, for $i = 0$, and $\texttt{tree}(s_i) = (\mathsf{cap}_i(s_i), M_i)$, for $i \neq 0$.

**Example 6.30.** The following picture shows the same term as in example 6.25 together with its corresponding tree for the e-status $\tau_0(a) = (\emptyset, 0)$.




**Remark 6.31.** From now on we assume we have an equational rewriting system $R/EQ$, such that the equations in $EQ$ are length-limited and variable-preserving, and do not contain the function symbol $a$. We consider the equational theory generated by $EQ$ over the set of terms $T(\mathcal{F}_\diamond, \mathcal{X})$, so $\mathcal{E}(R)/EQ$ is an equational rewrite system over $T(\mathcal{F}_\diamond, \mathcal{X})$. Furthermore we assume that $\mathcal{E}(R)$ is well-defined and that $\mathcal{E}(R)/EQ$ is terminating.

Since $\mathcal{E}(R)/EQ$ is terminating, as a consequence of theorem 2.86, we have:

**Lemma 6.32.** *The relation $\rightarrow^+_{\mathcal{E}(R)/EQ}$ is a well-founded partial order on $T(\mathcal{F}_\diamond, \mathcal{X})$, closed under contexts and substitutions, compatible with $\mathcal{E}(R)$ and compatible with $=_{EQ}$ (i. e. condition 2.1 is satisfied).*

**Remark 6.33.** The lemma above states that $\rightarrow^+_{\mathcal{E}(R)/EQ} \cup =_{EQ}$ is a well-founded quasi-order on $T(\mathcal{F}_\diamond, \mathcal{X})$. From now on we use this quasi-order and its tree lifting $\succeq \, = \, \succ \cup \approx$ on $Tr(T(\mathcal{F}_\diamond, \mathcal{X}))$ (actually any partial order with the properties stated in the lemma could be used for our purposes).

First we will see that $\approx$ is compatible with $=_{EQ}$, i. e., if $s, t \in T(\mathcal{F}_a, \mathcal{X})$ then $s =_{EQ} t \Rightarrow \mathtt{tree}(s) \approx \mathtt{tree}(t)$. Some auxiliary results are required.

**Lemma 6.34.** *Let $s, t$ be terms in $T(\mathcal{F}_a, \mathcal{X})$. If $s =_{EQ} t$ then $\mathsf{cap}_i(s) =_{EQ} \mathsf{cap}_i(t)$, for any $0 \le i \le N$.*

**Proof** We proceed by induction on the definition of $=_{EQ}$ (see definition 2.68). If $s = t$ the result obviously holds. Suppose now that $s = C[e_1\sigma]$ and $t = C[e_2\sigma]$, for some context $C$, equation $(e_1, e_2) \in EQ$ and substitution $\sigma : \mathcal{X} \rightarrow T(\mathcal{F}_a, \mathcal{X})$. For this case we have to do induction on the context. Suppose first that $C$ is the trivial context. Then $s = e_1\sigma$ and $t = e_2\sigma$. We have

$$\begin{array}{rcr}
\mathsf{cap}_i(e_1\sigma) & = & \text{(by lemma 6.18)} \\
\mathsf{cap}_i(e_1)\mathsf{cap}_i(\sigma) & = & (a \text{ does not occur in } e_1) \\
e_1\mathsf{cap}_i(\sigma) & =_{EQ} & \text{(by definition of } =_{EQ}) \\
e_2\mathsf{cap}_i(\sigma) & = & (a \text{ does not occur in } e_2) \\
\mathsf{cap}_i(e_2)\mathsf{cap}_i(\sigma) & = & \text{(by lemma 6.18)} \\
\mathsf{cap}_i(e_2\sigma) & &
\end{array}$$

Suppose now that $C = f(\ldots, D[\ ], \ldots)$ and that the result holds for context $D$. If $f \neq a$, we have

$$\begin{array}{rcr} \mathsf{cap}_i(s) & = & \\ \mathsf{cap}_i(f(\ldots, D[e_1\sigma], \ldots)) & = & \text{(by definition 6.16)} \\ f(\ldots, \mathsf{cap}_i(D[e_1\sigma]), \ldots) & =_{EQ} & \text{(by induction hypothesis)} \\ f(\ldots, \mathsf{cap}_i(D[e_2\sigma]), \ldots) & = & \text{(by definition 6.16)} \\ \mathsf{cap}_i(f(\ldots, D[e_2\sigma], \ldots)) & = & \\ \mathsf{cap}_i(t) & & \end{array}$$

If $f = a$ we have to distinguish several cases. If $i = 0$ then $\mathsf{cap}_i(s) = \diamond$ and $\mathsf{cap}_i(t) = \diamond$ and so $\mathsf{cap}_i(s) =_{EQ} \mathsf{cap}_i(t)$. Suppose now that $i \neq 0$. Then $\mathsf{cap}_i(s) = \mathsf{cap}_i(s_i)$, where $s_i$ is the $i$-th argument of $a$. If $D[\,]$ does not occur at position $i$ then $\mathsf{cap}_i(s) = \mathsf{cap}_i(s_i) = \mathsf{cap}_i(t)$ and the result holds. If $D[\ ]$ occurs at position $i$, then

$$\begin{array}{rcr} \mathsf{cap}_i(s) & = & \text{(by definition 6.16)} \\ \mathsf{cap}_i(D[e_1\sigma]) & =_{EQ} & \text{(by induction hypothesis)} \\ \mathsf{cap}_i(D[e_2\sigma]) & = & \text{(by definition 6.16)} \\ \mathsf{cap}_i(t) & & \end{array}$$

Suppose now that there is a term $u \in T(\mathcal{F}_a, \mathcal{X})$ such that $s =_{EQ} u$ and $u =_{EQ} t$ and $\mathsf{cap}_i(s) =_{EQ} \mathsf{cap}_i(t)$ and $\mathsf{cap}_i(t) =_{EQ} \mathsf{cap}_i(u)$, then the result follows from transitivity of $=_{EQ}$. □

**Lemma 6.35.** *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be an arbitrary term and $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ an arbitrary substitution. Let $\tau(a) = (X, i)$. Then $\mathtt{tree}(t\sigma) = (\mathsf{cap}_i(t\sigma), \bigsqcup_{x \in mvar(t)} M_x)$, where $\mathtt{tree}(\sigma(x)) = (\mathsf{cap}_i(\sigma(x)), M_x)$, for all $x \in mvar(t)$.*

**Proof** We proceed by induction on the term $t$. If $t = x$ the result holds since $\mathtt{tree}(t\sigma) = \mathtt{tree}(\sigma(x))$. If $t$ is a constant, then $t\sigma = t$ and since $mvar(t) = \emptyset$ and $\mathtt{tree}(t\sigma) = (\mathsf{cap}_i(t\sigma), \emptyset)$, the result also holds.

Suppose now that $t = f(t_1, \ldots, t_m)$, where $a \neq f$ and $a$ does not occur in $t_i$, for any $1 \leq i \leq m$. Then

$$\begin{array}{rcl} \mathtt{tree}(t\sigma) = \mathtt{tree}(f(t_1, \ldots, t_m)\sigma) & = & \mathtt{tree}(f(t_1\sigma, \ldots, t_m\sigma)) \\ & = & \left( \mathsf{cap}_i(f(t_1\sigma, \ldots, t_m\sigma)), \bigsqcup_{j=1}^{m} M_j \right) \end{array}$$

where for each $1 \leq j \leq m$, $\mathtt{tree}(t_j\sigma) = (\mathsf{cap}_i(t_j\sigma), M_j)$. We can apply the induction hypothesis to each $t_j$ to conclude that $M_j = \bigsqcup_{x \in mvar(t_j)} M_x$, where again $\mathtt{tree}(\sigma(x)) = (\mathsf{cap}_i(\sigma(x)), M_x)$, for each $x \in mvar(t_j)$.

We then have that $\bigsqcup_{j=1}^{m} M_j = \bigsqcup_{j=1}^{m} \left( \bigsqcup_{x \in mvar(t_j)} M_x \right)$, and since $mvar(t) = \bigsqcup_{j=1}^{m} mvar(t_j)$, the previous multiset coincides with $\bigsqcup_{x \in mvar(t)} M_x$, as we needed. □

**Lemma 6.36.** *Let* $s, t \in T(\mathcal{F}_a, \mathcal{X})$*; let* $\tau(a) = (X, i)$*. Then* $s =_{EQ} t \Rightarrow \mathtt{tree}(s) \approx \mathtt{tree}(t)$*.*

**Proof** Again we use induction on the definition of $=_{EQ}$. If $s = t$ then the result holds since $\approx$ contains syntactical equality on trees.

Suppose now that $s = C[e_1\sigma]$ and that $t = C[e_2\sigma]$ for some context $C$, equation $(e_1, e_2) \in EQ$ and substitution $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$. We do now some induction on the context. If $C$ is the trivial context then $s = e_1\sigma$ and $t = e_2\sigma$. Since $a$ does not occur in $e_1, e_2$, we can apply lemma 6.35 and write

$$\mathtt{tree}(e_1\sigma) = (\mathsf{cap}_i(e_1\sigma), \bigsqcup_{x \in mvar(e_1)} M_x),$$

where $\mathtt{tree}(\sigma(x)) = (\mathsf{cap}_i(\sigma(x)), M_x)$, for all $x \in mvar(e_1)$. Similarly $\mathtt{tree}(e_2\sigma) = (\mathsf{cap}_i(e_2\sigma), \bigsqcup_{x \in mvar(e_2)} M_x)$. Recall that all the equations in $EQ$ are variable-preserving. This means that $mvar(e_1) = mvar(e_2)$. Furthermore, by lemma 6.34, $\mathsf{cap}_i(e_1\sigma) =_{EQ} \mathsf{cap}_i(e_2\sigma)$, and from definition of tree lifting we conclude that $\mathtt{tree}(e_1\sigma) \approx \mathtt{tree}(e_2\sigma)$.

Suppose now that $C = f(v_1, \ldots, D[\ ], \ldots, v_m)$, where $D$ occurs at some position $1 \leq j \leq m$, and that the result holds for context $D$. If $f \neq a$ then

$$\begin{aligned}
\mathtt{tree}(s) &= \mathtt{tree}(f(v_1, \ldots, D[e_1\sigma], \ldots, v_m)) \\
&= (\mathsf{cap}_i(f(v_1, \ldots, D[e_1\sigma], \ldots, v_m)), \bigsqcup_{k=1}^{m} M_k) \\
&= (f(\mathsf{cap}_i(v_1), \ldots, \mathsf{cap}_i(D[e_1\sigma]), \ldots, \mathsf{cap}_i(v_m)), \bigsqcup_{k=1}^{m} M_k)
\end{aligned}$$

where, for $k \neq j$, $\mathtt{tree}(v_k) = (\mathsf{cap}_i(v_k), M_k)$ and $\mathtt{tree}(D[e_1\sigma]) = (\mathsf{cap}_i(D[e_1\sigma]), M_j)$. Similarly

$$\begin{aligned}
\mathtt{tree}(t) &= \mathtt{tree}(f(v_1, \ldots, D[e_2\sigma], \ldots, v_m)) \\
&= (\mathsf{cap}_i(f(v_1, \ldots, D[e_2\sigma], \ldots, v_m)), \bigsqcup_{k=1}^{m} M'_k) \\
&= (f(\mathsf{cap}_i(v_1), \ldots, \mathsf{cap}_i(D[e_2\sigma]), \ldots, \mathsf{cap}_i(v_m)), \bigsqcup_{k=1}^{m} M'_k)
\end{aligned}$$

where, for $k \neq j$, $M'_k = M_k$ and $\mathtt{tree}(D[e_2\sigma]) = (\mathsf{cap}_i(D[e_2\sigma]), M'_j)$. By induction hypothesis, $\mathtt{tree}(D[e_1\sigma]) \approx \mathtt{tree}(D[e_2\sigma])$ and this means that $M_j \approx_{mul} M'_j$. But then also $\bigsqcup_{k=1}^{m} M_k \approx_{mul} \bigsqcup_{k=1}^{m} M'_k$. Since $D[e_1\sigma] =_{EQ} D[e_2\sigma]$ and $=_{EQ}$ is closed under contexts, we can apply lemma 6.34 to conclude that

$$\mathsf{cap}_i(f(v_1, \ldots, D[e_1\sigma], \ldots, v_m)) =_{EQ} \mathsf{cap}_i(f(v_1, \ldots, D[e_2\sigma], \ldots, v_m))$$

and therefore that $\mathtt{tree}(s) \approx \mathtt{tree}(t)$.

If $f = a$ we consider two cases, namely $i = j$ and $i \neq j$. For $i \neq j$,

$$\begin{aligned}
&\mathtt{tree}(s)\\
&= (\mathsf{cap}_i(a(v_1, \ldots, D[e_1\sigma], \ldots, v_N)), \bigsqcup_{k=1,k\neq j,i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_1\sigma])\}\} \sqcup M_i)\\
&= \left(\mathsf{cap}_i(v_i), \bigsqcup_{k=1,k\neq j,i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_1\sigma])\}\} \sqcup M_i\right)
\end{aligned}$$

where if $i = 0$ then $M_i = \emptyset$ and $\mathsf{cap}_i(v_i) = \diamond$, and for $i \neq 0$, $\mathtt{tree}(v_i) = (\mathsf{cap}_i(v_i), M_i)$. Also

$$\begin{aligned}
&\mathtt{tree}(t)\\
&= (\mathsf{cap}_i(a(v_1, \ldots, D[e_2\sigma], \ldots, v_N)), \bigsqcup_{k=1,k\neq i,j}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_2\sigma])\}\} \sqcup M_i)\\
&= \left(\mathsf{cap}_i(v_i), \bigsqcup_{k=1,k\neq i,j}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_2\sigma])\}\} \sqcup M_i\right)
\end{aligned}$$

Since by induction hypothesis $\mathtt{tree}(D[e_1\sigma]) \approx \mathtt{tree}(D[e_2\sigma])$, we also have

$$\bigsqcup_{k=1,k\neq j}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_1\sigma])\}\} \sqcup M_i \quad \approx_{mul} \quad \bigsqcup_{k=1,k\neq j}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup \{\{\mathtt{tree}(D[e_2\sigma])\}\} \sqcup M_i$$

and consequently $\mathtt{tree}(s) \approx \mathtt{tree}(t)$ (recall that their roots are equal).

For $i = j$ (note that then $i \neq 0$), we have

$$\begin{aligned}
\mathtt{tree}(s) &= \left(\mathsf{cap}_i(a(v_1, \ldots, D[e_1\sigma], \ldots, v_N)), \bigsqcup_{k=1,k\neq i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M_i\right)\\
&= \left(\mathsf{cap}_i(D[e_1\sigma]), \bigsqcup_{k=1,k\neq i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M_i\right)
\end{aligned}$$

where $\mathtt{tree}(D[e_1\sigma]) = (\mathsf{cap}_i(D[e_1\sigma]), M_i)$. Also

$$\begin{aligned}
\mathtt{tree}(t) &= \left(\mathsf{cap}_i(a(v_1, \ldots, D[e_2\sigma], \ldots, v_N)), \bigsqcup_{k=1,k\neq i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M_i'\right)\\
&= \left(\mathsf{cap}_i(D[e_2\sigma]), \bigsqcup_{k=1,k\neq i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M_i'\right)
\end{aligned}$$

where $\mathtt{tree}(D[e_2\sigma]) = (\mathsf{cap}_i(D[e_2\sigma]), M'_i)$. Since by induction hypothesis

$$\mathtt{tree}(D[e_1\sigma]) \approx \mathtt{tree}(D[e_2\sigma])$$

we also have $M_i \approx_{mul} M'_i$ and consequently

$$\bigsqcup_{k=1,k\neq i}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M_i \approx_{mul} \bigsqcup_{k=1,k\neq j}^{N} \{\{\mathtt{tree}(v_k)\}\} \sqcup M'_i$$

Since by lemma 6.34 $\mathsf{cap}_i(D[e_1\sigma]) =_{EQ} \mathsf{cap}_i(D[e_2\sigma])$, we have that $\mathtt{tree}(s) \approx \mathtt{tree}(t)$.

Suppose now that there is a term $u \in T(\mathcal{F}_a, \mathcal{X})$ such that $s =_{EQ} u$ and $u =_{EQ} t$ and $\mathtt{tree}(s) \approx \mathtt{tree}(u)$ and $\mathtt{tree}(u) \approx \mathtt{tree}(t)$; transitivity of $\approx$ gives the result. □

We have now all the necessary results that allow us to infer equivalence on trees built from equivalent terms, so we turn to the case of inequality.

**Lemma 6.37.** *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \setminus \mathcal{X}$ and let $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ be any substitution. Suppose that $x \in var(t)$ is such that $t \to^{+}_{\mathcal{E}(R)/EQ} x$. Then $\mathtt{tree}(t\sigma) \succ \mathtt{tree}(\sigma(x))$, where the trees are defined with respect to the e-status of $a$ given by $\tau(a) = (X, i)$.*

**Proof** By definitions 6.29 and 6.16, and lemmas 6.18, 6.35,

$$\begin{aligned} \mathtt{tree}(t\sigma) &= (\mathsf{cap}_i(t\sigma), \bigsqcup_{y\in mvar(t)} M_y) \\ &= (\mathsf{cap}_i(t)\mathsf{cap}_i(\sigma), \bigsqcup_{y\in mvar(t)} M_y) \\ &= (t\ \mathsf{cap}_i(\sigma), \bigsqcup_{y\in mvar(t)} M_y) \end{aligned}$$

where $\mathtt{tree}(\sigma(y)) = (\mathsf{cap}_i(\sigma(y)), M_y)$, for all $y \in mvar(t)$. Since

$$t \to^{+}_{\mathcal{E}(R)/EQ} x \ \Rightarrow\ t\ \mathsf{cap}_i(\sigma) \to^{+}_{\mathcal{E}(R)/EQ} \mathsf{cap}_i(\sigma(x))$$

(recall that $\to^{+}_{\mathcal{E}(R)/EQ}$ is closed under substitutions) and $M_x \sqsubseteq \bigsqcup_{y\in mvar(t)} M_y$, we conclude that $\mathtt{tree}(t\sigma) \succ \mathtt{tree}(\sigma(x))$. □

Recall that the definition of the transformation applied to a rule $l \to r$ in $R$, gives eventually more than one rule of the form $\mathsf{cap}_i(l) \to_{\mathcal{E}(R)} u$, with $u \in E(r) \cup \mathtt{dec}(r)$. Obviously we want the tree interpretation of the terms in $T(\mathcal{F}_a, \mathcal{X})$ to satisfy $\mathtt{tree}(l\sigma) \succ \mathtt{tree}(r\sigma)$, for any substitution $\sigma : X \to T(\mathcal{F}_a, \mathcal{X})$. A first approximation to this result is to prove that $\mathtt{tree}(\mathsf{cap}_i(l)\sigma) \succ \mathtt{tree}(r\sigma)$ and in order to obtain this result we have to use the fact that $\mathsf{cap}_i(l) \to^{+}_{\mathcal{E}(R)/EQ} u$, for all $u \in E(r) \cup \mathtt{dec}(r)$. Abstracting from the actual rules of $R$, this amounts to prove that $\mathtt{tree}(s\sigma) \succ \mathtt{tree}(t\sigma)$ whenever

- $var(t) \subseteq var(s)$ and
- $s \rightarrow^+_{\mathcal{E}(R)/EQ} u$, for all $u \in E(r) \cup \mathtt{dec}(r)$

This can be done by induction on the structure of $t$, and for the case that $t$ is a variable it has already been shown in lemma 6.37. However, in order to be able to use the induction hypothesis on subterms of $t$, the order $\rightarrow^+_{\mathcal{E}(R)/EQ}$ has to satisfy the subterm property (modulo $EQ$). In general this will not be the case, but fortunately it is not difficult to extend $\rightarrow^+_{\mathcal{E}(R)/EQ}$ to another order on $T(\mathcal{F}_a, \mathcal{X})$, that while loosing closedness under contexts, maintains all the other nice properties enjoyed by $\rightarrow^+_{\mathcal{E}(R)/EQ}$ and also has the subterm property. So before proceeding further in the proof, we introduce this order which is an adaptation to the equational case of an order used by Kamin and Lévy in [55].

**Definition 6.38.** Let $> \cup =_E$ be a quasi-order on $T(\mathcal{F}_\diamond, \mathcal{X})$ such that $>$ is closed under contexts and substitutions and $=_E$ is the congruence generated by a set of equations $E$. We define a relation $\gg$ on $T(\mathcal{F}_\diamond, \mathcal{X})$ as follows: $s \gg t$ iff $s \neq_E t$ and there is a context $C$ such that $s > C[t]$ or $s =_E C[t]$.

We have the following result.

**Lemma 6.39.** *In the conditions of definition 6.38 and if $E$ is length-limited, we have that if $>$ is well-founded then $\gg$ is a partial well-founded order on $T(\mathcal{F}_\diamond, \mathcal{X})$ extending $>$, compatible with $=_E$, closed under substitutions and satisfying $C[t] \gg t$, for any non-trivial context $C$.*

**Proof** We see first that $\gg$ is a transitive relation. Suppose that $s \gg t$ and $t \gg u$. Then there are contexts $C$ and $D$ such that $s \geq C[t]$ and $t \geq D[u]$. Consequently $s \geq C[D[u]]$. To conclude that $s \gg u$ we still need to see that $s \neq_E u$. If $s =_E u$ then we would have $s \geq C[t] \geq C[D[u]] =_E C[D[s]]$. But $s > C[D[s]]$ contradicts well-foundedness of $>$, and if $s =_E C[D[s]]$, since $=_E$ is length-limited, we must have $C = D = \square$ and then we can write $s > t > u =_E s$ (recall that both $s \neq_E t$ and $t \neq_E u$) and then $s > s$, again contradicting well-foundedness of $>$. Consequently $s \neq_E u$ and so $s \gg u$.

We check now well-foundedness. Suppose that $\gg$ is not well-founded, then there is an infinite descending sequence

$$s_0 \gg s_1 \gg s_2 \gg \ldots \gg \ldots$$

For each $i \geq 0$, we have $s_i \neq_E s_{i+1}$ and $s_i \geq C_i[s_{i+1}]$, for some context $C_i$. So the above sequence can be written as

$$s_0 \geq C_0[s_1] \geq C_0[C_1[s_2]] \geq C_0[C_1[C_2[s_3]]] \geq \ldots$$

Since $>$ is well-founded, strict inequality can only occur finitely many times, so there is an index $j \geq 0$, such that for all $i \geq j$ we have

$$s_i =_E C_i[s_{i+1}] =_E C_i[C_{i+1}[s_{i+2}]] =_E \ldots$$

Now, since $s_i \neq_E s_{i+1}$, each $C_i$ has to be non-trivial and so the above sequence of equalities contradicts the fact that $=_E$ is length-limited.

To see that $> \subseteq \gg$, note that if $s > t$ then $s \neq_E t$ and by taking $C$ as the empty context in the definition of $\gg$, we conclude that $s \gg t$.

To see that $\gg$ and $=_E$ are compatible, we need to check that $=_E \circ \gg \circ =_E \subseteq \gg$. Suppose that $s =_E t \gg u =_E v$; then we also have $s =_E t \geq C[u] =_E C[v]$, for some context $C$. Since $>$ and $=_E$ are compatible we can write $s \geq C[v]$. Again because $t \neq_E u$, either $t > C[u]$ or $C$ has to be non-trivial. If $t > C[u]$ then we have $s > C[v]$ and $s =_E v$ would contradict well-foundedness of $>$. If $C$ is non-trivial and $t =_E C[u]$, then $s =_E C[v]$ and $s =_E v$ would contradict the length-limited property; so we can say that $s \gg u$.

Finally we check that $\gg$ is closed under substitutions. Let $\sigma : \mathcal{X} \to T(\mathcal{F}_\diamond, \mathcal{X})$ be any substitution and suppose that $s \gg t$. Then $s > C[t]$ or $s =_E C[t]$ for some context $C$. In the first case we also have $s\sigma > C\sigma[t\sigma]$ and in the second we have $s\sigma =_E C\sigma[t\sigma]$. Suppose that $s\sigma =_E t\sigma$. Then we would have $t\sigma =_E s\sigma > C\sigma[t\sigma]$, and since $>$ is closed under contexts, this contradicts well-foundedness of $>$; or $s\sigma =_E C\sigma[t\sigma] =_E C\sigma[s\sigma]$, contradicting the fact that $=_E$ is length-limited (note that in this case since $s =_E C[t]$ and $s \neq_E t$, $C$ has to be non-trivial). □

**Remark 6.40.** From now on we take $\gg$ to be the order obtained from definition 6.38 if $>$ is $\to^+_{\mathcal{E}(R)/EQ}$. Since $\to^+_{\mathcal{E}(R)/EQ}$ is well-founded, $\gg$ has all the properties stated in lemma 6.39. We consider the quasi-order $\gg \cup =_{EQ}$ and the trees over $T(\mathcal{F}_\diamond, \mathcal{X})$ with the tree lifting associated with $\gg \cup =_{EQ}$. We will denote this new tree lifting by $\sqsupset \cup \approx$. The notation is justified by the fact that, since we do not change the congruence $=_{EQ}$ then the equivalence on trees remains unchanged (this actually follows from lemma 6.5).

**Lemma 6.41.** *Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \setminus \mathcal{X}$ and $t \in T(\mathcal{F}_a, \mathcal{X})$ such that $var(t) \subseteq var(s)$ and $s \gg v$ for all $v \in E(t) \cup \mathtt{dec}(t)$. Let $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ be any substitution and suppose that $\mathtt{tree}(s\sigma) = (\mathtt{cap}_i(s\sigma), M_s)$, $\mathtt{tree}(t\sigma) = (\mathtt{cap}_i(t\sigma), M_t)$, where the e-status of $a$ is $\tau(a) = (X, i)$. Then for all $U \in M_t$ either $U \in M_s$ or $\mathtt{tree}(s\sigma) \sqsupset U$.*

**Proof** We proceed by induction on the structure of $t$. If $t = x \in \mathcal{X}$ then the result follows from lemma 6.37 (note that this lemma can be applied since $\to^+_{\mathcal{E}(R)/EQ} \subseteq \gg$ and so also $\succ \subseteq \sqsupset$).

If $t = f(t_1, \dots, t_m)$ then

$$\mathtt{tree}(t\sigma) = \mathtt{tree}(f(t_1\sigma, \dots, t_m\sigma)) = \left(\mathtt{cap}_i(f(t_1\sigma, \dots, t_m\sigma)), \bigsqcup_{j=1}^{m} M_j\right)$$

where $\mathtt{tree}(t_j\sigma) = (\mathtt{cap}_i(t_j\sigma), M_j)$, for all $1 \leq j \leq m$. Fix some such $j$. Since $\mathtt{dec}(t) = \bigcup_{k=1}^{m} \mathtt{dec}(t_k)$ and by hypothesis $s \gg v$ for all $v \in \mathtt{dec}(t)$, we also have that

$s \gg u$ for any $u \in \texttt{dec}(t_j)$. For analyzing the elements of $E(t_j)$ we distinguish two cases:

- $\tau(a) = (\emptyset, 0)$. Then $E(t) = \{\mathsf{cap}_0(t)\} = \{f(\mathsf{cap}_0(t_1), \ldots, \mathsf{cap}_0(t_m))\}$. Since by hypothesis $s \gg u$, for any $u \in E(t)$ and $\gg$ has the subterm property, we conclude that $s \gg \mathsf{cap}_0(t_j)$. Given that $E(t_j) = \{\mathsf{cap}_0(t_j)\}$, we have $s \gg u$ for any $u \in E(t_j)$.
- $\tau(a) = (X, i)$ with $X \neq \emptyset$ and $i \in X$. Then

$$\begin{aligned} E(t) &= \bigcup_{l \in X} E_l(t) = \bigcup_{l \in X} E_l(f(t_1, \ldots, t_m)) \\ &= \bigcup_{l \in X} \{f(u_1, \ldots, u_m) | u_k \in E_l(t_k), 1 \leq k \leq m\} \end{aligned}$$

  Since $E(t_j) = \bigcup_{l \in X} E_l(t_j)$, we conclude that for any $u \in E(t_j)$, $f(\ldots u \ldots) \in E(t)$. Again because $s \gg f(\ldots u \ldots)$ and $\gg$ has the subterm property, we also conclude that $s \gg u$. Given the arbitrariety of $u$, we have $s \gg u$ for all $u \in E(t_j)$.

We have just seen that $s \gg u$ for all $u \in E(t_j) \cup \texttt{dec}(t_j)$. Also $var(t_j) \subseteq var(t) \subseteq var(s)$, so we can apply the induction hypothesis to $t_j$ and conclude that given any $U \in M_j$ either $U \in M_s$ or $\texttt{tree}(s\sigma) \sqsupset U$. Since $U \in \bigsqcup_{k=1}^{m} M_k \Rightarrow U \in M_j$, for some $1 \leq j \leq m$, the result holds.

If $t = a(t_1, \ldots, t_N)$ then

$$\texttt{tree}(t\sigma) = \left(\mathsf{cap}_i(a(t_1\sigma, \ldots, t_N\sigma)), \left(\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\texttt{tree}(t_j\sigma)\}\!\}\right) \sqcup M_i\right)$$

where, for $i \neq 0$, $\texttt{tree}(t_i\sigma) = (\mathsf{cap}_i(t_i\sigma), M_i)$, and $M_0 = \emptyset$, and $\tau(a) = (X, i)$. We distinguish again two cases

- $\tau(a) = (\emptyset, 0)$. We need to see that for any $j$, $1 \leq j \leq N$, either $\texttt{tree}(t_j\sigma) \in M_s$ or $\texttt{tree}(s\sigma) \sqsupset \texttt{tree}(t_j\sigma)$. Fix any such index $j$. By lemma 6.18 we know that

$$\texttt{tree}(t_j\sigma) = (\mathsf{cap}_0(t_j\sigma), M_j) = (\mathsf{cap}_0(t_j)\mathsf{cap}_0(\sigma), M_j)$$

  (see definition 6.17 too). Also

$$\texttt{tree}(s\sigma) = (\mathsf{cap}_0(s\sigma), M_s) = (\mathsf{cap}_0(s)\mathsf{cap}_0(\sigma), M_s) = (s\ \mathsf{cap}_0(\sigma), M_s)$$

  By hypothesis $s \gg u$, for all $u \in \texttt{dec}(t)$ and since

$$\texttt{dec}(t) = \bigcup_{k=1}^{N} (\{\mathsf{cap}_0(t_k)\} \cup \texttt{dec}(t_k))$$

we can say that $s \gg u$ for all $u \in \texttt{dec}(t_j)$. Also $E(t_j) = \{\texttt{cap}_0(t_j)\} \subseteq \texttt{dec}(t_j)$, and so we can say that $s \gg u$ for all $u \in E(t_j)$. Further $var(t_j) \subseteq var(t) \subseteq var(s)$, so we can apply the induction hypothesis to $t_j$ and conclude that if $U \in M_j$ then either $U \in M_s$ or $\texttt{tree}(s\sigma) \sqsupset U$. Since by hypothesis $s \gg \texttt{cap}_0(t_j)$, and $\gg$ is closed under substitutions we conclude that $s\ \texttt{cap}_0(\sigma) \gg \texttt{cap}_0(t_j)\texttt{cap}_0(\sigma)$ and by definition 6.4 we have $\texttt{tree}(s\sigma) \sqsupset \texttt{tree}(t_j\sigma)$, as we wanted.

- $\tau(a) = (X, i)$, with $X \neq \emptyset$ and $i \in X$. We need to see that

$$\text{for any } U \in \left( \bigsqcup_{j=1, j\neq i}^{N} \{\!\{\texttt{tree}(t_j\sigma)\}\!\} \cup M_i \right) \text{ either } U \in M_s \text{ or } \texttt{tree}(s\sigma) \sqsupset U,$$

where $\texttt{tree}(t_i\sigma) = (\texttt{cap}_i(t_i\sigma), M_i)$. Consider $\texttt{tree}(t_j\sigma)$, for a fixed $1 \leq j \leq N$.[2] By lemma 6.18 and definition 6.17, we know that

$$\texttt{tree}(t_j\sigma) = (\texttt{cap}_i(t_j\sigma), M_j) = (\texttt{cap}_i(t_j)\texttt{cap}_i(\sigma), M_j).$$

Also $\texttt{tree}(s\sigma) = (\texttt{cap}_i(s\sigma), M_s) = (\texttt{cap}_i(s)\texttt{cap}_i(\sigma), M_s) = (s\ \texttt{cap}_i(\sigma), M_s)$. By hypothesis $s \gg u$ for all $u \in E(t) \cup \texttt{dec}(t)$ and since

$$\texttt{dec}(a(t_1, \ldots, t_N)) \;=\; \left(\bigcup_{k=1}^{N} \texttt{dec}(t_k)\right) \cup \left(\bigcup_{l \in \{1,\ldots,N\}, l \notin X} E(t_l)\right)$$

and $E(a(t_1, \ldots, t_N)) = \bigcup_{k \in X} E(t_k)$ we also have $s \gg u$ for all $u \in \texttt{dec}(t_j) \cup E(t_j)$. Further $var(t_j) \subseteq var(t) \subseteq var(s)$, so we can apply the induction hypothesis to $t_j$ and conclude that if $U \in M_j$ then either $U \in M_s$ or $\texttt{tree}(s\sigma) \sqsupset U$. If $j = i$, we are done, otherwise by lemma 6.24 we know that $\texttt{cap}_i(t_j) \in E(t_j)$. Since $E(t_j) \subseteq E(t)$, the hypothesis gives us $s \gg \texttt{cap}_i(t_j)$, and since $\gg$ is closed under substitutions we conclude that $s\ \texttt{cap}_i(\sigma) \gg \texttt{cap}_i(t_j)\texttt{cap}_i(\sigma)$ and, by definition 6.4, we have that $\texttt{tree}(s\sigma) \sqsupset \texttt{tree}(t_j\sigma)$, as we wanted.

□

**Lemma 6.42.** *Let the e-status of* $a$ *be* $\tau(a) = (X, i)$*. Let* $s \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \setminus \mathcal{X}$ *and* $t \in T(\mathcal{F}_a, \mathcal{X})$ *be terms such that* $var(t) \subseteq var(s)$ *and* $s \gg v$ *for all* $v \in \texttt{dec}(t) \cup E(t)$*. Finally let* $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ *be any substitution. Then* $\texttt{tree}(s\sigma) \sqsupset \texttt{tree}(t\sigma)$*.*

**Proof** By definition 6.29 and lemma 6.18 we can write

$$\texttt{tree}(s\sigma) = (\texttt{cap}_i(s\sigma), M_s) = (\texttt{cap}_i(s)\texttt{cap}_i(\sigma), M_s) = (s\ \texttt{cap}_i(\sigma), M_s)$$

Similarly $\texttt{tree}(t\sigma) = (\texttt{cap}_i(t\sigma), M_t) = (\texttt{cap}_i(t)\texttt{cap}_i(\sigma), M_t)$. By lemma 6.41 we conclude that for any $U \in M_t$ either $U \in M_s$ or $\texttt{tree}(s\sigma) \sqsupset U$. Since by lemma

[2] Note that we can have $i = j$.

6.24 $\mathtt{cap}_i(t) \in E(t)$, we also have $s \gg \mathtt{cap}_i(t)$ and because $\gg$ is closed under substitutions, we have $s\ \mathtt{cap}_i(\sigma) \gg \mathtt{cap}_i(t)\mathtt{cap}_i(\sigma)$; by definition 6.4 we conclude that $\mathtt{tree}(s\sigma) \sqsupset \mathtt{tree}(t\sigma)$. $\square$

**Lemma 6.43.** *Let the e-status of $a$ be $\tau(a) = (X, i)$. Let $l \to r$ be a rule in $R$ and $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ an arbitrary substitution. Then $\mathtt{tree}(\mathtt{cap}_i(l)\sigma) \sqsupset \mathtt{tree}(r\sigma)$.*

**Proof** From the definition of $\mathcal{E}(R)$ (see definition 6.26), we know that $\mathtt{cap}_i(l) \to u$, with $u \in E(r) \cup \mathtt{dec}(r)$, is a rule in $\mathcal{E}(R)$ and therefore $\mathtt{cap}_i(l) \gg u$ for any $u \in E(r) \cup \mathtt{dec}(r)$. Also $var(r) \subseteq var(l)$ and $a$ does not occur in $\mathtt{cap}_i(l)$, therefore all the hypothesis of lemma 6.42 are satisfied, so we can apply it to conclude that $\mathtt{tree}(\mathtt{cap}_i(l)\sigma) \sqsupset \mathtt{tree}(r\sigma)$. $\square$

The previous lemma allows us to conclude that $\mathtt{tree}(\mathtt{cap}_i(l)\sigma) \sqsupset \mathtt{tree}(r\sigma)$, for any rule $l \to r \in R$. However this is not good enough, since we need to have $\mathtt{tree}(l\sigma) \sqsupset \mathtt{tree}(r\sigma)$. To obtain this we need to be able to compare the tree of a term with the tree of its cap. The following result provides this comparison.

**Lemma 6.44.** *Let the e-status of $a$ be $\tau(a) = (X, i)$. Let $t \in T(\mathcal{F}_a, \mathcal{X})$ be any term and $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ be any substitution. Then $\mathtt{tree}(t\sigma) \sqsupseteq \mathtt{tree}(\mathtt{cap}_i(t)\sigma)$.*

**Proof** We proceed by induction. First note that for any term $s$, lemmas 6.19 and 6.18 say that $\mathtt{tree}(s\sigma)$ and $\mathtt{tree}(\mathtt{cap}_i(s)\sigma)$ have the same root. If $t = x$ then $\mathtt{cap}_i(t) = t$ so the result is valid. If $t = f(t_1, \ldots, t_m)$, with $m \geq 0$ and $f \neq a$, then

$$\mathtt{tree}(t\sigma) = \mathtt{tree}(f(t_1, \ldots, t_m)\sigma) = \left(\mathtt{cap}_i(f(t_1, \ldots, t_m)\sigma), \bigsqcup_{i=1}^{m} M_i\right)$$

where for each $1 \leq i \leq m$, $\mathtt{tree}(t_i\sigma) = (\mathtt{cap}_i(t_i\sigma), M_i)$. For $\mathtt{cap}_i(t)$, we have

$$\begin{array}{rcl}
\mathtt{tree}(\mathtt{cap}_i(t)\sigma) & = & \mathtt{tree}(\mathtt{cap}_i(f(t_1, \ldots, t_m))\sigma) \\
 & = & \mathtt{tree}(f(\mathtt{cap}_i(t_1), \ldots, \mathtt{cap}_i(t_m))\sigma) \\
 & = & \left(\mathtt{cap}_i(f(\mathtt{cap}_i(t_1), \ldots, \mathtt{cap}_i(t_m))\sigma), \bigsqcup_{i=1}^{m} M'_i\right) \\
 & = & \left(f(\mathtt{cap}_i(\mathtt{cap}_i(t_1)), \ldots, \mathtt{cap}_i(\mathtt{cap}_i(t_m)))\mathtt{cap}_i(\sigma), \bigsqcup_{i=1}^{m} M'_i\right) \\
\text{(by lemma 6.19)} & = & \left(f(\mathtt{cap}_i(t_1), \ldots, \mathtt{cap}_i(t_m))\mathtt{cap}_i(\sigma), \bigsqcup_{i=1}^{m} M'_i\right)
\end{array}$$

where for each $1 \leq i \leq N$, $\mathtt{tree}(\mathtt{cap}_i(t_i)\sigma) = (\mathtt{cap}_i(t_i)\mathtt{cap}_i(\sigma), M'_i)$. We can apply the induction hypothesis to each $t_i$ and conclude that $\mathtt{tree}(t_i\sigma) \sqsupseteq \mathtt{tree}(\mathtt{cap}_i(t_i)\sigma)$. Since these trees have the same root, we can say that $M_i \sqsupseteq_{mul} M'_i$. Consequently $\bigsqcup_{i=1}^{m} M_i \sqsupseteq_{mul} \bigsqcup_{i=1}^{m} M'_i$, and so $\mathtt{tree}(t\sigma) \sqsupseteq \mathtt{tree}(\mathtt{cap}_i(t)\sigma)$.

If $t = a(t_1, \dots, t_N)$ we have to distinguish two cases, namely $\tau(a) = (\emptyset, 0)$ and $\tau(a) = (X, i)$, with $X \neq \emptyset$ and $i \in X$.

- if $\tau(a) = (\emptyset, 0)$, then $\mathsf{cap}_i(t) = \diamond$ and $\mathtt{tree}(\mathsf{cap}_i(t)\sigma) = (\diamond, \emptyset)$. Since $\mathtt{tree}(t\sigma)$ has the same root, obviously $\mathtt{tree}(t\sigma) \sqsupseteq \mathtt{tree}(\mathsf{cap}_i(t)\sigma)$.
- if $\tau(a) = (X, i)$, then $\mathsf{cap}_i(t) = \mathsf{cap}_i(t_i)$ and

$$\begin{array}{rcl} \mathtt{tree}(t\sigma) & = & \mathtt{tree}(a(t_1\sigma, \dots, t_N\sigma)) \\ & = & \left(\mathsf{cap}_i(a(t_1\sigma, \dots, t_N\sigma)), \displaystyle\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\mathtt{tree}(t_i\sigma)\}\!\} \sqcup M_i\right) \\ & = & \left(\mathsf{cap}_i(t_i\sigma), \displaystyle\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\mathtt{tree}(t_i\sigma)\}\!\} \sqcup M_i\right) \\ & = & \left(\mathsf{cap}_i(t_i)\mathsf{cap}_i(\sigma), \displaystyle\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\mathtt{tree}(t_i\sigma)\}\!\} \sqcup M_i\right) \end{array}$$

where $\mathtt{tree}(t_i\sigma) = (\mathsf{cap}_i(t_i\sigma), M_i) = (\mathsf{cap}_i(t_i)\mathsf{cap}_i(\sigma), M_i)$. Also

$$\begin{array}{rcl} \mathtt{tree}(\mathsf{cap}_i(t)\sigma) & = & \mathtt{tree}(\mathsf{cap}_i(t_i)\sigma) \\ & = & (\mathsf{cap}_i(\mathsf{cap}_i(t_i)\sigma), M_i') \\ & = & (\mathsf{cap}_i(t_i)\mathsf{cap}_i(\sigma), M_i') \end{array}$$

By induction hypothesis, $\mathtt{tree}(t_i\sigma) \sqsupseteq \mathtt{tree}(\mathsf{cap}_i(t_i)\sigma)$ and consequently $M_i \sqsupseteq_{mul} M_i'$ (recall that the trees have the same root), so

$$\left(\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\mathtt{tree}(t_i\sigma)\}\!\} \sqcup M_i\right) \sqsupseteq_{mul} \left(\bigsqcup_{j=1, j\neq i}^{N} \{\!\{\mathtt{tree}(t_i\sigma)\}\!\} \sqcup M_i'\right)$$

Since the roots of the trees $t\sigma$ and $\mathsf{cap}_i(t)\sigma$ are the same, we have that $\mathtt{tree}(t\sigma) \sqsupseteq \mathtt{tree}(\mathsf{cap}_i(t)\sigma)$.

□

**Theorem 6.45.** *Let the e-status of $a$ be $\tau(a) = (X, i)$. Let $l \to r$ be any rule in $R$ and let $\sigma : \mathcal{X} \to T(\mathcal{F}_a, \mathcal{X})$ be any substitution. Then* $\mathtt{tree}(l\sigma) \sqsupset \mathtt{tree}(r\sigma)$.

**Proof** Combining lemmas 6.43 and 6.44 we have $\mathtt{tree}(l\sigma) \sqsupseteq \mathtt{tree}(\mathsf{cap}_i(l)\sigma) \sqsupset \mathtt{tree}(r\sigma)$, and so $\mathtt{tree}(l\sigma) \sqsupset \mathtt{tree}(r\sigma)$. □

We have seen that $l\sigma \to_R r\sigma$ implies that $\mathtt{tree}(l\sigma) \sqsupset \mathtt{tree}(r\sigma)$. We still have to check that if a reduction occurs within a non-trivial context, the same results holds, i. e., $C[l\sigma] \to_R C[r\sigma]$ implies $\mathtt{tree}(C[l\sigma]) \sqsupset \mathtt{tree}(C[r\sigma])$. For that we still need some auxiliary results.

**Lemma 6.46.** *Let $s, t \in T(\mathcal{F}_a, \mathcal{X})$. If $s \rightarrow_R t$ then $\mathsf{cap}_i(s) \rightarrow^{0,1}_{\mathcal{E}(R)} \mathsf{cap}_i(t)$, i. e., $\mathsf{cap}_i(s)$ reduces via $\mathcal{E}(R)$ to $\mathsf{cap}_i(t)$ in zero or one steps.*

**Proof** We proceed by induction on the definition of reduction. Suppose $s = l\sigma$ and $t = r\sigma$ for some rule $l \rightarrow r$ of $R$ and some substitution $\sigma : \mathcal{X} \rightarrow T(\mathcal{F}_a, \mathcal{X})$. By definition 6.16 and lemma 6.18, $\mathsf{cap}_i(l\sigma) = \mathsf{cap}_i(l)\mathsf{cap}_i(\sigma)$. Similarly $\mathsf{cap}_i(r\sigma) = \mathsf{cap}_i(r)\mathsf{cap}_i(\sigma)$. Since $\mathsf{cap}_i(l) \rightarrow \mathsf{cap}_i(r)$ is a rule in $\mathcal{E}(R)$ (recall that $\mathsf{cap}_i(r) \in E(r)$), we have $\mathsf{cap}_i(l)\ \mathsf{cap}_i(\sigma) \rightarrow^{1}_{\mathcal{E}(R)} \mathsf{cap}_i(r)\mathsf{cap}_i(\sigma)$, as we had to show.

Suppose $s \rightarrow_R t$, $\mathsf{cap}_i(s) \rightarrow^{0,1}_{\mathcal{E}(R)} \mathsf{cap}_i(t)$, $f(s_1, \ldots, s, \ldots, s_k) \rightarrow_R f(s_1, \ldots, t, \ldots, s_k)$. We have

$$\begin{array}{rcl} \mathsf{cap}_i(s) & \rightarrow^{0,1}_{\mathcal{E}(R)} & \mathsf{cap}_i(t) \\ & \Downarrow & \\ f(\mathsf{cap}_i(s_1), \ldots, \mathsf{cap}_i(s), \ldots, \mathsf{cap}_i(s_k)) & \rightarrow^{0,1}_{\mathcal{E}(R)} & f(\mathsf{cap}_i(s_1), \ldots, \mathsf{cap}_i(t), \ldots, \mathsf{cap}_i(s_k)) \\ & \Downarrow & \text{(by definition 6.16)} \\ \mathsf{cap}_i(f(s_1, \ldots, s, \ldots, s_k)) & \rightarrow^{0,1}_{\mathcal{E}(R)} & \mathsf{cap}_i(f(s_1, \ldots, t, \ldots, s_k)) \end{array}$$

If $a(s_1, \ldots, s, \ldots, s_N) \rightarrow_R a(s_1, \ldots, t, \ldots, s_N)$, we have to distinguish two cases.

- If $i = 0$, the result holds since by definition 6.16, $\mathsf{cap}_i(a(s_1, \ldots, s, \ldots, s_N)) = \diamond = \mathsf{cap}_i(a(s_1, \ldots, t, \ldots, s_N))$.
- If $i \neq 0$ then $\mathsf{cap}_i(a(s_1, \ldots, s, \ldots, s_N)) = \mathsf{cap}_i(s_i)$, where $s_i = s$ if $s$ occurs at position $i$. Also $\mathsf{cap}_i(a(s_1, \ldots, t, \ldots, s_N)) = \mathsf{cap}_i(s_i)$, where $s_i = t$ if $t$ occurs at position $i$. In any case the result holds.

□

**Lemma 6.47.** *Let $s, t \in T(\mathcal{F}_a, \mathcal{X})$ such that $s \rightarrow_R t$ and $\mathtt{tree}(s) \sqsupset \mathtt{tree}(t)$. Then, for any context $C$, $\mathtt{tree}(C[s]) \sqsupset \mathtt{tree}(C[t])$.*

**Proof** We proceed by induction on the context. If $C$ is the trivial context, the result holds by hypothesis. Suppose now that $D[\,]$ is a context for which the property holds and let $C[\,] = f(s_1, \ldots, D[\,], \ldots, s_k)$, with $D[\,]$ occurring at a fixed position $j$, $1 \leq j \leq k$, and $f \neq a$. By definitions 6.29 and 6.16 we have

$$\begin{array}{rcl} \mathtt{tree}(C[s]) & = & \mathtt{tree}(f(s_1, \ldots, D[s], \ldots, s_k)) \\ & = & \left( f(\mathsf{cap}_i(s_1), \ldots, \mathsf{cap}_i(D[s]), \ldots \mathsf{cap}_i(s_k)), \bigsqcup_{l=1}^{k} M_l \right) \end{array}$$

where $\mathtt{tree}(s_l) = (\mathsf{cap}_i(s_l), M_l)$ $(1 \leq l \leq k, l \neq j)$, $\mathtt{tree}(D[s]) = (\mathsf{cap}_i(D[s]), M_j)$. Similarly

$$\begin{array}{rcl} \mathtt{tree}(C[t]) & = & \mathtt{tree}(f(s_1, \ldots, D[t], \ldots, s_k)) \\ & = & \left( f(\mathsf{cap}_i(s_1), \ldots, \mathsf{cap}_i(D[t]), \ldots \mathsf{cap}_i(s_k)), \bigsqcup_{l=1}^{k} M'_l \right) \end{array}$$

where $M'_l = M_l$, for $1 \leq l \leq k$, $l \neq j$, and $\mathtt{tree}(D[t]) = (\mathtt{cap}_i(D[t]), M'_j)$. By hypothesis $s \rightarrow_R t$, so $D[s] \rightarrow_R D[t]$ and lemma 6.46 gives us $\mathtt{cap}_i(D[s]) \rightarrow^{0,1}_{\mathcal{E}(R)} \mathtt{cap}_i(D[t])$. As a consequence

$$f(\mathtt{cap}_i(s_1), \ldots, \mathtt{cap}_i(D[s]), \ldots \mathtt{cap}_i(s_k)) \rightarrow^{0,1}_{\mathcal{E}(R)} f(\mathtt{cap}_i(s_1), \ldots, \mathtt{cap}_i(D[t]), \ldots \mathtt{cap}_i(s_k))$$

If this reduction has $0$ steps then we also have $\mathtt{cap}_i(D[s]) \rightarrow^{0}_{\mathcal{E}(R)} \mathtt{cap}_i(D[t])$, and since $\mathtt{tree}(D[s]) \sqsupset \mathtt{tree}(D[t])$, definition 6.4 gives us $M_j$ $\mathtt{ord}(\sqsupset_{mul})$ $M'_j$. Then $\bigsqcup_{l=1}^{k} M_l$ $\mathtt{ord}(\sqsupset_{mul})$ $\bigsqcup_{l=1}^{k} M'_l$, and again by definition 6.4, we conclude that $\mathtt{tree}(C[s]) \sqsupset \mathtt{tree}(C[t])$.

If the reduction has one step then also $\mathtt{cap}_i(D[s]) \rightarrow^{1}_{\mathcal{E}(R)} \mathtt{cap}_i(D[t])$. Recall that $\rightarrow_{\mathcal{E}(R)} \subseteq \rightarrow_{\mathcal{E}(R)/EQ} \subseteq \gg$. Since $\mathtt{tree}(D[s]) \sqsupset \mathtt{tree}(D[t])$, this means that for any $U \in M'_j$ either there is an element $V \in M_j$ such that $V \sqsupseteq U$ or $\mathtt{tree}(D[s]) \sqsupset U$. In this last case, since $\mathtt{tree}(f(\ldots D[s] \ldots)) \sqsupset \mathtt{tree}(D[s])$ (as can easily be seen using definition 6.4 or lemma 6.7, and the fact that $\mathtt{cap}_i(f(\ldots D[s] \ldots)) \gg \mathtt{cap}_i(D[s])$), we can conclude that $\mathtt{tree}(f(\ldots D[s] \ldots)) \sqsupset U$, and so we can state that

$$\forall U \in \bigsqcup_{l=1}^{k} M'_l : \ (\mathtt{tree}(f(\ldots D[s] \ldots)) \sqsupset U) \text{ or } \left(\exists V \in \bigsqcup_{l=1}^{k} M_l : \ V \sqsupseteq U\right)$$

This together with

$$f(\mathtt{cap}_i(s_1), \ldots, \mathtt{cap}_i(D[t]), \ldots \mathtt{cap}_i(s_k)) \gg f(\mathtt{cap}_i(s_1), \ldots, \mathtt{cap}_i(D[t]), \ldots \mathtt{cap}_i(s_k))$$

gives us $\mathtt{tree}(f(\ldots D[s] \ldots)) \sqsupset \mathtt{tree}(f(\ldots D[t] \ldots))$.

Suppose now that $C = a(s_1, \ldots, D[\ ], \ldots, s_N)$, with $D[\ ]$ occurring at position $j$, for some fixed $1 \leq j \leq N$. First we consider the case $i = 0$. Then

$$\begin{aligned} \mathtt{tree}(C[s]) &= \mathtt{tree}(a(s_1, \ldots, D[s], \ldots s_N)) \\ &= \left(\diamond, \bigsqcup_{l=1, l \neq j}^{N} \{\!\{\mathtt{tree}(s_l)\}\!\} \sqcup \{\!\{\mathtt{tree}(D[s])]\}\!\}\right) \end{aligned}$$

and

$$\begin{aligned} \mathtt{tree}(C[t]) &= \mathtt{tree}(a(s_1, \ldots, D[t], \ldots s_N)) \\ &= \left(\diamond, \bigsqcup_{l=1, l \neq j}^{N} \{\!\{\mathtt{tree}(s_l)\}\!\} \sqcup \{\!\{\mathtt{tree}(D[t])\}\!\}\right) \end{aligned}$$

Since $\mathtt{tree}(D[s]) \sqsupset \mathtt{tree}(D[t])$ also

$$\bigsqcup_{l=1, l \neq j}^{N} \{\!\{\mathtt{tree}(s_l)\}\!\} \sqcup \{\!\{\mathtt{tree}(D[s])\}\!\} \ \mathtt{ord}(\sqsupset_{mul}) \ \bigsqcup_{l=1, l \neq j}^{N} \{\!\{\mathtt{tree}(s_l)\}\!\} \sqcup \{\!\{\mathtt{tree}(D[t])\}\!\}$$

and by definition 6.4 we conclude that $\texttt{tree}(C[s]) \sqsupset \texttt{tree}(C[t])$.

Suppose now that $i \neq 0$. Then

$$\begin{aligned}\texttt{tree}(C[s]) &= \texttt{tree}(a(s_1,\ldots,D[s],\ldots s_N)) \\ &= \left(\texttt{cap}_i(a(s_1,\ldots,D[s],\ldots s_N)), \bigsqcup_{l=1,l\notin\{i,j\}}^{N} \{\{\texttt{tree}(s_l)\}\} \sqcup M_i \sqcup M'\right)\end{aligned}$$

where $\texttt{tree}(s_i) = (\texttt{cap}_i(s_i), M_i)$, $M' = \emptyset$ if $i = j$ and $M' = \{\{\texttt{tree}(D[s])\}\}$ otherwise, and

$$\begin{aligned}\texttt{tree}(C[t]) &= \texttt{tree}(a(s_1,\ldots,D[t],\ldots s_N)) \\ &= \left(\texttt{cap}_i(a(s_1,\ldots,D[t],\ldots s_N)), \bigsqcup_{l=1,l\notin\{i,j\}}^{N} \{\{\texttt{tree}(s_l)\}\} \sqcup K_i \sqcup K'\right)\end{aligned}$$

where $\texttt{tree}(s_i) = (\texttt{cap}_i(s_i), K_i)$ and $K' = \emptyset$ if $i = j$ and $K' = \{\{\texttt{tree}(D[t])\}\}$ otherwise.

Since $s \to_R t$, also $D[s] \to_R D[t]$. By lemma 6.46 we have that $\texttt{cap}_i(D[s]) \to^{0,1}_{\mathcal{E}(R)} \texttt{cap}_i(D[t])$ and also

$$\texttt{cap}_i(a(s_1,\ldots,D[s],\ldots s_N)) \to^{0,1}_{\mathcal{E}(R)} \texttt{cap}_i(a(s_1,\ldots,D[t],\ldots s_N))$$

Suppose $i = j$; then $M' = K' = \emptyset$. If $\texttt{cap}_i(D[s]) = \texttt{cap}_i(D[t])$, since $\texttt{tree}(D[s]) = (\texttt{cap}_i(D[s]), M_i) \sqsupset (\texttt{cap}_i(D[t]), K_i) = \texttt{tree}(D[t])$, we must have $M_i \ \texttt{ord}(\sqsupset_{mul})\ K_i$ and consequently

$$\bigsqcup_{l=1,l\notin\{i,j\}}^{N} \{\{\texttt{tree}(s_l)\}\} \sqcup M_i \sqcup M' \ \texttt{ord}(\sqsupset_{mul}) \bigsqcup_{l=1,l\notin\{i,j\}}^{N} \{\{\texttt{tree}(s_l)\}\} \sqcup M_i \sqcup M'$$

Since $\texttt{cap}_i(a(s_1,\ldots,D[s],\ldots s_N)) = \texttt{cap}_i(D[s])$ and $\texttt{cap}_i(a(s_1,\ldots,D[t],\ldots s_N)) = \texttt{cap}_i(D[t])$, we conclude that $\texttt{tree}(C[s]) \sqsupset \texttt{tree}(C[t])$.

If $\texttt{cap}_i(D[s]) \neq \texttt{cap}_i(D[t])$ then $\texttt{cap}_i(D[s]) \gg \texttt{cap}_i(D[t])$. Take $U \in K_i$, then either there is an element $V \in M_i$ such that $V \sqsupseteq U$ or $\texttt{tree}(D[s]) \sqsupset U$. In the last case, since $\texttt{tree}(C[s]) \sqsupset \texttt{tree}(D[s])$[3], by transitivity we also get $\texttt{tree}(C[s]) \sqsupset U$. By definition 6.4 we conclude that $\texttt{tree}(C[s]) \sqsupset \texttt{tree}(C[t])$.

Suppose now that $i \neq j$. Then $M_i = K_i$, $M' = \{\{\texttt{tree}(D[s])\}\}$, $K' = \{\{\texttt{tree}(D[t])\}\}$. Since $\texttt{tree}(D[s]) \sqsupset \texttt{tree}(D[t])$, whether $\texttt{cap}_i(D[s]) = \texttt{cap}_i(D[t])$ or $\texttt{cap}_i(D[s]) \gg \texttt{cap}_i(D[t])$, it follows in a straightforward way from definition 6.4 that $\texttt{tree}(C[s]) \sqsupset \texttt{tree}(C[t])$. □

We can now prove our main result.

[3]Note that this is in general not true.

**Theorem 6.48.** *If $\mathcal{E}(R)/EQ$ terminates then $R/EQ$ terminates.*

**Proof** Suppose that $R/EQ$ does not terminate. Then we have an infinite sequence of the form

$$s_0 =_{EQ} s'_0 \to_R s_1 =_{EQ} s'_1 \to_R s_2 =_{EQ} s'_2 \ldots$$

Using lemmas 6.36, 6.47, theorem 6.45 and definition 2.54, this translates to the following sequence on trees

$$\texttt{tree}(s_0) \approx \texttt{tree}(s'_0) \sqsupset \texttt{tree}(s_1) \approx \texttt{tree}(s'_1) \sqsupset \texttt{tree}(s_2) \ldots$$

And since $\approx$ and $\sqsupset$ are compatible, this contradicts well-foundedness of $\sqsupset$. □

As we already mentioned, we can eliminate more than one function symbol simultaneously, though in this presentation we concentrate on eliminating one function symbol at a time. We can also present the transformations for the more general case of eliminating varyadic function symbols. Though it may seem that to present the theory in these more general settings would be better, the following example shows that that is not the case. When *dummy elimination* is the transformation chosen (i. e., the e-status of the function symbols to eliminate is $(\emptyset, 0)$), better results can be obtained if we consider a function symbol at a time. This is so because by introducing new constant symbols for each function symbol eliminated, instead of using one constant for all symbols, better results are obtained.

In the case of varyadic signatures, note that if we label the varyadic function symbols with its arity, thus obtaining a different set of fixed-arity symbols, there is a one-to-one correspondence between terms in the varyadic signature and terms in the fixed-arity signature. Furthermore termination of the original TRS is equivalent to termination of the labelled one. So by labelling the function symbol we want to eliminate we obtain a set of function symbols to eliminate. If we then apply the transformation (sequentially) to all the function symbols thus obtained we can infer termination of the original system from termination of the transformation of the fixed-arity version. The following example justifies these assertions.

**Example 6.49.** Consider the terminating TRS (with $EQ = \emptyset$)

$$f(x, x) \quad \to \quad f(a(x), b(x))$$

Suppose we want to eliminate function symbols $a$ and $b$. If we eliminate both symbols at the same time we get the system:

$$\begin{array}{lcl} f(x, x) & \to & f(\diamond, \diamond) \\ f(x, x) & \to & x \end{array}$$

which is clearly not terminating. But if we eliminate firs $a$ and then $b$ (or vice-versa) we get the system:

$$\begin{array}{lcl} f(x, x) & \to & f(\diamond_1, \diamond_2) \\ f(x, x) & \to & x \end{array}$$

which is terminating.

As for the varyadic case consider the terminating TRS (with $EQ = \emptyset$) over a varyadic signature and suppose that the function symbol $a$ admits arities $1$ and $2$:

$$f(x, x) \;\rightarrow\; f(a(x), a(x, x))$$

As far as termination is concerned, this system is equivalent to the system

$$f(x, x) \;\rightarrow\; f(a_1(x), a_2(x, x))$$

If we now eliminate first $a_1$ and then $a_2$, both with e-status $(\emptyset, 0)$, the result is the TRS

$$\begin{array}{lcl} f(x, x) & \rightarrow & f(\Diamond_1, \Diamond_2) \\ f(x, x) & \rightarrow & x \end{array}$$

which is still a terminating system. Eliminating $a_1$ and $a_2$ simultaneously (which amounts to the same as eliminating the varyadic function symbol $a$ in the original system) results in the following non-terminating TRS:

$$\begin{array}{lcl} f(x, x) & \rightarrow & f(\Diamond, \Diamond) \\ f(x, x) & \rightarrow & x \end{array}$$

So termination in $R$ could not be inferred from termination of $\mathcal{E}(R)$ in the last case.

## 6.4 Eliminating Constants

The transformations presented so far apply to the elimination of function symbols or arity $\geq 1$. Here we discuss the case where we want to eliminate a function symbol with arity $0$, i. e., a constant. There seems to be not much room for doing anything: if we replace the constant to eliminate by another fresh constant (as we do in the elimination of function symbols of greater arities) we do not improve our situation with respect to proving termination. So it seems that the only possible improvement is to throw out rules where this constant occurs. In general, however, this cannot be done: rules where the rhs is different from the constant cannot be eliminated. For example the rule $f(x) \rightarrow f(a)$ (where $a$ is the constant to eliminate) can clearly not be thrown out otherwise we may end up with a terminating system, while no rewrite system containing this rule is terminating. The best we can hope for is to eliminate rules where the rhs is equal to the constant to eliminate.

First we introduce some notation. Let $c$ be a constant. Given a TRS $(\mathcal{F}, \mathcal{X}, R)$ (where $c \in \mathcal{F}$) the TRS $R^c$ is given by the rules

$$f(x_1, \ldots, x_n) \rightarrow c$$

where $x_1, \ldots, x_n$ are pairwise diferent variables, and $f$ ranges over $\mathcal{F} \setminus \{c\}$.

The following lemma is not difficult to prove.

**Lemma 6.50.** *Let $EQ$ be a set of length-preserving equations where the constant $c$ does not occur. Then the $R^c/EQ$ is terminating.*

**Proof** Suppose that $R^c/EQ$ is not terminating. Then we have an infinite ground sequence of the form

$$s_0 =_{EQ} s'_0 \rightarrow_{R^c} s'_1 =_{EQ} s_1 =_{EQ} s''_1 \rightarrow_{R^c} \ldots$$

To each ground term we will associate a weight in the naturals in the following way. Every function symbol $f \in \mathcal{F} \setminus \{c\}$ has weight $1$ and to $c$ we give weight $0$. The weight of a ground term $t$, denoted by $w(t)$ is just the sum of the weights of the function symbols occurring in $t$ (actually $w(t) = |t| - \#_c(t)$).

The following fact is easy to prove by induction on the definition of $=_{EQ}$ (recall that $c$ does not occur in the equations of $EQ$):

For ground terms $s, t$, if $s =_{EQ} t$ then $w(s) = w(t)$.

Also by induction on terms it is not difficult to see that, for any ground term $t \neq c$, we have $w(t) >_{\mathbb{N}} w(c)$. Furthermore it is also easy to show by definition of reduction that for any ground terms $s, t$, if $s \rightarrow_{R^c} t$ then $w(s) >_{\mathbb{N}} w(t)$. Using all this, we have that for all ground terms $s, t, u, v$ if $s =_{EQ} u \rightarrow_{R^c} v =_{EQ} t$ then $w(s) >_{\mathbb{N}} w(t)$. Since $(\mathbb{N}, >_{\mathbb{N}})$ is well-founded, this means that no infinite rewrite sequence in $R^c/EQ$ can exist. □

Note that it is essential that both $c$ does not occur in the equations of $EQ$ and that the equations are length-preserving as the following example shows.

**Example 6.51.** Let $EQ = \{(f(f(x)), f(x))\}$ and let $R^c = \{f(x) \rightarrow c\}$. Then $R^c/EQ$ is not terminating since $f(c) =_{EQ} f(f(c)) \rightarrow_{R^c} f(c)$.

Let now $EQ = \{(a, c)\}$ and let $R^c$ contain the rule $a \rightarrow c$, then we have $a \rightarrow c =_{EQ} a$, so $R^c/EQ$ is not terminating.

Before proving the main lemma (lemma 6.54), we need some auxiliary results. The first lemma is an easy induction on the definition of $=_{EQ}$ so we refrain from presenting its proof. Note that the lemma is also valid if the equations are simply variable-preserving.

**Lemma 6.52.** *Let $EQ$ be a set of length-preserving (linear equations) and let $c$ be a constant not occurring in the equations of EQ. Then $s =_{EQ} t \Rightarrow \#_c(s) = \#_c(t)$, for any terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.*

**Lemma 6.53.** *Let $EQ$ be a set of (length-preserving) linear equations and let $c$ be a constant not occurring in the equations of $EQ$. Suppose that $s = C[c]$, for some context $C$ and let $t$ be a term such that $s = C[c] =_{EQ} t$. Then we can write $s = C[\tau(\overline{x})]$ and $t = D[\tau(\overline{x})]$, for some context $D$, where $\overline{x}$ is a fresh variable not occurring in $s$ or $t$, and $\tau$ is a substitution satisfying $\tau(\overline{x}) = c$. Furthermore the terms $C[\overline{x}]$ and $D[\overline{x}]$ are equal under $EQ$, i. e., $C[\overline{x}] =_{EQ} D[\overline{x}]$.*

**Proof** (Sketch) We proceed by induction on the definition of $=_{EQ}$.

First we mark the occurrence of $c$ we are interested in (or equivalently keep track of its position).

Suppose that $s = C[c] = e_1\sigma$ and $t = e_2\sigma$, for some equation $(e_1, e_2) \in EQ$ and substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. Since both $e_1$ and $e_2$ are linear terms and contain exactly the same variables, we can write $e_1 = H_1[x_1, \ldots, x_k]$ and $e_2 = H_2[x_1, \ldots, x_k]$, for some contexts $H_1, H_2$ and where $var(e_1) = var(e_2) = \{x_1, \ldots, x_k\}$ $(k \geq 1)$. Since $s = C[c] = e_1\sigma$ and the constant $c$ does not occur in $e_1$ it must be introduced by the substitution $\sigma$, i. e., there is a variable $x \in var(e_1)$ such that the marked occurrence of $c$ is in $\sigma(x)$. Let $x_j$ be that unique variable, we denote by $\overline{\sigma}(x_j)$ and $\overline{\sigma(x_j)}$, respectively, the term $\sigma(x)$, where the marked occurrence of $c$ has been replaced by a fresh variable $\overline{x}$ or the empty context $\square$, respectively. If we define the substitution $\pi$ by

$$\pi(x) = \begin{cases} \overline{\sigma}(x_j) & \text{if } x = x_j \\ \sigma(x) & \text{if } x = x_i,\ i \neq j \\ x & \text{otherwise} \end{cases}$$

Then

$$\begin{array}{rcl} e_1\pi & = & H_1[\pi(x_1), \ldots, \pi(x_j), \ldots, \pi(x_k)] \\ & = & H_1[\sigma(x_1), \ldots, \overline{\sigma}(x_j), \ldots, \sigma(x_k)] \\ & =_{EQ} & H_2[\sigma(x_1), \ldots, \overline{\sigma}(x_j), \ldots, \sigma(x_k)] \\ & = & e_2\pi \end{array}$$

And also

$$\begin{array}{rcl} e_1\sigma & = & H_1[\sigma(x_1), \ldots, \sigma(x_j), \ldots, \sigma(x_k)] \\ & = & H_1[\sigma(x_1), \ldots, \tau(\overline{\sigma}(x_j)), \ldots, \sigma(x_k)] \end{array}$$

where $\tau$ is a substitution satisfying $\tau(\overline{x}) = c$ (and being the identity at least on the other variables of $\tau(\overline{x_j})$), and the same holds for $e_2\sigma$. So we can write $s = C[\tau(\overline{x})]$, $t = D[\tau(\overline{x})]$, such that $e_1\pi = C[\overline{x}] =_{EQ} e_2\pi = D[\overline{x}]$, where

$$C = H_1[\sigma(x_1), \ldots, \overline{\sigma(x_j)}, \ldots, \sigma(x_k)] \text{ and } D = H_2[\sigma(x_1), \ldots, \overline{\sigma(x_j)}, \ldots, \sigma(x_k)].$$

Suppose now that $s = C[c] = F[e_1\sigma]$ and $t = F[e_2\sigma]$, for some non-trivial context $F$, some equation $(e_1, e_2) \in EQ$ and substitution $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. Since $C[c] = F[e_1\sigma]$, we have to consider several possibilities.

- the marked $c$ occurs in $e_1\sigma$; then since $e_1\sigma =_{EQ} e_2\sigma$, by the previous case, we can conclude that $C[c] = F[F_1[\tau(\overline{x})]] =_{EQ} F[F_2[\tau(\overline{x})]]$, for some fresh variable $\overline{x}$, some substitution $\tau$ satisfying $\tau(\overline{x}) = c$, and some contexts $F_1, F_2$, such that $e_1\sigma = F_1[\tau(\overline{x})]$ and $e_2\sigma = F_2[\tau(\overline{x})]$. Furthermore $F_1[\overline{x}] =_{EQ} F_2[\overline{x}]$ and so also $C[\overline{x}] = F[F_1[\overline{x}]] =_{EQ} F[F_2[\overline{x}]]$. By taking $D = F[F_2[]]$, the result holds.
- the marked $c$ occurs in $F$; then $F[e_1\sigma] = F'[c, e_1\sigma]$, for some context $F'$. Furthermore also $t = F'[c, e_1\sigma]$. Since $e_1\sigma =_{EQ} e_2\sigma$ and $=_{EQ}$ is closed under contexts,

we have that $F'[\overline{x}, e_1\sigma] =_{EQ} F'[\overline{x}, e_2\sigma]$, for any fresh variable $\overline{x}$. Furthermore $C[\overline{x}] = F'[\overline{x}, e_1\sigma]$ and by taking $D = F'[\square, e_2\sigma]$, also $D[\overline{x}] = F'[\overline{x}, e_2\sigma]$. Since $s = C[c] = C[\tau(\overline{x})]$ and $t = D[\tau(\overline{x})]$, for any substitution $\tau$ satisfying $\tau(\overline{x}) = c$, the result holds.

Finally suppose that $s = C[c] =_{EQ} u$ and that $u =_{EQ} t$. By induction hypothesis we can write $s = C[\tau(\overline{x})]$, $u = F_1[\tau(\overline{x})]$ and $C[\overline{x}] =_{EQ} F_1[\overline{x}]$, for some context $F_1$, fresh variable $\overline{x}$ and substitution $\tau$ satisfying $\tau(\overline{x}) = c$. Similarly we can write $u = F_1[\tau'(\overline{x}')]$, $t = D[\tau'(\overline{x}')]$ and $F_1[\overline{x}'] =_{EQ} D[\overline{x}']$, for some context $D$, fresh variable $\overline{x}'$ and substitution $\tau'$ satisfying $\tau'(\overline{x}') = c$. Without loss of generality we can assume that $\overline{x} \neq \overline{x}'$ and that both variables do not occur in the terms $s, u, t$. Let $\overline{y}$ be another fresh variable not occurring in $var(s) \cup var(u) \cup var(t) \cup \{\overline{x}, \overline{x}'\}$ and define the substitution $\gamma$ by:

$$\gamma(x) = \begin{cases} \overline{y} & \text{if } x = \overline{x} \text{ or } x = \overline{x}' \\ x & \text{otherwise} \end{cases}$$

Then

$$C[\overline{y}] = (C[\overline{x}])\gamma =_{EQ} (F_1[\overline{x}])\gamma = F_1[\overline{y}] = (F_1[\overline{x}'])\gamma =_{EQ} (D[\overline{x}'])\gamma = D[\overline{y}]$$

Also $s = C[c] = C[\pi(\overline{y})]$ and $t = D[\pi(\overline{y})]$, where $\pi$ is any substitution satisfying $\pi(\overline{y}) = c$. $\square$

**Lemma 6.54.** *Let $EQ$ be a set of length-preserving linear equations. Let $R$ be a left-linear TRS over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and suppose that the constant $c \in \mathcal{F}$ does not occur in the lhs of rules in $R$ nor in the equations of $EQ$. Then termination of $R/EQ$ implies termination of $(R \cup R^c)/EQ$.*

**Proof** We proceed by contradiction. Suppose that $R/EQ$ and $R^c/EQ$ (by lemma 6.50) are terminating and that $(R \cup R^c)/EQ$ is not terminating. Then there are infinite rewrite sequences in $(R \cup R^c)/EQ$. Consider a ground infinite $EQ$-rewrite sequence[4]

$$s_0 =_{EQ} s_0' \rightarrow_{R \cup R^c} s_1' =_{EQ} s_1 \rightarrow_{R \cup R^c} s_2' =_{EQ} s_2 \rightarrow_{R \cup R^c} \dots$$

Any infinite rewrite sequence will contain infinitely many rewrite steps of both $R/EQ$ and $R^c/EQ$, since both systems are terminating. Given such an infinite ground $EQ$-rewrite chain, we prove that we can push the $R^c/EQ$-steps forward, so that the infinite rewrite sequence will have an infinite initial segment of $R/EQ$-steps, contradicting termination of $R/EQ$. For that we prove that $\rightarrow_{R^c/EQ} \circ \rightarrow_{R/EQ} \subseteq \rightarrow_{R/EQ} \circ \rightarrow^*_{R^c/EQ}$ (i. e., that $R/EQ$ quasi-commutes over $R^c/EQ$), by a careful analysis of the position of the redexes.

First remark that if we have $C[v] = D[w]$, for some contexts $C, D$ and terms $v, w$, then we can have three possibilities:

[4] We need only consider ground $EQ$-reductions since any infinite $EQ$-rewrite sequence can be turned into a similar ground one.

(i) there is a context $C'$ such that $C[v] = D[C'[w]]$, i. e., $v = C'[w]$.

(ii) there is a context $D'$ such that $D[w] = C[D'[v]]$, i. e., $w = D'[v]$.

(iii) there is a context $F$ such that $D[w] = F[v, w]$ and $C[v] = F[v, w]$.

Suppose then we have $s =_{EQ} u \to_{R^c} v =_{EQ} w \to_R p =_{EQ} t$, for some terms $s, u, v, w, p, t$. This sequence can be rewritten as

$$s =_{EQ} C[s'] \to_{R^c} C[c] =_{EQ} D[l\sigma] \to_R D[r\sigma] =_{EQ} t$$

(i. e., $u = C[s']$, $v = C[c]$, $w = D[l\sigma]$ and $p = D[r\sigma]$), for some term $s'$, contexts $C, D$, substitution $\sigma$ and rule $l \to r$ of $R$.

If we apply lemma 6.53 to $C[c]$ and $D[l\sigma]$ we obtain $C[c] = C[\tau(\overline{x})]$, $D[l\sigma] = F[c] = F[\tau(\overline{x})]$ and $C[\overline{x}] =_{EQ} F[\overline{x}]$, for some context $F$, fresh variable $\overline{x}$, substitution $\tau$ satisfying $\tau(\overline{x}) = c$. From $F[c] = D[l\sigma]$ and by the observations made before, we have either (mark the special occurrence of $c$ we are interested in)

- $c$ occurs in $l\sigma$; since $c$ does not occur in $l$, then there must be a variable $x \in var(l)$ such that the marked occurrence of $c$ is in $\sigma(x)$. Since $l$ is linear, we can write $l$ as $L[x_1, \ldots, x_k]$, where $var(l) = \{x_1, \ldots, x_k\}$. Suppose that the marked $c$ occurs in $\sigma(x_j)$, with $1 \leq j \leq k$. Let $\overline{\sigma}(x_j)$ denote the term $\sigma(x_j)$ with the marked $c$ replaced by $\overline{x}$. Then $F[\overline{x}] = D[L[\sigma(x_1), \ldots, \overline{\sigma}(x_j), \ldots, \sigma(x_k)]]$. Define the substitution $\gamma$ by:

  $$\gamma(x) = \begin{cases} s' & \text{if } x = \overline{x} \\ x & \text{otherwise} \end{cases}$$

  Then $(F[\overline{x}])\gamma = F[s']$ and

  $$(D[L[\sigma(x_1), \ldots, \overline{\sigma}(x_j), \ldots, \sigma(x_k)]])\gamma = D[L[\sigma(x_1), \ldots, \overline{\overline{\sigma}(x_j)}, \ldots, \sigma(x_k)]],$$

  where $\overline{\overline{\sigma}(x_j)}$ is the term $\overline{\sigma}(x_j)$ with $\overline{x}$ replaced by $s'$. Also $C[s'] = (C[\overline{x}])\gamma =_{EQ} (F[\overline{x}])\gamma$, so $C[s'] =_{EQ} D[L[\sigma(x_1), \ldots, \overline{\overline{\sigma}(x_j)}, \ldots, \sigma(x_k)]$. Define now the substitution $\pi$ by:

  $$\pi(x) = \begin{cases} \sigma(x) & \text{if } x \in \{x_1, \ldots, x_k\} \text{ and } x \neq x_j \\ \overline{\overline{\sigma}(x_j)} & \text{if } x = x_j \\ x & \text{otherwise} \end{cases}$$

  Then $l\pi = L[\sigma(x_1), \ldots, \overline{\overline{\sigma}(x_j)}, \ldots, \sigma(x_k)] \to_R r\pi \to^*_{R^c} r\sigma$, where $r\pi \to^*_{R^c} r\sigma$ is obtained by reducing the term $s'$ to $c$ in all (possibly zero) occurrences of the variable $x_j$ in $r$.

  We now have $s =_{EQ} C[s'] =_{EQ} D[l\pi] \to_R D[r\pi] \to^*_{R^c} D[r\sigma] =_{EQ} t$, i. e., $s \to_{R/EQ} \circ \to^*_{R^c/EQ} t$, as we wanted.

- $c$ occurs in $D$; then we can write $F[c] = D'[c, l\sigma]$, $F[\overline{x}] = D'[\overline{x}, l\sigma]$ and $F[s'] = D'[s', l\sigma]$, for some context $D'$. Furthermore $D[r\sigma] = D'[c, r\sigma]$. Since $C[\overline{x}] =_{EQ} F[\overline{x}]$, we have $C[s'] = (C[\overline{x}])\gamma =_{EQ} (F[\overline{x}])\gamma = F[s']$, where the substitution $\gamma$ is defined as before. Consequently $C[s'] =_{EQ} D'[s', l\sigma]$. Since $D'[s', l\sigma] \to_R D'[s', r\sigma] \to_{R^c} D'[c, l\sigma] = D[r\sigma]$, we conclude that $s =_{EQ} C[s'] =_{EQ} D'[s', l\sigma] \to_R D'[s', r\sigma] \to_{R^c} D'[c, r\sigma] = D[r\sigma] =_{EQ} t$, so $s \to_{R/EQ} \circ \to^*_{R^c/EQ} t$, as we wanted.

□

We can now introduce a particular form of elimination of constants. Given a TRS $R$ where the constant $a$ may occur only in the rhs of the rules of $R$, the TRS $R^s$ denotes $R \setminus \{l \to a\}$, i. e., $R^s$ contains all the rules of $R$ that are not of the form $l \to a$. Then the following result holds:

**Theorem 6.55.** *Let $EQ$ be a set of length-preserving linear equations. Let $R$ be a left-linear TRS over $T(\mathcal{F}_a, \mathcal{X})$, where $a$ is a constant not occurring in the lhs of the rules of $R$ nor in the equations of $EQ$. Then termination of $R^s/EQ$ implies termination of $R/EQ$.*

**Proof** Since all conditions of lemma 6.54 are satisfied, we can apply it to conclude that $(R^s \cup R^a)/EQ$ is terminating. Since $\to^+_{R/EQ} \subseteq \to^+_{(R^s \cup R^a)/EQ}$, the result holds. □

Interesting enough, lemma 6.54 also enables us to improve the previous transformations, since in some cases we can eliminate the introduced constant $\diamond$. More precisely:

**Theorem 6.56.** *Let $EQ$ be a set of length-preserving linear equations (such that $\diamond$ does not occur in the equations of $EQ$). If $R$ is a left-linear TRS over $T(\mathcal{F}_a, \mathcal{X})$ and its transformed version $\mathcal{E}(R)$ is well-defined and is such that $\diamond$ does not occur in the lhs of rules of $\mathcal{E}(R)$, then termination of $\mathcal{E}(R)^s/EQ$ implies termination of $R/EQ$.*

**Proof** Apply lemma 6.54 with $c$ replaced by $\diamond$. Then we get that termination of $\mathcal{E}(R)^s/EQ$ implies termination of $\mathcal{E}(R)/EQ$, and by theorem 6.48 we get termination of $R/EQ$. □

Note that left-linearity of the TRS, linearity of $EQ$ and non-occurrence of $\diamond$ in the lhs of rewrite rules are essential as the following example shows.

**Example 6.57.** Suppose $EQ = \emptyset$ and let $R$ be

$$\begin{array}{rcl} R : f(x,x) & \to & f(0, a(1)) \\ 0 & \to & a(1) \end{array}$$

Let $a$ have e-status $\tau(a) = (\emptyset, 0)$. Then $\mathcal{E}(R)$ is given by

$$\begin{array}{rcl} \mathcal{E}(R) : f(x,x) & \to & f(0, \diamond) \\ f(x,x) & \to & 1 \\ 0 & \to & \diamond \\ 0 & \to & 1 \end{array}$$

and $\mathcal{E}(R)^s$ is given by

$$\begin{array}{rcl} \mathcal{E}(R)^s : f(x,x) & \rightarrow & f(0,\diamond) \\ f(x,x) & \rightarrow & 1 \\ 0 & \rightarrow & 1 \end{array}$$

We have that neither $R$ nor $\mathcal{E}(R)$ terminate, and it is not difficult to see that $\mathcal{E}(R)^s$ is terminating (even simply terminating).

It is also essential that the constant does not occur in the lhs of rules. For example, the TRS (again with $EQ = \emptyset$)

$$\begin{array}{rcl} R : f(a(x)) & \rightarrow & f(f(0)) \\ f(x) & \rightarrow & a(x) \end{array}$$

is not terminating. By eliminating $a$ (with e-status $(\emptyset, 0)$) and then taking $\mathcal{E}(R)^s$ we get the TRS

$$\begin{array}{rcl} \mathcal{E}(R)^s : f(\diamond) & \rightarrow & f(f(0)) \\ f(x) & \rightarrow & x \end{array}$$

which is terminating (in this case $\mathcal{E}(R)^s = \mathcal{E}(R) \setminus \{f(x) \rightarrow \diamond\}$).

Finally let $EQ$ be given by $\{(f(x,x), g(x,x))\}$ and let $R$ be:

$$\begin{array}{rcl} R : f(x,y) & \rightarrow & g(0,1) \\ 0 & \rightarrow & c \\ 1 & \rightarrow & c \end{array}$$

Then $R^s$ only contains the rule $f(x,y) \rightarrow g(0,1)$. We have that $R^s/EQ$ is terminating but $R/EQ$ is not terminating since $g(c,c) =_{EQ} f(c,c) \rightarrow g(0,1) \rightarrow^+ g(c,c)$.

## 6.5 Comparing Transformations

As mentioned before we can obtain different transformations associated with the same function symbol by changing the e-status of the symbol. The natural question is then what is the relationship between different transformations arising from different e-status of the same function symbol. Another interesting question is what happens with respect to other kinds of termination. In chapter 2 we described a hierarchy of types of termination. Here, for the purposes of comparing the transformations, we distinguish between *termination*, *simple termination*, *total termination* and *rpo-termination*. We are now interested in the effects of the transformation over these kinds of termination, i. e., if $\mathcal{E}(R)$ is simply, totally or rpo-terminating, what can be said about $R$? The reverse question may not be so interesting from a practical point of view, yet it is helpful for understanding the relation between transformations. We discuss these issues in this section. As mentioned in chapter 2, the concepts of simple and total termination carry over easily to the equational case and the results stated in this chapter for simple and/or total termination also hold for the equational case (where the equational theory satisfies the restriction of all equations being length-limited and variable-preserving). But since *rpo-termination* is not defined for the equational case and the equational theory in not really relevant for the comparison, in this presentation we will restrict ourselves to the case where the equational theory is just syntactical equality.

First we introduce some notation. In general $\tau_0$ will denote the e-status $(\emptyset, 0)$ and $\tau_N$ will denote some e-status $(\{1, \ldots, \texttt{arity}(a)\}, i)$, for any fixed $1 \leq i \leq \texttt{arity}(a)$. We will use the symbol $P$ to denote a property of TRS's. For example $P(R)$ means that the TRS $R$ has property $P$. We will also use the following abreviations: *SN* for termination (strong normalization), *ST* for simple termination, *TT* for total termination and *RPO* for rpo-termination.

The following tables provide answers for some of the questions mentioned above. We first consider the transformation associated to the e-status $\tau_0$.

| $\tau_0$ | | | |
|---|---|---|---|
| P | $P(R) \Rightarrow P(\mathcal{E}_{\tau_0}(R))$ | | $P(\mathcal{E}_{\tau_0}(R)) \Rightarrow P(R)$ |
| | $a \in$ lhs | $a \notin$ lhs | |
| *SN* | no (4) | no $(3)$ | yes $(1)$ |
| *ST* | no $(4)$ | no; yes if left-linear $(5)$ | no $(2)$ |
| *TT* | no $(4)$ | yes $(6)$ | no $(2)$ |
| *RPO* | no $(4)$ | yes $(7)$ | no $(2)$ |

We give now the justifications for the above table.

- $(1)$ follows from the main result proved in this chapter (theorem 6.48).
- for $(2)$, consider the following TRS

$$R : f(f(x)) \rightarrow f(a(f(x)))$$

We eliminate the function symbol $a$ with e-status $\tau_0$ and get

$$\begin{aligned} \mathcal{E}(R) : f(f(x)) &\rightarrow f(\diamond) \\ f(f(x)) &\rightarrow f(x) \end{aligned}$$

Now the system $R$ is terminating but not simply terminating: we have $f(f(x)) \rightarrow_R f(a(f(x))) \rightarrow_{\mathcal{E}mb_{\mathcal{F}_a}} f(f(x))$. Since total termination implies simple termination for finite signatures and rpo-termination implies total termination (see chapter 4), we have that $R$ is neither totally terminating nor rpo-terminating. However $\mathcal{E}(R)$ is rpo-terminating (just take standard *rpo* over a precedence $\triangleright$ satisfying $f \triangleright \diamond$), and so (see observations above) also totally terminating and simply terminating.

- for $(3)$, consider the following TRS (which is a modification of the famous Toyama's example)

$$\begin{aligned} R : f(0, 1, x) &\rightarrow f(x, x, x) \\ g(x, y) &\rightarrow a(x, y) \end{aligned}$$

This left-linear TRS is terminating; choose $A = \{0, 1\} \times \mathbb{N}_1$ with the order $\succ$ given by:

$$(a, m) \succ (b, n) \iff a = b \text{ and } m > n$$

where $>$ is the usual order on $\mathbb{N}$ and $\mathbb{N}_1$ contains only positive integers, and define the interpretations

$$\begin{array}{lcl} f_A((a,k),(b,m),(c,n)) & = & \begin{cases} (0, k+m+3n) & \text{if } (a,k) \neq (b,m) \\ (0, k+m+n) & \text{otherwise} \end{cases} \\ g_A((a,k),(b,m)) & = & (1, k+m+1) \\ a_A((a,k),(b,m)) & = & (1, k+m) \\ 0_A & = & (0,1) \\ 1_A & = & (1,1) \end{array}$$

The transformed version $\mathcal{E}(R)$ is not terminating.

$$\begin{array}{rcl} \mathcal{E}(R) : f(0,1,x) & \to & f(x,x,x) \\ g(x,y) & \to & \diamond \\ g(x,y) & \to & x \\ g(x,y) & \to & y \end{array}$$

Clearly $f(0,1,g(0,1)) \to f(g(0,1), g(0,1), g(0,1)) \to^+ f(0,1,g(0,1))$.

- for $(4)$, consider the TRS

$$R : f(a(1)) \to f(a(0))$$

This TRS is rpo-terminating (take standard *rpo* over a precedence $\rhd$ satisfying $1 \rhd 0$) and consequently also totally terminating and simply terminating. The transformed system $\mathcal{E}(R)$ is given by

$$\begin{array}{rcl} \mathcal{E}(R) : f(\diamond) & \to & f(\diamond) \\ f(\diamond) & \to & 0 \end{array}$$

which is clearly non-terminating.

Note that it is not essential that the function symbol $a$ occurs in the rhs. The following (non right-linear) TRS is rpo and thus totally and simply terminating, while its transformed version is not terminating.

$$R : f(a(x), y) \to f(y,y) \qquad\qquad \mathcal{E}(R) : f(\diamond, y) \to f(y,y)$$

- for $(5)$, consider the TRS

$$R : f(x,x) \to f(a(0), a(1))$$

This TRS is simply terminating. Choose $A_1$ to be $\mathbb{N}$ with the usual order $>$. In $A_1$ define the interpretations $0_{A_1} = 1_{A_1} = 0$, $a_{A_1}(x) = x$ and $f_{A_1}(x,y) = x+y+1$. Note that $A_1$ is compatible with $R_1$, the rewrite system containing the embbeding rules for $f$, and is a (well-founded monotone) quasi-model for $R_2 = R \cup \{a(x) \to x\}$. For compatibility with $R_2$, choose the well-founded monotone algebra $(A_2, \succ) = (\mathbb{N}, \succ)$ with

$$m \succ n \iff (n \text{ mod } 2 = m \text{ mod } 2) \text{ and } m > n$$

and define interpretations $0_{A_2} = 0, 1_{A_2} = 1$, $a_{A_2}(x) = x + 2$ and

$$f_{A_2}(x,y) \;=\; \begin{cases} 2(x+y+6) & \text{if } (x \bmod 2) = (y \bmod 2) \\ 2(x+y) & \text{otherwise} \end{cases}$$

Combination of these two algebras and theorems 2.99 and 2.106 give simple termination of $R$.

The transformed system is

$$\begin{aligned} \mathcal{E}(R) : f(x,x) &\rightarrow f(\diamond,\diamond) \\ f(x,x) &\rightarrow 0 \\ f(x,x) &\rightarrow 1 \end{aligned}$$

which is not even terminating.

If $R$ is left-linear, then $\mathcal{E}(R)$ is also left-linear for any possible transformation. In the next table we will see that for the e-status $\tau_N$, and if $a \notin lhs$, then $ST(R) \Rightarrow ST(\mathcal{E}_{\tau_N}(R)$. Later on (when comparing the transformation $\mathcal{E}_{\tau_0}$ with $\mathcal{E}_{\tau_N}$), we will prove that, given left-linearity and $a \notin lhs$, $ST(\mathcal{E}_{\tau_N}(R)) \Rightarrow ST(\mathcal{E}_{\tau_0}(R))$. Combining these two results we get $ST(R) \Rightarrow ST(\mathcal{E}_{\tau_N}(R)) \Rightarrow ST(\mathcal{E}_{\tau_0}(R))$.

- the justification for $(6)$ is as follows. Since $(R, \mathcal{F}_a, \mathcal{X})$ is totally terminating there is a monotone $\mathcal{F}$-algebra $(A, >)$, compatible with the rules of $R$ and such that $>$ is a well-order. Consider the order $>_A$ on $T(\mathcal{F}_a, \mathcal{X})$ induced by the interpretation $[\![\ ]\!] : T(\mathcal{F}_a, \mathcal{X}) \times A^{\mathcal{X}} \rightarrow A$ and the order $>$ (recall the definition from chapter 2). We now interpret $\diamond$ as the minimal element of $A$. Then $(A, >)$ is still a total and well-founded monotone algebra for $(R, \mathcal{F}_a \cup \{\diamond\}, \mathcal{X})$. By induction on terms it is not difficult to see that:

  **Lemma 6.58.** *Let* $\tau(a) = (\emptyset, 0)$. *Then*

  - $\forall t \in T(\mathcal{F}_a, \mathcal{X}) : \ t \geq_A \mathtt{cap}_0(t)$.
  - $\forall t \in T(\mathcal{F}_a, \mathcal{X})\ \forall u \in E(t) \cup \mathtt{dec}(t) : t \geq_A u$.

  Note that the minimality of $\diamond$ in the algebra, is essential for the proof of the lemma; also this lemma remains valid in the equational setting.

  Now, given the definition of $\mathcal{E}(R)$ and the properties above, it is easy to check that $(A, >)$ is also compatible with $\mathcal{E}(R)$, giving its total termination (note that to be able to conclude that $l > r$ for any rule $l \rightarrow r$ in $\mathcal{E}(R)$ it is essential that the eliminated symbol doesn't occur on the lhs).

- for $(7)$ we just have to extend the precedence $\triangleright$ on $\mathcal{F}_a$ to $\mathcal{F}_a \cup \{\diamond\}$ as follows $f \triangleright \diamond$, for all $f \in \mathcal{F}_a$. The rest is similar to $(6)$, with the order $>_A$ replaced by $>_{rpo}$.

We now consider the transformation associated to the e-status $\tau_N$.

| $\tau_N$ | | | |
|---|---|---|---|
| P | $P(R) \Rightarrow P(\mathcal{E}_{\tau_N}(R))$ | | $P(\mathcal{E}_{\tau_N}(R)) \Rightarrow P(R)$ |
| | $a \in$ lhs | $a \notin$ lhs | |
| *SN* | no (4) | no $(3)$ | yes $(1)$ |
| *ST* | no $(4)$ | yes $(5)$ | no $(2)$; yes if right-linear $(8)$ |
| *TT* | no $(4)$ | yes $(6)$ | yes $(9)$ |
| *RPO* | no $(4)$ | yes $(7)$ | no $(10)$ |

We give the justifications which are different from the corresponding ones in the previous table.

- for $(2)$ consider yet another variant of Toyama's TRS

$$R : f(0,1,x) \to f(x,x,a(0,1))$$

This TRS is not simply terminating. The transformed one is given by

$$\begin{aligned}\mathcal{E}(R) : f(0,1,x) &\to f(x,x,0)\\ f(0,1,x) &\to f(x,x,1)\end{aligned}$$

We can see that $\mathcal{E}(R)$ is simply terminating by choosing the following monotone algebra $A = \mathbb{N}_1 \times \{0,1\} \times \mathbb{N}_1$, where $\mathbb{N}_1$ represents the strictly positive natural numbers, with the well-founded order

$$(n,p,m) > (n',p',m') \iff \begin{array}{l}(n >_{\mathbb{N}_1} n') \text{ or}\\ (n = n' \text{ and } p = p' \text{ and } m >_{\mathbb{N}_1} m')\end{array}$$

and interpreting $0$ by $(1,0,1)$, $1$ by $(1,1,1)$ and $f$ by the monotone function

$$f((a,b,c),(k,m,n),(r,s,t)) = \begin{cases}(a+k+2r,0,c+n+3t) & \text{if } b \neq m\\ (a+k+2r,0,c+n+t) & \text{otherwise}\end{cases}$$

This interpretation is compatible with the TRS $\mathcal{E}(R) \cup \mathcal{E}mb_{\mathcal{F}}$ and therefore $\mathcal{E}(R)$ is simply terminating.

- for $(3)$, consider the same example as in $(3)$, $\tau_0$-table, the only difference being that the rule $g(x,y) \to \diamond$ does not occur in the transformed system associated with $\tau_N$.

- for $(4)$ consider the TRS

$$R : f(a(0)) \to f(0)$$

This TRS is rpo-terminating and so also totally and simply terminating. The transformed system $\mathcal{E}(R)$ is given by

$$\mathcal{E}(R) : f(0) \to f(0)$$

which is not terminating.

- for $(5)$ and $(6)$ we proceed similarly. Since $(R, \mathcal{F}_a, \mathcal{X})$ is simply, respectively totally, terminating, we know that there is a well-founded monotone algebra (total in the case of total termination) $(A, >)$ compatible with $(R \cup \mathcal{E}mb_{\mathcal{F}_a}, \mathcal{F}_a, \mathcal{X})$ (for total termination this requires the equivalence between total termination of $R$ and total termination of $R \cup \mathcal{E}mb_{\mathcal{F}}$; see theorem 2.111). We don't really need to consider the function symbol $\diamond$, however since $\diamond$ does not occur in the rules of $\mathcal{E}(R)$, introducing $\diamond$ in the alphabet is not problematic, we just interpret it by an arbitrary element of $A$. This said, lemma 6.58 does also hold for the transformation associated with $\tau_N$, where $\mathsf{cap}_0(t)$ is now replaced by $\mathsf{cap}_i(t)$. Now it is easy to show that $l >_A u$, for all rules $l \to u$ in $\mathcal{E}(R)$, so $\mathcal{E}(R)$ is simply (respectively totally) terminating (note that the embedding rules for $\mathcal{E}(R)$ are a subset of the embedding rules for $R$, so compatibility with the embedding rules of $\mathcal{E}(R)$ is guaranteed).

- $(8)$ and $(9)$ were already shown in Zantema [109] for the case that the function symbol $a$ does not occur in the lhs of $R$, but a similar proof applies if we remove this restriction, since $t \to^*_{\mathcal{E}mb_a} \mathsf{cap}_i(t)$, for any term $t$, where $\mathcal{E}mb_a$ consists of the embbeding rules for $a$.

- for $(10)$, consider the TRS $R$:

$$\begin{array}{llll} R: & g(h(x), h(y)) & \to & f(a(x, y)) \\ & f(x) & \to & g(x, x) \end{array}$$

  Then the transfomed TRS $\mathcal{E}(R)$ is given by:

$$\begin{array}{llll} \mathcal{E}(R): & g(h(x), h(y)) & \to & f(x) \\ & g(h(x), h(y)) & \to & f(y) \\ & f(x) & \to & g(x, x) \end{array}$$

  The TRS $\mathcal{E}(R)$ is rpo-terminating with $>_{rpo}$ taken over the precedence $\triangleright$ such that $h \triangleright f \triangleright g$. However $R$ is not rpo-terminating. The second rule of $R$ forces any precedence to satisfy $f \triangleright g$ and no subterm of $g(h(x), h(y))$ is greater in the $>_{rpo}$ order than $f(a(x, y))$, even if $h \triangleright f \triangleright a$, since different variables are incomparable. An even simpler example is provided by the TRS $f(g(f(x))) \to g(f(f(a(x))))$ (though this is not an orthogonal system).

What happens if the e-status $\tau$ is neither $\tau_0$ nor $\tau_N$? We then get a table "in-between" tables for $\tau_0$ and $\tau_N$, namely

| | $\tau$ | | |
|---|---|---|---|
| P | $P(R) \Rightarrow P(\mathcal{E}_\tau(R))$ | | $P(\mathcal{E}_\tau(R)) \Rightarrow P(R)$ |
| | $a \in$ lhs | $a \notin$ lhs | |
| *SN* | no | no $(3)$ | yes $(1)$ |
| *ST* | no $(4)$ | yes $(5)$ | no $(2)$ |
| *TT* | no $(4)$ | yes $(6)$ | no $(2)$ |
| *RPO* | no $(4)$ | yes $(7)$ | no $(2)$ |

Note that $(2)$ is the same as in the table for $\tau_0$ and $(5)$ is the same as in the table for $\tau_N$ (being the proof of it also the same). With respect to $(2)$, the example for $\tau_0$ can be modified to accomodate any e-status $\tau \neq \tau_N$. Let then $\tau(a) = (X, i)$ with $\emptyset \neq X \subsetneq \{1, \ldots, \mathtt{arity}(a)\}$, and $i$ an arbitrary element of $X$. Let $R$ be

$$R : f(f(x)) \to f(a(0, \ldots, f(x), \ldots, 0))$$

where the term $f(x)$, in the rhs, occurs at some position $j \in \{1, \ldots, \mathtt{arity}(a)\} \setminus X$. The system $\mathcal{E}(R)$ will be given by

$$\begin{aligned} \mathcal{E}(R) : f(f(x)) &\to f(0) \\ f(f(x)) &\to f(x) \end{aligned}$$

and eventually will also contain the rule $f(f(x)) \to 0$. Again $\mathcal{E}(R)$ is rpo-terminating, and thus totally and simply terminating, while $R$ is not even simply terminating.

Since we are interested in simplifying the task of proving termination of $\mathcal{E}(R)$, it would be useful to identify which e-status are more suitable for that purpose. We therefore consider what happens when a function symbol $a$ has e-status $\tau^1(a) = (X, i)$ and $\tau^2(a) = (Y, i)$ with $X \subset Y$. First we consider the case where $X \neq \emptyset$. Surprisingly enough we have that termination of $\mathcal{E}_{\tau^1}(R)$ does not imply termination of $\mathcal{E}_{\tau^2}(R)$ nor vice-versa. Consider the following examples

$$R : f(f(x)) \to f(a(0, f(x)))$$

Let $a$ have e-status $\tau^1(a) = (\{1\}, 1)$ and $\tau^2(a) = (\{1, 2\}, 1)$. Then the associated transformed systems are

$$\begin{aligned} \mathcal{E}_{\tau^1}(R) : f(f(x)) &\to f(0) \\ f(f(x)) &\to f(x) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{\tau^2}(R) : f(f(x)) &\to f(0) \\ f(f(x)) &\to f(f(x)) \end{aligned}$$

We easily see that $\mathcal{E}_{\tau^1}(R)$ is rpo-terminating and that $\mathcal{E}_{\tau^2}(R)$ is not terminating. Fix the same e-status for $a$ and consider now the TRS

$$\begin{aligned} R : f(g(0), 1) &\to f(2, 2) \\ 2 &\to g(a(0, 1)) \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E}_{\tau^1}(R) : f(g(0), 1) &\to f(2, 2) \\ 2 &\to g(0) \\ 2 &\to 1 \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_{\tau^2}(R) : f(g(0), 1) &\to f(2, 2) \\ 2 &\to g(0) \\ 2 &\to g(1) \end{aligned}$$

In this case we have the reversed situation: $\mathcal{E}_{\tau^1}(R)$ is not terminating while $\mathcal{E}_{\tau^2}(R)$ is terminating.

We now compare the transformations associated with $\tau_0$ and any other e-status $\tau$. In Ferreira and Zantema [35] we suggested that, although the transformation associated with $\tau_N$ and the transformation associated with $\tau_0$ were not really comparable, that the later seemed stronger than the former. The following table confirms our intuition. Let $\tau$ be any e-status different from $\tau_0$, then

| P | $a \notin lhs, \tau \neq \tau_0$ | |
|---|---|---|
| | $P(\mathcal{E}_\tau(R)) \Rightarrow P(\mathcal{E}_{\tau_0}(R))$ | $P(\mathcal{E}_{\tau_0}(R)) \Rightarrow P(\mathcal{E}_\tau(R))$ |
| *SN* | no $(2)$ | no $(1)$ |
| *ST* | no; yes if $R$ left-linear $(3)$ | no $(1)$ |
| *TT* | yes $(4)$ | no $(1)$ |
| *RPO* | yes $(5)$ | no $(1)$ |

The justifications are as follows:

- for $(1)$, consider again the TRS

  $$R : f(f(x)) \to f(a(f(x)))$$

  Then $\mathcal{E}_{\tau_0}(R)$ is given by

  $$\begin{array}{rcl} \mathcal{E}_{\tau_0}(R) : f(f(x)) & \to & f(\diamond) \\ f(f(x)) & \to & f(x) \end{array}$$

  For any other e-status $\tau$ we have that $\mathcal{E}_\tau(R)$ is given by

  $$\mathcal{E}_\tau(R) : f(f(x)) \to f(f(x))$$

  We have that $\mathcal{E}_{\tau_0}(R)$ is rpo-terminating (thus also totally and simply terminating) while $\mathcal{E}_\tau(R)$ is not terminating.

- for $(2)$ consider the TRS

  $$\begin{array}{rcl} R : f(0,1) & \to & f(2,2) \\ 2 & \to & g(a(0,1)) \end{array}$$

  The system $\mathcal{E}_{\tau_0}(R)$ is given by

  $$\begin{array}{rcl} \mathcal{E}_{\tau_0}(R) : f(0,1) & \to & f(2,2) \\ 2 & \to & g(\diamond) \\ 2 & \to & 0 \\ 2 & \to & 1 \end{array}$$

The systems $\mathcal{E}_\tau(R)$ are the following (the indices give the set $X$ of the e-status $(X, i)$):

$$\begin{array}{rcl} R_{\{1\}} : f(0,1) & \to & f(2,2) \\ 2 & \to & g(0) \\ 2 & \to & 1 \end{array} \qquad \begin{array}{rcl} R_{\{2\}} : f(0,1) & \to & f(2,2) \\ 2 & \to & 0 \\ 2 & \to & g(1) \end{array}$$

$$\begin{array}{rcl} R_{\{1,2\}} : f(0,1) & \to & f(2,2) \\ 2 & \to & g(0) \\ 2 & \to & g(1) \end{array}$$

Clearly $\mathcal{E}_{\tau_0}(R)$ is not terminating, while all TRS's $\mathcal{E}_\tau(R)$ are terminating, for $\tau \neq \tau_0$.

- consider $(3)$; this property was first remarked by Middeldorp and Ohsaki (personal communication) for the case $\tau = \tau_N$. We have that $\mathcal{E}_\tau(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)}$ is terminating, for some *pwo* $\succ$ on $\mathcal{F}_\diamond$. Since $R$ is left-linear, so is $\mathcal{E}_\tau(R)$. Recall the definition of $\mathcal{E}_\tau(R)^\diamond$ from section 6.4. Since both $\mathcal{E}_\tau(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)}$ and $\mathcal{E}_\tau(R)^\diamond$ are terminating, $\mathcal{E}_\tau(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)}$ is left-linear and $\diamond$ does not occur in the lhs of the rules of $\mathcal{E}_\tau(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)}$, we can apply lemma 6.54 and conclude that $\mathcal{E}_\tau(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)} \cup \mathcal{E}_\tau(R)^\diamond$ is terminating. Let us call this TRS $\mathcal{Q}$.

  We define the following order $\sqsupset$ on $T(\mathcal{F}_\diamond, \mathcal{X})$:

  $$s \sqsupset t \iff \forall \sigma : \mathcal{X} \to T(\mathcal{F} \cup \{\diamond\}) : \; s\sigma \to^+_{\mathcal{Q}} t\sigma$$

  Note that $\sqsupset$ is a well-founded partial order on $T(\mathcal{F}_\diamond, \mathcal{X})$, closed under contexts and substitutions.

  We can prove the following lemma by induction on terms ($\texttt{dec}(t)_\tau$ indicates that the decomposition of $t$ is taken with respect to e-status $\tau$):

  **Lemma 6.59.** *For all terms* $t \in T(\mathcal{F} \cup \{a\}, \mathcal{X})$*:*

  – $\texttt{cap}_i(t) \sqsupseteq \texttt{cap}_0(t)$.
  – $\forall u \in E_{\tau_0}(t) \cup \texttt{dec}(t)_{\tau_0} \; \exists v \in E_\tau(t) \cup \texttt{dec}(t)_\tau : \; v \sqsupseteq u$.

  Take an arbitrary rule $l \to u$ of $\mathcal{E}_{\tau_0}(R)$. Then there is a rule $l \to r$ of $R$ such that $u \in E_{\tau_0}(r) \cup \texttt{dec}(r)_{\tau_0}$. Applying lemma 6.59, we conclude that there is a term $v \in E_\tau(r) \cup \texttt{dec}(r)_\tau$, such that $v \sqsupseteq u$. Since $l \to v$ is a rule in $\mathcal{E}_\tau(R)$, we have that $l \sqsupset u$, i. e., $\to^+_{\mathcal{E}_{\tau_0}(R) \cup \mathcal{E}mb_{(\mathcal{F}_\diamond, \succ)}} \subseteq \to^+_{\mathcal{Q}}$, so $\mathcal{E}_{\tau_0}(R)$ is simply terminating.

  Left-linearity is essential. The TRS with the rule $f(x,x) \to f(a(0), a(1))$ originates the only possible transformed TRS's $\mathcal{E}R_\tau$ and $\mathcal{E}R_{\tau_0}$:

  $$\mathcal{E}R_\tau : f(x,x) \;\to\; f(0,1) \qquad \begin{array}{rrcl} \mathcal{E}R_{\tau_0} : & f(x,x) & \to & f(\diamond, \diamond) \\ & f(x,x) & \to & 0 \\ & f(x,x) & \to & 1 \end{array}$$

  Then $\mathcal{E}R_\tau$ is simply terminating while $\mathcal{E}R_{\tau_0}$ is not even terminating.

- for $(4)$ note that in a total well-founded monotone algebra compatible with $\mathcal{E}_\tau(R)$, $\diamond$ is not necessarily interpreted as the minimal element. However it is not difficult to see that, since the constant $\diamond$ does not occur in the rules of $\mathcal{E}_\tau(R)$, we can force the interpretation of $\diamond$ to be the minimal element while keeping the interpretation of the other function symbols, and the resulting algebra will still be compatible with the TRS. This said we assume that the interpretation of $\diamond$ is indeed the minimal element of the algebra. We have:

  **Lemma 6.60.** *For all terms $t \in T(\mathcal{F}_a, \mathcal{X})$:*

  - $\mathtt{cap}_i(t) \geq_A \mathtt{cap}_0(t)$.
  - $\forall u \in E_{\tau_0}(t) \cup \mathtt{dec}(t)_{\tau_0}\ \exists v \in E_\tau(t) \cup \mathtt{dec}(t)_\tau :\ v \geq_A u$.

  where $>_A$ is the order on $T(\mathcal{F}_\diamond, \mathcal{X})$ induced by the algebra (see definition 2.92). Now we can proceed as for $(3)$ above, replacing lemma 6.59 by lemma 6.60 and $\sqsupset$ by $>_A$ and considering only rules of $\mathcal{E}_{\tau_0}(R)$, to conclude that if $l \to u$ is such a rule then $l >_A u$, so $\mathcal{E}_{\tau_0}(R)$ is totally terminating.

- for $(5)$ first note that if $(\mathcal{F}_\diamond, \mathcal{X}, \mathcal{E}_\tau(R))$ is rpo-terminating, since the constant $\diamond$ does not occur in the rules of $\mathcal{E}_\tau(R)$, then $(\mathcal{F}_\diamond, \mathcal{X}, \mathcal{E}_\tau(R))$ is still rpo-terminating for a precedence $\triangleright$ satisfying $f \triangleright \diamond$, for all function symbols $f \in \mathcal{F}$. We fix such a $>_{rpo}$. By double induction on $s$ and $t$, we can prove that:

  **Lemma 6.61.** *For all terms $s \in \mathcal{T}(\mathcal{F}, \mathcal{X}), t \in T(\mathcal{F}_a, \mathcal{X})$:*

  - $s >_{rpo} \mathtt{cap}_i(t) \Rightarrow s >_{rpo} \mathtt{cap}_0(t)$ *(with $i \neq 0$)*
  - $(\forall u \in E_\tau(t) \cup \mathtt{dec}(t)_\tau :\ s >_{rpo} u) \Rightarrow (\forall u \in E_{\tau_0}(t) \cup \mathtt{dec}(t)_{\tau_0} :\ s >_{rpo} u)$

  Let $l \to u$ be a rule in $\mathcal{E}_{\tau_0}(R)$. Then there is a rule $l \to r$ in $R$ such that $u \in E_{\tau_0}(r) \cup \mathtt{dec}(r)_{\tau_0}$. But since $l \to v$, for any $v \in E_\tau(r) \cup \mathtt{dec}(r)_\tau$, is a rule in $\mathcal{E}_\tau(R)$ and so $l >_{rpo} v$ for all such elements, we conclude using the previous lemma, that $l >_{rpo} u$, for all $u \in E_{\tau_0}(r) \cup \mathtt{dec}(r)_{\tau_0}$, proving that $\mathcal{E}_{\tau_0}(R)$ is rpo-terminating.

### 6.5.1 Eliminating Distribution Rules

We now consider the elimination of function symbols in the presence of distribution rules for that symbol.

In Zantema [108] *distribution elimination* was introduced. It consists of a transformation on terms which is applied to TRS's that may contain distribution rules for the eliminated function symbol. From Zantema [108] we recall the definition of distribution rule.

**Definition 6.62.** Let $a$ be a fixed function symbol of arity $N \geq 1$. A distribution rule for $a$ is a rule of the form

$$C[a(x_1, \ldots, x_N)] \to a(C[x_1], \ldots, C[x_N])$$

for any non-trivial context $C$ where $a$ does not occur.

The transformation on terms used in Zantema [108] is denoted by $E_a$ and is similar to our transformation associated with the e-status $\tau(a) = (\{1, \ldots, N\}, -)$, the difference being that with $E_a$ we get all possible combinations (or crossed) subterms, while with $\mathcal{E}$ we only get the combinations arising from always taking the same branch of alien terms. In other words, if we fix $\tau_N = (\{1, \ldots, N\}, i)$ (with $1 \leq i \leq N$ arbitrary), then, for any term $t \in T(\mathcal{F}_a, \mathcal{X})$ we have $(E(t) \cup \mathtt{dec}(t)) \subseteq E_a(t)$, and in general the inclusion is strict. As a consequence termination of $\mathcal{E}(R)$ follows from termination of $E_a(R)$ and the former transformation is more general. However we have until now left out the possibility of existence of distribution rules for $a$ in $R$. Here we discuss that case. From Zantema [108] we recall:

**Theorem 6.63.** *Let $R$ be a TRS over $T(\mathcal{F}_a, \mathcal{X})$, such that each rule is either a distribution rule for $a$ or a rule where $a$ does not occur in the lefthand-side. Then*

1. *if $E_a(R)$ is terminating and right-linear then $R$ is terminating.*
2. *if $E_a(R)$ is right-linear then $E_a(R)$ is simply terminating if and only if $R$ is simply terminating.*
3. *$E_a(R)$ is totally terminating if and only if $R$ is totally terminating.*

The following example, suggested by Aart Middeldorp, shows that in the presence of distribution rules, our transformations are not sound with respect to termination, even if no equational theory is considered.

**Example 6.64.** Let $EQ = \emptyset$ and let $R$ be given by:

$$\begin{array}{rcl} f(0,1) & \rightarrow & f(a(0,1), a(0,1)) \\ f(a(x,y), z) & \rightarrow & a(f(x,z), f(y,z)) \\ f(x, a(y,z)) & \rightarrow & a(f(x,y), f(x,z)) \end{array}$$

This system is not terminating since it allows the infinite rewrite sequence

$$\underline{f(0,1)} \rightarrow f(a(0,1), a(0,1)) \rightarrow a(f(0, a(0,1)), f(1, a(0,1))) \rightarrow$$
$$a(a(f(0,0), \underline{f(0,1)}), f(1, a(0,1))) \rightarrow \ldots$$

All possible transformations of $R$, with elimination of the distribution rules are shown below

| $\tau$ | $(\emptyset, 0)$ | $(\{1\}, 1)$ |
|---|---|---|
| | $f(0,1) \rightarrow f(\Diamond, \Diamond)$<br>$f(0,1) \rightarrow 0$<br>$f(0,1) \rightarrow 1$ | $f(0,1) \rightarrow f(0,0)$<br>$f(0,1) \rightarrow 1$ |
| $\tau$ | $(\{2\}, 2)$ | $(\{1,2\}, -)$ |
| | $f(0,1) \rightarrow f(1,1)$<br>$f(0,1) \rightarrow 0$ | $f(0,1) \rightarrow f(0,0)$<br>$f(0,1) \rightarrow f(1,1)$ |

All systems presented above are terminating. We therefore conclude that only the transformation $E_a$ presented in Zantema [108] remains sound with respect to termination when distribution rules are allowed.

## 6.6 Final Remarks

We concentrated our efforts on the elimination of function symbols. We can ask ourselves whether it is possible to apply the same kind of reasoning to patterns. That is not so. Even if the pattern is a normal form, elimination of patterns does not provide a sound transformation. The following example supports this assertion.

**Example 6.65.** Let $R$ be:

$$\begin{array}{lcl} f(x,x) & \rightarrow & f(a(0),a(1)) \\ 1 & \rightarrow & 0 \end{array}$$

Note that this system is not terminating. Suppose we eliminate the pattern $a(0)$ and replace it by the constant $\diamond$. The result is:

$$\begin{array}{lcl} f(x,x) & \rightarrow & f(\diamond,a(1)) \\ 1 & \rightarrow & 0 \end{array}$$

which is a terminating system.

There is yet another problem with patterns and that is that since patterns may be dynamically created during reductions, eliminating a pattern from the reduction rules does not ensure that the pattern does not occur later.

We can also wonder whether we can weaken the condition necessary to prove theorem 6.48. It is still an open problem whether theorem 6.48 holds if we remove the restrictions imposed on the equational system, i. e., based solely on the hypothesis of termination of $\mathcal{E}(R)/EQ$. Due to the technical tool used for proving theorem 6.48, namely the tree construction and the tree lifting of a quasi-order, those restrictions are necessary. The restriction $mvar(e_1) = mvar(e_2)$, for any equation $(e_1, e_2)$, is necessary to ensure that $s =_{EQ} t \Rightarrow \mathtt{tree}(s) \approx \mathtt{tree}(t)$. If an equation $(e_1, e_2)$ would not satisfy this then lemma 6.35 and lemma 6.5 would imply that $\mathtt{tree}(e_1) \not\approx \mathtt{tree}(e_2)$. But length-limited cannot be weakened to variable-preserving and we now see why. In order to be able to prove lemma 6.47, we need to have an order on terms that is compatible with some form of subterm property (modulo $EQ$). But we also want such an order to be well-founded. The problem is that ensuring that the quasi-order "subterm modulo $EQ$" is well-founded is not possible in general, as the following example shows.

**Example 6.66.** Let $EQ = \{(f(g(f(x))), g(x))\}$. We define $\rhd \;=\; \unrhd \setminus \unlhd$, where $s \unrhd t \iff \exists C : \; s =_{EQ} C[t]$ (here $\unrhd$ is a form of subterm property modulo $EQ$). Then we have $g(f^i(x)) \rhd g(f^{i+1}(x))$, for all $i \geq 0$, so $\rhd$ is not well-founded.

The best way of ensuring well-foundedness of an order having the subterm property modulo $EQ$ we could find was imposing the length-limited condition on the equations of $EQ$.

We were not able to find neither another proof nor a counterexample for equational theories not satisfying our restriction and it seems that termination of $\mathcal{E}(R)/EQ$ is a strong enough condition to ensure validity of theorem 6.48. However that remains to be proved.

With respect to practical application of the techniques presented in this chapter, we believe they can be quite useful, especially in the field of automatic termination proofs, since these techniques can easily be implemented and used to pre-process the (equational) term rewrite systems to be proven terminating. So far dummy elimination has been used in Zantema and Geser [112], in the context of term rewriting, and in Fokkink and Zantema [36], for equational rewriting, this last application being in the field of Process Algebra.

# Appendix A

# A note on Well-founded Orders

## A.1 The Problem

It is well-known that given a partially ordered set, it is always possible to extend that partial order to a total one. It is also known that given any set, it is possible to define a total and well-founded order on that set (Zermelo's Theorem; see Kuratowski and Mostowski [65]). We turn now to the problem of extending a partial well-founded order. We remark that these three problems are closely related to the Axiom of Choice (see Kuratowski and Mostowski [65]). Indeed Zermelo's Theorem is equivalent to it and so is Zorn's lemma (see Davey and Priestley [18]), the result used in the proof of the first problem and also in the proof we present here.

First let us make clear what we mean by *well-founded* (see definition 2.14). Given a partially ordered set $(P, >)$, we define an *$\omega$-descending chain* in $P$ as a function $\alpha$ from the naturals to $P$, such that $i < j \Rightarrow \alpha(i) > \alpha(j)$. By a *finite descending chain* we mean that $\alpha$ is defined only on a finite subset of the natural numbers. If $(P, >)$ admits no $\omega$-descending chains, we say that it is *well-founded*.

It is interesting to remark that we can characterize well-foundedness without using the natural numbers, since we have the following result (also equivalent to the Axiom of Choice):

$$(P, >) \text{ is well-founded } \iff \forall \emptyset \neq S \subseteq P: \; S \text{ has a minimal element.}$$

We state now the essential tool for our solution.

**Lemma A.1.** **(Zorn's Lemma)** *Let $(P, >)$ be a non-empty partially ordered set such that every (ascending) chain has an upper bound. Then $(P, >)$ has a maximal element.*

Recall that a an ascending chain has the form: $x_0 > x_1 > \ldots > x_n \ldots$. Recall also that given a set $X \subseteq P$ an element $m \in P$ is an upper bound for $X$ if for all $x \in X$ we have $m \geq x$.

We state our problem.

**Theorem A.2.** *Let $(A, >)$ be a partially ordered set such that $>$ is well-founded. Then it is possible to extend the order $>$ to a total and well-founded order on $A$.*

In the case that $A$ is finite the total well-founded extension is called a *topological sort* (see Knuth [62]).

We will prove this result as follows: first a convenient set (denoted by $K$) of partial orders over subsets of $A$ will be defined and with it an appropriate notion of inclusion $\sqsupset$; then we check that $(K, \sqsupset)$ satisfies the conditions of Zorn's lemma; finally we see that a maximal element of $K$ (whose existence is ensured by Zorn's lemma), is an element which complies with our requisites, i. e., it is a total well-founded order extending the original one.

The intuitive idea behind the proof is that we want to extend an initially well-founded order, so we can start with a finite descending chain of that order and try to extend that chain. However for that extension to be well-founded it is necessary that we do not extend the chain from below (at least not in an indiscriminate manner). One can achieve this by fixing (an initial segment of) the chain and add elements at the top.

## A.2 The Proof

We fix the set $A$ and the well-founded partial order $>$ on $A$. Define $K$ as the set of partial orders $(P, >_P)$ satisfying the following conditions:

1. $P \subseteq A$.
2. If $x, y \in P$ and $x > y$ then $x >_P y$.
3. $>_P$ is total and well-founded (in $P$).
4. If $x > y$ and $x \in P$ then $y \in P$.

We note that $K$ is non-empty since (for example) every singleton set containing a minimal element of $(A, >)$ with the trivial order, is in $K$.

The first three conditions state that we are interested in total well-founded orders respecting the original order but defined only on a subset of $A$. The fourth condition means that $P$ has to be an ideal for the original order.[1]

We now turn $K$ into a partially ordered set defining, for any $T, R \in K$

$$R \sqsupset T \iff T \text{ is an initial segment of } R.$$

That is,

(a) $T \subset R$ (as sets),

(b) if $x >_R y$ and $x \in T$ then $y \in T$ and $x >_T y$.

It is easy to see that $\sqsupset$ is indeed a partial order in $K$.

[1] Recall that an ideal $I$ of a poset $(P, \succ)$ is a set satisfying $x \in I$ and $x \succ y \Rightarrow y \in I$.

We remark that condition (b) and the fact that both $>_R$ and $>_T$ are total orders, imply that $>_R$ subsumes $>_T$, i. e., $x >_T y \Rightarrow x >_R y$.

The next step is to verify that $K$ satisfies the conditions of Zorn's lemma. We already saw that $K$ is non-empty. Now we take an ascending chain $S_0 \sqsubset S_1 \sqsubset S_2 \sqsubset \ldots$ in $K$ (where as usual $\sqsubset$ is the inverse relation of $\sqsupset$, i. e., $S \sqsubset T \iff T \sqsupset S$) and define $(S, >_S) = (\bigcup_{n \geq 0} S_n, \bigcup_{n \geq 0} >_{S_n})$.

We shall see now that $(S, >_S) \in K$. First we note that $>_S$ is indeed a partial order on $S$ since:

- irreflexivity: suppose we have an element $s \in S$ such that $s >_S s$; then there is an index $i \geq 0$ such that $s \in S_i$ and $s >_{S_i} s$, contradicting irreflexivity of $>_{S_i}$.
- transitivity: $x >_S y$ and $y >_S z \Rightarrow \exists i, j :\ x >_{S_i} y$ and $y >_{S_j} z$. Taking $m = max\{i, j\}$ we have that $S_i \sqsubseteq S_m$ and $S_j \sqsubseteq S_m$. So $x >_{S_m} y$ and $y >_{S_m} z$ and since $>_{S_m}$ is transitive we conclude that $x >_{S_m} z$ implying that $x >_S z$.

It is also easy to see that $(S, >_S)$ is a upper bound for every element in the chain. Indeed for any $n$:

(a) $S_n \subseteq S$.

(b) Suppose $x >_S y$ and $x \in S_n$, then there is an index $k$ such that $x >_{S_k} y$. Now either $S_n \sqsubseteq S_k$ or the converse holds and in both cases $x >_{S_n} y$, because of the definition of $\sqsupset$.

We check now the conditions defining $K$.

- Condition 1 is trivial since we take a union of subsets of $A$.
- For condition 2, if $x, y \in S$ then there are indexes $i, j$ such that $x \in S_i$ and $y \in S_j$. Taking $m = max\{i, j\}$, then $x, y \in S_m$. Now, since $S_m$ satisfies condition 2, if $x > y$ then $x >_{S_m} y$ and therefore $x >_S y$.
- For condition 3, let us take $x,\ y \in S$. Again there must be indexes $i,\ j$ such that $x \in S_i$ and $y \in S_j$ and consequently $x, y \in S_m$, where $m = max\{i, j\}$. Since $S_m$ is total then either $x \geq_{S_m} y$ or the converse must hold, and in any case $x$ and $y$ are related in $S$, so $S$ is total. To see that $(S, >_S)$ is well-founded, we suppose there is an $\omega$-descending chain and then reach a contradiction. Let then $x_0 >_S x_1 >_S x_2 \ldots$ be an $\omega$-descending chain. Since $x_0 >_S x_1$ there must be an index $i_0$ such that $x_0, x_1 \in S_{i_0}$ and $x_0 >_{S_{i_0}} x_1$. Also $x_1 >_S x_2 \Rightarrow \exists i_1 :\ x_1 >_{S_{i_1}} x_2$. Since $\{S_i\}_{i \geq 0}$ is a chain, we have either $S_{i_1} \sqsubseteq S_{i_0}$ or $S_{i_0} \sqsubseteq S_{i_1}$. In the first case we have $x_1, x_2 \in S_{i_0}$ and $x_1 >_{S_{i_0}} x_2$ (see remark after the definition of $\sqsupset$). In the second case we have $x_2 <_{S_{i_1}} x_1 \in S_{i_0}$ and $S_{i_0} \sqsubseteq S_{i_1}$, so, by condition (b) in the definition of $\sqsupset$, we conclude that also $x_2 \in S_{i_0}$ and $x_2 <_{S_{i_0}} x_1$.

  By induction we prove that $\forall i \geq 0 :\ \ x_i \in S_{i_0}$ and $x_i >_{S_{i_0}} x_{i+1}$. Therefore the $\omega$-descending chain above is contained in $S_{i_0}$ contradicting its well-foundedness.

- Finally the fourth condition is also trivially verified since if $x > y$ and $x \in S$, then there is some index $i$ such that $x \in S_i$ and since $S_i$ satisfies condition $4$, we have $y \in S_i$ and therefore $y \in S$.

We have just seen that $(S, >_S) \in K$. Furthermore $(S, >_S)$ is an upper bound for the chain, so we can apply Zorn's lemma to conclude that $K$ has a maximal element we shall denote by $(\overline{S}, \succ)$.

In order to show that this is the total well-founded order we are looking for, it is enough to see that $\overline{S} = A$.

Let us suppose that that is not the case, then $A \setminus \overline{S} \neq \emptyset$. Since $>$ is well-founded in $A$, $A \setminus \overline{S}$ has a minimal element (with respect to $>$). We denote that element by $m$.

We now define $(S', >')$ by letting $S' = \overline{S} \cup \{m\}$, and $>'$ be given by:

$$x >' y \iff (x = m \text{ and } y \neq m) \text{ or } x \succ y$$

It is not difficult to see that $>'$ is well-defined and also that $(\overline{S}, \succ) \sqsubset (S', >')$. If we now can prove that $(S', >') \in K$, we reach a contradiction and therefore are able to conclude that $A = \overline{S}$. We check the conditions defining $K$.

1. Obviously $S' \subseteq A$.

2. Suppose $x, y \in S'$ and $x > y$. We have the following cases:

   - $x, y \in \overline{S}$; then $x \succ y$ (since $(\overline{S}, \succ) \in K$) implying $x >' y$.
   - $y \in \overline{S}$ and $x = m$; then $x >' y$ by definition of $>'$.
   - $y = m$ and $x \in \overline{S}$; this case cannot occur since by condition 4 of the definition of $K$ and the fact that $(\overline{S}, \succ) \in K$ we would get $m \in \overline{S}$.

3. $>'$ is total in $S'$ by definition. As for well-foundedness, suppose there is an $\omega$-descending chain $\Omega$ in $(S', >')$. If $m$ does not occur in $\Omega$ then $\Omega$ is an $\omega$-descending chain in $(\overline{S}, \succ)$ contradicting the fact that $(\overline{S}, \succ)$ is well-founded. If $m$ occurs in $\Omega$ then it occurs only once (since the chain is strictly decreasing and no element of $S'$ is greater, with respect to $>'$, than $m$); removing $m$ from $\Omega$ we get again an $\omega$-descending chain in $(\overline{S}, \succ)$, again contradicting well-foundedness of $(\overline{S}, \succ)$. Therefore no $\omega$-descending chains exist in $(S', >')$, so $(S', >')$ is well-founded.

   Alternatively, to establish well-foundedness of $(S', >')$ we can verify that every non-empty subset of $S'$ has a minimal element. Let $B$ be such a set. If $B \cap \overline{S} \neq \emptyset$ then we can take the minimal element of $B \cap \overline{S}$ in $(\overline{S}, \succ)$ since $(\overline{S}, \succ)$ is well-founded. From the definition of $(S', >')$ we see that this element is also a minimal element of $B$ in $(S', >')$. If $B \cap \overline{S} = \emptyset$ then $B = \{m\}$ and $m$ is the minimal element of $B$.

4. Suppose that $x > y$ and $x \in S'$. If $x = m$ then $y \in \overline{S}$, since $m$ is minimal in $A \setminus \overline{S}$, and then $y \in S'$. If $x \neq m$ then $x \in \overline{S}$ and since $(\overline{S}, \succ) \in K$ also $y \in \overline{S} \subseteq S'$.

We have just seen that $(S', >') \in K$ contradicting the maximality of $(\overline{S}, \succ)$. Therefore $A = \overline{S}$ and $\succ$ is a total, well-founded extension of $>$.

# A.3 Another Solution

Another way to prove this result is by using ordinals and Zermelo's Theorem. Let $>$ be a partial well-founded order on a set $A$. By Zermelo's Theorem we can ensure the existence of $\succ$, a *total* well-founded order in $A$. In general $\succ$ will not be an extension of $>$ so we have to do something else.

Let $Ord$ denote the class of ordinals well-ordered by the order $>_o$. We define the following function $\phi : A \to Ord$ by

$$\phi(x) = \left(\bigvee_{a<x} \phi(a)\right) + 1$$

where the supremum over an empty set is the unique minimal ordinal $0$ and with the operation $\alpha + 1$ giving the successor ordinal of ordinal $\alpha$.

Note that $\phi$ is well-defined and is order-preserving, i. e., it satisfies $x > y \Rightarrow \phi(x) >_o \phi(y)$, since if $x > y$ then $\phi(y) \in \{\phi(a) |\ a < x\}$ so $\phi(y) \leq_o \bigvee_{a<x} \phi(a) \Rightarrow \phi(y) <_o \left(\bigvee_{a<x} \phi(a)\right) + 1$.

In $A$ we define the relation $\sqsupset$ by

$$x \sqsupset y \iff (\phi(x) >_o \phi(y)) \text{ or } (\phi(x) = \phi(y) \text{ and } x \succ y)$$

It is not difficult to see that $\sqsupset$ is a partial order on $A$. Furthermore, since both $>_o$ and $\succ$ are well-founded on, respectively, $Ord$ and $A$, also $\sqsupset$ is well-founded on $A$: an $\omega$-descending chain with respect to $\sqsupset$ would translate into either an $\omega$-descending chain in $(Ord, >_o)$ or $(A, \succ)$, contradicting well-foundedness of these orders.

Since $\phi$ is order-preserving, we conclude that $\sqsupset$ is an extension of $>$. Finally, in order to see that $\sqsupset$ is the order we want, we need to see that it is total on $A$. Let then $x, y$ be two arbitrary different elements of $A$. Since the order $>_o$ is total in $Ord$, we must have either $\phi(x) >_o \phi(y)$, $\phi(y) >_o \phi(x)$ or $\phi(x) = \phi(y)$. In the first two cases we conclude that $x \sqsupset y$ or vice-versa, respectively. In the last case, totality of $\succ$ implies that $x \succ y$ or $y \succ x$. In any case we conclude that the elements $x$ and $y$ are related under $\sqsupset$.

# A.4 Remarks

To conclude this note, we remark that the solutions presented came out of some discussions with Hans Zantema and Luís Monteiro. Thanks to Narciso Martí-Oliet for proof-reading part of this appendix and suggestions.

We would also like to remark that in Wechler [104] (Theorem 19) another proof of theorem A.2 is given using also a mapping from the set $A$ to ordinals. The solution presented there is different from ours and is based on an enumeration of the elements of $A$, starting from an arbitrary minimal element. In that solution not only the extension is obtained but also its order type.

For more information on this topic, see Fraïssé [37].

# Appendix B

# Undecidability of termination

In the following simulation of Turing Machines by Term rewriting systems and the undecidability of termination of those are discussed. The main subject of this appendix is the uniform halting problem for which two different solutions will be given. The first approach presented is similar to the one followed by Klop [61] and differs from the approach of Huet and Lankford [47] essentially in the way the states of the machine are interpreted in the rewriting system. The second approach is new and is based on the notion of many-sorted term rewriting systems as presented in Zantema [106].

## B.1 Turing Machines

We define Turing Machines as follows:

**Definition B.1.** A Turing Machine is a tuple $M = \langle Q,\ \Sigma,\ \delta\rangle$ such that:

- $Q = \{q_0, \ldots, q_n\}$ is the set of states of the machine.
- $\Sigma = \{s_0, \ldots, s_m\}$ is a set of symbols constituting the alphabet of the machine. The symbol $s_0$ represents the blank symbol.
- $\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 1\}$ is a function giving the transitions of the machine, where $\delta(q_l, s_i) = (q_k, s_j, \sigma)$ means "if in state $q_l$ the symbol $s_i$ is read then (over)write symbol $s_j$, go to state $q_k$ and advance $\sigma$ cells in the tape" (where $\sigma \in \{-1, 1\}$).
- associated to the machine there is a tape, infinite in both directions and blank everywhere except in a finite (consecutive part) in which a word over $\Sigma$ is writen.

Some authors consider the possible moves in the tape to include $0$ (no move). The definition obtained is equivalent to the one presented. Also we omit the initial and final states since they are not relevant for our purposes.

Additionally a snapshot of the Turing Machine is characterized by the contents of the tape and the state the machine is in. Such a snapshot is called *configuration* and its definition is:

**Definition B.2.** A configuration of a Turing Machine is a triple of the form $\langle w_0, q_l, w_1\rangle$, where $w_0, w_1 \in \Sigma^*$ and $q_l \in Q$. $w_0$ and $w_1$ represent the left and the right parts of the tape respectively. The machine is in state $q_l$ and we conventionalize that the head is positioned on the first symbol of $w_1$. We denote by $\mathcal{C}$ the set of all configurations of a Turing Machine.

**Remark B.3.** If $w_1 = \lambda$ (empty word), the head of the Turing Machine has reached the right extremity of the tape. This means that all subsequent symbols are blanks, so actually $\lambda$ stands for an infinite sequence of blanks. Similarly for $w_0 = \lambda$ and the left part of the tape.

In the following we will use a dot to denote both the concatenation of elements and strings and the concatenation of strings. Since concatenation is associative the order taken is irrelevant.

The transition function $\delta$ induces a transition relation between the configurations. Namely, if $I$ and $I'$ are configurations, we say that there is a transition from $I$ to $I'$ and write $I \rightarrow_M I'$, if and only if:

- $I = \langle w_0, q_l, w_1\rangle$, $\delta(q_l, s_i) = (q_k, s_j, \sigma), \texttt{first}(w_1) = s_i$, and
  - $I' = \langle w_0 \cdot s_j, q_k, \texttt{tail}(w_1)\rangle$, for $\sigma = 1$, or
  - $I' = \langle \texttt{init}(w_0), q_k, \texttt{last}(w_0) \cdot s_j \cdot \texttt{tail}(w_1)\rangle$, for $\sigma = -1$.

where `first`, `last`, `init` and `tail` are functions from $\Sigma^*$ to $\Sigma^*$, defined as:

- `first`($\lambda$)$= s_0$ and `first`($s \cdot w$)$= s$, for $s \in \Sigma$ and $w \in \Sigma^*$.
- `last`($\lambda$)$= s_0$ and `last`($w \cdot s$)$= s$, for $s \in \Sigma$ and $w \in \Sigma^*$.
- `init`($\lambda$)$= \lambda$ and `init`($w \cdot s$)$= w$, for $s \in \Sigma$ and $w \in \Sigma^*$.
- `tail`($\lambda$)$= \lambda$ and `tail`($s \cdot w$)$= w$, for $s \in \Sigma$ and $w \in \Sigma^*$.

## B.2 Defining the TRS

We want to simulate the behaviour of the Turing Machine (TM for short) by an appropriate term rewriting system (TRS). For defining the TRS, we have to define its alphabet and its reduction rules. Then we have to translate the TM to the TRS in such a way that transitions in the TM correspond to reductions in the TRS.

We start by defining the alphabet $\mathcal{F}$ of the TRS. We need to find analogues for the symbols of $\Sigma$ and $Q$ in order to be able to represent the tape. A possible choice is to interpret the symbols of $\Sigma$ as unary symbols in the TRS, and the symbols of $Q$ as binary symbols. A justification for such a choice is that we can think of a state $q_l$ as a function of two arguments: the left and right parts (with respect to the head) of the tape. So for each state $q_i \in Q$, we define a binary function symbol $Q_i$, thus yielding a set of binary function symbols $\mathcal{F}_2 = \{Q_0, \ldots, Q_n\}$. For each symbol $s_i \in \Sigma$, we define an unary function symbol $S_i$, so we have a set of unary function symbols $\mathcal{F}_1 = \{S_0, \ldots, S_m\}$. We define also $\mathcal{F}_0$, the set of constant symbols, that contains

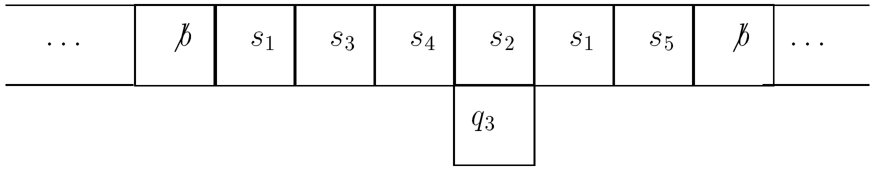


Figure B.1: A Turing Machine's tape. The symbol $\not b$ marks the beginning of the infinite blank part of the tape.

only the symbol $\Delta$, intended to represent the empty tape. Therefore the alphabet of the TRS is given by $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_0$.

Furthermore we have a set of variables $\mathcal{X}$. Now $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is, as usual, the set of terms over $\mathcal{F}$ and $\mathcal{X}$.

To be able to simulate the TM by the TRS we need some way of translating the configurations of the machine to terms in the TRS (and, of course, provide the reduction rules). Informally, a tape as the one presented in Figure B.1, which has as configuration the tuple $I = \langle s_1 \cdot s_3 \cdot s_4, q_3, s_2 \cdot s_1 \cdot s_5 \rangle$, gives rise to a term $t_I = Q_3(X_1, X_2)$, where $X_1$ codes the left part of the tape and $X_2$ the right part. For $X_2$ we expect a term like $S_2(S_1(S_5(\Delta)))$ but for $X_1$ things are a little different. Since the backward movement of the tape is from right-to-left, we are actually interested in the left part of the tape reversed. That is, we take $X_1 = S_4(S_3(S_1(\Delta)))$. So the term corresponding to this configuration is $t_I = Q_3(S_4(S_3(S_1(\Delta))), S_2(S_1(S_5(\Delta))))$.

More formally, we define the following function:

$$\begin{array}{rccc} \Gamma : & \mathcal{C} & \to & \mathcal{T}(\mathcal{F}, \mathcal{X}) \\ & \langle w_0, q_l, w_1 \rangle & \mapsto & Q_l(\tau(\gamma(w_0)^{-1}), \tau(\gamma(w_1))) \end{array}$$

where the exponent $-1$ represents string inversion, $\gamma : \{s_0, \ldots, s_n\}^* \to \mathcal{F}_1 \cup \mathcal{F}_0$ is given by:

$$\begin{array}{rcl} \gamma(\lambda) & = & \Delta \\ \gamma(s_i \cdot w) & = & S_i \cdot \gamma(w) \end{array}$$

and $\tau(w)$ is the ground term obtained from the string $w \in \mathcal{F}1^* \cdot \mathcal{F}_0$ as follows: $\tau(\Delta) = \Delta$ and $\tau(S_i \cdot w) = S_i(\tau(w))$.

The reasons behind the definition of $\mathcal{F}$ should now be clearer. The function symbols $Q_i$ represent the head of the Turing Machine, its first parameter represents the left part of the tape (and therefore is reversed) and its second parameter represents the right part of the tape.

**Remark B.4.** From the definition of $\Gamma$ it is clear that configurations are coded by only a certain set of ground terms, namely the ground terms of the form $Q_i(W_l, W_r)$ for some $Q_i \in \mathcal{F}_2$, and where $W_0, W_1$ are the images under $\tau$ of some words over $\mathcal{F}_1^* \cdot \{\Delta\}$. In the following we use the notation $t_I$ to stand for a term corresponding to the translation of a configuration $I$, that is $t_I = \Gamma(I)$.

We now introduce the rewriting rules of the TRS. These rules are derived from the transition function $\delta$ in the following way:

- If $\delta(q_i, s_j) = (q_k, s_l, 1)$, we introduce in the TRS the rules:
  - $Q_i(x, S_j(y)) \to Q_k(S_l(x), y)$,
  - $Q_i(x, \Delta) \to Q_k(S_l(x), \Delta)$, if $j = 0$.
- If $\delta(q_i, s_j) = (q_k, s_l, -1)$, then we add the rules:
  - $Q_i(S(x), S_j(y)) \to Q_k(x, S(S_l(y)))$, for each $S \in \mathcal{F}_1$,
  - $Q_i(S(x), \Delta) \to Q_k(x, S(S_l(\Delta)))$, for each $S \in \mathcal{F}_1$ and if $j = 0$,
  - $Q_i(\Delta, S_j(y)) \to Q_k(\Delta, S_0(S_l(y)))$,
  - $Q_i(\Delta, \Delta) \to Q_k(\Delta, S_0(S_l(\Delta)))$, if $j = 0$.

We will denote the TRS defined above by $\mathcal{R}_M$. The reduction relation will be denoted by $\to_{\mathcal{R}_M}$ or simply by $\to$.

It is clear from the definitions of $\to_M$ and $\to_{\mathcal{R}_M}$, that $I \to_M I' \iff t_I \to_{\mathcal{R}_M} t_{I'}$, where $I$ and $I'$ are configurations and $t_I$ and $t_{I'}$ their respective images under $\Gamma$. Consequently we have the following lemma:

**Lemma B.5.** *$M$ has an infinite sequence of configurations starting with $I_0$ if and only if $t_{I_0} = \Gamma(I_0)$ has an infinite reduction in $\mathcal{R}_M$.*

As a corollary we have:

**Corollary B.6.** *Given a term $t$ and a TRS $R$, the problem of determining if $t$ admits infinite reductions in $R$ is undecidable.*

**Proof** Given a Turing Machine $M$ and input tape $I$ , the problem of determining if $M$ stops with input $I$ reduces to determine if $t_I$ has no infinite reductions in $\mathcal{R}_M$. Since the former problem is undecidable, so is the second. □

The previous results does not allow us to conclude anything on the uniform halting problem for TRS's since not all terms in $\mathcal{R}_M$ correspond to valid configurations of the machine. However, as we will see, a general term can be viewed as a set of replicas of the same Turing Machine, each of them acting on a different tape.

A general term in $\mathcal{R}_M$ may contain more than one $Q$ symbol, looking something like:

$$S_{i_1}(\dots S_{i_k}(Q_{j_1}(\dots Q_{j_2}(\dots Q_{j_3}(\dots)), \dots Q_{j_l}(\dots))))$$

where $i_k, j_l \geq 0$ and where ... may contain $S$ and/or $Q$ symbols.

We want to identify and separate in such a term the parts which relate to valid configurations of the machine, that is, subterms of the form

$$S_{i_1}(\dots S_{i_k}(\sigma)\dots) \text{ and } Q_j(S_{l_1}(\dots S_{l_p}(\sigma_l)\dots), S_{r_1}(\dots S_{r_s}(\sigma_r)\dots))$$

where $\sigma, \sigma_l, \sigma_r \in \{\Delta\} \cup \mathcal{X}$ and $k, p, s \geq 0$.

First we introduce the notions of S-term and Q-term. Intuitively, S-terms are terms which consist of zero or more applications of $S_i$ symbols to the constant $\Delta$, and Q-terms are terms which have one and only one occurrence of a $Q_j$ symbol and whose arguments are S-terms. More formally we define:

$$\begin{array}{lcl} \mathsf{S} & = & \{t \in \mathcal{T}(\mathcal{F}) | \; t = S_{i_1}(\dots S_{i_k}(\Delta)\dots), \text{ for some } k \geq 0\} \\ \mathsf{Q} & = & \{t \in \mathcal{T}(\mathcal{F}) | \; t = Q_l(T_1, T_2), \text{ for some } Q_l \in \mathcal{F}_2 \text{ and } T_1, T_2 \in \mathsf{S}\} \end{array}$$

We now define a function that will allow us to break a term into its S and Q-constituents. Let then $f = (f_1, f_2) : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to \mathsf{S} \times \mathcal{M}(\mathsf{Q})$ (where $\mathcal{M}$ represents the finite multisets[1]), be defined inductively as follows:

- $f(\sigma) = (\Delta, \emptyset)$, if $\sigma \in \{\Delta\} \cup \mathcal{X}$.
- $f(S_i(w)) = (S_i(u), P)$, if $f(w) = (u, P)$.
- $f(Q_j(w_l, w_r)) = (\Delta, \{\!\{Q_j(u_l, u_r)\}\!\} \sqcup P_l \sqcup P_r)$, if $f(w_r) = (u_r, P_r)$ and $f(w_l) = (u_l, P_l)$.

The idea behind this definition is to get in the second coordinate the possible configurations contained in the term, the first coordinate being used to build the arguments of those configurations.

**Example B.7.** Let the term $t$ be

$$S_k(S_r(Q_i(S_s(Q_j(S_l(\Delta), \Delta)), Q_t(y, S_u(\Delta)))))$$

This term is showed in Figure B.2 and gives the following partition: $f(t) = (S_k(S_r(u)), P)$, and

$$\begin{array}{lcl} (u, P) & = & f(Q_i(S_s(Q_j(S_l(\Delta), \Delta)), Q_t(y, S_u(\Delta)))) \\ & = & (\Delta, \{\!\{Q_i(u_1, u_2)\}\!\} \sqcup P_1 \sqcup P_2) \end{array}$$

where $(u_1, P_1) = f(S_s(Q_j(S_l(\Delta)), \Delta))$ and $(u_2, P_2) = f(Q_t(y, S_u(\Delta))) = (\Delta, \{\!\{Q_t(u_3, u_4)\}\!\} \sqcup P_3 \sqcup P_4)$. But

$$\begin{array}{l} (u_3, P_3) = f(y) = (\Delta, \emptyset) \\ (u_4, P_4) = f(S_u(\Delta)) = (S_u(\Delta), \emptyset). \end{array}$$

Also

$$(u_1, P_1) = f(S_s(Q_j(S_l(\Delta), \Delta))) = (S_s(\Delta), \{\!\{Q_j(u_5, u_6)\}\!\} \sqcup P_5 \sqcup P_6)$$

[1]Since we have to account for possible repetition of Q-terms, sets are not enough but actually we don't need to use multisets. Lists, trees or (meta)words, for example, would also solve our problem.

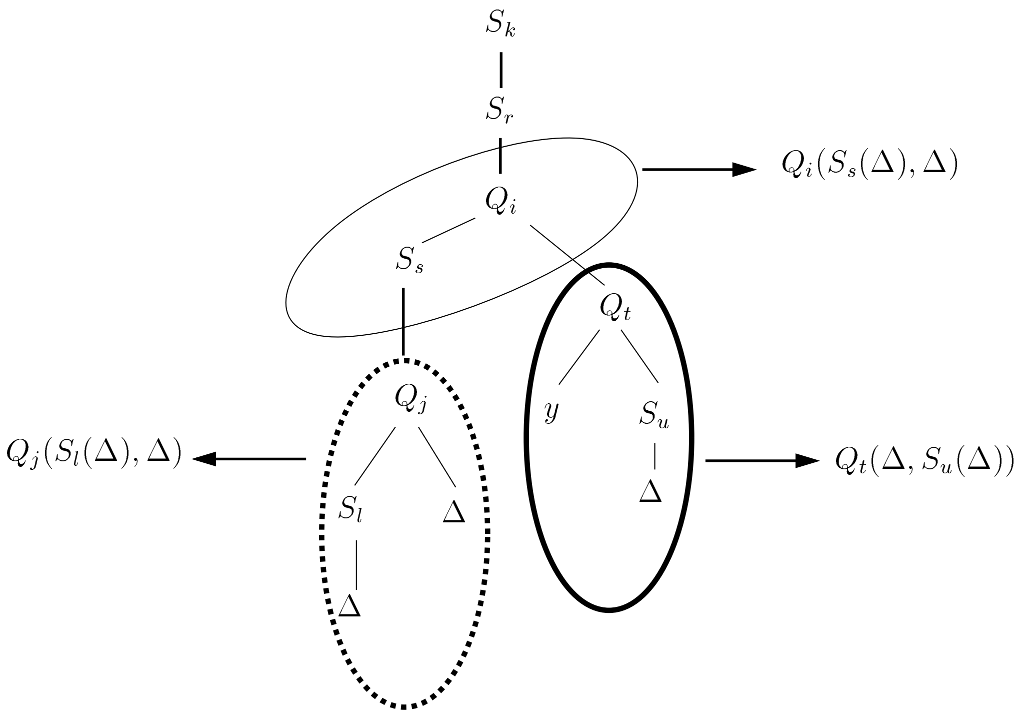



Figure B.2: Partitioning a term.

where

$$(u_5, P_5) = f(S_l(\Delta)) = (S_l(\Delta), \emptyset)$$
$$(u_6, P_6) = f(\Delta) = (\Delta, \emptyset).$$

Making all the replacements we get

$$f(t) = (S_k(S_r(\Delta)), \{\!\{Q_i(S_s(\Delta), \Delta), Q_j(S_l(\Delta), \Delta), Q_t(\Delta, S_u(\Delta))\}\!\})$$

We have the following lemma:

**Lemma B.8.** *Suppose* $t \to_{\mathcal{R}_M} t'$ *then there are* Q*-terms* $q \in f_2(t)$, $q' \in f_2(t')$ *such that* $q \to_{\mathcal{R}_M} q'$. *Furthermore* $f_2(t) \setminus \{\!\{q\}\!\} = f_2(t') \setminus \{\!\{q'\}\!\}$.

**Proof** Suppose $t \to_{\mathcal{R}_M} t'$, then there is a subterm $u$ of $t$ such that $u$ is of the form $Q_i(t_a, t_b)$ matching one of the rewriting rules.

It is easily seen by induction on the structure of a term that if $u$ is a subterm of $t$ then $f_2(u) \sqsubseteq f_2(t)$. So we have $f_2(Q_i(t_a, t_b)) \sqsubseteq f_2(t)$. But $f_2(Q_i(t_a, t_b)) = \{\!\{Q_i(u_a, u_b)\}\!\} \sqcup P_a \sqcup P_b$, where $(u_a, P_a) = f(t_a)$ and $(u_b, P_b) = f(t_b)$.

It can be seen by case analysis that the following diagram holds:

$$u = \quad Q_i(t_a, t_b) \quad \stackrel{r}{\to} \quad Q_k(t'_a, t'_b) \quad = u'$$

$$Q_i(u_a, u_b) \quad \stackrel{r}{\to} \quad Q_k(u'_a, u'_b)$$

that is, if $Q_i(t_a, t_b)$ matches a rule $r$ then $Q_i(u_a, u_b)$ matches the same rule and its reduct $Q_k(u'_a, u'_b)$ is in $f_2(Q_k(t'_a, t'_b))$.

We will check it for the rules of the type:

$$Q_i(x, S_j(y)) \to Q_k(S_l(x), y), \text{ for any } j \in \{0, \ldots, m\}$$

For the other rules it is done in a similar way.

Suppose then that $Q_i(t_a, t_b)$ matches $Q_i(x, S_j(y))$. Then there is a substitution $\sigma$ such that $\sigma(x) = t_a$ and $S_j(\sigma(y)) = t_b$. Thus we can write $(u_a, P_a) = f(\sigma(x))$ and $(u_b, P_b) = f(S_j(\sigma(y))) = (S_j(w), P_w)$ where $(w, P_w) = f(\sigma(y))$. Consequently $Q_i(u_a, u_b) = Q_i(u_a, S_j(w))$ and this term matches the same rule with substitution $\sigma'$ satisfying $\sigma'(x) = u_a$, $\sigma'(y) = w$.[2] Applying the rule on both redexes, we get the resulting terms $Q_k(S_l(t_a), \sigma(y))$ and $Q_k(S_l(u_a), w)$. Furthermore

$$f(Q_k(S_l(t_a), \sigma(y))) = (\Delta, \{\!\{Q_k(S_l(u_a), w)\}\!\} \sqcup P_a \sqcup P_w).$$

So $Q_k(S_l(u_a), w) \in f_2(Q_k(S_l(t_a), \sigma(y))) \sqsubseteq f_2(t')$.

Recalling that $f_2(Q_i(t_a, t_b)) = \{\!\{Q_i(u_a, u_b)\}\!\} \sqcup P_a \sqcup P_w$ (since $P_b = P_w$), we have that $f_2(u) \setminus \{\!\{Q_i(u_a, u_b)\}\!\} = f_2(u') \setminus \{\!\{Q_k(S_l(u_a), w)\}\!\}$. Note that also $f_1(u) = f_1(u')$.

We still have to see that $f_2(t)$ and $f_2(t')$ differ only in the redex and reduct (in the following this will be denoted by $f_2(t) \rightsquigarrow f_2(t')$). We will see then that:

$$t \to_{\mathcal{R}_M} t' \quad \Rightarrow \quad f_1(t) = f_1(t') \text{ and } f_2(t) \rightsquigarrow f_2(t') \qquad \text{(B.3)}$$

Let $u$ be the reduced subterm, that is $t = C[u]$ and $t' = C[u']$ with $u \to u'$. We prove B.3 by induction on the context. If $C = \square$ then the result holds (as seen in the previous case analysis). If $t = S_i(C[u])$ then $f(t) = f(S_i(C[u])) = (S_i(a), P_a)$ where $(a, P_a) = f(C[u])$. Also $t' = S_i(C[u'])$ and $f(t') = (S_i(a'), P_{a'})$ where $(a', P_{a'}) = f(C[u'])$. By induction hypothesis $a = a'$ and $P_a \rightsquigarrow P_{a'}$, so the result follows.

If $t = Q_i(t_a, C[u])$ then $f(t) = (\Delta, \{\!\{Q_i(u_a, u_b)\}\!\} \sqcup P_a \sqcup P_b)$, where $(u_a, P_a) = f(t_a)$ and $(u_b, P_b) = f(C[u])$. Also $t' = Q_i(t_a, C[u'])$ and $f(t') = (\Delta, \{\!\{Q_i(u_a, u'_b)\}\!\} \sqcup P_a \sqcup P_{b'})$, with $(u'_b, P_{b'}) = f(C[u'])$. Again by induction hypothesis, $u_b = u_{b'}$ and $P_b \rightsquigarrow P_{b'}$, giving the result. The case $t = Q_i(C[u], t_a)$ is symmetrical. $\square$

Now if $t$ is a ground term which has an infinite reduction then one of the Q-terms of its decomposition must also be rewritten infinitely many times, since the number of Q-terms in the decomposition is finite from the start and that number remains constant during the reductions. So we can state:

**Lemma B.9.** *If $t$ is a ground term that has an infinite reduction then there is a term $q \in f_2(t)$ that has an infinite reduction.*

[2]Note that there are no variable clashes since both $u_a$ and $w$ are ground terms.

If $t$ is a general term that has an infinite reduction, then $t$ also has an infinite ground reduction, so the following result holds.

**Theorem B.10.** *The TRS $\mathcal{R}_M$ is terminating if and only if $M$ stops on any input.*

Consequently we have:

**Corollary B.11.** *Termination of Term rewriting systems is undecidable.*

## B.3 Using Many-sorted TRS's

As remarked before not all terms in $\mathcal{R}_M$ correspond to valid configurations in $M$. Following the approach of Zantema [106], we can go around this problem by defining a sorted TRS such that the well-defined terms in this TRS contain at most one $Q$ symbol. First we introduce some basic notions from sorted TRS's (all this notions can be found in Zantema [106]).

Let $\mathcal{S}$ be a set of sorts and $\mathcal{X}_\mathcal{S}$ an $\mathcal{S}$-sorted set of variables. Let $\mathcal{F}$ be a set of function symbols such that with each symbol there is associated a sort and an arity, given respectively by the functions:

$$\begin{aligned} \texttt{st}: &\ \mathcal{F} \to \mathcal{S} \\ \texttt{ar}: &\ \mathcal{F} \to \mathcal{S}^* \end{aligned}$$

The $\mathcal{S}$-sorted set of terms, denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})$, is defined by:

- $\mathcal{X}_a \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_a$, for any $a \in \mathcal{S}$ (we remark that all sets $\mathcal{X}_a$ are disjoint),
- $f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_a$, for $f \in \mathcal{F}$ with $\texttt{ar}(f) = s_1 \cdots s_k$, $\texttt{st}(f) = a$ and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_{s_i}$, for $i = 1, \ldots, k$.

Note also that substitutions now have to respect the sort of variables, i. e., a substitution $\sigma$ is a function $\sigma : \mathcal{X}_\mathcal{S} \to \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})$ such that $\sigma(\mathcal{X}_a) \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_a$ for all sorts $a \in \mathcal{S}$.

An $\mathcal{S}$-sorted TRS has an $\mathcal{S}$-sorted set of rules $R_\mathcal{S} = \bigcup_{s \in \mathcal{S}} R_s$ such that $R_s \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_s \times \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_s$, for any $s \in \mathcal{S}$. In the following we denote this rewrite system, as well as its rewrite relation, by $R_\mathcal{S}$.

**Definition B.12.** The rewrite (or reduction) relation of an $\mathcal{S}$-sorted TRS $R$ is the $\mathcal{S}$-sorted relation $\to_{R_\mathcal{S}} = \bigcup_{s \in \mathcal{S}} \to_{R_s}$ on $\mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})$, where $\to_{R_s}$ is given by:

- $l\sigma \to_{R_s} r\sigma$, for every rule $(l, r) \in R_s$ and every substitution $\sigma$.
- $f(t_1, \ldots, t_k, \ldots, t_n) \to_{R_s} f(t_1, \ldots, t'_k, \ldots, t_n)$, for every $f \in \mathcal{F}$ with $\texttt{ar}(f) = s_1 \cdots s_k$, $\texttt{st}(f) = s$, $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_{s_i}$, $i = 1, \ldots, n$, $t'_k \in \mathcal{T}(\mathcal{F}, \mathcal{X}_\mathcal{S})_{s_k}$, and $t_k \to_{R_{s_k}} t'_k$.

Now let us build our particular sorted TRS. In our case $\mathcal{S} = \{s, q\}$ and $\mathcal{X}_\mathcal{S}$ is the $\mathcal{S}$-sorted set of variables such that $\mathcal{X}_q = \emptyset$. The set of operation symbols is given by $\mathcal{F} = \{Q_0, \ldots, Q_n, S_0, \ldots, S_m, \Delta\}$ with the following arities and sorts:

- $\mathtt{ar}(\Delta)=\lambda$; $\mathtt{st}(\Delta)=s$,
- $\mathtt{ar}(Q_i)=s\cdot s$; $\mathtt{st}(Q_i)=q$, for all $0\leq i\leq n$,
- $\mathtt{ar}(S_i)=s$; $\mathtt{st}(S_i)=s$, for all $0\leq i\leq m$.

We have only two kinds of terms, namely terms of sort $s$ and terms of sort $q$, such that

- $t\in\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_s \iff t=S_{i_1}(\dots S_{i_k}(\sigma)\dots)$, for $k\geq 0$ and $\sigma\in\{\Delta\}\cup\mathcal{X}_s$,
- $t\in\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_q \iff t=Q_i(t_1,t_2)$, for some $Q_i\in\mathcal{F}$ and $t_1,t_2\in\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_s$.

As for the reduction rules, our sorted system has two set of rules $R_s$ and $R_q$ with $R_s=\emptyset$ and $R_q=\mathcal{R}_M=R_{\mathcal{S}}\subseteq\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_q\times\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_q$.

The $\mathcal{S}$-sorted TRS $R_{\mathcal{S}}$ is terminating if and only if for every $a\in\mathcal{S}$, there is no infinite reduction of the reduction relation $\rightarrow_{R_a}$. In this case since $R_s=\emptyset$, $\rightarrow_{R_s}$ gives no reductions, so the system is terminating if and only if $\rightarrow_{R_q}$ has no infinite reductions.

Now associated with any sorted TRS $R$, there is a one-sort TRS obtained from the sorted version by ignoring the sort information. This new TRS, that we denote by $\Theta(R)$, is given by:

- $\mathcal{F}'=\{f'|f\in\mathcal{F}\}$
- $\mathcal{X}'=\bigcup_{s\in\mathcal{S}}\mathcal{X}_s$
- $(l',r')\in\Theta(R) \iff (l,r)\in\bigcup_{s\in\mathcal{S}}R_s$ and $l'=\Theta(l)$ and $r'=\Theta(r)$, where $\Theta:\bigcup_{s\in\mathcal{S}}\mathcal{T}(\mathcal{F},\mathcal{X}_{\mathcal{S}})_s\rightarrow\mathcal{T}(\mathcal{F}',\mathcal{X}')$ is defined inductively by:
  - $\Theta(x)=x$, for all $x\in\mathcal{X}_s$, for all $s\in\mathcal{S}$ (this definition poses no problem since all sets $\mathcal{X}_s$ are disjoint).
  - $\Theta(f(t_1,\dots,t_n))=f'(\Theta(t_1),\dots,\Theta(t_n))$, for all $f\in\mathcal{F}$ and terms $t_i$, $1\leq i\leq n$ ($n\geq 0$), of the appropriate sort.

We remark that $t\rightarrow_{R_s}t' \iff \Theta(t)\rightarrow_{\Theta(R)}\Theta(t')$.

From Zantema [106] we know that if $R$ is an $\mathcal{S}$-sorted TRS without collapsing rules then $R$ is terminating if and only if $\Theta(R)$ is terminating. Since by construction we have $\mathcal{R}_M=\Theta(R_{\mathcal{S}})$, we can establish our main result:

**Theorem B.13.** *$\mathcal{R}_M$ is terminating if and only if $M$ stops on any input $I$.*

**Proof** From the above observation and since $R_{\mathcal{S}}$ contains no collapsing rules, we know that $\mathcal{R}_M$ is terminating if and only if $R_{\mathcal{S}}$ is terminating. So it suffices to establish that

$R_{\mathcal{S}}$ is terminating if and only if $M$ stops on any input $I$

and this is an easy consequence of the following facts:

- for any configuration $I$, $\Gamma(I)$ is a $R_{\mathcal{S}}$-term of sort $q$,

- any reducible $R_{\mathcal{S}}$-term has to have the form

$$Q_i(S_{l_1}(\dots S_{l_k}(\sigma_l)\dots), S_{r_1}(\dots S_{r_p}(\sigma_r)\dots))$$

where $k, p \geq 0$, $S_{l_i}, S_{r_j} \in \mathcal{F}$ for any $1 \leq i \leq k$ and $1 \leq j \leq p$, and $\sigma_l, \sigma_r \in \{\Delta\} \cup \mathcal{X}_s$. Any ground instance of such a term corresponds to a configuration of the Turing Machine.

□

## B.4 Huet and Lankford's approach

We will comment briefly on the approach presented in Huet and Lankford [47] comparing it with the approach presented here.

Essentially these approaches differ in the way the symbols from the alphabet of the Turing Machine are interpreted in the TRS. In Huet and Lankford's approach, the TRS contains no binary symbols, both states of the machine and symbols of the alphabet are interpreted as unary function symbols. However it is still necessary to distinguish between the right and the left parts of the tape and for that purpose a direction is introduced in the tape symbols. That is, for each $s_i \in \Sigma$ two symbols are created in $\mathcal{F}$, namely $\overleftarrow{S_i}$ and $\overrightarrow{S_i}$ representing, respectively, an occurrence of $s_i$ to the right and to the left of the head of the Turing Machine. The signature $\mathcal{F}$ also contains another unary function symbol $L$ representing the left part of the tape, and a constant $R$ representing the right side of the tape (in our model both $L$ and $R$ collapse to $\Delta$). Thus the alphabet $\mathcal{F}$ is given by $\mathcal{F}_1 \cup \mathcal{F}_0$ where:

- $\mathcal{F}_1 = \{Q_0, \dots, Q_n, \overleftarrow{S_0}, \dots, \overleftarrow{S_m}, \overrightarrow{S_0}, \dots, \overrightarrow{S_m}, L\}$,
- $\mathcal{F}_0 = \{R\}$.

Configurations are defined in the same way but their correspondence with terms is different as the following example shows.

**Example B.14.** The configuration $I = \langle s_1 \cdot s_3 \cdot s_4, q_3, s_2 \cdot s_1 \cdot s_5\rangle$, from Figure B.1, gives rise to the term

$$t_I = L(\overrightarrow{S_1}\ (\overrightarrow{S_3}\ (\overrightarrow{S_4}\ (Q_3(\overleftarrow{S_2}\ (\overleftarrow{S_1}\ (\overleftarrow{S_5}\ (R)))))))$$

In general a configuration $I = \langle w_0, q_j, w_1\rangle$ will give rise to a term

$$t_I = L(\overrightarrow{W_0}\ (Q_j(\overleftarrow{W_1}\ (R)\dots),$$

where $\overrightarrow{W_0}$ consists of the symbols of $w_0$ in capitals and with the rightarrow on top, and similarly for $\overleftarrow{W_1}$.

The rules of the TRS have a somewhat simpler formulation:

- if $\delta(q_i, s_j) = (q_k, s_l, 1)$, we introduce in the TRS the rules:
  - $Q_i(\overleftarrow{S_j}(x)) \rightarrow \overrightarrow{S_l}(Q_k(x))$,
  - $Q_i(R) \rightarrow \overrightarrow{S_l}(Q_k(R))$, if $j = 0$.
- if $\delta(q_i, s_j) = (q_k, s_l, -1)$, we introduce in the TRS the rules:
  - $\overrightarrow{S_r}(Q_i(\overleftarrow{S_j}(x))) \rightarrow Q_k(\overleftarrow{S_r}(\overleftarrow{S_l}(x)))$, for all $r \in \{0, \ldots, m\}$,
  - $\overrightarrow{S_r}(Q_i(R)) \rightarrow Q_k(\overleftarrow{S_r}(\overleftarrow{S_l}(R)))$, if $j = 0$ and for any $r \in \{0, \ldots, m\}$,
  - $L(Q_i(\overleftarrow{S_j}(x))) \rightarrow L(Q_k(\overleftarrow{S_0}(\overleftarrow{S_l}(x))))$,
  - $L(Q_i(R)) \rightarrow L(Q_k(\overleftarrow{S_0}(\overleftarrow{S_l}(R))))$, if $j = 0$.

We also have a "reduction" relation on configurations induced by the transition function $\delta$, and it also holds that $I \rightarrow_M I' \iff t_I \rightarrow_R t_{I'}$.

Lemma 3 holds, giving us undecidability of termination for TRS and a given term. Undecidability of termination is a consequence of the following lemma:

**Lemma B.15.** *If $t$ has an infinite reduction in $\mathcal{R}_M$ then there is a configuration $I$ admiting an infinite reduction in $M$.*

Before concluding the section we want to remark that we can not interpret the symbols of $Q$ and $\Sigma$ as unary symbols and forget about the position in which those symbols appear, thus reducing $\mathcal{F}_1$ to the set $\{Q_0, \ldots, Q_n, S_0, \ldots, S_m, L\}$, as the following example shows.

**Example B.16.** Suppose we have the following Turing Machine:

- $\delta(q_0, 0) = (q_1, 1, -1)$,
- $\delta(q_0, 1) = (q_2, 0, 1)$,
- $\delta(q_1, \sigma) = (q_3, \sigma, 1)$, for $\sigma \in \{0, 1\}$,
- $\delta(q_2, \sigma) = (q_4, \sigma, -1)$, for $\sigma \in \{0, 1\}$,
- $\delta(q_3, 0) = (q_1, 1, -1)$,
- $\delta(q_4, 1) = (q_2, 0, 1)$.

The alphabet and the states are built from the symbols appearing in the definition above.

"Operationally", the effect of the rules defined above is to change 1 into 0 and vice-versa in certain states while maintaining the value in other states. The initial state is $q_0$.

It is easy to see that this Turing machine terminates for any input. Now consider the derived TRS, with the rules:

1. $0(Q_0(0(x))) \to Q_1(0(1(x)))$,
2. $1(Q_0(0(x)) \to Q_1(1(1(x)))$,
3. $Q_0(1(x)) \to 0(Q_2(x))$,
4. $Q_1(0(x)) \to 0(Q_3(x))$,
5. $Q_1(1(x)) \to 1(Q_3(x))$,
6. $0(Q_2(x)) \to Q_4(0(x))$,
7. $1(Q_2(x)) \to Q_4(1(x))$,
8. $0(Q_3(0(x))) \to Q_1(0(1(x)))$,
9. $1(Q_3(0(x))) \to Q_1(1(1(x)))$,
10. $Q_4(1(x)) \to 0(Q_2(x))$.

It is also easy to see that any term with at most one occurrence of a $Q$ symbol has only finite rewritings, but for a term with more than one such occurrence that is not necessarily so. For instance, the rewriting sequence (where $\stackrel{i}{\to}$ indicates that the rule used in the reduction step was rule $i$):

$$\begin{aligned} 0(Q_3(1(Q_2(0(x))))) &\stackrel{7}{\to} 0(Q_3(Q_4(1(0(x))))) \stackrel{10}{\to} 0(Q_3(0(Q_2(0(x))))) \\ &\stackrel{8}{\to} Q_1(0(1(Q_2(0(x))))) \stackrel{4}{\to} 0(Q_3(1(Q_2(0(x))))) \end{aligned}$$

is a cyclic sequence and therefore infinite.

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# Index

# Samenvatting

Dit proefschrift gaat over eigenschappen van terminatie van herschrijfsystemen. We zullen eerst aan de hand van een voorbeeld dat gaat over optellen van de getallen de belangrijkste concepten proberen uit te leggen. Vervolgens vatten we de inhoud van het proefschrift kort samen.

Een bekende eigenschap van optellen ("add") is dat optellen met het getal nul een neutrale bewerking is. We kunnen dit in een vergelijking als volgt formuleren:

$$\mathsf{add}(0,x) \;=\; x; \qquad \mathsf{add}(x,0) \;=\; x$$

waarbij $x$ een variabele is die een willekeurig natuurlijke getal voorstelt.

Een andere optelwet gaat over de volgorde van berekeningen: $\mathsf{add}(x,\mathsf{add}(y,1))$ kan ook worden verkregen door eerst $\mathsf{add}(x,y)$ te berekenen en dan bij dit resultaat $1$ op te tellen. Stellen we de natuurlijke getallen voor door $0$, $s(0)$, $s(s(0))$, etc. ($s$ betekent "successor") dan luidt deze wet in formulevorm:

$$\mathsf{add}(x,s(y)) \;=\; s(\mathsf{add}(x,y))$$

Op analoge wijze vinden we ook de vergelijking:

$$\mathsf{add}(s(x),y) \;=\; s(\mathsf{add}(x,y))$$

Nu zijn we in staat om $1+2$ op formele wijze te berekenen. Daartoe schrijven we eerst $1+2$ als $\mathsf{add}(s(0),s(s(0)))$ en proberen we vervolgens de laatste formule te vereenvoudigen door optelwetten toe te passen. Een mogelijke manier om dit te doen is als volgt:

$$\begin{aligned} \mathsf{add}(s(0),s(s(0))) &= s(\mathsf{add}(s(0),s(0))) \\ &= s(s(\mathsf{add}(s(0),0))) \\ &= s(s(s(0))) \end{aligned}$$

In dit voorbeeld zien we duidelijk dat de vergelijkingen voor add in een bepaalde richting werden gebruikt. Dit "gericht" gebruik van vergelijkingen heet herschrijven.

In het algemeen herschrijven we termen uit een zg. termalgebra. Een termalgebra wordt verkregen uit een gegeven verzameling van variabelen, zeg $\mathcal{X}$, en een verzameling van functiesymbolen, zeg $\mathcal{F}$. Bij elk functiesymbool hoort een natuurlijke getal, de ariteit, die het aantal argumenten van het functiesymbool aangeeft. Variabelen hebben ariteit nul.

Termen worden inductief opgebouwd door functiesymbolen toe te passen op andere termen. Uiteraard dient hierbij de ariteit te worden gerespecteerd. De verzameling van termen wordt genoteerd door $\mathcal{T}(\mathcal{F},\mathcal{X})$. In het optelvoorbeeld geldt: "$s$" heeft ariteit $1$, "add" heeft ariteit $2$ en "$0$" heeft ariteit $0$, en $\mathsf{add}(x,s(y))$ is een term van de termalgebra $\mathcal{T}(\{\mathsf{add},s,0,\},\{x,y\})$.

Nu we weten wat de objecten zijn waarmee we herschrijven kunnen we afspreken hoe we gaan herschrijven. We gaan ervan uit dat we een aantal vergelijkingen hebben waarin we een linker- en een rechterdeel onderscheiden bestaande uit termen van een termalgebra. Een vergelijking $l = r$ leidt tot de herschrijfregel $l \to r$, waarbij de pijl de richting van het gebruik van de vergelijking aangeeft. Zo'n verzameling regels heet een termherschrijfsysteem (afgekort tot TRS). Bijvoorbeeld:

$$
\begin{aligned}
\mathsf{add}(0,x) &\to x && (1)\\
\mathsf{add}(x,0) &\to x && (2)\\
\mathsf{add}(x,s(y)) &\to s(\mathsf{add}(x,y)) && (3)\\
\mathsf{add}(s(x),y) &\to s(\mathsf{add}(x,y)) && (4)
\end{aligned}
$$

is een TRS.

Een TRS induceert een herschrijfrelatie $\to_R$ (of $\to$) in de verzamelingen van termen. Een term $s$ herschrijft tot een term $t$ (notatie $s \to_R t$) indien we in $s$ een deel $g$ herkennen dat correspondeert met een linkerdeel van een regel in $R$ én $t$ wordt uit $s$ verkregen door $g$ te vervangen door het overeenkomstige rechterdeel van de gevonden regel.

Veronderstel dat we de term $\mathsf{add}(s(0),s(0))$ willen herschrijven. We kunnen bovenstaande regels (3) en (4) gebruiken. Natuurlijke leidt dit tot de vraag: "wanneer verschillende regels toegestaan zijn, is het uiteindelijke resultaat onafhankelijk van de gekozen regel?". Systemen waarvoor het antwoord "ja" is voldoen aan de *Church-Rosser eigenschap*; ze worden ook wel *confluent* genoemd. Niet elk TRS is echter confluent.

Een andere belangrijk vraag is: "als we een willekeurige term herschrijven, kunnen we dan garanderen dat we na een eindig aantal herschrijvingen tot een term komen waarop geen regel toepasbaar is (normaal vorm)?". In het algemeen is hierop geen bevestigend antwoord te geven. Systemen die gegarandeerd leiden tot een normaalvorm heten *terminerend*. Terminatie is onbeslisbaar, d.w.z. er is geen procedure die uitsluitsel geeft over terminatie van een TRS. Desalniettemin bestaan er vele bruikbare methoden die behulpzaam zijn bij het geven van een bewijs van terminatie. Ruwweg kunnen we twee soorten methoden onderscheiden (beiden worden in dit proefschrift behandeld):

- syntactische methoden,
- semantische methoden.

De syntactische methoden maken alleen gebruik van de syntactische structuur van termen om tot een terminatie uitspraak te komen. Voorbeeld van deze methoden zijn de zogenaamde pad ordeningen. In de semantische methoden worden termen compositioneel geïnterpreteerd in een algebra om zo terminatie te bewijzen; dit betekent dat we een verzamelingen $A$, een partiële ordening $>$, en operaties $f_A$ voor elke functiesymbool $f$ in $\mathcal{F}$, moeten definiëren. Elke term in $A$ kan worden geïnterpreteerd door een toekenning van waardes van $A$ aan variabelen, De ordening $>$ mits welgefundeerd kan worden gebruikt om terminatie van het systeem te bewijzen.

Beiden soorten van methoden gebruiken in essentie het concept "welgefundeerde ordening". Een ordening is een binaire relatie $>$ (lees groter dan) met de eigenschappen irreflexibiliteit ($s \not> s$ voor elke $s$, d.w.z. geen element is groter dan zichzelf) en transitiviteit (als $s > t$ en $t > u$ dan ook $s > u$). Een welgefundeerde ordening is een ordening $>$ waarin geen oneindige rijen van de vorm $s_0 > s_1 > s_2 > \ldots$, bestaan. Als voor een TRS $R$ een welgefundeerde ordening $>$ bestaat zodanig dat uit $s \to_R t$ volgt $s > t$ dan termineert $R$. Dus welgefundeerdheid van ordeningen is een zeer belangrijk en relevant onderwerp in de studie van terminatie.

Zoals reeds eerder gezegd gaat dit proefschrift over terminatie van herschrijfsystemen.

In hoofdstuk 1 beschrijven we in het kort de concepten TRS en terminatie van TRS.

Hoofdstuk 2 bevat een uitvoerige samenvatting van definities, notaties and resultaten opdat het proefschrift op zichzelf staande is.

In hoofdstuk 3 bestuderen we welgefundeerdheid van ordeningen gedefinieerd op de verzameling van termen. Welgefundeerdheid van ordeningen is in het algemeen moeilijk te bewijzen, het is daarom gewenst om eenvoudige criteria te hebben die de welgefundeerdheidseigenschap kunnen controleren. Zulke criteria worden in dit hoofdstuk gegeven, waardoor welgefundeerheid van bekende ordeningen zoals *rpo* geconcludeerd kan worden. Een belangrijk voordeel van deze criteria is dat ze gelden voor alle terminerende TRSen in tegenstelling tot bv. de stelling van Kruskal.

Hoofdstuk 4 is verdeeld in twee delen. In het eerste deel bestuderen we het algemene probleem van het definiëren van recursieve pad ordeningen op termen. In het tweede deel kijken we naar een andere belangrijke eigenschap van ordeningen nl. totaliteit. Totale ordeningen hebben de eigenschap dat elk tweetal (verschillende) elementen uit de verzameling waarover de ordening is gedefinieerd vergelijkbaar is. We tonen aan dat bekende ordeningen zoals *rpo* of *kbo* in essentie totaal zijn. Dit betekent dat TRSen waarvan we terminatie met deze ordeningen kunnen bewijzen ook geïnterpreteerd kunnen worden in totale welgefundeerde monotone algebra's. Dit type terminatie noemen we totale terminatie.

In hoofdstuk 5 gaan we verder in op totale terminatie. We kijken naar eigenschappen van algebra's die in bewijzen voor totale terminatie gebruikt kunnen worden. Het blijkt dat de interessante algebra's precies gekarakteriseerd kunnen worden, nl. ze zijn algebra's die overeenkomen met multiverzamelingen over een verzameling. We gebruiken in dit hoofdstuk eigenschappen van de ordinalen en we zijn in staat om enige interessante resultaten over TRSen af te leiden.

In het laatste hoofdstuk introduceren we enige transformaties gedefinieerd op termen (en dus ook op TRSen) die de taak om terminatie te bewijzen vergemakkelijken. Een gegeven TRS kan worden getransformeerd tot een nieuwe TRS met in het algemeen meer regels maar met een eenvoudigere syntactische structuur. We bewijzen de opmerkelijke eigenschap dat terminatie van de oorspronkelijke TRS volgt uit terminatie van de getransformeerde TRS. Dit geeft een techniek die eenvoudig aan bestaande automatische terminatiebewijssystemen kan worden toegevoegd die daarmee terminatie van meer TRSen kunnen bewijzen. Deze techniek blijft van toepassing voor herschrijven modulo vergelijkingen.

In de appendix geven we bewijzen van bekende resultaten. Deze bewijzen zijn of nieuw of bestaand maar dan moeilijk te vinden in de literatuur. Daardoor is dit proefschrift meer op zichzelf staand.

# Curriculum Vitae

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