This appendix collects several technical details about the solution of the Toda equation constructed in section 5.1. In the first part we prove that $m_n \geq -2$. The proof that $\alpha = 0$ is given the second part. In the third part the derivation of the one-instanton solution is given.

The lower bound on $m_n$

In this subsection we establish $m_n \geq -2$. Our starting point is the ansatz (5.9), which we substitute into the Toda equation (5.10). This results in the following power series expansion

$$0 = \sum_{n,m} r^{-m/2 + \alpha + 1} e^{-2n\sqrt{r}} [(\Delta + n^2)f_{n,m} + (n a_{m+1} r^{-1/2} + b_{m+2} r^{-1})f_{n,m}]$$

$$+ \sum_{n,m} \sum_{n',m'} r^{-(m+m')/2 + 2\alpha} e^{-2(n+n')\sqrt{r}} \left[ f_{n',m'}(\Delta + 2n^2)f_{n,m} - \nabla f_{n,m} \cdot \nabla f_{n',m'} + 2(a_{m+1} r^{-1/2} + b_{m+2} r^{-1})f_{n,m} f_{n',m'} \right]$$

$$+ \sum_{n,m} \sum_{n',m'} \sum_{n'',m''} r^{-(m+m'+m'')/2 + 3\alpha - 1} e^{-2(n+n'+n'')\sqrt{r}} f_{n,m} f_{n',m'} f_{n'',m''} \times \left[ n^2 + n a_{m+1} r^{-1/2} + b_{m+2} r^{-1} \right] ,$$

where we have extended the definitions for $a_m$, $b_m$ given in (5.12) to non-zero $\alpha$:

$$a_m = \frac{1}{2} (2m - 4\alpha - 1) , \quad b_m = \frac{1}{4} (m - 2\alpha)(m - 2\alpha - 2) .$$

In order to obtain a bound on $m_n$ (for which the $f_{n,m_n} \neq 0$), we extract the leading order contributions in the $r$-expansion arising from the single, double and triple sum in (G.1). Starting at $n = 1$ and working iteratively

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1 Here we have not performed the splitting into instanton sectors yet.
towards higher values $n = 2, 3, \ldots$, we find that at a fixed value of $n$ these contributions are proportional to
\begin{align}
\text{single sum} & \propto r^{-m_n/2+\alpha+1} \\
\text{double sum} & \propto r^{-m_n+2\alpha} \\
\text{triple sum} & \propto r^{-3m_n/2+3\alpha-1}.
\end{align}
\tag{G.3}

Investigating the $m_n$-dependence of these relations, we find that for $m_n \leq -3$ the leading order term in $r$ arises from the triple sum, which decouples from all the other terms in (G.1).

We now assume that for a fixed value $n$ there exists an $f_{n,m_n} \neq 0$ for $m_n \leq -3$. Extracting the equation leading in $r$ from (G.1), we find that
\begin{equation}
n^2 f_{n,m_n}^3 = 0, \quad m_n \leq -3,
\end{equation}
\tag{G.4}
which has $f_{n,m_n} = 0$ as its only solution. Hence, we establish the lower bound
\begin{equation}
m_n \geq -2
\end{equation}
\tag{G.5}
for all values of $n$ or, equivalently, all instanton sectors\(^2\).

**Fixing the parameter $\alpha$**

When making the ansatz (5.9) in order to describe membrane instanton corrections to the universal hypermultiplet, we included the parameter $\alpha \in [0, 1/2)$ to allow for the possibility that the leading term in the instanton solution occurs with a fractional power of $g_s$. Based on the plausible assumption that the perturbation series around the instanton gives rise to a power series in $g_s$ (and not fractional powers thereof) we now give a proof that a consistent solution of the Toda equation requires $\alpha = 0$.

Splitting (G.1) into instanton sectors gives us the following analogue of

\(^2\)Notice that this argument is not quite sufficient to also fix $\alpha = 0$, as for $\alpha = 1/4$ the single and triple sums do not decouple, which has been crucial in establishing (G.4).
Based on this equation we can now make several observations. First, we find that the $N = 1$ sector of (G.6) still gives rise to (5.13), with the coefficients $a_m, b_m$ now replaced by (G.2). To lowest order, $m = m_1$, this is just the equation

\[(\Delta + 1)f_{1,m_1}(u, v) = 0.\] (G.7)

Second, we observe that the equation describing the $N = 2$ sector is modified to

\[0 = (\Delta + 4)f_{2,m} + 2a_m f_{2,m-1} + b_m f_{2,m-2} + \sum_{m'} r^{2\alpha} e^{-2(n'+n'')\sqrt{\tau}} f_{n',m'} f_{n'',m''} [n^2 f_{n'',m'-m''-2} + n a_{m'+1} f_{n'',m'-m''-3} + b_{m'+2} f_{n'',m'-m''-4}] + \nabla f_{1,m_1}(u, v) \cdot \nabla f_{1,m_1}.\] (G.8)

Note that for $\alpha = 0$, the sum appearing in the second line is just an inhomogeneous term to the equations determining $f_{2,m}$. For $\alpha \neq 0$, however, the sum decouples due to the different powers in $r$. Therefore, in the case $\alpha \neq 0$, the sum gives rise to an additional constraint equation, which is absent for $\alpha = 0$. Since the sum contains the $f_{1,m}$ only, this additional relation imposes a restriction on the $N = 1$ instanton solution. Upon using (G.7), this additional constraint reads at the lowest level

\[f_{1,m_1}^2 - (\nabla f_{1,m_1})^2 = 0.\] (G.8)

For $\alpha \neq 0$ a non-trivial 1-instanton solution has to satisfy both (G.7) and (G.8), so that for establishing $\alpha = 0$ it suffices to show that these equations have no common non-trivial solution.

Suppose that $f_{1,m_1} \neq 0$, which by definition of $f_{1,m_1}$ has to hold. We then multiply (G.7) with $f_{1,m_1}$, giving

\[0 = f_{1,m_1} \Delta f_{1,m_1} + f_{1,m_1}^2 = f_{1,m_1} \Delta f_{1,m_1} + (\nabla f_{1,m_1})^2 = \frac{1}{2} \Delta f_{1,m_1}^2,\]
where we have used (G.8) in the first step. In terms of complex coordinates $z = u + iv$ it is $\Delta = 4\partial_z \partial_{\bar{z}}$, and the general solution reads

$$f_{1,m_1}^2(z, \bar{z}) = g(z) + \bar{g}(\bar{z}) .$$

Substituting this back into (G.7), we find

$$0 = (\Delta + 1)f_{1,m_1} = f_{1,m_1}^{-3}[-\partial_z g(z) \partial_{\bar{z}} \bar{g}(\bar{z}) + (g(z) + \bar{g}(\bar{z}))^2] ,$$

which is equivalent to

$$\partial_z g(z) \partial_{\bar{z}} \bar{g}(\bar{z}) = g(z)^2 + 2g(z)\bar{g}(\bar{z}) + \bar{g}(\bar{z})^2 .$$

Since the right-hand side of this expression contains terms which are (anti-) holomorphic, whereas the left-hand side does not, we find that the only solution is given by $g(z) = ic$ with $c \in \mathbb{R}$ constant. Thus $f_{1,m_1} = 0$, which contradicts our assumption and shows that the ansatz (5.9) does not give rise to a one-instanton sector if $\alpha \neq 0$. Conversely, a non-trivial one-instanton sector exists for $\alpha = 0$ only, which then fixes $\alpha = 0$.

The one-instanton solution

The general one-dimensional solution in the one-instanton sector was given in (5.14). The functions $G_s(x)$ introduced there are defined by

$$G_s(x) = x^{s+1}h_{s-1}(x) ,$$

where $h_s(x) = j_s(x) + iy_s(x)$ are the spherical Bessel functions of the third kind. For $s \geq 0$ the $G_s(x)$ have no poles, they read

$$G_0(x) = e^{ix} , \quad G_{s>0}(x) = 2^{-s}e^{ix} \sum_{k=1}^{s} \frac{(2s - k - 1)!}{(s - k)! (k - 1)!} (-2ix)^k .$$

Using the properties

$$x^2h''_s + 2xh'_s + [x^2 - s(s + 1)]h_s = 0 , \quad h'_s + \frac{s + 1}{x} h_s = h_{s-1} ,$$

we easily verify the relation

$$(\partial^2_x + 1) G_s(x) = 2s G_{s-1}(x) .$$
The proof of (5.14) is now simple:

\[
(\partial_x^2 + 1)f_{1,m}(x) = \text{Re} \sum_{s \geq 0} \frac{1}{s! (-2)^s} k_{1,m}(s) (\partial_x^2 + 1)G_s(x)
\]

\[= -\text{Re} \sum_{s \geq 1} \frac{1}{(s-1)! (-2)^{s-1}} k_{1,m}(s) G_{s-1}(x)\]

\[= -\text{Re} \sum_{s \geq 0} \frac{1}{s! (-2)^s} k_{1,m}(s+1) G_s(x)\]

\[= -\text{Re} \sum_{s \geq 0} \frac{1}{s! (-2)^s} \left[ a_m k_{1,m-1}(s) + b_m k_{1,m-2}(s) \right] G_s(x)\]

\[= -a_m f_{1,m-1}(x) - b_m f_{1,m-2}(x).\]  

\[\text{(G.9)}\]

For the general \((u,v)\)-dependent solution given in (5.17), the proof is almost identical.