In the previous chapter we have shown how the tensor multiplet (and its equivalent formulation the DTM and UHM) arises in Calabi-Yau compactifications. We will proceed to demonstrate explicitly how NS 5-branes and membranes appear in the DTM closely following [84, 31]. We derive two Bogomol’nyi bounds for the DTM. Then we shall construct NS 5-brane solutions corresponding to the first bound and membrane solutions corresponding to the second Bogomol’nyi bound.

3.1 The Bogomol’nyi bound

It is convenient to use form notation in deriving the Bogomol’nyi bound. The Euclidean DTM Lagrangian (2.36) is then written as

\[ \mathcal{L}^E_{DT} = d^4x \sqrt{g} R + \frac{1}{2} |d\phi|^2 + \frac{1}{2} e^{-\phi} |d\chi|^2 + \frac{1}{2} M^{IJ} H_I \wedge H_J. \]

(3.1)

Note that we have dropped the term \( \frac{1}{4} F_{\mu
u} F^{\mu\nu} \), since from now on we will choose a vanishing graviphoton and focus our attention exclusively on the scalar-tensor sector. Defining

\[ H \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad E \equiv \begin{pmatrix} \frac{d\phi}{e^{-\phi/2} d\chi} \\ \frac{e^{-\phi/2}}{d\chi} \end{pmatrix}, \quad N \equiv \begin{pmatrix} 0 & e^{\phi/2} \\ 1 & -\chi \end{pmatrix}, \]

such that \( N^T N = M \) one can rewrite (3.1) as

\[ \mathcal{L}^E_{DT} = d^4x \sqrt{g} R + \frac{1}{2} (N^* H + O E)^T \wedge (N^* H + O E) + H^T \wedge N^T O E, \]

where \( O \) is some scalar dependent orthogonal matrix. We explicitly include this matrix since \( N \) and \( E \) are defined only up to \( O(2) \) rotations. This formulation shows that the action is bounded from below by

\[ S^E \geq \int_M \left( d^4x \sqrt{g} R + H^T \wedge N^T O E \right), \]

(3.2)
The instanton solutions

which is a topological term because it is independent of the spacetime metric. The action attains its lowest value for configurations satisfying the Bogomol’nyi bound

\[ *H = -N^{-1}OE \]  

Naturally this condition must imply the equations of motion, which will fix the matrix \( O \). Furthermore, field configurations satisfying (3.3) have vanishing energy-momentum tensor which allows us to restrict ourselves to the case \( g_{\mu\nu} = \delta_{\mu\nu} \). The equations of motion for the tensors, i.e. \( d(M^*H) = 0 \), are satisfied if

\[ d \left( N^T OE \right) = 0 , \]

which also guarantees that the topological term in (3.2) is closed and can therefore be written as a total derivative. Consequently this term does not affect the equations of motion for the scalars, which are determined by demanding that the Bianchi identity holds \( (dH = 0) \) which gives upon using (3.3)

\[ d \left( N^{-1}O^* E \right) = 0 . \]

We have enough information to determine the possible \( O' \)'s. The first solution is given by

\[ O_1 = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]

The second solution is given by

\[ O_2 = \pm \frac{1}{|\tau'|} \begin{pmatrix} \text{Re}\tau' & -\text{Im}\tau' \\ \text{Im}\tau' & \text{Re}\tau' \end{pmatrix} , \]

with \( \tau' \equiv (\chi - \chi_0) + 2i e^{\phi/2} \) and \( \chi_0 \) a real integration constant.

3.2 The NS 5-brane

Using (3.4) the Bogomol’nyi bound for the 5-brane takes the form

\[ \begin{pmatrix} H_{\mu 1} \\ H_{\mu 2} \end{pmatrix} = \pm \partial_\mu \begin{pmatrix} e^{-\phi} \chi \\ e^{-\phi} \end{pmatrix} \]

and as this configuration defines a lower bound of the action, it is the background we will be expanding around. The + corresponds to instantons and the – to anti-instantons. The second equation in (3.6) comes from the
3.2 The NS 5-brane

NS sector and specifies the NS 5-brane, compare with (2.10). The first equation determines the R-R background in which the 5-brane lives. It is often useful to use the basis of (2.30) in which the Bogomol’nyi bound takes the form \( \hat{H}_{\mu 1} = \pm e^{-\phi} \partial_\mu \chi \).

Since the tensors have to obey the Bianchi identity, the scalars in (3.6) have to obey Laplace-like equations. One could either use source terms for these equations or excise points \( \{ x_i \} \) from the flat spacetime \( \mathbb{R}^4 \), we will use the latter method. These excised points correspond to the locations of the instantons. We find multi-centered solutions of the form

\[
e^{-\phi} = e^{-\phi_\infty} + \sum_i \frac{|Q_{2i}|}{4\pi^2 (x - x_i)^2}, \tag{3.7}
\]

\[
e^{-\phi} \chi = e^{-\phi_\infty} \chi_\infty + \sum_i \frac{Q_{1i}}{4\pi^2 (x - x_i)^2}, \tag{3.8}
\]

where \( Q_{1i}, Q_{2i}, \chi_\infty \) and \( \phi_\infty \) are independent integration constants. In equation (3.7) we write \(|Q_{2i}|\) because the exponential function has to be positive everywhere in spacetime. In our conventions the string coupling constant is identified as \( g_s \equiv e^{-\phi_\infty}/2 \). As already anticipated, the two charges are defined by integrating the tensor field strengths \( H_{\mu \nu \rho I} = -\varepsilon_{\mu \nu \rho \sigma} H_{\sigma I} \) over 3-spheres at infinity,

\[
Q_I = \int_{S^3_\infty} H_I, \quad I = 1, 2 \tag{3.9}
\]

which are related to (3.7) and (3.8) via (3.6). Explicitly calculating (3.9) gives

\[
Q_2 = \mp \sum_i |Q_{2i}|, \quad Q_1 = \mp \sum_i Q_{1i},
\]

which means that for instantons \( Q_2 \) is negative and for anti-instantons positive. \( Q_1 \) can be anything since the combination \( e^{-\phi} \chi \) does not have to be positive.

The (anti-)instanton action (3.2) is found to be

\[
S_{\text{cl}} = \pm \int_{\partial M} \left( \chi H_1 - \left( e^\phi + \frac{1}{2} \chi^2 \right) H_2 \right), \tag{3.10}
\]

where the integral is over the boundaries of \( \mathbb{R}^4 \); \( \partial M = S^3_\infty \cup \sum_i S^3_i \), i.e. the sphere at infinity together with the infinitesimal spheres around the excised points. This action is finite only if \( \chi(x) \) is finite near the excised points:

\[
\chi_i \equiv \lim_{x \to x_i} \chi(x) = \frac{Q_{1i}}{|Q_{2i}|}, \tag{3.11}
\]
which is finite if $Q_{2i} \neq 0$, for nonvanishing $Q_{1i}$. This implies that the integrated Heisenberg invariants (2.30) vanish,

$$\hat{Q}_{1i} \equiv Q_{1i} - \chi_i |Q_{2i}| = 0 .$$

(3.12)

The result for the action is

$$S_{cl} = \frac{|Q_2|}{g_s^2} + \frac{1}{2} \sum_i |Q_{2i}| (\chi_\infty - \chi_i)^2 , \quad \chi_\infty \equiv \lim_{x \to \infty} \chi(x) .$$

(3.13)

This action has an inversely quadratic dependence on $g_s$ which is precisely as expected for a 5-brane wrapped around a CY 3-fold, [71].

If we consider a single-centered instanton around $x_0$, the action simplifies to

$$S_{cl} = \frac{|Q_2|}{g_s^2} \left(1 + \frac{1}{2} g_s^2 (\Delta \chi)^2\right) ,$$

(3.14)

where $\Delta \chi \equiv \chi_\infty - \chi_0$. In contrast to the dilaton, $\chi$ remains finite and thus regular at the origin, so no source term can be associated to it. We can therefore regard this as a R-R background in which the instanton lives. Consequently, the ‘bare’ instanton configuration is obtained by turning this background off, which lowers the value of the action. Turning it off means taking $\chi_\infty = \chi_0$ and using (3.8) one finds that this implies that $\chi(x)$ is constant everywhere: $\chi(x) = \chi_0$.

The actual instanton calculation performed in chapter 4 will be for a single-centered instanton with action (3.14). The multi-centered solutions\(^1\) can then be obtained by making a multipole expansion around the single-centered one, the dominant term of which corresponds to the single-centered one.

To complete action (3.14) we have to combine it with the theta-angle terms of (2.35) which we can rewrite as

$$S_{surf} = i \varphi Q_1 + i \sigma Q_2 = \mp i (\sigma + \chi_0 \varphi) |Q_2| + i \varphi \hat{Q}_1 ,$$

(3.15)

the total single-centered instanton action thus becomes

$$S_{inst}^\pm = S_{cl} + S_{surf} .$$

(3.16)

The reason for rewriting (3.15) in this way is because we will associate $\hat{Q}_1$ to the membrane charge in the following section. The surface term allows us to distinguish instantons from anti-instantons in the action, again with the + denoting the former and − the latter.

\(^1\)In chapter 4 we demonstrate that these solutions can be constrained further by requiring them to preserve half of the supersymmetries.
3.3 The membrane

The second Bogomol’nyi bound is somewhat more complicated and becomes, using (3.5),

\[
\begin{pmatrix}
H_{\mu 1} \\
H_{\mu 2}
\end{pmatrix} = \pm \frac{1}{|\tau'|} \left( \begin{array}{c}
\chi (\chi - \chi_0) \partial_\mu e^{-\phi} + e^{-\phi} (\chi + \chi_0) \partial_\mu \phi + 2e^\phi \partial_\mu e^{-\phi} \\
(\chi - \chi_0) \partial_\mu e^{-\phi} + 2e^{-\phi} \partial_\mu \chi
\end{array} \right).
\]

(3.17)

Remember that \( \tau' \equiv (\chi - \chi_0) + 2ie^{\phi/2} \) and \( \chi_0 \) is an arbitrary constant which will be identified below with the previously introduced \( \chi_0 \) for a single-centered instanton. It is convenient to consider the Heisenberg combination

\[
H_{\mu 1} - \chi_0 H_{\mu 2} = \pm \partial_\mu h \quad h \equiv e^{-\phi} |\tau'|,
\]

(3.18)

the Bianchi identities imply that \( h \) must be harmonic and positive everywhere. We can write \( \chi \) as follows:

\[
\chi - \chi_0 = e^\phi \sqrt{h^2 - 4e^{-\phi}},
\]

(3.19)

where we have taken the positive branch, the negative branch only differs by some unimportant minus signs in the following calculations. The Bogomol’nyi equation for \( H_{\mu 2} \) now becomes

\[
H_{\mu 2} = \pm \frac{1}{\sqrt{h^2 - 4e^{-\phi}}} \left( 2e^{-\phi} \partial_\mu h - h \partial_\mu e^{-\phi} \right)
\]

(3.20)

and together with the Bianchi identities, the fact that \( h \) is harmonic gives

\[
(h^2 - 4e^{-\phi}) \partial_\mu \partial^{\mu} e^{-\phi} + 2\partial_\mu e^{-\phi} \partial^{\mu} e^{-\phi} - 2h \partial_\mu h \partial^{\mu} e^{-\phi} - 2e^{-\phi} \partial_\mu h \partial^{\mu} h = 0.
\]

(3.21)

For simplicity we limit ourselves to spherically symmetric solutions\(^2\) for \( h \):

\[
h = e^{-\phi|\chi_0^\prime|} + \frac{|\hat{Q}_1|}{4\pi^2 (x-x_0)^2},
\]

(3.22)

which validates the assumption that the dilaton depends on the coordinates only through \( h \).

Differentiating (3.21) allows us to solve for \( \phi \):

\[
e^{-\phi} = ah^2 + bh + c,
\]

where \( a, b, c \) are three integration constants. Combining this with (3.20) yields \( c = -\beta^2 \) where \( \beta \equiv \pm Q_2/|\hat{Q}_1|, b = -\beta\sqrt{1-\alpha} \) and \( \alpha = 4a \). Since

\(^2\text{Some multi-centered solutions were constructed in [85].}\)
$H_{\rho 2}$ is real and $e^{-\phi}$ must be positive this requires that $0 \leq \alpha \leq 1$ where
$\alpha = 0$ must be treated separately.

Evaluating $\Delta \chi$ using (3.19) fixes $\alpha$ in terms of the charges and asymptotic values of the fields,

$$\alpha = 1 - \frac{(\Delta \chi - 2\beta e^{\phi_{\infty}})^2}{|\tau'_{\infty}|^2} \quad |\tau'_{\infty}|^2 = (\Delta \chi)^2 + \frac{4}{g_s^2}.$$  

Furthermore, the solution for $\chi$ can be directly read off from (3.19), using the information obtained above one can check that $\lim_{x \to x_0} \chi(x)$ is indeed $\chi_0$, as in the case of the 5-brane instanton.

Note that, contrary to the 5-brane system, $\chi$ does require a source at the excised point. Since $\alpha$ has to lie in the interval $[0, 1]$ we must have

$$\frac{\Delta \chi - |\tau'_{\infty}|}{2e^{\phi_{\infty}}} \leq \beta \leq \frac{\Delta \chi + |\tau'_{\infty}|}{2e^{\phi_{\infty}}},$$  

(3.23)

for fixed $g_s$ and (positive) $\Delta \chi$.

All the integration constants of the solution have now been expressed in terms of the charges and the asymptotic values of the scalar fields. The formula for the action (3.10) gives us

$$S = |\tau'_{\infty}| \left( |\hat{Q}_1| + \frac{1}{2} \Delta \chi Q_2 \right),$$

which is manifestly positive by virtue of (3.23). The simplest form of the action is obtained by switching off the R-R background by taking $Q_2 = \Delta \chi = 0$ and including the appropriate theta-angle term\(^3\), which gives

$$S_{\text{inst}} = \frac{2|Q_1|}{g_s} + i\phi^s Q^s_1,$$  

(3.24)

where $\phi^s$ is either $\varphi$ or $\chi$ and $Q^s_1$ either $Q^\varphi_1$ or $Q^\chi_1$. Note the factor of 2 in (3.24), which will be very nicely confirmed in chapter 5. As discussed in section 2.4 the imaginary term is related to a surface term that arises in the dualization process. In the dual UHM formulation the parameter $\varphi$ (or $\chi$) in (3.24) can be identified with the value of $\varphi$ (or $\chi$) at infinity. Its presence breaks the shift symmetry in $\varphi$ (or $\chi$) to a discrete subgroup.

The explanation of the choice of theta-angle terms in (3.24) is as follows. Instead of dualizing from the tensor multiplet to the double tensor multiplet using the shift symmetry in $\varphi$, as we did in section 2.4, we could as well dualize using the shift symmetry in $\chi$. Or in the case of going from the UHM to the DTM, dualizing over $\sigma$ and $\chi$.

The point is that dualizing over $\chi$, i.e. using the shift generated by $\gamma$ (see

\(^3\)Remember that in the DTM formulation the $\phi^s$ (i.e. $\varphi$ or $\chi$) are just parameters.
would have given a (formally) different 2-form with associated field strength and consequently a different charge. This means that there are two different charges: $Q_1^\varphi$ and $Q_1^\chi$, the first charge corresponds to the shift $\gamma$ and the second to $\beta$. Hence they are associated to two different membranes (see also [52]).

This means that we can either dualize to a DTM in which the Bogomol’nyi bound leads to a membrane configuration with charge $Q_1^\varphi$, or to a configuration with charge $Q_1^\chi$. We cannot dualize over the two scalars simultaneously, since the shift symmetries in $\chi$ and $\varphi$ do not commute. This fact will be re-derived from a string theory point of view in chapter 5 (it will have to do with the fact that the membrane can wrap either along the one or the other supersymmetric cycle). The 5-brane charge $Q_2$ corresponds to the shift in $\sigma$ by $\alpha$.

This action also has exactly the right dependence on $g_s$, [71], for a membrane instanton with charge $\hat{Q}_1$, note that $\hat{Q}_1 = Q_1$ if $Q_2 = 0$.

One could add 5-brane charge which raises the action until, for fixed $g_s$ and $\Delta \chi$, one reaches $\alpha = 0$ in (3.23). From that point on, the solution is no longer valid and one must restrict to the 5-brane, without membranes.

**A short summary and outlook**

We have constructed the 5-brane and membrane solutions in $N = 2$ supergravity in 4 dimensions. These solutions can be identified with the 5-brane and membrane respectively, since we know from string theory, [5, 86, 71], what their actions should look like in four dimensions (in terms of $g_s$ anyway) and we find agreement. As explained above, we will perform an instanton calculation for the 5-brane in chapter 4 and compute in this way the instanton corrections to the UHM. To do this it is necessary to calculate the (Euclidean) supersymmetry transformations, Killing spinors and instanton measures, for this particular background. All of this will be done in the subsequent chapter.

In principle one could try to do the same for the membrane solution, that is, perform an instanton calculation as for the 5-brane. Instead we will use knowledge of the isometries of the the UHM to directly construct nonperturbative corrections to the UHM, without the usual instanton calculations. Comparison to string theory permits us to identify these corrections with loop expansions around membrane instantons. This will be the subject of chapter 5.