Introduction

String theory

In recent decades string theory has become a leading candidate for a unified theory of particle physics, gravity and quantum mechanics, see [1, 2, 3, 4]. As the name suggests, the starting point of string theory is to consider strings instead of point particles. One can consider closed or open strings, which both sweep out two-dimensional surfaces as they move through spacetime. Such a surface is called the world-sheet.

String theory has one dimensionful coupling constant called $\alpha'$ with dimension of length squared. The vibrations of the string correspond to particles in spacetime. For example, in closed string theory there is a massless rank two traceless symmetric tensor which we must identify with the graviton if we view string theory as a unified theory. There is also a massless rank two antisymmetric tensor. Furthermore, there is a whole tower of massive states, with masses proportional to $\sqrt{1/\alpha'}$.

When considering string theory as a unified theory, as we are, $\alpha'$ is of the order of the natural scale determined by the fundamental constants of gravity and quantum mechanics. This scale is given by $M_P^{-2}$ where $M_P \equiv \sqrt{\hbar c/G_N}$ is the Planck mass, roughly $1.2 \times 10^{19}$ GeV. Put differently, $\sqrt{\alpha'}$ which sets the scale of the string length is of the order of the Planck length $l_P \equiv \sqrt{\hbar G_N/c^3}$, which is about $1.6 \times 10^{-33}$ cm.

Furthermore, there is a dimensionless parameter called the string coupling constant $g_s$. This constant organizes the perturbative expansion of string theory. Just as in normal field theory we can set up a perturbation theory with the difference that in- and outgoing lines are now tubes, in the case of the closed string. A one-loop diagram, for a four-point amplitude for example, takes the form of a donut with four tubes attached. This diagram comes with a higher power of $g_s$, compared with the tree level diagram (a sphere with four tubes attached). Higher order loop diagrams come with higher powers of $g_s$, this parameter therefore counts the order of the diagram. In general the perturbative expansion of string theory is a genus expansion in the world-sheet where each genus comes with a certain power of $g_s$. The result is a power series in $g_s$: $\sum_n g_s^{2n} A^{(n)}$. Here $A^{(n)}$ denotes the diagram, that is, $A^{(0)}$ the tree level diagram, $A^{(1)}$ the one-loop
diagram (the donut) and so on. We see that such an expansion makes sense as long as $g_s \ll 1$.

The parameter $g_s$ is not an arbitrary parameter. It is related to the vacuum expectation value of the dilaton, a scalar field which is one of the massless modes. However, in general it is not clear what the vacuum expectation value of the massless scalar fields should be. In chapter 5 we will consider a scenario in which we can dynamically provide a vacuum expectation value for the dilaton. We will come back to this issue below on page xiv.

If one considers only the coordinates which describe the embedding of the string in spacetime, one has bosonic string theory. A next step is to add fermions on the world-sheet. This gives rise to the so-called superstring theories, which contain additional massless states besides the massless states of the bosonic string. There are actually five different superstring theories, which seems to be a bit of a setback if one would like string theory to be a unique and unifying theory. About ten years ago though, it became clear that these five theories are actually different phases of a more general theory dubbed M theory, [5, 6]. However, not only is it as yet unclear what this theory looks like, it is also unclear where the ‘M’ stands for: dial M for mystery!

Without going into the specifics of these five string theories, we remark that we will be concerned only with one of them in this thesis, namely with the ‘type IIA’ theory. For reasons of consistency, the superstring theories have to live in 10 spacetime dimensions and the bosonic string in 26. Naturally this seems excessive and has to be remedied. We will come back to this issue in a moment.

The name ‘string’ theory has become a misnomer since the discovery [7] of ‘D-branes’. These are extended objects on which strings can end. For example, in type IIA theory there are membranes and four-branes which sweep out three- and six-dimensional world-volumes respectively.

Such D-branes are nonperturbative objects, like solitons in field theory, because these object are heavy when the string coupling constant is small, but become lighter when it increases. The reason is that their tension is inversely proportional to $g_s$, so when the perturbative expansion breaks down (large $g_s$) these branes are light.

Moreover, string theory contains also a NS five-brane. This object is dual to the fundamental string in the same sense that a magnetic monopole is dual to an electric charge by means of electric-magnetic duality. The tension of the NS five-brane is proportional to $1/g_s^2$, this NS five-brane will be very important to us and we will discuss it more extensively in chapter 2.

For some general references on D-branes see [3, 4, 8, 9].

These branes are solitonic objects, but below we will see that there are sit-
uations in which the soliton will appear in a lower dimensional spacetime as an instanton.

This reminds us of nonperturbative effects in Yang-Mills theories. Apart from the trivial vacuum, (Euclidean) Yang-Mills theory has a vacuum which consists of an infinite number of topologically distinct vacua. The instanton solution represents a transition from one vacuum class to another. The tunnelling amplitude for such a transition is proportional to \( \exp(-S_E) \), with \( S_E \) the Euclidean action. The action for a Yang-Mills instanton is proportional to \( 1/g_{ym}^2 \), with \( g_{ym} \) the Yang-Mills coupling constant. So when one computes amplitudes in the presence of such instanton configurations in doing a semiclassical approximation, one finds an additional weight factor of the form \( \exp(-1/g_{ym}^2) \). We therefore see that instanton contributions to physical processes are heavily suppressed for small \( g_{ym} \). However, when \( g_{ym} \) becomes large this factor becomes more important. Thus starting from the perturbative description of Yang-Mills theory, one can construct solutions which give insight into nonperturbative effects. Instantons are, for instance important in the context of CP violating processes, see [10] for a review and references. For a very nice introduction to instantons in Yang-Mills theory see [11, 12], for a clear account of the topological details of instantons see [13].

In this thesis we will approach the NS five-brane in a similar fashion. That is, in chapter 4 we will use the semiclassical approximation to compute its contribution to correlation functions. We should stress that we will use an effective description of the NS five-brane. We will not make use of a worldvolume description of the brane. How this brane can be effectively described is discussed below. For more information on the semiclassical approach see for instance [12].

**Effective theories**

As remarked above, the massive states of the string theories have masses proportional to the Planck mass. This means that only when considering processes with extremely high energies the massive states start to play a role. Such energies are far beyond the reach of any man-made accelerator. Therefore one can restrict to the massless modes only and describe them by an effective theory. As long as one considers processes with energies much lower than the Planck mass, this is a good approximation. The low energy effective theories of the superstring theories take the form of supergravity theories. The low energy theory of the type IIA superstring, for instance, is type IIA supergravity which is a theory with two local supersymmetry generators. The universal (since it occurs in other superstring theories as
well) part of its bosonic sector is given by

$$\int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{-\phi} H^2 \right).$$

$H$ is the field strength of the massless rank two antisymmetric tensor and $\phi$ is the dilaton. The Einstein-Hilbert term describes the graviton. This action captures the massless states already present in the bosonic string theory. In type IIA superstring theory there are in addition massless tensor- and fermionic fields. The nonperturbative objects from string theory, i.e., the D-branes are then described by so-called $p$-branes which are solitonic solutions of the supergravity theory. For references on branes from a supergravity point of view see [14, 15].

We can illustrate this by the well-known example in the context of type IIB string theory and its low energy effective theory, type IIB supergravity. This is the example of the D-instanton. It is a D-brane with no spatial extension and which is moreover located in time, hence the name instanton. In the context of type IIB it is a BPS solution which preserves half of the spacetime supersymmetries. In [16] the supergravity solution corresponding to the D-instanton was found. Consider the following bosonic subsector of the Euclidean type IIB action

$$\int d^{10}x \sqrt{g} \left( R - \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} e^{2\phi} (\partial a)^2 \right).$$

In flat Euclidean space we can solve the equation of motion for $\phi$ by

$$e^\phi = e^{\phi_\infty} + \frac{c}{r^8},$$

with $e^{\phi_\infty}$ and $c$ integration constants. The axion $a$ is determined in terms of $\phi$. The above solution for the dilaton solves the field equation $\partial^2 e^\phi = 0$ except at the origin where a delta function source term is needed. Consequently we can add the source term $\delta^{(10)}(x) e^{-\phi}$ to the action which cancels the singularity in the field equation. This source term tells us that we have a new object in the theory at $x = 0$, namely the D-instanton. So we see that we can use supergravity to describe the objects in string theory. The action of the D-instanton is given by

$$S_{\text{D-inst}} = \frac{|Q|}{g_s},$$

where we have defined $g_s \equiv e^{\phi_\infty}$, note the $1/g_s$ behaviour. This action derives from the fact that one has to consider an additional boundary term to the action. The charge $Q$ is due to the existence of a conserved
current and is related to the integration constant $c$, see also [17]. By studying certain amplitudes in the background of the D-instanton one can compute the effective vertices due to this instanton, [18]. For a review of D-instantons and more references see [19]. We will make a similar effort in this thesis for the NS five-brane and two-brane. However, we will not be working in ten dimensions but in four dimensions, as we will discuss first.

**Compactification**

The superstring theories, and therefore their low energy effective theories, live in ten spacetime dimensions. Of course we are interested in the case of four spacetime dimensions. One way to go from ten to four spacetime dimensions, is to compactify six space dimensions on some internal manifold. If these six dimensions are taken to be very small, they are effectively not seen anymore in experiment. Consequently, the fields have to be expanded in the compact directions. This leads to massless states and a tower of massive states, determined by the manifold the theory is compactified on. The mass is inversely proportional to the characteristic radius $R_c$ of the compactified dimensions. This defines a compactification scale $m_c = 1/R_c$, the characteristic mass of states with momentum in the compactified directions. When $R_c$ is sufficiently small, $m_c$ is high enough to enable us to work with the massless modes only. This means that a four-dimensional theory can be constructed: compactify the low energy effective action of string theory on some (small) internal manifold. This then gives a theory in four spacetime dimensions. If $m_c$ is high enough this theory can be approximated by an effective theory which only describes the massless modes. This is done by integrating out the heavy degrees of freedom (which can give corrections to the massless degrees of freedom). The idea of compactifying higher dimensional theories to four dimensions was initiated by Kaluza and Klein. See [20] for a historical overview.

The effective theory in four dimensions can be (depending on the internal manifold) rather complicated. An interesting choice for manifolds to compactify on are the so-called Calabi-Yau manifolds, which we will describe in chapter 1. In the eighties it was found that compactifying the heterotic string theory on a Calabi-Yau manifold gives rise to phenomenologically interesting models, see [21]. Furthermore, these manifolds preserve supersymmetry. For example, compactifying type IIA supergravity in ten dimensions on a six-dimensional Calabi-Yau manifold gives an $N = 2$ supergravity theory with vector and hypermultiplets [22]. This theory describes the massless degrees of freedom. The fact that it has (local) supersymmetry means that the theory is more constrained, thus making it easier to
work with. In chapter 2 we will review the process of compactifying type IIA supergravity on a Calabi-Yau manifold of six dimensions. To summarize: the reason why we study type IIA supergravity on a Calabi-Yau manifold is that this allows us to work with a theory which has just enough supersymmetry. In contrast, $N = 4$ or $N = 8$ (which does not even allow for matter multiplets) supersymmetry is too constraining and $N = 1$ supersymmetry is not constraining enough, see also [23].

The aim of this thesis can now be formulated as follows: to study the nonperturbative effects of NS five-branes and membranes on the underlying $N = 2$ supergravity theory in four dimensions.

Just as one can describe these branes in the supergravity approximation in ten dimensions, so can one construct corresponding solutions which describe them in four dimensions. Intuitively this can be understood as follows. Consider the five-brane with its six dimensional worldvolume$^1$, which we can embed entirely in the six-dimensional internal Calabi-Yau manifold. This means that the five-brane is then completely localized in four-dimensional spacetime. From a four-dimensional point of view it will then appear as an instanton. This will also be described in chapter 2. In chapter 4 we will perform a semiclassical computation to compute the instanton effects. Contrary to the case of Yang-Mills instantons, we will not consider instantons as giving tunnelling amplitudes between different vacua, because such an interpretation is not clear in our case. Instead instantons will be regarded as solutions to the equations of motion which have finite (Euclidean) action. These solutions furthermore saturate lower bounds of the action and will be derived using Bogomol’nyi equations in chapter 3.

Similarly to the five-brane, the three-dimensional worldvolume of the membrane can be wrapped around a three-dimensional submanifold of the Calabi-Yau. This will also correspond to an instanton solution in four dimensions which will be constructed in chapter 3 as well. In chapter 5 we will then consider the effect of these membranes on the four-dimensional supergravity theory.

**Geometry**

As said above, we will be working with the supergravity theory describing the massless sector of the theory which arises when compactifying type IIA supergravity on a Calabi-Yau manifold. This is an $N = 2$ supergravity

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$^1$Assume for the moment that we have performed a Wick rotation, such that we are wrapping an Euclidean brane around the Calabi-Yau manifold.
theory coupled to hypermultiplets and vector multiplets. Such a theory can be thought of in geometrical terms. We can consider the various scalars, appearing in these multiplets, as maps from space-time to some target space manifold. The scalars are then coordinates on this manifold. The kinetic terms of the scalars function as the metric on such a target space. Supersymmetry relates the various terms in the action and puts constraints on them. This means that the ‘metrics’ are also constrained to be of a certain type, thus determining the geometry of the target space.

This has a (relatively) long history. For instance, $N = 1$ matter-coupled supergravity in four dimensions has a number of complex scalar fields. These fields are governed by Kähler geometry: they parametrize a Kähler manifold, as discovered by Zumino and others [24, 25].

The geometric interpretation of hypermultiplets coupled to $N = 2$ supergravity in four dimensions was emphasized by Bagger and Witten [26]. For more information on the interplay between supergravity and geometry see for instance [27] and its references.

This geometrical formulation of the constraints obeyed by the scalars of the hypermultiplets coupled to $N = 2$ supergravity will be very important to us, because this is the sector of the theory we will be working with. The target space of the hypermultiplet sector is a so-called quaternionic manifold, which we will review in chapter 1. The crucial point is that the conditions for unbroken supersymmetry force the hypermultiplet sector of the supergravity theory to have such a target space. So when we compute our instanton corrections to this theory, the corrections will in general perturb the target space but in such a way that it is still quaternionic. In chapter 4 we will establish that after computing the five-brane instanton effects on the supergravity theory, and thus on the target space, this target space is still of the quaternionic type.

We will be focussing on the hypermultiplet sector of the theory because this is where the instanton effects appear, as we will discuss. Furthermore, since the vector multiplet sector decouples, we will not consider it in our calculations.

In chapter 1 we will present some background material on the quaternionic geometry. Furthermore, we will (briefly) consider the geometry corresponding to the vector multiplets, for the sake of completeness. These geometries will reappear in chapter 2 where we sketch how the $N = 2$ supergravity theory emerges when compactifying type IIA supergravity theory on a Calabi-Yau manifold.

We can now reformulate the aim of this thesis: to compute the nonperturbative corrections induced in the quaternionic target space of the hyper-
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multiple sector of the four-dimensional $N=2$ supergravity theory.

Moduli

The massless scalar fields resulting from a compactification are sometimes referred to as ‘moduli’. The reason is that they are closely related to the parameters labelling the geometry of the internal manifold, which are also called moduli\(^2\). The point is that the various couplings in four dimensions are determined by the internal geometry and typically depend on the vacuum expectation values of the scalar fields in four dimensions. As long as these vacuum expectation values are undetermined, the couplings are free as well and the four-dimensional theory has no predictive power. In other words, every vacuum expectation value corresponds to a different ground-state of the four-dimensional theory. One would like to find a mechanism that (spontaneously) fixes these moduli, this is called moduli stabilization. One such mechanism is that of ‘compactification with fluxes’. When compactifying to four dimensions one can switch on fluxes in the internal manifold. This alters the four-dimensional theory and in particular gives rise to a potential for the scalar fields, i.e., the moduli.

Alternatively one can describe such effects by staying in four dimensions and gauging certain isometries in the supergravity theory. This also gives rise to a potential which can in principle stabilize (some of) the moduli, we will come back to this in chapter 5. This subject has attracted considerable and renewed attention in recent years. For some recent reviews see [28, 29]. In chapter 5 we will gauge an isometry in the hypermultiplet\(^3\) which will generate a potential. The membrane instanton corrections to the potential make it possible to stabilize all the fields in the hypermultiplet. The membrane instanton correction furthermore makes it possible for the potential to have a positive minimum. This value appears in the action as a cosmological constant term. This is very interesting because a positive cosmological constant corresponds to a de Sitter universe and at the moment our own universe seems to be of that type, see for instance the review [30] and references therein.

Outline of this thesis

After having introduced the main topics of this thesis, let us for clarity’s sake summarize them and specify where they will be discussed. This will

\(^2\)In chapter 2 we will closely investigate this relation for Calabi-Yau manifolds

\(^3\)We will consider only one hypermultiplet. This is related to the fact that we choose a simple type of Calabi-Yau compactification, as explained in chapter 2.
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hopefully make the relations between the various chapters and sections more clear.

• **Chapter 1**  
  Because quaternionic geometry will be important in this thesis, we devote the first half of this chapter to reviewing the geometry relevant to quaternionic manifolds and Calabi-Yau compactifications in general. The second part will be a short review of Calabi-Yau manifolds and some differential geometry, which will be especially relevant for chapters 2 and 5.

• **Chapter 2**  
  We discuss the process of compactifying type IIA supergravity on a Calabi-Yau manifold. We shall derive the bosonic field content of the resulting four-dimensional supergravity theory. We will concentrate in particular on the massless fields obtained by compactifying on a simple type of Calabi-Yau manifold.

• **Chapter 3**  
  By making use of Bogomol’nyi bounds, we construct solutions to the equations of motion that provide local minima to the action and as such can be interpreted as instantons. This will be done for a certain description of a hypermultiplet that is introduced in chapter 2. We construct two types of instanton solutions. The first corresponds to the NS five-brane and the second to the membrane. Their actions, which agree with results from string theory, and theta angles will be given.

• **Chapter 4**  
  This is the chapter where we will perform a detailed instanton calculation for the NS five-brane in the context of the four-dimensional effective supergravity theory. This is a fairly traditional field theoretic semiclassical approximation, although it does contain (apart from the final results) some novel features. The result is the effective $N = 2$ supergravity theory. Effective now means that the theory can reproduce at tree level the corrections to the action due to the five-brane instanton. We will examine the consequences for the quaternionic target space, in particular the consequences for its isometries.

• **Chapter 5**  
  A chapter in which the membrane instanton corrections to the four-dimensional action are derived. The material from chapter 1 is used and elaborated on. Where we can compare our results with string
theory we find beautiful agreement. Furthermore, the effect of these membrane instanton corrections to the potential obtained by gauging a certain isometry is studied. This is the mechanism referred to above, it will be explained in this chapter. The result can lead to a de Sitter vacuum in which all the moduli of the universal hypermultiplet sector are stabilized. The effect of the membrane instanton is to make it possible for the minimum to be positive.

- **Appendices**

Because there are a number of technical details we have collected them in a few appendices. These contain further information on calculations or extra background material. They will be referred to in the text as needed.

We must stress again that, apart from a section in chapter 5, we will always perform our calculations within the supergravity approximation. This means that when we speak of branes we think of them in a supergravity sense. Moreover, they will always be treated from a four-dimensional point of view, where they appear as instanton field configurations. Furthermore, this thesis is not about supergravity as such. We shall use it as a tool to consider nonperturbative effects. Consequently we will treat it as such. We will neither explain nor construct the $N = 2$ supergravity theory we will be working with from first principles, as this would warrant another thesis. Instead we will present the necessary information such as supersymmetry transformation rules and such when needed, especially in chapter 4. We will mainly focus on the geometry of the target space manifold of the theory.

Lastly let us indicate which chapters contain new material. Chapter 3 is largely based on [31], chapter 4 on [32] and chapter 5 on [33].