4

The NS 5-brane

In this chapter we compute the instanton effects originating from the NS 5-brane, using the DTM formulation. The underlying idea is that, for small coupling constant $g_s$, the path integral is dominated by the configurations of lowest Euclidean action and one may proceed by expanding around these configurations. Naturally, the simplest such configuration is the ordinary perturbative vacuum of the theory, which has $S_E = 0$. However, as we have seen in chapter 3, there are other minima of the action we have to expand about. These minima are the ones corresponding to the 5-brane and membrane. For reasons that will become clear later on, we will not follow such a program for the membrane and we will restrict ourselves to the 5-brane only.

We expand the action up to second order around the instanton configuration, which will be discussed further in section 4.4. Ideally one would like to compute the determinant of the resulting quadratic operator acting on the fluctuations, thus doing a one-loop computation. However, we are dealing with a difficult non-linear sigma model coupled to supergravity and the resulting quadratic operator is rather complicated, therefore we will not compute its determinant.

The instanton solutions (3.7) and (3.8) break the translation invariance of the theory, since they are located at specific points $\{x_i\}$, these are the so-called collective coordinates. Furthermore, the instanton solution partially breaks supersymmetry, which gives rise to fermionic collective coordinates. We will discuss the relation between broken symmetries and collective coordinates in section 4.3.

As we shall see in section 4.4, the collective coordinates are related to zero modes of the quadratic operator acting on the fluctuations. This means that one has to be careful in constructing the path integral measure. In section 4.4 we will construct the path integral measure suitable for a (1-loop) calculation in the presence of the NS 5-brane instanton. We shall see that we have to trade the integration over a certain set of quantum fluctuations (zero modes of the quadratic operator) for integrations over...
the collective coordinates. Apart from the integration over the position (the bosonic collective coordinates) there are also the integrations over the fermionic collective coordinates, these are Berezin integrals. Such integrals are very restrictive because they will only be nonzero if we compute correlation functions which contain enough fermions to ‘soak up’ the Berezin integrals.

This leads us to consider specific correlators in section 4.5. By computing these correlators we can construct the effective action\(^1\). To be precise: the action of the DTM (with \(N = 2\) local supersymmetry) with 5-brane instanton corrections.

In section 4.6 we shall consider the consequences for the moduli space of the UHM, which is what we were after all along, and examine the breaking of certain isometries of the Heisenberg algebra to a discrete subgroup.

We have to keep in mind that we are approximating a NS 5-brane wrapped along the Calabi-Yau by this instanton. Stated differently, in string theory the 5-brane instanton is described by an embedding of the 6-dimensional worldvolume into the 10-dimensional space \(\mathbb{R}^4 \times Y_3\) such that the worldvolume ends up entirely on the Calabi-Yau. The embedding maps are then thought of as the collective coordinates of the 5-brane and performing a genuine path integral would involve doing an integration over these maps. This amounts to a path integral over the worldvolume theory in the supergravity background \(\mathbb{R}^4 \times Y_3\), as advocated in [71], see also [87].

Due to the complicated and somewhat mysterious nature of that worldvolume theory, this would be difficult. We will not try to include such worldvolume effects and limit ourselves to an integration over the collective coordinates in their capacity as positions in Euclidean space.

First we will present some background material, in section 4.1, on the general \(N = 2\) supergravity theory coupled to tensors and scalars. This is the theory into which the DTM fits. In section 4.2 we will restrict ourselves to the case of only the DTM for which we will be doing the calculations of the rest of the chapter. Many technical details and calculations have been directed to the appendices, they will be referred to as needed.

## 4.1 Supersymmetry

In chapter 2 we have explained that we will be performing the NS 5-brane instanton calculation in the Euclidean DTM. One can Wick rotate to Lorentzian signature and dualize back to the UHM, so that we do not

\(^1\)Part of the effective action actually, namely the corrections to the kinetic terms of the scalars and tensors and some vertices. To compute corrections to the other terms in the action (4.1) one can apply supersymmetry.
need the (Euclidean) supersymmetric UHM. The most general form of the action for scalars and tensors coupled to $N=2$ supergravity has been constructed in [76] for spacetimes with Lorentzian signature. In Euclidean space the action is given by

$$e^{-1} \mathcal{L} = \frac{1}{2 \kappa^2} R + \frac{1}{4} \mathcal{F}^{\mu \nu} \mathcal{F}_{\mu \nu} + \frac{1}{2} \mathcal{G}_{AB} \hat{D}^A \hat{D}^B + \frac{1}{2} M^{IJ} \mathcal{H}^{I \bar{J}} \mathcal{H}_{\mu I}$$

$$- i A_I^a H_I^a \partial_\mu \phi^A + i \varepsilon^{\mu \nu \rho \sigma} (D_\mu \psi^i_\nu \sigma \psi_\sigma i + \psi^i_\sigma \sigma \rho \partial_\rho \psi_\sigma i)$$

$$+ \frac{i}{2} h_{a \bar{a}} (\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a - D_\mu \lambda^a \sigma^\mu \bar{\lambda}^a) + i \kappa M^{IJ} \mathcal{H}^{I \bar{J}} (g_{Ja} \psi^i_\mu \lambda^a + c.c.)$$

$$- \kappa \mathcal{G}_{AB} (\hat{D}_\mu \phi^A + \partial_\mu \phi^A) (\gamma^B \lambda^a \sigma^\mu \psi^i_\nu + c.c.) + i \kappa M^{IJ} \mathcal{H}^{I \bar{J}} (g_{Ja} \psi^i_\mu \lambda^a + c.c.)$$

$$- \frac{i \kappa}{2 \sqrt{2}} (\tilde{\mathcal{F}}^{\mu \nu} + \tilde{\mathcal{F}}^{\nu \mu}) (\psi^i_\mu \psi_\nu i + \bar{\psi}^i_\nu \bar{\psi}_\mu i) + \frac{i \kappa}{2 \sqrt{2}} \mathcal{F}_{\mu \nu} (\mathcal{E}_{ab} \lambda^a \sigma^\mu \lambda^b - c.c.)$$

$$+ i M^{IJ} k_{Ja} \lambda^a \sigma^\mu \bar{\lambda}^\bar{a} [\mathcal{H}_{\mu I} + i \kappa (g_{Jb} \psi^i_\mu \lambda^b + c.c.)]$$

$$+ \kappa^2 M^{IJ} (g_{Jb} \psi^i_\mu \lambda^a + c.c.) (g_{Ja} \lambda^b \sigma^\mu \psi^j_\nu + c.c.)$$

$$+ \frac{\kappa^2}{8} (\mathcal{E}_{ac} \mathcal{E}_{bd} \lambda^a \lambda^b \lambda^c \lambda^d + c.c.) - \frac{1}{4} V_{ab \bar{a} \bar{b}} \lambda^a \lambda^b \bar{\lambda}^\bar{a} \bar{\lambda}^\bar{b}. \quad (4.1)$$

This action is based on $n_T$ tensors $B_{\mu I}$, $I = 1, \ldots, n_T$ and $4n - n_T$ scalars $\phi^A$, $A = 1, \ldots, 4n - n_T$, together with $2n$ 2-component spinors $\lambda^a$, $a = 1, \ldots, 2n$, called the hyperinos. Furthermore there are the fields from the supergravity multiplet: the vielbeins $e^m_\mu$, the graviphoton $A_\mu$ and the gravitinos $\psi^i_\mu$, $i = 1, 2$. We have performed the Wick rotation as discussed in appendix B.

Our conventions are such that all the field dependent quantities, such as $M^{IJ}$, $\mathcal{G}_{AB}$ and so on, are the same as in the Lorentzian case. Sign changes or different factors of $i$ are never absorbed, but always written explicitly. The condition that the supersymmetry algebra closes and that the action is invariant, imposes constraints on (and relations between) the various quantities appearing in the action. They are the same as in the Lorentzian case. We list a number of them in appendix C, where we will also specify the various covariant derivatives. For more details see [76].

In this chapter we only need transformation rules and supersymmetry relations for the instanton background. We will work up to linear order in the fermions. The supersymmetry transformations of the scalars are given by

$$\delta_\epsilon \phi^A = \gamma^A_{ia} \epsilon^i \lambda^a + \bar{\gamma}^a_{i \bar{a}} \bar{\epsilon}_i \bar{\lambda}^\bar{a}. \quad (4.2)$$
The transformation rules for the fermions (up to linear order in the fermions) are
\[
\begin{align*}
\delta_\epsilon \lambda^a &= (i \partial_\mu \phi^A W^a_A + H_{\mu I} f^{Ia} ) \sigma^\mu \bar{\epsilon}_i + \ldots \\
\delta_\epsilon \bar{\lambda}^i &= (i \partial_\mu \phi^A W^{-a}_A + H_{\mu I} f^{-Ia} ) \bar{\sigma}^\mu \epsilon^i + \ldots ,
\end{align*}
\] (4.3)
with scalar dependent functions \(\gamma, W\) and \(f\). We have left out higher order terms in the fermions, denoted by the ellipses.

The transformation of the tensors is given by
\[
\delta_\epsilon B_{\mu\nu I} = 2i g_{Iia} \epsilon^i \sigma_{\mu
u} \lambda^a - 4\kappa^{-1} \Omega_{I j} \epsilon^i \sigma_{\mu\nu} \bar{\psi}_{[\nu} \psi_{i]} + \text{c.c.} .
\] (4.4)

The transformations of the supergravity multiplet are given by
\[
\begin{align*}
\delta_\epsilon e^m_{\mu} &= i\kappa (\epsilon^i \sigma^m \bar{\psi}_{\mu i} - \psi^i_{\mu} \sigma^m \bar{\epsilon}_i) \\
\delta_\epsilon A_\mu &= i\sqrt{2} (\epsilon^i \bar{\psi}^i_{\mu} + \bar{\epsilon}^i \psi_{\mu i}) \\
\delta_\epsilon \psi^i_{\mu} &= \kappa^{-1} D_\mu \epsilon^i + \frac{1}{\sqrt{2}} \varepsilon^{ij} F^+_{\mu\nu} \sigma^\nu \bar{\epsilon}_j - i\kappa^{-1} H_{\mu I} \Gamma^{Ij}_i \epsilon^j + \ldots \\
\delta_\epsilon \bar{\psi}_{\mu i} &= \kappa^{-1} D_\mu \bar{\epsilon}_i + \frac{1}{\sqrt{2}} \varepsilon_{ij} F^-_{\mu\nu} \bar{\sigma}^\nu \epsilon^j + i\kappa^{-1} H_{\mu I} \Gamma^{Ij}_i \bar{\epsilon}_j + \ldots ,
\end{align*}
\]
where, in the last two lines, we have denoted the (anti-) selfdual graviphoton field strengths by \(F^\pm_{\mu\nu} = \frac{1}{2} (F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \) and we have dropped fermion bilinears.

The quantities \(g_{Iia}, W_A^i\), etc. appearing in the above equations are functions of the scalar fields and satisfy the relations (C.1)–(C.9). Moreover, we have the relation
\[
\Gamma^{Ij}_i = M^{IJ} \Omega_{J j}^i ,
\] (4.5)
between the coefficients which appear in the supersymmetry transformations of the gravitinos and tensors, respectively.

Note that the fermions (and gravitini) are no longer related to each other by complex conjugation. If we write ‘c.c.’ in the action or anywhere else, we mean the analytic continuation of the complex conjugated expressions in Lorentzian signature. We work in the so-called 1.5 order formalism. This means that the spin connection is a function of other fields which is determined by its own (algebraic) field equation.

### 4.2 The supersymmetric DTM

We will discuss the supersymmetry transformation rules of the DTM at linear order in the fermions. Action (4.1) is a general action for scalars...
and tensors coupled to $N = 2$ supergravity. The action of the DTM is a specific example of action (4.1). We can obtain it by taking $n = 1$ and $n_T = 2$ so there are 2 scalars and 2 tensors, as discussed in section 2.3. Furthermore there are 2 spinors ($\lambda^1$, $\lambda^2$), 2 gravitini ($\psi^1_\mu$, $\psi^2_\mu$) and their conjugate counterparts. Specifying, as in section 2.3, to

$$M^{IJ} = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix}, \quad G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\phi} \end{pmatrix}, \quad A^I_A = 0$$

we obtain the bosonic part of the DTM. From now on we will work in units of $\kappa^2 = \frac{1}{2}$ and rescale the supersymmetry parameters $\epsilon^i$ by a factor of $\sqrt{2}$ for convenience.

Note that we have set $2A^I_A$ equal to zero. Strictly speaking we could have allowed for a nonvanishing connection $A^I_A$ with trivial field strength $F_{AB} = 2\partial_A A^I_B = 0$. Such connections are pure gauge and lead to total derivatives in the action, which could be dropped in perturbation theory. Nonperturbatively they can be nonvanishing and lead to imaginary theta-angle-like terms. We have discussed and included such terms separately in (2.35) and (3.15), so it suffices to set $A^I_A = 0$. In appendix D we list the functions $g_{ia}$, $W^a_A$, etc. for the DTM.

The linearized Euclidean supersymmetry transformations of the fermions can be written as

$$\delta_\epsilon \lambda^a = i\sqrt{2} E^{ai}_\mu \sigma^\mu \bar{\epsilon}_i, \quad \delta_\epsilon \bar{\psi}_\mu^i = 2\bar{D}_\mu^i \bar{\epsilon}_j + \varepsilon_{ij} F^{-}_{\mu \nu} \bar{\sigma}^\nu \epsilon^j$$

$$\delta_\epsilon \bar{\lambda}^{\bar{a}} = i\sqrt{2} \bar{E}^{\bar{a}i}_\mu \bar{\sigma}^\mu \bar{\epsilon}_i, \quad \delta_\epsilon \psi^i_\mu = 2D^i_\mu \bar{\epsilon}_j + \varepsilon^{ij} F^+_{\mu \nu} \sigma^\nu \bar{\epsilon}_j,$$

where we have introduced

$$E^{ai}_\mu = \partial_\mu \phi^A W^{ai}_A - iH_{iA} f^{Aai}, \quad \bar{D}^i_\mu = \delta^i_\mu \nabla_\mu - \partial_\mu \bar{\phi}^A \Gamma^A_{ij} + iH_{iA} \Gamma_{ij}$$

$$\bar{E}^{\bar{a}i}_\mu = \partial_\mu \bar{\phi}^A \bar{W}^{\bar{a}i}_A - i\bar{H}_{i\bar{A}} f^{\bar{A}ai}, \quad D^i_\mu = \delta^i_\mu \nabla_\mu + \bar{\partial}_\mu \phi^A \Gamma^A_{ij} - i\bar{H}_{i\bar{A}} \Gamma_{ij},$$

with $\nabla_\mu$ the Lorentz-covariant derivative. The observation that $\bar{E}_\mu$ and $D_\mu$ are related to their counterparts $E_\mu$ and $\bar{D}_\mu$ according to

$$\bar{E}^{\bar{a}i}_\mu = -h^{\bar{a}b} \varepsilon_{ab} E^{bi}_\mu \varepsilon_{ij}, \quad D^i_\mu \varepsilon_{ij} = \varepsilon^{ik} \bar{D}^k_\mu \varepsilon_{ij},$$

will prove useful.

The first identity is due to the relation (C.9), while the second is a consequence of $SU(2)$-covariant constancy of $\varepsilon_{ij}$.

---

1. $A^I_A \equiv M^{IJ} G_{JA}$ and arises in the dualization process from the UHM to the DTM, as discussed in section 2.4.

2. Note that in the second identity the covariant derivatives in $D_\mu$ and $\bar{D}_\mu$ are in the same representation of Spin(4), whereas in (4.7) they are not.
More explicitly, we have, at the linearized level,

\[
\delta_{\epsilon} \left( \frac{\lambda^1}{\lambda^2} \right) = i \left( e^{-\phi/2} \partial_{\mu} \chi - e^{\phi/2} \hat{H}_{\mu 1} \right) \left( \partial_{\nu} \phi + e^{\phi} H_{\nu 2} \right) \left( \sigma^\mu \bar{\epsilon}_1 \right) + \ldots
\]

\[
\delta_{\epsilon} \left( \frac{\bar{\lambda}^1}{\bar{\lambda}^2} \right) = i \left( e^{-\phi/2} \partial_{\mu} \chi + e^{\phi/2} \hat{H}_{\mu 1} \right) \left( \partial_{\nu} \phi - e^{\phi} H_{\nu 2} \right) \left( \bar{\sigma}^\mu \epsilon_1 \right) + \ldots
\]

for the matter fermions, and

\[
\delta_{\epsilon} \left( \frac{\psi^1_{\mu}}{\psi^2_{\mu}} \right) = \left( 2\nabla_{\mu} + \frac{1}{2} e^\phi H_{\mu 2} \right) \left( -e^{-\phi/2} \partial_{\mu} \chi + e^{\phi/2} \hat{H}_{\mu 1} \right) \left( \epsilon^1 \right) + \ldots
\]

\[
\delta_{\epsilon} \left( \frac{\bar{\psi}_{\mu 1}}{\bar{\psi}_{\mu 2}} \right) = \left( 2\nabla_{\mu} - \frac{1}{2} e^\phi H_{\mu 2} \right) \left( -e^{-\phi/2} \partial_{\mu} \chi - e^{\phi/2} \hat{H}_{\mu 1} \right) \left( \bar{\epsilon}_1 \right) + \ldots
\]

for the gravitinos, where we have omitted the graviphoton terms.

We end this section by giving the fermionic equations of motion, at the linearized level. For the hyperinos we find

\[
i\sigma^{\mu \rho} D_{\mu} \lambda^a + H_{\mu I} \Gamma^I_{ab} \sigma^{\mu} \bar{\lambda}^b + \frac{i}{2} h^{\bar{a}}_{ab} E_{\alpha \beta} F^{\alpha \mu \nu} \sigma^{\mu \nu} \lambda^b = -\frac{1}{\sqrt{2}} \sigma^{\nu} \bar{E}^{a}_{\mu} \bar{\sigma}^{\mu} \psi^i \]

\[
i\bar{\sigma}^{\mu} D_{\mu} \lambda^a + H_{\mu I} \Gamma^I_{ab} \bar{\sigma}^{\mu} \lambda^b - \frac{i}{2} h^{\bar{a}}_{ab} \bar{E}_{\alpha \beta} F_{\mu \nu}^{\alpha \beta} \bar{\sigma}^{\mu \nu} \bar{\lambda}^b = -\frac{1}{\sqrt{2}} \bar{\sigma}^{\nu} \bar{E}^{ai}_{\mu} \sigma^{\mu} \bar{\psi}_{i}.
\]

(4.12)  

(4.13)

For the definition of the covariant derivative $D_{\mu}$ see appendix C. What makes these different from the usual Dirac-like equation is the presence of the mixing term with the (anti-) selfdual graviphoton field strength and the inhomogeneous gravitino term originating from its coupling to the rigid supersymmetry current of the double-tensor multiplet. This will become important in the discussion of the fermionic zero modes. The gravitino field equations read

\[
i\tilde{z}^{\mu \nu \rho \sigma} \sigma_{\mu} D_{\sigma} \bar{\psi}_{\nu \alpha} - \tilde{z}^{\mu \nu \rho \sigma} H_{\sigma I} \Gamma^I_{\alpha j} \sigma_{\rho} \bar{\psi}_{\nu j} - i F^{\mu \nu -} \psi_{\nu i} = \frac{1}{2\sqrt{2}} h_{\alpha \bar{a}} \bar{E}^{a}_{\nu i} \sigma^{\nu} \sigma^{\mu} \lambda^a
\]

\[
i\tilde{z}^{\mu \nu \rho \sigma} \bar{\sigma}_{\rho} D_{\sigma} \psi^i_{\nu} - \tilde{z}^{\mu \nu \rho \sigma} H_{\sigma I} \Gamma^I_{\alpha j} \sigma_{\rho} \psi^j_{\nu} + i F^{\mu \nu +} \bar{\psi}^i_{\nu} = -\frac{1}{2\sqrt{2}} h_{\alpha \bar{a}} E^{ai}_{\nu} \tilde{\sigma}^{\nu} \sigma^{\mu} \bar{\lambda}^a.
\]

(4.14)  

(4.15)

Note that one can combine the first two terms on the left-hand side into the operator $D_{\mu} \tilde{z}^{i}_{\nu j}$, as defined in (4.8).
4.3 (Un)broken supersymmetry

Often a specific solution to the equations of motion will break most (or all) of the symmetries of the theory. The symmetries which are not broken by the solution are called unbroken of residual symmetries and form a symmetry group. The symmetries which are broken by the solution are called the broken symmetries and can be used to generate new solutions.

For instance, the single-centered\(^4\) instanton solution (4.23) is not translationally invariant because it is located at a specific point \(\{x_0\}\). This is sometimes phrased by saying that the instanton ‘breaks translational invariance’. Because the underlying theory is translationally invariant, this will manifest itself in the degeneracy of the solutions related to each other by translations. Indeed, the action (3.14) does not depend on \(\{x_0\}\). These solutions can be translated into each other by acting with the broken symmetry. This leads to the notion of collective coordinates. These are the coordinates generated by the broken symmetries, in this case simply the position \(\{x_0\}\).

Similarly, the solutions of a supergravity theory are in general not invariant under the supersymmetry transformations that leave the theory invariant. The solutions that are preserve part of the supersymmetries are often called BPS\(^5\) solutions.

Schematically (local) supersymmetry transformations take the form

\[
\delta_\epsilon B \sim \epsilon F
\]
\[
\delta_\epsilon F \sim \partial \epsilon + B \epsilon ,
\]

(4.16)

where \(B\) stands for bosons and \(F\) for fermions. Typically one is interested in purely bosonic configurations. According to the general definition given above, the bosonic solutions will be supersymmetric if there is an \(\epsilon(x)\) such that (4.16) vanishes. The bosonic fields are trivially invariant (because the fermionic fields are zero) and one only needs to examine the equation

\[
\delta_\kappa F \sim (\partial + B)\kappa = 0 .
\]

(4.17)

The commutator of 2 such spinors will give an (infinitesimal) Killing vector which generates an isometry of the bosonic background. Equation (4.17) is often called the Killing spinor equation.

As we shall see in the next subsection, the instanton configuration preserves only part of the supersymmetries. The supersymmetric vacuum is

\(^4\)The multicentered case is similar, but we will mainly use the single-centered one in this chapter.

\(^5\)After Bogomol’nyi-Prasad-Sommerfield [88, 89]. For more information on BPS solutions and supersymmetry see for example [90, 15] and references therein.
the trivial one with all fields equal to zero. The broken supersymmetries will generate new solutions characterized by their fermionic collective coordinates. Thus if we start with a purely bosonic configuration which is a solution to the equations of motion, acting with the broken supersymmetries generates fermionic fields. Naturally, this new configuration is also a solution to the equations of motion because the broken supersymmetries are symmetries of the theory.

There is a lot more to say about this subject, and we refer to the literature, e.g. [91, 92, 15] and the references therein. In the following we present a detailed inspection of the (un)broken supersymmetries in the background of our instanton solution. The case of the anti-instanton is similar.

4.3.1 Unbroken supersymmetry

Which supersymmetries leave the instanton solution (3.7, 3.8, 3.6) invariant, or equivalently, which supersymmetries are left unbroken by this solution? The background is determined by the instanton solution for the bosonic fields as in (3.7, 3.8, 3.6). The fermionic fields are all equal to zero. The supersymmetry transformation rules of the bosonic fields contain the fermionic fields, which means that the bosons are always invariant under supersymmetry transformations. To find the unbroken supersymmetries we therefore only have to examine the conditions

\[ \delta_\epsilon \lambda^a = \delta_\epsilon \bar{\lambda}^{\bar{a}} = \delta_\epsilon \psi^i_\mu = \delta_\epsilon \bar{\psi}^{\bar{i}}_{\mu} = 0 \]  

where \( \{a, \bar{a}, i\} = 1, 2 \). This will put certain constraints on the supersymmetry parameters \( \bar{\epsilon}_i(x) \) and \( \epsilon^i(x) \). We will focus on the \( \bar{\epsilon}_i(x) \) because the \( \epsilon^i(x) \) can easily be obtained from those as we shall see.

It is convenient to consider \( \delta_\epsilon \bar{\psi}^{\bar{i}}_{\mu} \) first. Using (4.11) and (3.6) we obtain

\[ 0 = \frac{1}{2} \delta_\epsilon \bar{\psi}_{\mu}^{\bar{2}} = \left( \partial_\mu - \frac{1}{4} \partial_\mu \phi \right) \bar{\epsilon}_2 , \]

defining \( \bar{\epsilon}_2 \equiv e^{1/4\phi} \bar{\eta}_2 \) gives

\[ e^{\phi/4} \partial_\mu \bar{\eta}_2 = 0 , \]

which means that \( \bar{\eta}_2 \) is a constant spinor. We can rewrite the \( \lambda^a \) variations (4.10) as

\[ \delta_\epsilon \lambda^1 = -2i \sigma^\mu \left( \partial_\mu - \frac{1}{2} \partial_\mu \phi - \frac{1}{4} e^\phi H_{\mu} \right) \bar{\epsilon}_2 + i \sigma^\mu \delta_\epsilon \bar{\psi}^{\bar{i}}_{\mu} \]  

\[ \delta_\epsilon \lambda^2 = +2i \sigma^\mu \left( \partial_\mu - \frac{1}{2} \partial_\mu \phi + \frac{1}{4} e^\phi H_{\mu} \right) \bar{\epsilon}_1 - i \sigma^\mu \delta_\epsilon \bar{\psi}^{\bar{i}}_{\mu} . \]
Upon using (3.6) these equations simplify. Equation (4.19) becomes
\[ \delta_{\epsilon} \lambda^1 = -2i\sigma^\mu \left( \partial_\mu - \frac{1}{4} \partial_\mu \phi \right) e^{\phi/4} \bar{\eta}_2 = 0 , \]
where we used that \( \bar{\epsilon}_2 \equiv e^{1/4\phi} \bar{\eta}_2 \) and the fact that \( \delta_{\bar{\epsilon}_2} \psi_{\mu 2} = 0 \). Because this equation is identically zero, it imposes no new constraints. Next on the list is \( \delta_{\epsilon} \bar{\psi}_{\mu 1} \) which is, using (3.6) and the result for \( \bar{\epsilon}_2 \)
\[ 0 = \frac{1}{2} \delta_{\epsilon} \bar{\psi}_{\mu 1} = \left( \partial_\mu + \frac{1}{4} \partial_\mu \phi \right) \bar{\epsilon}_1 - e^{-\phi/4} \partial_\mu \chi \bar{\eta}_2 = e^{-\phi/4} (\partial_\mu \bar{\eta}_1 - \partial_\mu \chi \bar{\eta}_2) \]
where we have defined \( \bar{\epsilon}_1 \equiv e^{-\phi/4} \bar{\eta}_1 \) in the last step, \( \bar{\eta}_1 \) is some spinor. We have defined \( \bar{\eta}_1 = e^{-\phi/4} \bar{\eta}_1 + \partial_\nu h \sigma^\nu \xi \), with \( h \) a harmonic function and \( \xi \) a constant spinor. However we still have to deal with equation (4.21) and we did not find any non-trivial solutions for the \( \partial_\mu h \bar{\alpha}^\nu \xi \) part. Therefore we only consider \( \bar{\eta}_1 = e^\phi \bar{c}_1 \). Relation (4.21) then becomes
\[ e^\phi \partial_\mu \phi \bar{c}_1 = \partial_\mu \chi \bar{\eta}_2 , \]
which is only satisfied for
\[ \chi = ae^\phi + b , \]
with some constants \( a, b \) and \( \bar{c}_1 = a\bar{\eta}_2 \). In the calculations above we have not used single-centeredness of our instanton solutions, so \( \phi \) and \( \chi \) were still determined by (3.7) and (3.8). Now we see that equation (4.22) imposes a new constraint because we can only write \( \chi \) in this particular form if, see (3.11), all \( \chi_i \) are equal. Note that (4.22) follows automatically, for some constants \( a \) and \( b \) that we will determine in a moment, if one imposes spherical symmetry as in that case.
the harmonic function $e^{-\phi}\chi$ depends linearly on the harmonic function $e^{-\phi}$. This means that we are dealing with a single-centered instanton. Equations (3.7) and (3.8) then simplify:

$$e^{-\phi} = e^{-\phi}\infty + \frac{|Q_2|}{4\pi^2(x-x_0)^2}$$

$$e^{-\phi}\chi = e^{-\phi}\chi\infty + \frac{Q_1}{4\pi^2(x-x_0)^2}$$

and the last equation can indeed be rewritten as

$$\chi = e^{\phi\infty}\Delta\chi e^{\phi} + \chi_0 = g_s^2\Delta\chi e^{\phi} + \chi_0 ,$$

with $\Delta\chi \equiv \chi\infty - \chi_0$ and $\chi_0 = \frac{Q_1}{|Q_2|}$, see (3.11). We can thus make the identifications $a = g_s^2\Delta\chi$ and $b = \chi_0$. The action for multi-centered instantons (3.13) reduces to single-centered one (3.14). In the following we will always consider single-centered instantons, or equivalently, spherically symmetric ones.

The same analysis can be performed for anti-instantons, the total result is

$$\bar{\epsilon}_1 = e^{\phi/2}(g_s^2\Delta\chi)^{(1\pm 1)}e^{\pm\phi/4}\bar{\eta}$$

$$\bar{\epsilon}_2 = \pm e^{\phi/2}(g_s^2\Delta\chi)^{(1\mp 1)}e^{\mp\phi/4}\bar{\eta} ,$$

where the (lower) upper sign corresponds to (anti-)instantons. We have given $\bar{\eta}_2$ the new name $\bar{\eta}$. We can write this result concisely as $\bar{\epsilon}_i(x) = u_i(x)\bar{\eta}$, where the $u_i(x)$ can be read off from (4.25) and (4.26).

The Killing spinors of opposite chirality are then given by $\epsilon^i(x) = \epsilon^{ij}u_j(x)\eta$ with $\eta$ another (unrelated) constant spinor. This immediately follows from (4.9): if $u_i$ are (spinless) zero modes of $E_\mu$ and $\bar{D}_\mu$, then $\epsilon^{ij}u_j$ are zero modes of $\bar{E}_\mu$ and $D_\mu$. The conclusion of this analysis is that the NS 5-brane instanton in flat space leaves one half of the supersymmetries unbroken. In other words, the instanton solution is a BPS configuration.

Although saying that the instanton preserves half of the supersymmetries is the standard way in which to phrase this result, it is slightly misleading. Namely, we go from 4 arbitrary spinors $(\bar{\epsilon}_i, \epsilon^i)$ to 2 constant spinors $(\bar{\eta}, \eta)$ together with a very specific space dependence via $e^\phi$. So we really go from 4 times ‘infinitely many’ supersymmetries, to 2 very specific ones. Note that the trivial solution with both $e^{-\phi}$ and $\chi$ constant, preserves all the supersymmetries. Thus the instanton configuration tends asymptotically to the maximally supersymmetric vacuum.

**The membrane**

For the sake of completeness we briefly discuss the unbroken supersymmetries in the case of the membrane instanton configuration in a similar
fashion, although we will not need them explicitly. As explained in section 3.3, the simplest membrane instanton solution is characterized by taking $Q_2 = \Delta \chi = 0$ which means that $H_{\mu 2} = 0$. The solution is then specified by

$$H_{\mu 1} = \pm \partial_\mu h ,$$

with $h$ given by (3.22). To find the unbroken supersymmetries we again have to examine the conditions (4.18), but now in the membrane background. For convenience we define

$$\lambda^\pm \equiv \frac{1}{2} (\lambda^1 \pm \lambda^2) \quad \psi^\pm_\mu \equiv \frac{1}{2} (\psi^1_\mu \pm \psi^2_\mu) \quad \epsilon^\pm \equiv \frac{1}{2} (\epsilon^1 \pm \epsilon^2)$$

and similarly for barred quantities. The supersymmetry variations (4.10) and (4.11) become

$$\delta \lambda^+ = -2i \partial_\mu \phi \bar{\sigma}^\mu \epsilon^- \quad \delta \lambda^- = 0$$

$$\delta \psi^+_\mu = 2 (\partial_\mu - \frac{1}{2} \partial_\mu \phi) \epsilon^+ \quad \delta \psi^-_\mu = 2 (\partial_\mu + \frac{1}{2} \partial_\mu \phi) \epsilon^- .$$

The unbroken supersymmetries are then given by

$$\epsilon^+ = e^{\phi/2} \eta^+ \quad \epsilon^- = 0 ,$$

with $\eta^+$ a constant spinor. This means that the membrane instanton preserves half of the supersymmetries as well, it is also a BPS solution. The broken supersymmetries can be used to generate solutions for the fermions as will be shown in the case of the 5-brane instanton in the next section. We will not demonstrate this explicitly for the membrane.

### 4.3.2 Broken supersymmetries

Now we will turn our attention to the broken symmetries. As we have explained, these will generate new solutions. Specifically, starting from a purely bosonic background (as determined by the instanton solution (3.7, 3.8, 3.6)) the broken supersymmetries will generate solutions for the fermions. Alternatively, if we are going to generate new (classical) solutions for the fermions, we can also try to construct them by directly solving the equations of motion (linear in the fermions).

This will be the plan for this section: we will solve the equations of motion (up to linear order in the fermions) for the fermions and then show how to obtain these solutions by means of the broken supersymmetries. The solutions therefore will depend on fermionic collective coordinates. These will appear as integration constants if one uses the equations of motion and as supersymmetry parameters in the other approach. The importance of these solutions will become clear in the next section, where we will see that a more apt name for them is zero modes.
The hyperinos

First we shall solve the equations of motion for the hyperinos, which are coupled to the gravitinos, (4.12, 4.14). For the DTM we have (see appendix D) $\Gamma^{a}_{b} = \Gamma^{1a}_{b} = 0$ and $\Gamma^{2a}_{b}$, is given by (D.2). Using (3.6) we obtain\(^6\)

$$i\sigma^{\mu} \begin{pmatrix} \partial_{\mu} \lambda^{1} - \frac{3}{4} \partial_{\mu} \phi \bar{\lambda}^{1} \\ \partial_{\mu} \bar{\lambda}^{2} + \frac{3}{4} \partial_{\mu} \phi \lambda^{2} \end{pmatrix} = -\sigma^{\nu} \sigma^{\mu} \begin{pmatrix} e^{-\phi/2} \partial_{\mu} \chi \psi_{\nu}^{1} + \partial_{\mu} \phi \psi_{\nu}^{2} \\ 0 \end{pmatrix}$$

(4.27)

and similarly for the $\lambda^{a}$:

$$i\sigma^{\mu} \begin{pmatrix} \partial_{\mu} \lambda^{1} + \frac{3}{4} \partial_{\mu} \phi \lambda^{1} \\ \partial_{\mu} \lambda^{2} - \frac{3}{4} \partial_{\mu} \phi \lambda^{2} \end{pmatrix} = \sigma^{\nu} \sigma^{\mu} \begin{pmatrix} 0 \\ \partial_{\mu} \phi \bar{\psi}_{\nu}^{1} - e^{-\phi/2} \partial_{\mu} \chi \bar{\psi}_{\nu}^{2} \end{pmatrix} .$$

(4.28)

It is convenient to start with $\bar{\lambda}^{2}$ and $\lambda^{1}$ because they decouple from the gravitinos. To this end, consider the more general operator

$$\mathcal{D}^{k} \equiv \sigma^{\mu} (\partial_{\mu} - k \partial_{\mu} \phi) = e^{k\phi} \partial / e^{-k\phi} , \quad k \in \mathbb{R} .$$

(4.29)

The zero modes of $\mathcal{D}^{k}$ are in one to one correspondence to those of $\partial$. But whereas the zero modes $\bar{\zeta}$ of $\partial$ ($\partial \bar{\zeta} = 0$) are not normalizable\(^7\) the corresponding modes of $\mathcal{D}^{k} (\lambda = e^{k\phi} \bar{\zeta})$ are only normalizable for appropriate values of $k$.

In flat Euclidean space the only solution for $\bar{\zeta}$ is a constant spinor. However, when the origin is cut out, as it is in our case, there is a nontrivial solution:

$$\bar{\zeta}(x) = 2i \partial_{\mu} h(x) \bar{\sigma}^{\mu} \xi ,$$

where $\xi$ is a constant spinor, $h$ is a harmonic function and we included the factor of $2i$ for later convenience. This is the only solution as one can show by writing $\bar{\zeta}$ as

$$\bar{\zeta}(x) = 2i f_{\mu}(x) \bar{\sigma}^{\mu} \xi$$

for an arbitrary real function $f_{\mu}$. The equation $\partial \bar{\zeta} = 0$ then imposes

$$\partial_{\mu} f_{\nu} = \partial_{\mu} f^{\mu} = 0 .$$

The constant solution cannot lead to a normalizable solution and must therefore be discarded. The possible normalizable solutions to $\mathcal{D}^{k} \bar{\lambda} = 0$ are given by

$$\bar{\lambda} = 2i e^{k\phi} \partial_{\mu} h \bar{\sigma}^{\mu} \xi .$$

\(^{6}\)Unless stated otherwise we choose the instanton background, i.e., the plus sign. The calculation with the minus sign is of course similar.

\(^{7}\)Normalizable zero modes $Z$ must satisfy $\int_{0}^{\infty} dr r^{3} |Z|^{2} < \infty$. This implies that $Z$ must go to zero as $r^{-5/2}$ or faster at infinity and it may not diverge faster that $r^{-3/2}$ at the origin.
As we are dealing with spherical solutions (4.23, 4.24), the only harmonic function available is given by $e^{-\phi}$. Thus the normalizable zero modes to $D_k$ are given by

$$\bar{\lambda} = 2ie^{k\phi}\partial_{\mu}e^{-\phi}\bar{\sigma}^\mu\xi \quad k \geq \frac{3}{4},$$

where $k \geq 3/4$ can be found by inspecting the asymptotic behaviour.

Using these results we can now easily write down the solutions for the hyperinos:

$$\bar{\lambda}_2 = 0 \quad \lambda_1 = 0.$$  \hspace{1cm} (4.30)

Because their equations of motion correspond to $k = -3/4$, which would lead to non-normalizable zero modes, they must be set to zero.

If there would be no gravitinos present in (4.27, 4.28) the solutions for $\bar{\lambda}_1$ and $\lambda_2$ would be given by

$$\bar{\lambda}_1 = 2ie^{3\phi/4}\partial_{\mu}e^{-\phi}\bar{\sigma}_1 \quad \lambda^2 = 2ie^{3\phi/4}\partial_{\mu}e^{-\phi}\sigma^\mu\bar{\xi}_2,$$  \hspace{1cm} (4.31)

with $\xi_1$ and $\bar{\xi}_2$ constant spinors.

As we will demonstrate now, there are in fact no normalizable solutions for the gravitinos.

**The gravitinos**

In the instanton background the equations of motion for the gravitinos (4.14, 4.15) become, using appendix D,

$$i\varepsilon^{\mu\nu\rho\sigma}\sigma_\rho \left( \partial_\sigma \bar{\psi}_{\nu 1} + \frac{1}{4}\partial_\sigma \phi \bar{\psi}_{\nu 1} - e^{-\phi/2}\partial_\sigma \chi \bar{\psi}_{\nu 2} \right) = \frac{1}{2} \left( e^{-\phi/2}\partial_\nu \chi \right) \sigma^\nu \bar{\sigma}^\mu \lambda^1.$$  \hspace{1cm} (4.32)

We notice that only $\lambda^1$ couples to the gravitino, but we just (4.30) concluded that $\lambda^1 = 0$. This means that the gravitinos decouple. If we define

$$\bar{\psi}_{\mu 2} \equiv e^{\phi/4}\bar{\xi}_{\mu 2},$$

we can, using (B.2), rewrite the second equation in (4.32) as

$$\sigma^\mu \left( \partial_\mu \bar{\xi}_{\nu 2} - \partial_\nu \bar{\xi}_{\mu 2} \right) = 0.$$  \hspace{1cm} (4.33)

There are many solutions to this equation, e.g. $\bar{\xi}_{\mu 2} = \partial_\mu \bar{\xi}_2$, but we still have to impose a (supersymmetry) gauge. We choose to work with the standard gauge

$$\bar{\sigma}^\mu \bar{\psi}_{\mu i} = \sigma^\mu \bar{\psi}_{\mu i} = 0.$$  \hspace{1cm} (4.34)
in particular $\sigma^\mu \zeta_{\mu 2} = 0$. Equation (4.33) simplifies to $\partial \zeta_{\mu 2} = 0$ which has the solution

$$\zeta_{\mu 2} = \partial_\mu \partial_\nu h \bar{\sigma}^\nu \xi ,$$

(4.35)

with $\xi$ a constant spinor and $h$ a harmonic function. For a given solution one can still act with a supersymmetry transformation. These supersymmetry transformations can either transform the solution for the gravitino in such a way that it still respects the gauge (4.34), or not. We must therefore check if there are residual supersymmetry transformations that produce new solutions for the gravitino that still obey the gauge condition (4.34).

The supersymmetry transformations for the gravitinos (4.11) simplify to

$$\delta_\epsilon \bar{\psi}_{\mu 1} = 2 \left( \partial_\mu + \frac{1}{4} \partial_\mu \phi \right) \bar{\epsilon}_1 - 2 e^{-\phi/2} \partial_\mu \chi \bar{\epsilon}_2$$

$$\delta_\epsilon \bar{\psi}_{\mu 2} = 2 \left( \partial_\mu - \frac{1}{4} \partial_\mu \phi \right) \bar{\epsilon}_2 ,$$

(4.36)

where we again have used the values for the instanton background and appendix D. We can write $\delta_\epsilon \bar{\psi}_{\mu 2}$ as

$$\delta_\epsilon \bar{\psi}_{\mu 2} = 2 \partial_\mu \left( e^{-\phi/4} \bar{\epsilon}_2 \right) ,$$

a total derivative, which vanishes in (4.33). We should also impose the gauge condition, i.e.

$$0 = \sigma^\mu \left( \delta_\epsilon \bar{\psi}_{\mu 2} \right) = \partial \left( e^{-\phi/4} \bar{\epsilon}_2 \right) .$$

(4.37)

The solutions are given by a constant or by the derivative of a harmonic function. Acting with such a residual supersymmetry transformation gives a solution which is contained within the class of solutions (4.35). Therefore the only solution for $\bar{\psi}_{\mu 2}$ is given by

$$\bar{\psi}_{\mu 2} = e^{\phi/4} \partial_\mu \partial_\nu h \bar{\sigma}^\nu \xi ,$$

where we can again choose $h = e^{-\phi}$ for spherically symmetric harmonic functions.

However, this solution is not normalizable because it diverges too fast at the origin. Consequently there is no normalizable solution for $\bar{\psi}_{\mu 2}$ and we must set it to zero in the hyperino equations of motion.

A similar analysis shows that there are no normalizable solutions for $\bar{\psi}_{\mu 1}$ and the unbarred gravitinos either, so we set them to zero in (4.27) and (4.28). Summarizing, we have the following results

$$\bar{\psi}_{\mu i} = \psi^i_{\mu} = \lambda^1 = \bar{\lambda}^2 = 0$$

$$\lambda^1 = 2ie^{3\phi/4} \partial_\mu e^{-\phi} \bar{\sigma}^\mu \xi \quad \lambda^2 = 2ie^{3\phi/4} \partial_\mu e^{-\phi} \sigma^\mu \xi ,$$

(4.37)
with two constant spinors $\xi$ and $\bar{\xi}$.

**Using the broken supersymmetries**

We have constructed all the solutions to the equations of motion (linear in the fermions) with certain boundary conditions. We shall now show that we can generate these solutions with the broken supersymmetries.

As it turns out, the broken supersymmetries are contained in the residual supersymmetry transformations left after gauge fixing the gravitinos, i.e. those $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ satisfying $\sigma^\mu \delta_\epsilon \bar{\psi}_{\mu 1} = \sigma^\mu \delta_\epsilon \bar{\psi}_{\mu 2} = 0$. The nontrivial solutions to this equation are given by

$$\tilde{\epsilon}_2 = e^{\phi/4} \bar{\eta}, \quad (4.38)$$

with $\partial_\mu \bar{\eta} = 0$. This is precisely the Killing spinor from (4.26), so it leaves the hyperinos invariant. The broken supersymmetries for $\tilde{\epsilon}_1$ are determined by those transformations that leave $\bar{\psi}_{\mu 1}$ invariant and preserve the gauge condition (4.37) but act nontrivially on the hyperinos. Using (4.36) and (4.38) we find

$$\tilde{\epsilon}_1 = e^{-\phi/4} (\chi \bar{\eta} + \tilde{\xi}'), \quad (4.39)$$

where $\tilde{\xi}'$ is another constant spinor. If we insert $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ into the transformations of the $\lambda^a$ we find that (after a redefinition $\tilde{\xi}' = \tilde{\xi} - \lambda_0 \bar{\eta}$) the spinor proportional to $\bar{\eta}$ is precisely the Killing spinor (4.25). The spinor proportional to $\tilde{\xi}$ does generate a new solution for $\lambda^2$:

$$(1)^1 \lambda^1 = 0 \quad (1)^2 \lambda^2 = -2ie^{-\phi/4} \partial_\mu \phi \sigma^\mu \tilde{\xi}, \quad (4.40)$$

where we included a factor of $2i$ for convenience. These are precisely the solutions obtained earlier, see (4.30, 4.31). Equation (4.9) relates the Killing spinors of opposite chirality, which gives

$$\epsilon^1 = 0 \quad \epsilon^2 = -e^{-\phi/4} \xi. \quad (4.41)$$

These generate

$$(1)^1 \bar{\lambda}^1 = -2ie^{-\phi/4} \partial_\mu \phi \bar{\sigma}^\mu \xi \quad (1)^2 \bar{\lambda}^2 = 0, \quad (4.42)$$

---

8Note that we use a slightly different notation now, compare with (4.31). The spinors $\xi$ and $\bar{\xi}$ are unrelated.

9Namely that they are normalizable, which imposes certain conditions on their asymptotic behaviour.

10The supersymmetry parameters which generate new solutions for the hyperinos are summarized in (4.42).

11Remember that is really just a very concise way of doing the same calculation for the barred sector.
The NS-brane also a perfect match with (4.31).
Note that these solutions depend on $x$ and the collective coordinate $\{x_0\}$.
We have introduced some new notation: the superscript $(1)$ left of the $\lambda$’s indicates that this solution is obtained by applying a supersymmetry transformation once. The resulting function therefore contains one Grassmann collective coordinate (GCC). We omit the superscript $(0)$.
Later on we will construct the scalar superpartner of the solutions for the hyperinos found above. The solution for the superpartner can be found by using (4.2) and the solutions for the hyperinos and the supersymmetry parameter which generated them. These solutions for scalars will carry the label $(2)$ because they contain two fermionic collective coordinates (one from the supersymmetry parameters and one from the hyperinos).
We can perform a similar analysis for the anti-instanton in which case we find
\[
(1)\lambda^2 = (1)\bar{\lambda}^1 = 0 \\
(1)\lambda^1 = -2ie^{-\phi/4}\partial_\mu \phi \sigma^\mu \bar{\xi} \\
(1)\bar{\lambda}^2 = +2ie^{-\phi/4}\partial_\mu \bar{\phi} \bar{\sigma}^\mu \bar{\xi}.
\]
For future convenience, we will introduce a very compact notation that links the hyperino labels ‘1’ and ‘2’ to the (anti-)instanton labels $(-)+$.
This is done in such a way that the hyperino labels 1 and 2 are denoted by upper and lower indices respectively. These indices are then further specified by indicating the background, i.e. instanton or anti-instanton.
In this notation the absence of fermionic zero modes is expressed by the equations $(1)\lambda^\pm = 0$ where the upper index is associated to the first hyperino in the instanton $(+)$ background and the lower label to the second hyperino in the anti-instanton $(-)$ background. Similarly we have $(1)\bar{\lambda}^\mp = 0$. For the broken supersymmetry parameters we have $\epsilon^\pm = \bar{\epsilon}^\mp = 0$ and for the $\lambda$’s we can write
\[
(1)\lambda^\mp = -2ie^{-\phi/4}\partial_\mu \phi \sigma^\mu \bar{\xi}^\mp \\
(1)\bar{\lambda}^\pm = \mp2ie^{-\phi/4}\partial_\mu \bar{\phi} \bar{\sigma}^\mu \bar{\xi}^\mp.
\]
These are generated by
\[
\epsilon^\mp = -e^{-\phi/4}\xi^\mp \\
\bar{\epsilon}^\pm = \pm e^{-\phi/4}\bar{\xi}^\pm,
\]
where the fermionic collective coordinates $\bar{\xi}^\pm$ are two independent constant spinors which distinguish between instantons and anti-instantons, similarly for $\xi^\pm$.
Where it is clear from the context what is meant, we shall drop the ± indices on $\xi$ and $\bar{\xi}$ for clarity.
As a last remark in this section, note the difference with Yang-Mills theories where fermionic zero modes only appear in one chiral sector. In the situation described above they are evenly distributed over the two (barred and unbarred) sectors.
4.4 The instanton measure

In the previous section we have seen how broken symmetries and collective coordinates are related. In this section we shall examine the consequences collective coordinates have for the path integral.

Consider a generic bosonic system without gauge invariance, with fields $\Phi^M$ and Euclidean action $S[\Phi]$. We can expand the fields around the instanton solution (which solves the equation of motion) as follows:

$$\Phi^M(x) = \Phi_{cl}^M(x, C) + \Phi_{qu}^M(x, C).$$  \hspace{1cm} (4.43)

The collective coordinates are collectively denoted by $C^{i_0}$ and the fluctuations $\Phi_{qu}^M$ depend on them as well. Note however that the field as a whole, i.e. $\Phi^M(x)$ does not. For a general field $\Phi^M(x)$, collective coordinates do not exist as such. These coordinates only become meaningful when the fields are expanded around a configuration which minimizes the action. Only then can some symmetries be broken, giving rise to collective coordinates. The collective coordinates in turn parametrize (by construction) only the classical solution and the fluctuations. Expanding the action around the instanton background gives

$$S = S_{cl} + \frac{1}{2} \int d^4 x \, \Phi_{qu}^M M_{MN} \Phi_{qu}^N + \mathcal{O}(\Phi^3),$$  \hspace{1cm} (4.44)

where $M_{MN}$ results from taking the second variation of the action with respect to the fields and is given by $M_{MN} = \mathcal{G}_{MP} \Delta_{PN}$. $\Delta_{PN}$ is a Hermitian operator with respect to the inner product for the fields defined by $\mathcal{G}_{MN}$ and can explicitly be calculated by expanding in the fluctuations. Being Hermitian it possesses a basis of eigenfunctions $F^M_i$ in which we can expand the fluctuations:

$$\Delta_{MN} F^N_i = \epsilon_i F^M_i$$  \hspace{1cm} (4.45)

$$\Phi_{qu}^M = \sum_i \xi_i F^M_i.$$  \hspace{1cm} (4.46)

There is a big caveat however, because the operator $\Delta$ is guaranteed to have zero modes, which we can see by starting with the equations of motion $\delta S \big|_{\Phi_{cl}} = 0$ and deriving with respect to the collective coordinates.

$$0 = \frac{\delta}{\delta C^{i_0}} \frac{\delta S}{\delta \Phi^N} \bigg|_{\Phi_{cl}} = \frac{\delta^2 S}{\delta \Phi^M \delta \Phi^N} \frac{\partial \Phi_{cl}^M}{\partial C^{i_0}},$$

which means that $Z^N_{i_0} \equiv \frac{\partial \Phi^N_{cl}}{\partial C^{i_0}}$ is a null vector\footnote{Admittedly, the question whether there are zero modes which cannot be obtained in this fashion remains an unsolved problem.} of $\Delta$.

We see that if we have a solution to the equations of motion and take
the derivative with respect to the collective coordinates, we obtain a zero mode. In the previous section we concluded that we can either generate fermionic solutions to the equations of motion by directly solving them or by applying the broken supersymmetries, naturally the results agreed. The collective coordinates were nothing but the fermionic parameters $\xi$ and $\bar{\xi}$ and for reasons which have just become clear we referred to those solutions as zero modes.

The following general treatment of zero modes in the path integral and the relation with collective coordinates focuses on bosonic fields for simplicity. The fermionic case is similar.

Some of the modes in (4.45) and (4.46) will have zero eigenvalue $\epsilon$ but nonzero fluctuation coefficient $\xi$. It is convenient to split (4.46) into two sets:

$$\Phi^M_{qu} = \sum_{i_0} \xi_{i_0} F^M_{i_0} + \sum_{i_q} \xi_{i_q} F^M_{i_q},$$

such that the $\{i_0\}$ run over the zero mode fluctuations $F^M_{i_0}$ which have $\epsilon_{i_0} = 0$ but $\xi_{i_0} \neq 0$ and the $\{i_q\}$ run over the non-zero mode fluctuations $F^M_{i_q}$. We see that we can identify $Z^M_{i_0} = F^M_{i_0}$.

We can define an inner product for the fluctuations as follows:

$$U_{ij} \equiv \int d^4x F^M_i G_{MN} F^N_j.$$

The action (4.44) then becomes

$$S = S_{cl} + \frac{1}{2} \sum_{i,j} \xi_{i} \xi_{j} U_{ij} = S_{cl} + \frac{1}{2} \left( \sum_{i_0,j_0} \xi_{i_0} \xi_{j_0} U_{i_0j_0} + \sum_{i_q,j_q} \xi_{i_q} \xi_{j_q} U_{i_qj_q} \right).$$

The path integral measure is defined as

$$\int [d\Phi^M] \equiv \int \sqrt{\det U_0} \prod_{i_0} \frac{d\xi_{i_0}}{\sqrt{2\pi}} [d\Phi^A].$$ (4.47)

We have separated the zero and non-zero fluctuations. The measure of the latter is indicated by $[d\Phi^A]$, corresponding to an integral over the fluctuation coefficients $\xi_{i_q}$. Including the zero modes in this measure would

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13These sets are mutually orthogonal, in fact we can always choose an entirely orthogonal basis for the fluctuations, because $\Delta$ is a Hermitian operator.

14The part involving the zero mode fluctuations is zero, because $\epsilon_{i_0} = 0$. Thus these fluctuations are also ‘zero modes’ in the sense that they represent physical fluctuations in field space which do not change the value of the action.
lead to a determinant of $M$ which would equal zero, thus invalidating this approach. Instead, the non-zero modes now lead to an *amputated* determinant $\det' \Delta$. The $U_0$ is the inner product over the zero modes only, it actually defines a metric on the moduli space of collective coordinates.

The integral over the zero modes must be treated separately, by converting it into an integral over the collective coordinates by an Faddeev-Popovish trick, [93, 94]. This means that we insert the number 1, suitably represented, into the path integral measure (4.47). A suitable representation for 1 is

\[
1 = \int \prod_{i_0} dC^{i_0} \left| \det \frac{\partial f_{i_0}}{\partial C^{j_0}} \right| \prod_{j_0} \delta [f_{j_0}(C)] ,
\]

which holds for any set of (invertible) functions $f_{i_0}$. We identify the $C$ with the collective coordinates (as in (4.43)) and we let the labels run over the zero mode set. An apt choice for the functions $f_{i_0}$ is

\[
f_{i_0} \equiv - \int d^4 x \Phi_M^i G_{MN} F_{i_0}^N = \xi_{j_0} U_{j_0 i_0} .
\]

Taking the derivative yields

\[
\frac{\partial f_{i_0}(C)}{\partial C^{k_0}} = - \int d^4 x \left\{ \frac{\partial \Phi_M^i}{\partial C^{k_0}} G_{MN} F_{i_0}^N + \Phi_M^i \partial_{C^{k_0}} (G_{MN} F_{i_0}^N) \right\}
\]

and because $\Phi^M$ does not depend on the collective coordinates $C$, see (4.43), we can replace $\Phi_M^i$ by $-\Phi_{cl}^i$ in the first term, which gives

\[
\frac{\partial f_{i_0}(C)}{\partial C^{k_0}} = - \int d^4 x \left\{ -F_{k_0}^M G_{MN} F_{i_0}^N + \Phi_{cl}^i \partial_{C^{k_0}} (G_{MN} F_{i_0}^N) \right\}
\]

\[
= U_{k_0 i_0} - \int d^4 x \Phi_{cl}^i \partial_{C^{k_0}} (G_{MN} F_{i_0}^N) .
\]

Equation (4.48) can thus be expressed as

\[
1 = \int \prod_{i_0} dC^{i_0} \left| \det \left\{ U_{k_0 i_0} - \int d^4 x \Phi_{cl}^i \partial_{C^{k_0}} (G_{MN} F_{i_0}^N) \right\} \right| \prod_{j_0} \delta \left[ \sum_{i_0} \xi_{i_0} U_{i_0 j_0} \right]
\]

which has to be inserted into (4.47). The delta function over the $\xi_{i_0}$’s forces them to be zero, because $U_{i_0 j_0}$ is invertible and thus has no zero eigenvectors. This means that $\Phi_{cl}^i$ in the second term in the determinant only contains the genuine quantum fluctuations which gives this term an extra factor of $\hbar$ (had we kept $\hbar$ in the game) meaning that it contributes at 2-loop instead of 1-loop. We will therefore drop this term, see also
The total measure\(^\text{15}\) becomes
\[
\int [d\Phi^M] e^{-S} = \int \prod_{i_0} \frac{dC_{i_0}}{\sqrt{2\pi}} \sqrt{\text{det} U_{i_0}} e^{-S_{i_0}} (\text{det}' \Delta)^{-\frac{1}{2}}.
\]

This is the general procedure to obtain the one-loop measure in the presence of instantons, for reviews of instanton calculations in supersymmetric theories, see [98, 17].

### 4.4.1 The bosonic measure

The general treatment above can be directly applied to the NS 5-brane instanton solution. Let us start with the bosonic collective coordinates. For the single centered instanton there are 4 collective coordinates, namely the location of the instanton \(x_0^\mu\) in \(\mathbb{R}^4\). This means that the moduli space is four dimensional and has a 4-dimensional metric which we shall denote by \((U_0)_{\mu\nu}\). This metric receives contributions from both the scalars and the tensors. In the above treatment we considered a generic bosonic system with fields \(\Phi^M\). For the instanton solution we must consider the fields \((\phi^A, B_{\mu\nu}I)\), so we can identify
\[
\Phi^M = \{\phi^A, B_{\mu\nu}I\} \quad \text{and} \quad G_{MN} = \left( \begin{array}{cc} G_{AB}(\phi_d) & 0 \\ 0 & M^{IJ}(\phi_d) \end{array} \right).
\]

The Hermitian operator reads
\[
\Delta_{MN}^M = \left( \begin{array}{cccc} -\delta_A^B \partial^2 + \ldots & \ldots & \ldots \\ \ldots & -\delta_J^I \partial^2 + \ldots \end{array} \right),
\]

the ellipses stand for operators at most linear in derivatives. In the scalar sector we have a block diagonal metric \(G_{AB}\), see (2.33), hence we can compute the contributions coming from \(\phi\) and \(\chi\) separately. For the case of the dilaton zero mode, i.e. \(\frac{\partial \phi}{\partial x_0^\mu}\), we have to compute
\[
U^{(\phi)}_{\mu\nu} = \int d^4 x \frac{\partial \phi}{\partial x_0^\mu} \frac{\partial \phi}{\partial x_0^\nu} = \int d^4 x \frac{x_{\mu} x_{\nu}}{r^2} e^{-2\phi} (\partial_\tau e^{-\phi})^2 = \frac{1}{4} \delta_{\mu\nu} \int d^4 x e^{2\phi} (\partial_\tau e^{-\phi})^2
\]

where we used \(\frac{\partial \phi}{\partial x_0^\mu} = e^\phi \partial_\mu e^{-\phi}\) and the spherical symmetry of the single-centered instanton. To compute the last expression in (4.50) we use the

\(^{15}\)Note that his measure is invariant under general coordinate transformations on the moduli space.
4.4 The instanton measure

slightly more general integral

\[ I_p \equiv \int d^4x h^{-p}(\partial_r h)^2 \quad h \equiv h_\infty + \frac{Q}{4\pi^2 r^2} \]

for some power \( p > 1 \). Using \( \frac{dr}{r^3} = -2\pi^2 \frac{dh}{Q} \) this integral becomes

\[ I_p = \left( \frac{Q}{2\pi^2} \right)^2 \text{Vol}(S^3) \int_0^\infty dr r^{-3} h^{-p} = Q \int_{h_\infty}^\infty dh h^{-p} = \frac{Q}{p-1} h_\infty^{1-p} , \]

which diverges for \( p \leq 1 \). Applying this to (4.50) yields the result

\[ U_{\mu\nu}^{(\phi)} = \frac{|Q_2|}{4g_s^2} \delta_{\mu\nu} . \]

Similarly, we have for the \( \chi \) zero mode,

\[ \frac{\partial \chi}{\partial x_0} = g_s^2 \Delta \chi e^{2\phi} \partial_\mu e^{-\phi} , \]

the metric

\[ U_{\mu\nu}^{(\chi)} = \int d^4x e^{-\phi} \frac{\partial \chi}{\partial x_0} \chi = \frac{|Q_2|}{8} (\Delta \chi)^2 \delta_{\mu\nu} , \]

remember that \( G_{\chi\chi} = e^{-\phi} \).

The tensors are slightly more subtle. Tensors have gauge symmetries which have to be gauge-fixed, we choose the background gauge condition

\[ \partial_\mu \left( M^{IJ} B_{\mu \nu J} \right) = 0 . \tag{4.51} \]

The instanton configurations are solutions to the classical, gauge invariant equations of motion and taking derivatives with respect to \( x_0^\mu \) do not yield zero modes of the gauge-fixed operator \( \Delta_{\mu\nu}^{(M)} \) in general. Therefore we consider the following zero modes

\[ Z_{\mu\nu I\rho} \equiv \frac{\partial B_{\mu\nu I}}{\partial x_0^\rho} - 2\partial_\mu \Lambda_{\nu I} \delta_{\rho} - H^\sigma_{\mu I} H_{\nu J} - H_{\mu I} H_{\nu J} \]

where we added the second part to make sure the \( Z \) obey the background gauge condition (4.51). If we then choose \( \Delta_{\nu I} = B_{\nu I} \) we obtain

\[ Z_{\mu\nu I\rho} = -H_{\mu \nu I} = \varepsilon_{\mu \nu \rho \sigma} H_I^\sigma , \]

which manifestly satisfies (4.51) because of the classical tensor field equations, see section 3.1. The tensorial part of the metric consequently becomes

\[ U_{\mu\nu}^{(B)} = \frac{1}{2} \int d^4x M^{IJ} Z_{\mu I}^\rho Z_{\nu J}^\sigma = \int d^4x M^{IJ} \left( \delta_{\mu I} H_\rho \partial J - H_{\mu I} H_{\nu J} \right) , \]
spherical symmetry and the Bogomol’nyi equation (3.6) give

\[ U_{\mu\nu}^{(B)} = \frac{3}{4} \delta_{\mu\nu} \int d^4x \left( e^\phi \hat{H}_\rho^\mu \hat{H}_\rho^\nu + e^{2\phi} H_\rho^\mu H_\rho^\nu \right) = 3 \left( U_{\mu\nu}^{(\chi)} + U_{\mu\nu}^{(\phi)} \right), \]

where \( \hat{H}_\mu^I \) was defined in (2.30). The total bosonic metric becomes

\[ (U_0)_{\mu\nu} = U_{\mu\nu}^{(\phi)} + U_{\mu\nu}^{(\chi)} + U_{\mu\nu}^{(B)} = S_{cl} \delta_{\mu\nu}, \]

with \( S_{cl} \) given in (3.14). This compact result is similar to the case of Yang-Mills instantons.

With these results we can finally write down the bosonic part of the single-centered (anti-)instanton measure:

\[ \int \frac{d^4x_0}{(2\pi)^2} (\det U_0) e^{-S_{\pm}^{\text{inst}} (\det' \Delta)^{-\frac{1}{2}}} = \int \frac{d^4x_0}{(2\pi)^2} S_{cl}^2 e^{-S_{\pm}^{\text{inst}} (\det' \Delta)^{-\frac{1}{2}}} \quad (4.52) \]

### 4.4.2 The fermionic measure

In section 4.3.2 we have shown that there are (nonzero) solutions for some of the hyperinos and none for the gravitinos. This means that the equations of motion (linear in the fermions) are given by (4.27, 4.28) with zeroes on the right hand sides. We can reproduce these equations of motion by starting from the action

\[ S = \int d^4x i \lambda^a (\not{\partial}_{3/4})_{ab} \bar{\lambda}^b, \]

where

\[ (\not{\partial}_{3/4})_{ab} \equiv \begin{pmatrix} \partial_{3/4} & 0 \\ 0 & \partial_{-3/4} \end{pmatrix}, \]

and \( \not{\partial}_{\pm3/4} \) has been defined in (4.29).

When expanding the fermionic fields as in (4.43), the quadratic part of this action (which includes the zero modes) becomes

\[ S_2 = \int d^4x i \lambda^a_{qu} (\not{\partial}_{3/4})_{ab} \bar{\lambda}^b_{qu}. \]

We have to proceed along the same lines as for the bosons. Contrary to \( \Delta_M^N \) however, \( \not{\partial}_k \) is not an Hermitian operator, but it satisfies\(^{16}\)

\[ (\not{\partial}_k)^\dagger = \not{\partial}_{-k}. \]

\(^{16}\)This is different from one’s usual twisted Dirac operator, because we twist with an anti-Hermitian connection. As a result, it is not clear how to calculate the index of such an operator.
On the other hand, the operators

\[ M_k \equiv \bar{\varphi}_k \varphi_k \quad \text{and} \quad \bar{M}_k \equiv \bar{\varphi}_k \varphi_k \]

are Hermitian. Furthermore, the spectrum of nonzero modes of \( M_{3/4} \) and \( \bar{M}_{3/4} \) is identical, similarly for \( M_{-3/4} \) and \( \bar{M}_{-3/4} \). This can be seen as follows, let \( F_i^1 \) and \( F_i^2 \) denote a basis of eigenfunctions of \( M_{-3/4} \) and \( M_{3/4} \) respectively. \( \bar{F}_i^1 \) and \( \bar{F}_i^2 \) denote a basis of eigenfunctions of \( \bar{M}_{-3/4} \) and \( \bar{M}_{3/4} \).

The eigenfunctions of \( M_{-3/4} \) and \( \bar{M}_{3/4} \) are then related, with the same eigenvalue \( \varepsilon_i^1 = \bar{\varepsilon}_i^1 \neq 0 \), by

\[ \bar{F}_i^1 = (\varepsilon_i^1)^{-1/2} \bar{\varphi}_{-3/4} F_i^1 \quad F_i^1 = (\varepsilon_i^1)^{-1/2} \varphi_{3/4} \bar{F}_i^1 . \]

Similarly, the spectrum of nonzero modes of \( M_{3/4} \) and \( \bar{M}_{-3/4} \) is identical and the relation between the eigenfunctions is given by

\[ \bar{F}_i^2 = (\varepsilon_i^2)^{-1/2} \bar{\varphi}_{3/4} F_i^2 \quad F_i^2 = (\varepsilon_i^2)^{-1/2} \varphi_{-3/4} \bar{F}_i^2 . \]

Here we assumed for simplicity that the eigenvalues are positive. Bearing in mind that both \( M_{3/4} \) and \( \bar{M}_{3/4} \) have zero modes, together with the fact that the fermion zero modes are in \( \lambda^2 \) and \( \bar{\lambda}^1 \), we can expand the fermions in a basis of eigenfunctions (suppressing spinor indices),

\[ \lambda^a_{\text{qu}} = \sum_i \xi^a_i F_i^a , \quad \bar{\lambda}^a_{\text{qu}} = \sum_i \bar{\xi}^a_i \bar{F}_i^a , \quad (4.53) \]

with \( \xi^a_i \) and \( \bar{\xi}^a_i \) anticommuting (there is no sum over \( a \)). Substituting this into the action and using the relation between the different eigenfunctions as discussed above we get

\[ S_2 = i \sum_{a,i,j} \xi^a_i U^{aa}_{ij} (\varepsilon^a_j)^{1/2} \bar{\xi}^b_j , \quad U^{ab}_{ij} \equiv \int d^4x \ F_i^a F_j^b . \quad (4.54) \]

We then define the fermionic part of the path-integral measure as (up to a sign from the ordering of the differentials)

\[ [d\lambda] [d\bar{\lambda}] \equiv \prod_a \prod_i d\xi^a_i d\bar{\xi}^a_i (\det U^{aa})^{-1} , \quad (4.55) \]

such that the fermion integral gives the Pfaffians of \( \bar{\varphi}_{3/4} \) and \( \varphi_{3/4} \) in the nonzero mode sector. In the zero mode sector, we are left over with an integral over the four GCCs (Grassmann collective coordinates). These are combined into two spinors, multiplied by the inverses\(^{17}\) of the norms

---

\(^{17}\)Note that to ensure invariance under reparametrizations of the GCCs we have to use the inverse determinant on the moduli space of GCCs, instead of the square root, as in the case of bosonic collective coordinates (4.47).
of the zero modes. The zero mode eigenfunctions have the form $Z^2_{\alpha\beta'} = \partial^{(1)} \lambda^{2}_{\alpha}/\partial \xi^{\beta'}$ given in (4.39), so that we find for their inner product

$$U^{22}_{\alpha'\beta'} = \int d^{4}x \, Z^{2\alpha}_{\alpha'} \, Z^{2}_{\alpha\beta'} = -4 \int d^{4}x \, e^{-\phi/2} \partial_{\mu} \phi \partial_{\nu} \phi (\varepsilon^{\mu\nu})_{\alpha'\beta'}$$

$$= 4 \varepsilon_{\alpha'\beta'} \int d^{4}x \, e^{3\phi/2} (\partial_{r} e^{-\phi})^{2} = \frac{8 |Q_{2}|}{g_{s}} \varepsilon_{\alpha'\beta'} . \quad (4.56)$$

The fermionic measure on the moduli space of collective coordinates then is

$$\int d^{2}\xi \, d^{2}\bar{\xi} \left(\frac{g_{s}}{8 |Q_{2}|}\right)^{2} (\det' M_{3/4} \, \det' \bar{M}_{3/4})^{1/2} . \quad (4.57)$$

The convention is that $d^{2}\xi \equiv d\xi_{1} d\xi_{2}$.

The complete measure is then given by combining (4.57) with (4.52) (which contains the determinant due to the fluctuations in the hypermultiplet) this gives

$$\int\frac{d^{4}x_{0}}{(2\pi)^{2}} \int d^{2}\xi \, d^{2}\bar{\xi} \left(\frac{g_{s} S_{\text{cl}}}{8 |Q_{2}|}\right)^{2} K^{\pm 1}\text{-loop} \, e^{-S_{\text{inst}}^{\pm}} . \quad (4.58)$$

Although the classical values in the instanton background of the gravitational and vector multiplets are trivial (flat metric and vanishing vector multiplets) their quantum effects cannot be ignored. This is a complicated calculation which is beyond the scope of our computations. Moreover, these loop effects would have to be computed in the full ten-dimensional string theory. We will be pragmatic and denote with $K^{\pm 1}\text{-loop}$ the ratio of all fermionic and bosonic determinants in the one-(anti-)instanton background.

### 4.5 Correlation functions

In this section instanton effects to the effective action will be computed by calculating correlators that receive instanton corrections. The (single-centered anti-)instanton measure (4.58) is used to compute the instanton contributions to the correlators. We again notice that it contains an integral over the four GCCs. Hence a generic correlation function $\langle A \rangle$ will only be nonzero if the ‘$A$’ is able to saturate the fermionic measure. Clearly there will be at least a nonzero four-point fermion correlation function, which will correct the four point vertex. Diagrammatically, such a four-point vertex consists of four fermion zero modes connected to an instanton at position $x_{0}$ which is integrated over. If one computes this diagram, one can read off the four-index tensor that determines the four-fermi terms in the effective action, i.e., one can see how $V_{a'b'a'b}$ would get corrected, see (4.1).

This would be difficult to do in practice due to the fact that we are working
in the 1.5 order formalism as explained in section 4.1. Additional four-fermi terms are hidden in the spacetime curvature scalar $R(\omega)$ as a function of the spin connection. Moreover, the four-fermi correlator would merely give information about the target-space curvature-like terms rather than the fundamental objects $M^{IJ}$, $G_{AB}$ and $A^I_A$. The latter are much more interesting because they determine the metric of the hypermultiplet, see (2.34).

We can obtain instanton corrections to these fundamental quantities by studying the GCC dependence of the scalars and tensors. For instance, the correlator

$$\langle \phi^A(x)\phi^B(y) \rangle$$

will give one-loop corrections to $G_{AB}$ in the presence of an instanton. But what to insert for $\phi^A(x)$?

Obviously this $\phi^A(x)$ must be something with two GCCs. We must again use the broken supersymmetries which generate fluctuations that are related by supersymmetry to the purely bosonic instantons and are genuine zero modes which leave the action unchanged as we saw in section 4.3.2. We observed that applying them once (at linear order) generated the fermionic zero modes, see (4.41) and (4.42). To these solutions for the hyperinos correspond bosonic superpartners. These are obtained by acting twice with the broken supersymmetries on the scalars. Put differently, substitute the values for the hyperinos and the broken supersymmetry parameters ((4.41) and (4.42)) into (4.2). This induces a quadratic GCC dependence in the scalars, i.e., the scalar superpartner contains two GCCs. Similarly, by using (4.4) the tensor superpartners can be constructed. The relevant correlators to study will thus be 2-point functions of scalars and tensors.

At second order in the GCCs the scalars are given by (see (4.2))

$$^{(2)}\phi^A = \frac{1}{2} \delta^2 \phi^A|_{cl} = \frac{1}{\sqrt{2}} \left( \gamma^A_{ia}(\phi|_{cl}) \epsilon_i^{(1)} \lambda^a + \tilde{\gamma}^{iA}_{a}(\phi|_{cl}) \tilde{\epsilon}_i^{(1)} \tilde{\lambda}\right),$$

where we write $\frac{1}{2} \delta^2 \phi^A|_{cl}$ because on has to exponentiate the infinitesimal (broken) supersymmetry transformations\textsuperscript{18}. In section 4.3.2 we have seen that $^{(1)}\lambda^\pm = ^{(1)}\bar{\lambda}^\mp = 0$ and for the broken supersymmetry parameters $\epsilon^\pm = \tilde{\epsilon}^\mp = 0$. This means that only terms proportional to $\gamma^A_{\pm}$ and $\tilde{\gamma}^{\pm A}$ contribute, they are both zero for the dilaton. Only $\chi$ gets corrected (at this order)

$$^{(2)}\chi = 2i \partial_\mu \phi \xi \sigma^\mu \bar{\xi}, \quad ^{(2)}\phi = 0.$$  \hspace{1cm}(4.59)

Due to our conventions for the fermionic zero modes chosen in (4.41), this expression for $\chi$ is the same in the instanton and anti-instanton background.

\textsuperscript{18}Remember that we have rescaled the supersymmetry parameters by a factor of $\sqrt{2}$. 
Analogously, the second order corrections of the tensors follow from (4.4). The instanton and anti-instanton cases yield, up to a sign, the same answer,

\[ (2)B_{\mu\nu1} = \mp 2i \varepsilon_{\mu\nu\rho\sigma} \partial^\rho e^{-\phi} \xi^\sigma \bar{\xi}, \quad (2)B_{\mu\nu2} = 0, \]  

(4.60)

notice again that only the R-R sector is turned on. One can easily check that the Bogomol’nyi equation (3.6) still holds at this order in the GCCs:

\[ (2)H_{\mu1} = \pm \partial_\mu (e^{-\phi} (2)\chi), \]  

(4.61)

the second equation in (3.6) is trivially satisfied. It may surprise those familiar with instanton calculus, that the equations of motion are satisfied without any fermion-bilinear source term. One would expect such a source term to be present, since (4.59) and (4.60) are obtained by acting with those broken supersymmetries that also generate the fermionic zero modes. This is typically what happens with the Yukawa terms in \( N = 2 \) or \( N = 4 \) SYM theory in flat space; in that theory the adjoint scalar field is found by solving the inhomogeneous Laplace equation with a fermion-bilinear source term. The fermionic zero modes in the presence of a YM instanton then determine the profile and GCC dependence of the adjoint scalar field. Some references where this is discussed in more detail are given in [99, 98, 100].

In the case at hand, the fermion bilinear source term actually vanishes when the zero modes are plugged in. To see this, let us consider the tensors, for which the full equations of motion read

\[
e^{-1} \delta S \delta B_{\mu\nu1} = \varepsilon^{\mu\nu\rho\sigma} \partial_\rho [M^{IJ} \mathcal{H}_{\sigma J I} - iA_{\lambda}^I \partial_\sigma \phi^A + \frac{i}{\sqrt{2}} M^{IJ} (\gamma_{j a} \psi^j I \lambda^a + \text{c.c.}) + iM^{IJ} k_{j a} \lambda^a \sigma \bar{\lambda} \bar{a}]. \]  

(4.62)

The fermionic zero modes we have found above do not enter these equations directly, because \( (1)\psi^i_\mu = (1)\bar{\psi}^a_{\bar{\mu}} = 0 \), and the two matrices \( M^{IJ}, k_{j a} \) are diagonal (actually zero for \( I = 1 \)) but for \( a = \bar{a} \) either \( (1)\lambda^a \) or \( (1)\bar{\lambda} \bar{a} \) vanishes. Hence, up to second order in the GCC, only the bosonic fields contribute. This is consistent with the fact that the BPS condition still holds at this order. A similar analysis can be done for the equations of motion for the scalars.

### 4.5.1 2-point functions

We have seen above that the objects of prime interest are bilinears in the R-R fields \( (2)\chi \) and \( (2)H_{\mu} \). In this section we will compute correlators of these R-R combinations. In the next section we will compute combinations of one R-R field with the zero modes \( (1)\lambda^{\tau} \) and \( (1)\bar{\lambda}^{\pm} \). The total measure for the single-centered (anti-) instanton has been found
in the previous section, see (4.58). Using this measure, one computes a correlator by integrating the fields over the collective coordinates. However, computing the correlator of two $\chi$’s for instance (as given in (4.59)) is not straightforward due to the integration over $x_0$. For simplicity we therefore take a large distance limit. We then express the fields (for convenience) in terms of propagators, which will enable us to read off the effective vertices from the correlation functions by stripping off the external legs. For the bosons we find

\[
(2)\chi(x) = -2i |Q_2| g_s^{-2} \xi \partial_\mu \xi \partial_\mu G(x, x_0) \left(1 + \ldots\right)
\]

\[
(2)H^\mu_1(x) = \mp 2i |Q_2| \xi \partial^\nu \xi \left(\partial_\mu \partial_\nu - \delta_\mu^\nu \partial^2\right) G(x, x_0)
\]

(4.63)

where $G(x, x_0) = 1/4\pi^2(x-x_0)^2$ is the massless scalar propagator.

In the first equation in (4.63) we only keep the leading term in the large distance expansion valid when $(x-x_0)^2 \gg |Q_2|/4\pi^2 g_s^2$. In this limit the dilaton is effectively given by $e^{-\phi} \approx e^{-\phi_{\text{inst}}} = g_s^2$ and similarly $\chi \approx \chi_{\text{inst}}$. So the fields are replaced by their asymptotic values and these will be used to describe the asymptotic geometry of the moduli space in a next section. The second equation in (4.63) is exact: for correlators involving $(2)H^\mu_1$ it is not necessary to make a large distance approximation. For correlators involving the fermions it is also necessary to make a large distance approximation. In this approximation the fermion zero modes are given by

\[
(1)\lambda_\alpha^\pm(x) = -2 |Q_2| g_s^{-3/2} S_{\alpha \beta'}(x, x_0) \bar{\xi} \beta' \left(1 + \ldots\right)
\]

\[
(1)\bar{\chi}^\pm_{\beta'}(x) = \pm 2 |Q_2| g_s^{-3/2} \xi^\alpha S_{\alpha \beta'}(x, x_0) \left(1 + \ldots\right)
\]

(4.64)

where the ellipses again indicate terms of higher order in the large distance expansion and $S(x, x_0) = -i\theta G(x, x_0)$ is the $\lambda\bar{\lambda}$ propagator.

Let us begin with the purely bosonic correlators. With the GCC measure $\mathrm{d}\mu_\xi \equiv d^2\xi \delta^2\xi \left(g_s/8|Q_2|\right)^2$ from (4.57) and the Fierz identity $\xi_{\alpha \mu} \bar{\xi} \xi_{\nu} \bar{\xi} = -\frac{1}{2} \delta_{\mu \nu} \xi \xi \bar{\xi} \xi$, we find in the large distance limit

\[
\int \mathrm{d}\mu_\xi (2)\chi(x) (2)\chi(y) = \frac{1}{8g_s^2} \partial_\mu^0 G(x, x_0) \partial_\mu^0 G(y, x_0)
\]

where we have replaced $\partial_\mu$ with $-\partial_\mu^0 \equiv -\partial/\partial x_0^\mu$, denoting the derivative with respect to the bosonic collective coordinates. Using $(2)\phi = 0$, we then obtain for the leading semiclassical contribution to the correlation function of two scalars in the one-(anti-)instanton background

\[
\langle \phi^A(x) \phi^B(y) \rangle_{\text{inst}} = g_s^{-2} \delta^A_\chi \delta^B_\chi \int d^4x_0 \ Y_\pm (\partial_\mu^0 G(x, x_0) \partial_\mu^0 G(y, x_0))
\]

\[
= g_s^{-2} \ Y_\pm \delta^A_\chi \delta^B_\chi G(x, y)
\]

(4.65)
Here we denote (remember that the difference between $S_{cl}$ and $S_{\text{inst}}^\pm$ is given by the surface terms (3.16, 3.15))

$$Y_\pm \equiv \frac{1}{32\pi^2} S_{cl}^2 e^{-S_{\text{inst}}^\pm} K_{1}\text{-loop}$$

which is small for small string coupling constant $g_s$. Since translation invariance implies that neither $S_{cl}$ nor $K_{1}\text{-loop}$ depends on the collective coordinates $x_0$, we were allowed to integrate by parts and use $\partial_0^2 G(x, x_0) = -\delta(x - x_0)$. There is no boundary term because the domain of integration covers all of $\mathbb{R}^4$ with no points excised (the instanton can be located at any point) and the integrand vanishes at infinity.

The result (4.65) is to be compared with the propagator derived from an effective action with instanton and anti-instanton corrected metric $G_{\text{eff}}^{AB} = G_{AB} + G_{\text{inst}}^{AB}$, with $G_{AB}$ as in (4.6). Similarly we write for the inverse $G_{\text{eff}}^{AB} = G^{AB} + G_{\text{inst}}^{AB}$, with $G_{AC} G^{CB} = \delta^B_A$. At leading order in $Y_\pm$, we find

$$G_{\text{inst}}^{AB} = \begin{pmatrix} 0 & 0 \\ 0 & g_s^{-2}(Y_+ + Y_-) \end{pmatrix}$$

Note that since $Y_- = (Y_+)^*$, instanton and anti-instanton contributions combine into a real correction. This result receives of course corrections from perturbation theory and from terms that become important beyond the large distance approximation in which $e^{-\phi} \approx g_s^2$. Such terms play a role when inverting the result of (4.67) to obtain the effective metric $G_{\text{eff}}^{AB}$. They correspond to higher order powers in $Y_\pm$ and interfere with multi-centered (anti-)instanton effects. Dropping all these subleading terms we find

$$G_{\text{eff}}^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\phi} - g_s^2(Y_+ + Y_-) \end{pmatrix}$$

for more details see appendix E.

Having computed the correlator of two R-R scalars in the (anti-)instanton background, we now turn to the correlator of two R-R tensors using (4.63). This will give instanton corrections to the tensor metric $M^{IJ}$. The integration over the GCCs gives

$$\int d\mu_\xi (2) H_{\mu_1}(x) (2) H_{\nu_1}(y) = \frac{g_s^2}{8} G_{\mu\rho}(x, x_0) G_{\nu_\rho}^\rho(y, x_0)$$

\footnote{We are assuming here that $K_- = (K_+)^*$. Presumably, the one-loop determinants $K_\pm$ only differ by a phase coming from the fermionic determinants. If this phase can be absorbed in the corresponding surface terms (3.15), the instanton and anti-instanton determinants are real and equal.}
where \( G(\mu \nu(x, x_0) = (\partial_{\mu} \partial_{\nu} - \delta_{\mu \nu} \partial^2) G(x, x_0) \) is the gauge-invariant propagator of the dual tensor field strengths. We then have to integrate over \( x_0 \), use \( H_{\mu 2} = 0 \) and the convolution property\(^{20}\) \( G_{\mu \rho} * G_{\rho \nu} = G_{\mu \nu} \), it follows that

\[
\langle H_{\mu I}(x) H_{\nu J}(y) \rangle_{\text{inst}} = g_s^2 Y_+ \delta_I^1 \delta_J^1 G_{\mu \nu}(x, y) .
\] (4.70)

From the right-hand side we read off the (anti-) instanton correction to the inverse metric \( M_{IJ} \), which multiplies the tensor propagators. We find for the sum

\[
M_{IJ}^{\text{inst}} = g_s^2 (Y_+ + Y_-) \delta_I^1 \delta_J^1 ,
\] (4.71)

In the large distance approximation (for which \( \chi \approx \chi_\infty \)) we then obtain

\[
M_{IJ}^{\text{eff}} = M_{IJ} - g_s^2 (Y_+ + Y_-) \left( \begin{array}{cc} 1 - \chi_\infty & -\chi_\infty \\ -\chi_\infty & \chi_\infty^2 \end{array} \right) ,
\] (4.72)

with \( M_{IJ} \) as in (4.6). This seems to suggest that both R-R and NS-NS sectors get corrections in front of the tensor kinetic terms. However, when expressed in terms of \( \hat{H}_1 = H_1 - \chi H_2 \), the tensor kinetic terms in the effective action simplify to

\[
e^{-1} L_{\text{eff}} = \frac{1}{2} \left( e^\phi - g_s^{-2} (Y_+ + Y_-) \right) \hat{H}_1 \hat{H}_1 + \frac{1}{2} e^{2\phi} H_2^\mu H_2^\nu + \ldots ,
\] (4.73)

In this basis, which is the one to distinguish between fivebrane and membrane instantons (see the discussion in section 3.3), the NS-NS sector does not receive any instanton corrections.

There is also the combination of the tensor and scalar (see (4.63)) we can consider. The GCC integration over this mixed bosonic combination yields

\[
\int d\mu_\xi (^{(2)} \hat{H}_{\mu 1}(x)(^{(2)} \chi)(y) = \mp \frac{1}{8} G_{\mu \nu}(x, x_0) \partial_0^\mu G(y, x_0) ,
\] (4.74)

which vanishes when integrated over \( x_0 \) thanks to the Bianchi identity \( \partial^\mu G_{\mu \nu} = 0 \). We conclude that

\[
\langle H_{\mu I}(x) \phi^A(y) \rangle = 0 .
\] (4.75)

This was to be expected, since for constant coefficients \( A_I^A \) the vertex

\[
-i A_I^A H_1^\mu \partial_\mu \phi^A
\]
is a total derivative and therefore does not contribute to the propagator. However, we will argue later that instantons must induce such a vertex with field-dependent coefficients. To determine this vertex explicitly we would have to go beyond the leading and large distance expansion, see appendix F.

The calculation of the bosonic 2-point functions is sufficient to determine the instanton-corrected moduli space metric, which we present in the next section. The full geometry, or the full effective action, does not follow from the metric alone, but also from various connections that appear e.g. in the supersymmetry transformation rules. As we now show, these connections can be read off from the three-point functions.

4.5.2 3-point functions

First we compute $\Gamma^I_{\alpha b}$ in the instanton background. It appears in the effective action (4.1) through the relation (C.7) and measures the strength of the coupling between the tensors and the fermions. We therefore compute

$$\langle \lambda^a_\alpha(x) \bar{\lambda}^b_\beta(y) H_{\mu I}(z) \rangle$$

$$= -i \frac{|Q_2|}{g_s} Y_+ \delta^a_+ \delta^b_+ \delta^1 \int d^4 x_0 \left[ S(x, x_0) \bar{\sigma}^\mu S(y, x_0) \right]_{\alpha \beta'} G_{\mu \nu}(z, x_0) .$$

These two correlators induce an effective vertex $-h_{a\bar{a}}(\Gamma^I_{\text{inst}})^a_b \chi^b_{\sigma \mu} \bar{\lambda}^a H_{\mu I}$ with coefficients

$$(\Gamma^I_{\text{inst}})^a_b = -i \frac{|Q_2|}{g_s} M_{\infty}^{IJ}(Y_+ \delta^a_+ h_{b1} + Y_- \delta^a_- h_{b2}) = -i \frac{|Q_2|}{g_s} M_{\infty}^{I1} \begin{pmatrix} 0 & Y_- \\ Y_+ & 0 \end{pmatrix} .$$ (4.76)

Here we have used that $h_{a\bar{a}}$ is not corrected at leading order\textsuperscript{21}. We also used the notation that $M_{\infty}^{IJ}$ stands for $M^{IJ}$ with the fields replaced by their asymptotic values at infinity.

The last two correlators contribute to the connection $\Gamma_A^{a b}$, which appears in the covariant derivative on the fermions (C.12). This connection was zero on tree-level, see (D.1), but it receives instanton corrections as follows from

$$\langle \lambda^a_\alpha(x) \bar{\lambda}^b_\beta(y) \phi^A(z) \rangle$$

$$= \mp i \frac{|Q_2|}{g_s} Y_+ \delta^a_+ \delta^b_+ \delta^A \int d^4 x_0 \left[ S(x, x_0) \bar{\sigma}^\mu S(y, x_0) \right]_{\alpha \beta'} \partial_\mu G(z, x_0) .$$

\textsuperscript{21}The 2-point function of two fermion insertions vanishes in the semiclassical limit.
This corresponds to an effective vertex \(-i h_{\bar{a}a} (\Gamma_{A}^{\text{inst}})^a_b \lambda^b \sigma^\mu \tilde{\chi} \partial_\mu \phi^A\) with

\[
(\Gamma_{A}^{\text{inst}})^a_b = \frac{|Q_2|}{g^3_s} \Gamma^{\infty}_{A\chi} \begin{pmatrix} 0 & -Y_- \\ Y_+ & 0 \end{pmatrix} .
\]

The above connections induce instanton corrections to the curvature tensors that appear in the four-fermi couplings of the effective action (4.1). Indeed, the fact that these curvatures receive instanton corrections also follows from the computation of 4-point functions of fermionic insertions. These results should be consistent with the instanton corrections to the curvatures as determined by the connections. For reasons explained in the beginning of this section, checking this consistency may be a complicated task.

Note however, that there is another four-fermi term in (4.1) proportional to the product of two antisymmetric tensors \(\mathcal{E}_{ab}\). It is easy to see that this tensor cannot receive instanton corrections since it multiplies only \(\lambda^a\) in the action, not \(\bar{\lambda}^a\). Due to the even distribution of fermionic zero modes among \(\lambda^a\) and \(\bar{\lambda}^a\) there are thus no non-vanishing correlation functions that could induce an effective vertex involving \(\mathcal{E}_{ab}\). A similar argument shows that the connections \(\Gamma^{H_{I}}_{i j}\) do not get corrected: they occur in the action only in combination with gravitinos. For example in the vertex \(2\Gamma^{H_{I}}_{i j} H_{I}^{\mu \nu \rho} \psi_{J}^{\mu} \sigma^{\nu} \bar{\psi}_{\rho i}\), hidden in the square of the supercovariant field strengths of the tensors (C.11), which have no zero modes to lowest order in the GCC. Correlation functions of fields corresponding to vertices involving \(\Gamma^{H_{I}}_{i j}\) then do not saturate the GCC integrals and vanish. If we were to continue the procedure of sweeping out solutions by applying successive broken supersymmetry transformations to the fields, the gravitinos could obtain a GCC dependence at third order, but then the number of GCCs in the correlators of interest exceeds the number of degrees of freedom and they therefore vanish as well. However, due to (C.13) the coefficients \(\Omega_{I}^{i j}\) do get corrected:

\[
(\Omega_{I}^{\text{eff}})^i_j = M_{i j}^{\text{eff}} \Gamma^{H_{I}}_{i j} = \Omega_{I}^{i j} + g^2_s (Y_+ + Y_-) \delta_1^I \Gamma^{H_{I}}_{\infty j} .
\]

These quantities appear in the supersymmetry transformations of the tensors (4.4).

4.6 The universal hypermultiplet moduli space

In order to determine the instanton corrections to the universal hypermultiplet, we first Wick-rotate back to Lorentzian signature and then dualize the tensors \(H_{I}\) into two pseudoscalars \(\phi^I = (\varphi, \sigma)\), using the same notation as in section 2.3, i.e. \(\varphi\) is a R-R field and \(\sigma\) the NS axion. If we combine
the latter and \( \phi^A = (\phi, \chi) \) into a four-component field \( \phi^A = (\phi^A, \phi^I) \), then in this basis the universal hypermultiplet metric reads, see (2.34)

\[
G^A_B = \begin{pmatrix}
G_{AB} + A^I_M A_J^M & A^K_M M_K^J \\
M_K^A A^K_B & M_{IJ}
\end{pmatrix}.
\]

Using (4.68), (4.71) and \( A^I_A = 0 \), we find for the asymptotic effective Lagrangian

\[
e^{-1} L_{UH} = -\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^{-\phi}(1 - g_s e^\phi Y) (\partial_\mu \chi)^2
\]

\[
- \frac{1}{2} e^{-\phi}(1 + g_s e^\phi Y) (\partial_\mu \varphi)^2 - \frac{1}{2} e^{-2\phi}(\partial_\mu \sigma + \chi \partial_\mu \varphi)^2 + \ldots,
\]

where the ellipses stands for subleading terms and \( Y = Y_+ + Y_- \) is the sum of the instanton and anti-instanton contributions, as introduced in (4.66). It can be written as

\[
Y = \frac{1}{32 \pi^2} S_{cl}^2 e^{-S_{cl}} (e^{i\hat{\sigma} Q_2} K_{1-\text{loop}}^+ + e^{-i\hat{\sigma} Q_2} K_{1-\text{loop}}^-) = \frac{1}{16 \pi^2} S_{cl}^2 e^{-S_{cl}} K_{1-\text{loop}} \cos(\hat{\sigma} Q_2),
\]

where we introduced \( \hat{\sigma} \equiv \sigma + \chi_0 \varphi \) such that \( Y \) is periodic in \( \hat{\sigma} \). The second equality in (4.80) holds only under the reality assumption made in footnote 19. Furthermore note that only the R-R sector receives corrections from the NS5-brane instanton.

### 4.6.1 The metric and isometries

In this section we present the instanton corrected line element of the quaternionic target space of the UHM. First we write down the general form of the line element, which is given by (using \( A^I_A = 0 \))

\[
ds_{UH}^2 = G^A_B d\phi^A \otimes d\phi^B = G_{AB} d\phi^A \otimes d\phi^B + M_{IJ} d\phi^I \otimes d\phi^J.
\]

We remind the reader that the classical metric is given by the line element

\[
ds_{UH}^2 = d\phi^2 + e^{-\phi} d\chi^2 + e^{-\phi} d\varphi^2 + e^{-2\phi}(d\sigma + \chi d\varphi)^2,
\]

and describes the homogeneous quaternion-Kähler space \( SU(1,2)/U(2) \), see (2.22). As discussed in section 2.3, the isometry group \( SU(1,2) \) can be split into three categories. First, there is a Heisenberg subgroup of shift isometries,

\[
\phi \to \phi, \quad \chi \to \chi + \gamma, \quad \varphi \to \varphi + \beta, \quad \sigma \to \sigma - \alpha - \gamma \varphi,
\]

where the ellipses stands for subleading terms and \( Y = Y_+ + Y_- \) is the sum of the instanton and anti-instanton contributions, as introduced in (4.66). It can be written as

\[
Y = \frac{1}{32 \pi^2} S_{cl}^2 e^{-S_{cl}} (e^{i\hat{\sigma} Q_2} K_{1-\text{loop}}^+ + e^{-i\hat{\sigma} Q_2} K_{1-\text{loop}}^-) = \frac{1}{16 \pi^2} S_{cl}^2 e^{-S_{cl}} K_{1-\text{loop}} \cos(\hat{\sigma} Q_2),
\]

where we introduced \( \hat{\sigma} \equiv \sigma + \chi_0 \varphi \) such that \( Y \) is periodic in \( \hat{\sigma} \). The second equality in (4.80) holds only under the reality assumption made in footnote 19. Furthermore note that only the R-R sector receives corrections from the NS5-brane instanton.
4.6 The universal hypermultiplet moduli space

where $\alpha, \beta, \gamma$ are real (finite) parameters. This Heisenberg group is preserved in perturbation theory [73, 74]. We have not discussed these perturbative corrections, which only appear at one-loop in the string frame, here. They are discussed in [72, 75] and should be added to our final result for the metric.

Second, there is a U(1) symmetry (2.26) that acts as a rotation on $\varphi$ and $\chi$, accompanied by a compensating transformation on $\sigma$,

$$\varphi \rightarrow \cos(\delta) \varphi + \sin(\delta) \chi \quad \chi \rightarrow \cos(\delta) \chi - \sin(\delta) \varphi ,$$

$$\sigma \rightarrow \sigma - \frac{1}{4} \sin(2\delta) (\chi^2 - \varphi^2) + \sin^2(\delta) \chi \varphi . \quad (4.83)$$

We now present the instanton corrected moduli space metric. As shown above, instanton effects are proportional to $Y$, given by (4.80), and depend on the instanton charge $Q_2$ and the R-R background specified by $\chi_0$. Moreover, also the asymptotic values of the fields, $g_s$ and $\chi_\infty$, appear. They are treated as coordinates in the asymptotic regime of the moduli space, i.e., where $\chi = \chi_\infty$ and $e^{-\phi} = g_s^2$. For fixed values of $\chi_0$ and $Q_2$, the moduli space metric is given by

$$ds_{\text{UH}}^2 = d\phi^2 + e^{-\phi} (1 - Y) d\chi^2 + e^{-\phi} (1 + Y) d\varphi^2 + e^{-2\phi} (d\sigma + \chi d\varphi)^2 , \quad (4.84)$$

up to subleading terms. This metric therefore satisfies the constraints from quaternionic geometry only up to leading order\textsuperscript{22}. It remains to be seen to what extent the quaternionic structure can fix these subleading corrections. The result written in (4.84) depends on $Q_2$ and on the chosen R-R background. To obtain the full moduli space metric, one must sum over all instanton numbers $Q_2$. It would be very interesting to do this sum explicitly and to see of which function we have the asymptotic limit. Unfortunately, for that we need more knowledge of the one-loop determinants and the subleading corrections, which is not available at present.

We can also deduce the leading-order instanton corrections to the vielbeins and other geometric quantities. These can be computed from the vielbeins that determine the double-tensor multiplet geometry, which we give in appendix F.

What happens to the isometries (4.82) and (4.83)? For the Heisenberg group, this amounts to investigating which isometries are broken by the quantity $Y$, as the other terms are invariant. First we focus on the $\gamma$-shift in $\chi$. For a given, fixed R-R background $\chi_0$, the $\gamma$-shift is broken completely. This is because $Y$ is proportional to $S_{cl}$, which contains $\Delta \chi = \chi_\infty - \chi_0$, see (3.14). However, this symmetry can be restored if we simultaneously

\textsuperscript{22}As we have checked explicitly.
change the background as $\chi_0 \to \chi_0 + \gamma$. Since $\chi_0$ is subject to a quantization condition (see section 3.2), this induces a quantization condition on the possible values for $\gamma$. This means that the $\gamma$-shift is broken to a discrete subgroup.

With this in mind, we find that under the action of a generic element in the Heisenberg group the metric is invariant only if the following quantization condition is satisfied:

$$\alpha - (\chi_0 + \gamma)\beta = \frac{2\pi n}{|Q_2|}, \quad (4.85)$$

with $n$ an integer. As explained before, the $\gamma$-dependence is not relevant here since we could shift the R-R background again. As for the other two isometries generated by $\alpha$ and $\beta$, only a linear combination is preserved. Stated differently, the $\beta$-isometry is preserved as a continuous isometry if we accompany it by a compensating $\alpha$-shift, where $\alpha$ is determined from (4.85).

If we solely perform an $\alpha$-transformation, only a discrete $\mathbb{Z}_{|Q_2|}$ subgroup survives as a symmetry. In fact, since the full metric includes a sum over $Q_2$, only shifts with $\alpha = 2\pi n$ are unbroken. In conclusion, for the Heisenberg group one isometry remains continuous and two are broken to discrete subgroups. This is precisely in line with the proposal made in [75].

The remaining isometry we discuss is (4.83). Since the last term in (4.84) is invariant by itself, we should only look at the R-R sector. Due to the fact that $Y$ is independent of $\varphi$, but depends on $\chi^2$, this continuous rotation symmetry is broken. In fact, the terms proportional to $Y$ break this isometry down to the identity $\delta = 0$ and the discrete transformation with $\delta = \pi$,

$$\chi \rightarrow -\chi, \quad \varphi \rightarrow -\varphi, \quad \sigma \rightarrow \sigma. \quad (4.86)$$

This conclusion is different from [52], where also $\delta = \pi/2$ was claimed to survive as an isometry. It is not excluded though, that the full answer may have more symmetries. This full answer should contain (all the) subleading corrections and perhaps membranes. At the end of appendix F we in fact argue that the connection $A^I_A$ becomes nonzero at subleading order, away from the asymptotic region. After dualization, this induces new terms in the metric, as follows from (4.78), so one has to reanalyze the breaking of isometries. Clearly, this is an interesting point that deserves further study. Finally note the existence of another discrete isometry which changes the sign in $\chi$ (or $\varphi$) together with a sign flip in $\sigma$. This is because the (leading) instanton plus anti-instanton corrections are even in $\chi$ and $\sigma$. This discrete isometry is however not part of (a discrete subgroup of) $SU(1,2)$. 
4.7 Short summary and outlook

Having reached our goal, (4.84), we need to reflect on its meaning. In the range of approximations we are working in, using supergravity and the large distance approximation for the fields (see (4.63) and (4.64)), this is a solid result. Note that one should actually combine our result with the one-loop correction found in [72]. We shall elaborate on this one-loop correction in the next chapter.

To make further progress, one should really perform this calculation in string theory, which then includes the worldvolume theory of the 5-brane, and calculate the one-loop determinants. Another generalization is the case of more hypermultiplets, which corresponds to more general Calabi-Yau manifolds with \( h^{(1,2)} \neq 0 \). To advance in the UHM case, one should try to take a closer look at the constraints coming from the quaternionic geometry. We saw in section 1.4 that the definition of a four (real) dimensional quaternionic manifold is that it is Einstein and has (anti-)selfdual Weyl curvature. It is easy to check that the classical metric, (4.81) obeys these conditions. If we include the 5-brane contributions as in (4.84), these conditions are still obeyed up to leading order in \( Y \). In fact, this already imposes some constraints on \( Y \). One could try to systematically analyze these conditions and find the corrections to the metric which obey the constraints. This is exactly what we will do in the next chapter where we will construct the possible deformations of the metric allowed by the quaternionic constraints.