Chapter 2

Getting Started

The aim of this chapter is to introduce the mathematical objects we are going to study in this thesis. These are defined here, and various notation and conventions that I follow are explained.

2.1 Definitions

The first step is to introduce the diophantine equations we are going to study and give the definition of a parameterization.

Definition 2.1.1 ($\mathcal{D}(r,d)$). *Let R be a ring. For any $r \in \{3,4,5\}$ and any $d \in R$ we define $\mathcal{D}(r,d)(R)$ to be the set of triples $(X,Y,Z) \in R^3$ that satisfy*

$$X^2 + Y^3 = dZ^r. \tag{2.1}$$

Definition 2.1.2 (Parameterization). *Suppose R is an entire ring and $r \in \{3,4,5\}$. Let K be the quotient field of R and $\bar{K}$ an algebraic closure of K. We assume that x, y are algebraically independent and transcendental over $\bar{K}$.*

A parameterization of $\mathcal{D}(r,d)(R)$ is a triple $\chi := (\hat{X}, \hat{Y}, \hat{Z})$ in $\mathcal{D}(r,d)(K[x,y])$ such that the equation

$$\hat{X}^2 + \hat{Y}^3 = d\hat{Z}^r$$

is homogeneous in the x, y of some degree $n > 0$. The integer n is called the order of the parameterization. If $\hat{X}, \hat{Y}, \hat{Z}$ are co-prime in $\bar{K}[x,y]$ we say that the parameterization is co-prime.

Definition 2.1.3 (Specialization). *Suppose that χ is a parameterization of $\mathcal{D}(r,d)(R)$. We say that $(X,Y,Z) \in \mathcal{D}(r,d)(R)$ is obtained by an R-specialization of χ if it can be obtained from χ by specializing the variables x, y to values in R.*

2.2 G-spaces

Much of this thesis will involve analyzing the action of a group G on a space X. We say that $x, x' \in X$ are G-equivalent if there is a $g \in G$ so that $x' = gx$. We call the classes of G-equivalent elements of X the G-orbits. If the whole space consists of a single G-orbit we say that X is a *homogeneous G-space.*

For any $x \in X$, the set $\mathrm{Stab}_G(x)$ is defined to be those $g \in G$ such that $gx = x$. Maps between G-spaces that commute with the action of G are called G-equivariant maps.

In most cases G will be a subgroup of $\mathrm{GL}_2(K)$, where K is a field. The group $\mathrm{GL}_2(K)$ acts on elements of K^2 and $K[x, y]$ as follows.

Definition 2.2.1 (Actions of $\mathbf{GL}_2(K)$). *Suppose that*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K).$$

We let $GL_2(K)$ act on K^2 by

$$(x, y) \mapsto (ax + by, cx + dy).$$

We extend this to an action on $K[x, y]$ by

$$f(x, y) \mapsto g \cdot f := f(g^{-1}(x, y)).$$

This induces an action of $GL_2(K)$ on parameterizations by

$$\chi := (\hat{X}, \hat{Y}, \hat{Z}) \mapsto g \cdot \chi := (g \cdot \hat{X}, g \cdot \hat{Y}, g \cdot \hat{Z}).$$

2.3 A First Lemma

Lemma 2.3.1. *Fix a ring R. Then any $\lambda \in R^*$ induces a bijection:*

$$\begin{aligned} \mathcal{D}(r, d)(R) &\to \mathcal{D}(r, \lambda^{r-6} d)(R), \\ (X, Y, Z) &\mapsto (\lambda^3 X, \lambda^2 Y, \lambda Z). \end{aligned}$$

We will be interested in looking at how many parameterizations are needed to specialize to all co-prime $(X, Y, Z) \in \mathcal{D}(r, d)(R)$. If R is a domain, the map can also be applied to the ring $Q(R)[x, y]$, where $Q(R)$ is the quotient field of R. As this map commutes with R-specialization, the 'theory of parameterizations' depends only on d modulo $R^{*(6-r)}$.

2.4 Reserved Symbols

Throughout this thesis the symbols r, k, N will have a special meaning. The symbol r will always represent a number in the set $\{3, 4, 5\}$. This will be the exponent of Z in the equation $X^2 + Y^3 = dZ^r$. We associate $k \in \{4, 6, 12\}$ and $N \in \{12, 24, 60\}$ to these equations depending on which of the $r \in \{3, 4, 5\}$ is currently under consideration.

We will be associating a platonic solid to each of the $r \in \{3, 4, 5\}$. The solid is the tetrahedron, the octahedron, and the icosahedron respectively. To motivate these numbers I include the following table.

Symbol	Values	Interpretation
r	$\{3, 4, 5\}$	The exponent of Z in $X^2 + Y^3 = dZ^r$.
k	$\{4, 6, 12\}$	The number of vertices of the associated platonic solid.
N	$\{12, 24, 60\}$	The order of the group of rotational symmetries of the Solid.