Appendix B

Fields of Low Characteristic

Certain propositions about $\mathcal{C}(r,d)$ that are trivial when $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > k$ become false or, if true, need extra arguments in low non-zero characteristics. This Appendix contains proofs of these properties.

Lemma B.0.1. *Suppose $d \in K^*$ and $a \in K^*$. Then*

$$\hat{\varphi}(a) := \begin{cases} [0, a, 0, 0, 4a^{-2}d] & \textit{Tetrahedron}, \\ [0, a, 0, 0, 0, 12a^{-1}d, 0] & \textit{Octahedron}, \\ [0, a, 0, 0, 0, 0, \frac{144d}{7}, 0, 0, 0, 0, -a^{-1}(144d)^2, 0] & \textit{Icosahedron} \end{cases}$$

is an element of $\mathcal{C}(r,d)(K)$. In particular $\mathcal{C}(r,d)(K) \neq \emptyset$.

Proof. This is verified by calculation. □

Proposition B.0.2. *Suppose K is a field and $2, d \in K^*$. If $(X,Y,Z) \in \mathcal{D}(r,d) - (0,0,0)$, there is a $\varphi \in \mathcal{C}(r,d)(K)$ so that $\pi(\varphi) = (X,Y,Z)$.*

Proof. If $Z = 0$, we let $\varphi := \hat{\varphi}(-X/Y)$, where $\hat{\varphi}$ is the function defined in Lemma B.0.1. One checks that $\pi(\varphi) = (X,Y,Z)$.

If $Z \neq 0$, define

$$\varphi := [Z, 0, \frac{Y}{Z}, \frac{2X}{Z}, \dots],$$

where the omitted terms are uniquely determined by the first $k-3$ defining equations of $\mathcal{C}(r,d)$. The coefficients of φ can be expressed as elements in the ring $\mathbb{Z}[X,Y,Z,Z^{-1}]$. With the help of PARI we get the algebraic identities

$$\boldsymbol{\tau}_4(\varphi) = 0, \quad 7\boldsymbol{\tau}_6(\varphi) = -360\left(\frac{X^2+Y^3}{Z^5}\right),$$
$$7\boldsymbol{\tau}_{12}(\varphi) = 3110400\left(\frac{X^2+Y^3}{Z^5}\right)^2,$$

when $r = 5$, and similar identities when $r = 3, 4$. This shows that φ is a 'generic' lift of the element (X,Y,Z). The result follows. □

Proposition B.0.3. *Suppose K a field and $d \in K^*$. If $g \in GL_2(K)$ and $\lambda \in K^*$ then*

$$\varphi \in \mathcal{C}(r,d) \Rightarrow \lambda g \cdot \varphi \in \mathcal{C}(r,d'),$$

where $d' = \lambda^{6-r}\det(g)^{-6}d$.

Proof. The claim about multiplication by $\lambda \in K^*$ is clear by examining the defining equations of $\mathcal{C}(r,d)(K)$. Furthermore, any $g \cdot f$ can be written as $g \cdot f = \lambda g'.f$ with $g' \in \mathrm{SL}_2(K)$ and $\lambda \in K^*$ satisfying $\lambda^{6-r} = \det(g)^6$. Therefore we can assume $g \in \mathrm{SL}_2(K)$.

We are left to show that if $g \in \mathrm{SL}_2(K)$ and $\varphi \in \mathcal{C}(r,d)(K)$ then $g \cdot \varphi \in \mathcal{C}(r,d)$. The subsets of the equations defining $\mathcal{C}(r,d)$ that are equivalent to the vanishing of a polynomial combination of covariants of φ, necessarily remain valid by the definition of a covariant. This is true for all equations, except the equations derived from the coefficients of the 4th covariant $\boldsymbol{\tau}_4(\varphi)$, and the icosahedral equation labelled D_4^*.

The equations obtained by requiring that $\boldsymbol{\tau}_4$ vanish remain valid after an $\mathrm{SL}_2(K)$ substitution because of the covariance of $\boldsymbol{\tau}_4$. However, there is an added complication since the equations for $\mathcal{C}(r,d)$ are obtained from the coefficients of $\boldsymbol{\tau}_4(f)$ after dividing out by their content. Let V be the algebraic set defined by these equations.

Note that

$$\begin{pmatrix} \kappa & 0 \\ 0 & \kappa^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}, \tag{B.1}$$

where $\kappa \in K^*$ and $\nu \in K$, generate $\mathrm{SL}_2(K)$. By inspection V is closed under the action of the first two generators. A calculation (e.g. with the help of a computer package) shows that it is also closed under the action of the last generator.

Finally, we must check the equations remain closed under $\mathrm{SL}_2(K)$-substitutions if we also require that D_4^* is satisfied in the icosahedral case. This is also done by checking the claim on the generators (B.1) of $\mathrm{SL}_2(K)$.

□

Lemma B.0.4. *Suppose $d \in K^*$ and $\varphi \in \mathcal{C}(r,d)(\bar{K})$. Then there is a $g \in SL_2(\bar{K})$ so that $a_0(g \cdot \varphi) = 0$. If $N \in K^*$, there is a $g \in SL_2(K)$ so that $g \cdot \varphi = [0,1,0,\dots]$.*

Proof. Let $f := f(\varphi)$. We can find $g \in \mathrm{SL}_2(\bar{K})$ so that the binary form $g \cdot f$ has a zero at ∞. Since $\varphi \mapsto f(\varphi)$ is $\mathrm{SL}_2(\bar{K})$-equivariant we have $a_0(g \cdot \varphi) = 0$.

If $N \in K^*$, f has no multiple roots, as the discriminant of f is non-zero. Therefore, by equivariance, $a_1(g \cdot \varphi) \neq 0$. Twisting with an $\mathrm{SL}_2(K)$ matrix of the shape $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ allows us to suppose $g \cdot \varphi = [0,1,0,\dots]$.

□

Lemma B.0.5. *Suppose $N, d \in K^*$. Then $\mathcal{C}(r,d)(\bar{K})$ is a homogeneous $SL_2(\bar{K})$-space and $\mathcal{C}(r)(\bar{K})$ is a homogeneous $GL_2(\bar{K})$-space.*

Proof. Suppose $\varphi \in \mathcal{C}(r,d)(\bar{K})$. By Lemma B.0.4, we can assume that $\varphi = [0,1,0,\dots]$. However, as $N \in K^*$, the missing coefficients are determined by the defining equations of $\mathcal{C}(r,d)$. We have shown that there is a single element of $\mathcal{C}(r,d)$ to which every $\varphi \in \mathcal{C}(r,d)$ is $\mathrm{SL}_2(\bar{K})$-equivalent. This means that $\mathcal{C}(r,d)(\bar{K})$ is a homogeneous $\mathrm{SL}_2(\bar{K})$-space. As a corollary $\mathcal{C}(r)(\bar{K})$ is a homogeneous $\mathrm{GL}_2(\bar{K})$-space. □

Remark B.0.6. *Lemma B.0.5 is not true without some restrictions on the characteristic of K. For instance, if K has char$(K) = 2$ then $[0,1,0,0,1]$ and $[0,1,0,1,0]$ are elements of $\mathcal{C}(3,1)(K)$. These lie in distinct $SL_2(\bar{K})$-orbits, since being of the shape $[0,*,0,*,0]$ is an $SL_2(\bar{K})$-invariant property.*

Proposition B.0.7. *Suppose K is a field and $N \in K^*$. Suppose $(X,Y,Z) \in \mathcal{D}(r,d) - (0,0,0)$. If $\varphi \in \mathcal{C}(r,d)(K)$ and $\pi(\varphi) = (X,Y,Z)$ then there is a unique parabolic $g \in SL_2(K)$ so that*

$$g \cdot \varphi := \begin{cases} [Z, 0, \frac{Y}{Z}, \frac{2X}{Z}, \dots] & \text{if } Z \neq 0, \\ [0, -X/Y, 0, \dots] & \text{if } Z = 0, \end{cases}$$

where the omitted terms are uniquely determined by the defining equations of $\mathcal{C}(r,d)$.

Proof. (Existence) Since $N \in K^*$, there is a parabolic $g \in \mathrm{SL}_2(K)$ so that the φ has the shape $[*,0,\dots]$ if $Z \neq 0$, and $[0,*,0,\dots]$ if $Z = 0$. Since $\pi(\varphi) = \pi(g \cdot \varphi) = (X,Y,Z)$, the initial coefficients of φ agree with the coefficients in the announcement of the proposition. The defining equations of $\mathcal{C}(r,d)$ determine the rest of Ω_r.

(Uniqueness) Suppose φ has the canonical form given. Since $N \in K^*$, there is no non-zero parabolic $g \in \mathrm{SL}_2(K)$ that fixes φ.

□

Lemma B.0.8. *If $N, d \in K^*$ and $\varphi \in \mathcal{C}(r,d)$ then $\#\Gamma(\varphi) = 2N$. Furthermore, if φ corresponds to the coefficients of one of the forms chosen in § 4.3, then the explicit description of the group given in § 4.3 is the group $\Gamma(\varphi)(\bar{K})$.*

Proof. Since $N \in K^*$, the space $\mathcal{C}(r)(K)$ is a homogeneous $\mathrm{GL}_2(K)$-space by Lemma B.0.5. This means that we can assume that φ corresponds to one of the Klein forms mentioned in § 4.3. Let Γ' denote the group of symmetries mentioned in § 4.3, and $\Gamma := \Gamma(\varphi)$ the full group of symmetries. Clearly $\Gamma' \subset \Gamma(\varphi)$. Indeed, the set of equations witnessing the truth of the statement $\Gamma' \subset \Gamma(\varphi)$ in $\overline{\mathbb{Q}}$ can be written as a set of polynomial equations in the ring generated by $\mathbb{Z}$ and the entries of elements of Γ'. These identities remain valid in the field K.

Suppose that $g \in \Gamma(\varphi)$. We will show that $g \in \Gamma'$. Let $f := f(\varphi)$. Calculating the discriminant we see that the roots of f are distinct if $N, d \in K^*$. Since Γ' is transitive on the roots of f, we can multiply g by an element of Γ' and assume that g fixes the root at ∞. Therefore, $g = \begin{pmatrix} \kappa & \nu \\ 0 & \kappa^{-1} \end{pmatrix}$ for some $\kappa \in K^*$, $\nu \in K$. Since $\varphi = [0, *, 0, \dots]$ and $N \in K^*$ we calculate that $\nu = 0$. The matrix g is diagonal and we deduce that $g \in G'$. Conclusion: $\Gamma(\varphi) = \Gamma'$.

Finally, we must show that $\#\Gamma' = 2N$. I.e. that Γ' has no kernel when we reduce from $\mathrm{SL}_2(\overline{\mathbb{Q}})$ to $\mathrm{SL}_2(\bar{K})$. Let $f := \boldsymbol{f}(\varphi)$. Suppose $g \in \Gamma'(\overline{\mathbb{Q}})$ is not $\pm I$. Then g induces a non-trivial permutation of the roots of $f \in \overline{\mathbb{Q}}[x, y]$. As $N, d \in \bar{K}^*$, we have $\mathrm{disc}(f) \neq 0$, so the roots of f remain distinct in $\bar{K}$. Therefore the kernel is contained in $\{\pm I\}$. As $2 \in \bar{K}^*$, the matrix $-I$ is not in the kernel. Conclusion: $\#\Gamma(\varphi)(K) = \#\Gamma(\varphi)(\overline{\mathbb{Q}}) = 2N$.

□

Remark B.0.9. *Consider* $\varphi = [0, 144, 0, 0, 0, 0, \frac{144}{7}, 0, 0, 0, 0, -144, 0]$. *We have* $f := f(\varphi) = 1728d\ xy(x^{10} + 11x^5y^5 - y^{10})$

Suppose K *is a field with* $char(K) = 11$. *Then* f *has distinct roots, as the discriminant is non-zero. However,* $\Gamma(f)$ *contains the cyclic group of order* 11 *generated by* $g := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *This means that* $\Gamma(f) \neq \Gamma(\varphi)$. *Set* $\varphi' := g \cdot \varphi$. *Then* φ, φ' *are distinct elements of* $\mathcal{C}(5, d)$ *with* $f(\varphi) = f(\varphi')$.