Appendix C

Twisted Conjugacy Classes

This Appendix contains proofs of various claims about twisted conjugacy classes. These objects act in many ways like the usual conjugacy classes of Group Theory. The proofs have been placed in this appendix for completeness.

In this appendix we assume that $H \leq G$ are arbitrary finite groups. We assume that ψ is an automorphism of G that induces an automorphism of H. See § 8.4 for the definitions of (ψ, H)-conjugacy, $[g]_{(\psi,H)}$ and $C_{(\psi,H)}(g)$.

Lemma C.0.1. *Take any $g \in G$. Then $C_{(\psi,H)}(g)$ is a subgroup of H. We have*

$$\#[g]_{(\psi,H)} \ \ \#C_{(\psi,H)}(g) \ = \ \#H.$$

Proof. $C_{(\psi,H)}(g)$ is clearly a subgroup of H. For $h_1, h_2 \in H$ we have

$$\begin{aligned} h_1^{-1} g\psi(h_1) = h_2^{-1} g\psi(h_2) \Leftrightarrow \quad & g = (h_1 h_2^{-1})^{-1} g\psi(h_2 h_1^{-1}) \\ \Leftrightarrow \quad & h_1 h_2^{-1} \in C_{(\psi,H)}(g), \end{aligned}$$

so that the distinct elements of $[g]_{(\psi,H)}$ are in $1-1$ correspondence with the right cosets of $C_{(\psi,H)}$. The result follows. □

Proposition C.0.2. *Suppose $H \lhd G$ are groups and G/H is a cyclic group of order n. Suppose ψ is the automorphism of H given by conjugation $a \mapsto sas^{-1}$ for some $s \in G$ whose image in G/H generates this cyclic group. Then*

$$H \to Hs, \quad g \mapsto gs$$

is a bijection that maps (ψ, H)-conjugacy classes of H to G-conjugacy classes of the coset Hs of G. Furthermore

$$\#C_{(\psi,H)}(g) = \frac{1}{n} \ \#C_G(gs).$$

Proof. Suppose $g, g' \in H$. If $h \in H$ then

$$g = h^{-1}g'\psi(h) \Leftrightarrow gs = h^{-1}g'sh.$$

Therefore $[g]_{(\psi,H)} = [gs]_H \subset [gs]_G$. The proof will be complete if we can show that $\#C_G(gs) = n\#C_H(gs)$, since Lemma C.0.1 then implies that $\#[gs]_H = \#[gs]_G$. The element $h \in C_H(gs)$ if and only if $(gs)h \in C_G(gs)$. Since gs generates G/H we have $\#C_G(gs) = n\#C_H(gs)$. □

Proposition C.0.3. *Suppose $-I \in H$, that $H \leq G \leq SL_2(K)$ and that $\psi_1, \psi_2 \in Aut(G)$ satisfy $\psi_i(-I) = -I$. Denote projective versions of objects by tilde and suppose that $\widetilde{\psi_1} = \widetilde{\psi_2} \in Aut(\widetilde{G})$. Then either $\psi_1 = \psi_2$ or there is a subgroup $G' \leq G$ such that*

$$[G : G'] = [\widetilde{G} : \widetilde{G}'] = 2 \quad \text{and } \psi_1 = \epsilon\psi_2,$$

where ϵ is the unique non-trivial character $G/G' \to \{\pm 1\}$.

Proof. Consider the map

$$\rho : G \to \pm 1, \quad g \mapsto \psi_1(g)\psi_2(g)^{-1}.$$

This is a group character. Let $G' := \ker(\rho)$. The two cases correspond to whether ρ is trivial or not. Since $\psi_i(-I) = -I$, the mapping ρ takes the same value on $\pm g$. Therefore $[G : G'] = 2$ implies that $[\widetilde{G} : \widetilde{G}'] = 2$. □

Lemma C.0.4. *Suppose $-I \in H$, that $H \leq G \leq SL_2(K)$ and that $\psi \in Aut(G)$ satisfies $\psi(-I) = -I$. Then the following holds.*

- *For any $g \in G$, $\#C_{(\psi,H)}(g)$ is even.*
- $C_{(\psi,H)}(1) = \{h \in H \mid h = \psi(h)\}$.
- *Suppose that H has unique H-conjugacy class of trace 0 and that ψ restricts to an automorphism of H. Then there is a $g \in H$ such that $[-g]_{(\psi,H)} = [g]_{(\psi,H)}$.*

Proof. For the first claim, note that if $h \in C_G(g)$ then so is $-h$. The second is clear. For the third, take any h in the unique H-conjugacy class of trace 0. Then $-\psi(h)$ also has trace 0, so $-\psi(h)$ is H-conjugate to h. This means that there is a $g \in H$ with $h = -g\psi(h)g^{-1}$. The result follows. □

Lemma C.0.5. *Suppose G is a subgroup of $SL_2(K)$, that $-I \in H$ and that $\psi \in Aut(G)$ satisfies $\psi(-I) = -I$. Denote projective versions of objects by tilde. Fix $g \in G$. Then exactly one of the following two situations holds.*

	Situation 1	*Situation* 2
$\exists h \in H$ *such that* $-g = h^{-1}g\psi(h)$*?*	*YES*	*NO*
$[g]_{(\psi,H)} = [-g]_{(\psi,H)}$*?*	*YES*	*NO*
$\#C_{(\psi,H)}(g)$ *equals ...*	$\#C_{(\psi,\widetilde{H})}(g)$	$2\#C_{(\psi,\widetilde{H})}(g)$

Proof. An element h such that $-g = h^{-1}g\psi(h)$ is exactly the witness needed to show that $[g]_{\psi,H} = [-g]_{\psi,H}$. Since $\#H = 2\#\widetilde{H}$, Lemma C.0.1 implies that

$$\#C_{\psi,H}(g) = 2\frac{\#[g]_{\psi,\widetilde{H}}}{\#[g]_{\psi,H}}\#C_{\psi,\widetilde{H}}(g).$$

Since

$$\#[g]_{\psi,H} = \begin{cases} 2\#[g]_{\psi,\widetilde{H}} & \text{if } [g]_{\psi,H} = [-g]_{\psi,H}, \\ \#[g]_{\psi,\widetilde{H}} & \text{otherwise,} \end{cases}$$

the result follows. □

Proposition C.0.6. *Suppose H is a group and $\psi_i \in Aut(H)$ for $i = 1, 2$. If there is an $s \in H$ so that $\psi_1 = s^{-1}\psi_2 s$, then*

$$H \to H, \quad g \mapsto gs^{-1}$$

is a bijection that maps (ψ_1, H)-conjugacy classes to (ψ_2, H)-conjugacy classes.

Proof. Suppose $x, y, h \in H$. Then

$$\begin{aligned} x = h^{-1}y\psi_1(h) &\Leftrightarrow x = h^{-1}ys^{-1}\psi_2(h)s \\ &\Leftrightarrow xs^{-1} = h^{-1}ys^{-1}\psi_2(h). \end{aligned}$$

□