The moment maps

In this appendix we will elaborate on the quaternionic geometry introduced in section 1.4. Since we will apply the quaternionic geometry to supergravity we will call the coordinates $\phi^A$, $A = 1, \ldots, 4n$ with $n$ the number of hypermultiplets. For example, the quaternionic 1-forms are now expressed as $V_i^a \equiv V_i^a d\phi^A$. We can rewrite the hyperkähler 2-forms of equation (1.13) as

$$K^r = \frac{i}{2} G_{\bar{a}b} V^\bar{b}_i \wedge \bar{V}^{j\bar{a}} (\tau^r)_{ij} ,$$

(H.1)

with $\tau^r$ the Pauli matrices, $r = 1, 2, 3$, $i = 1, 2$ and $a, \bar{a} = 1, \ldots, 2n$. These 2-forms satisfy the quaternionic algebra (1.12). As in (1.15) we have the $SU(2)$ connection 1-form $\omega^r = \omega^r_A d\phi^A$ with $SU(2)$ curvature

$$\Omega^r = d\omega^r - \frac{1}{2} \varepsilon^{rst} \omega^s \wedge \omega^t .$$

(H.2)

In the conventions of [48, 49] we write the Einstein property as

$$R_{A\bar{B}} = \frac{1}{4n} G_{A\bar{B}} R ,$$

with $R$ the constant Ricci scalar. The relation between the $SU(2)$ curvature 2-forms and the hyperkähler 2-forms (1.16) reads

$$\Omega^r = \nu K^r \quad \text{with} \quad \nu \equiv \frac{1}{4n(n+2)} R .$$

(H.3)

The $SU(2)$ curvature obeys

$$d\Omega^r = \varepsilon^{rst} \omega^s \wedge \Omega^t ,$$

(H.4)

\footnote{The convention for the $SU(2)$ connection and curvature is chosen to be the same as e.g. in [77]. With respect to [48, 49], our $SU(2)$ connection is chosen (minus) twice the one in [48, 49], and therefore also the $SU(2)$ curvature is (minus) twice as large.}
compare with (1.14) and (1.16). For the potential we need the moment maps. They are defined from
\[ K^r_{\hat{A}\hat{B}} k^I = D_A P^r_I = \partial_A P^r_I - \varepsilon^{rst} \omega^s_A \wedge P_I^t , \]
where \( I \) labels the various isometries and \( D_A \) is the \( SU(2) \) covariant derivative, as in (1.14). One can solve this relation for the moment maps which yields [77]
\[ P^r_I = -\frac{1}{2n
u} K^r_{\hat{A}\hat{B}} D^\hat{A} k^I \hat{B} . \] (H.5)

In supergravity, the value of \( \nu \) is fixed in terms of the gravitational coupling constant. If we normalize the kinetic terms of the graviton and scalars in the supergravity action as
\[ e^{-1} \mathcal{L}_{\text{kin}} = -\frac{1}{2\kappa^2} R(e) - \frac{1}{2} G_{\hat{A}\hat{B}} \partial_\mu \phi^\hat{A} \partial^\mu \phi^\hat{B} , \] (H.6)
then local supersymmetry fixes \( \nu = -\kappa^2 \). This is in accordance with [48, 49] and with [26] after a rescaling of the metric \( G_{\hat{A}\hat{B}} \) with a factor \( 1/2 \). For the universal hypermultiplet, we will work with conventions in which \( \kappa^2 = 1/2 \), so we set \( \kappa^2 = 1/2 \) below. To compare with [77], we have to multiply the Lagrangian (H.6) by 2 and then set \( \kappa^2 = 2 \).

We now include the scalar potential that arises after gauging a single isometry, so we can leave out the subscript \( I \). The isometry can then be gauged by the graviphoton and in the absence of any further vector multiplets, the relevant terms in the Lagrangian are
\[ e^{-1} \mathcal{L} = -\frac{1}{2\kappa^2} R - \frac{1}{2} G_{\hat{A}\hat{B}} D_\mu \phi^\hat{A} D^\mu \phi^\hat{B} - V , \] (H.7)
the potential \( V \) is given by
\[ V = (2\kappa^{-2} G_{\hat{A}\hat{B}} k^\hat{A} k^\hat{B} - 3 \vec{P} \cdot \vec{P}) . \] (H.8)
\( D_\mu \) is the covariant derivative with respect to the gauged isometry that corresponds to the Killing vector \( k^\hat{A} \). The factor of \( \kappa \) has to appear on dimensional grounds. For \( \kappa^2 = 2 \) this agrees precisely with the result in [77], we will set \( \kappa^2 = 1/2 \).

Our conventions are chosen such that they apply to the universal hypermultiplet metric and the conventions used in [32, 76]. At the classical level we have
\[ ds^2 = G_{\hat{A}\hat{B}} d\phi^\hat{A} \otimes d\phi^\hat{B} = d\phi^2 + e^{-\phi}(d\chi^2 + d\varphi^2) + e^{-2\phi}(d\sigma + \chi d\varphi)^2 . \]
\(^2\)Our definition of the moment map is the same as in [77]. This normalization is different from [48, 49], and our moment maps are (minus) two times the ones defined in [48, 49].
For the corresponding Ricci tensor we find
\[ R_{\bar{A}\bar{B}} = \frac{3}{2} G_{\bar{A}\bar{B}} . \]
The Ricci scalar is then \( R = -6 \) and therefore we have \( \nu = -1/2 \). This implies that in these conventions we should set \( \kappa^2 = 1/2 \), which is equivalent to a cosmological constant \( \Lambda = -3/2 \), see (5.4) on the quaternionic manifold. Note that the constant \( \nu \) is identical to the \( \lambda \) introduced in (1.16).

**Quaternionic geometry of the PT metric**

We present the quaternionic geometry of the UHM in the PT framework. The quaternionic properties of the PT metric can be demonstrated by constructing the corresponding quaternionic 1-form vielbeins which we parameterize as
\[ V_i^a = \left( \bar{a} \quad -\bar{b} \right) . \]
Substituting this ansatz into (1.18) we obtain
\[ ds^2 = a \otimes \bar{a} + b \otimes \bar{b} + \text{c.c.} . \]
Comparing this expression with the PT metric (5.2) leads to
\[ a = \frac{1}{\sqrt{2}r} \left( f^{1/2} dr + if^{-1/2} (dt + \Theta) \right) , \quad b = \frac{1}{\sqrt{2}r} \left( fe^{ih} \right)^{1/2} (du + i dv) . \]
The computation of the quaternionic 2-forms (H.1) then yields
\[ K^1 = -i(a \wedge b - \bar{a} \wedge \bar{b}) , \quad K^2 = a \wedge b + \bar{a} \wedge \bar{b} , \quad K^3 = -i(a \wedge \bar{a} + b \wedge \bar{b}) . \]
These satisfy the quaternionic algebra (1.12). Using (H.3) and (H.4), we then determine the \( SU(2) \) connection for the PT metric,
\[ \omega^1 = \frac{1}{r} e^{h/2} dv , \quad \omega^2 = \frac{1}{r} e^{h/2} du , \quad \omega^3 = -\frac{1}{2r} (dt + \Theta) - \frac{1}{2} (\partial_t h \, du - \partial_u h \, dv) . \]
The PT metric has a shift symmetry in \( t \). In coordinates \((r,u,v,t)\) the corresponding Killing vector is given by
\[ k^A = (0, 0, 0, e_0)^T . \]
The moment maps of this shift symmetry can be computed from (H.5). The result is *independent* of the functions \( f, h, \) and \( \Theta \) and reads
\[ P^1 = 0 , \quad P^2 = 0 , \quad P^3 = \frac{e_0}{r} . \]