The Wick rotation

In this appendix we clarify the Wick rotation introduced in section 2.3, furthermore the calculations in chapter 4 will be performed in Euclidean space and require the use of appropriate fermions.

First let us Wick rotate the DTM Lagrangian (2.32). In Minkowskian space the action appears in the path integral as $e^{iS}$ with $\eta_{\mu\nu}=$diag$(-+++)$). We rotate $t \rightarrow it \equiv x_4$ which implies $\frac{\partial}{\partial t} = i \frac{\partial}{\partial x_4}$ and the integration measure becomes $d^4x = -id^4x_E$. This means that a massless scalar appears in the path integral as

$$\exp\{i \int d^4x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \} \rightarrow \exp\{- \int d^4x_E \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \} \equiv \exp\{-S_E\}$$

where in the second half the indices run over 1 to 4. We will always extract an overall minus sign out of the action giving a damped exponential, since the action has become positive definite.

Having seen how scalars transform we turn to the double tensor term:

$$L_{dt} = \frac{1}{2} M^{IJ} H^\mu_I H^\mu_J , \quad H^\mu_I = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} I = \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}$$

Using $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\delta} = -3! \delta_{[\nu}^{[\delta} \delta_{\mu]}^{\rho]}$, the conventions $\varepsilon^{0123} = -\varepsilon_{0123} = 1$ and $\varepsilon^{1234} = 1$, we can rewrite this (up to unimportant $I, J$ quantities) as

$$-H_{\nu\rho\sigma} H^{\nu\rho\sigma} = 3 H_{0ij} H^{0ij} + H_{ijk} H^{ijk} ,$$

where $i, j = 1, 2, 3$. Now we Wick rotate according to

$$B_{0i} = i B_{4i} \rightarrow H_{0ij} = i H_{4ij}$$

and since $H_{0ij} H^{0ij} = H_{4ij} H^{4ij}$ the double tensor term becomes

$$L_{dt}^E = \frac{1}{2} H_{\mu I}^E H^E_{\mu J} M^{IJ} .$$
Then there is the scalar-tensor term
\[ L_{st} = -A_I H_I^\mu \partial_\mu \phi^A \]
\[ \sim -\frac{1}{2} \left( \varepsilon^{0\nu\rho\sigma} H_{\nu\rho\sigma} \partial_0 \phi + 3 \varepsilon^{i0\rho\sigma} H_{0\rho\sigma} \partial_i \phi \right) . \]

Using \( \varepsilon^{0\nu\rho\sigma} = -\varepsilon^{A\nu\rho\sigma} \) and \( \varepsilon^{i0\rho\sigma} H_{0\rho\sigma} = -i \varepsilon^{i4\rho\sigma} H_{4\rho\sigma} \) we obtain
\[ L_{st}^E = -i A_I H_I^\mu E \partial_\mu \phi^A . \] (B.1)

When using these expressions we will leave out the superscript \( E \) and understand that \( \mu = 1, \ldots, 4 \).

Lastly there is the matter of the spinors. The spinors we work with are Lorentzian 2-component Weyl spinors, say \( \lambda_\alpha, \bar{\lambda}^{\dot{\alpha}} \), related to each other by complex conjugation. However, in Euclidean space the Lorentz group factorizes as \( \text{Spin}(4) \cong SU(2) \times SU(2) \) and to each \( SU(2) \) belongs a 2-component spinor, \( \lambda_\alpha \) and \( \bar{\lambda}^{\dot{\alpha}} \) which are not related to each other by complex conjugation, they constitute inequivalent representations of the Lorentz group. The same remarks hold for the supersymmetry parameters \( \epsilon^i \) and \( \bar{\epsilon}^{\dot{i}} \). The sigma matrices \( \sigma^\mu \) and \( \bar{\sigma}^{\dot{\mu}} \) have lower and upper indices respectively, i.e., \( (\sigma^\mu)_{\alpha\dot{\alpha}} \) and \( (\bar{\sigma}^{\dot{\mu}})^{\dot{\alpha}\alpha} \) where we instead of dotted indices use indices with a \( ' \) to denote the inequivalent representations. In Euclidean space these matrices take the form
\[ \sigma^\mu = (\vec{\sigma}, -i) \quad \bar{\sigma}^{\dot{\mu}} = (-\vec{\sigma}, -i) \]
where \( \vec{\sigma} \) are the Pauli matrices, note that this is consistent with \( \sigma^0 = -i \sigma^4 \).

We have the properties (slightly different from Lorentzian signature)
\[ \sigma^\mu \bar{\sigma}^{\dot{\nu}} = -g^{\mu\nu} + 2 \sigma^{\mu\nu} \quad \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = \sigma^{\mu\nu} , \]
where \( \sigma^{\mu\nu} \equiv \frac{1}{2} \sigma^{\mu}[\sigma^{\nu}] \) and
\[ \sigma^{\mu\nu} \sigma^\rho = g^{\mu\rho} \sigma^\nu - g^{\nu\rho} \sigma^\mu - g^{\mu\nu} \sigma^\rho + \varepsilon^{\mu\nu\rho\sigma} \sigma_\sigma . \]

Other often used properties of the sigma matrices are
\[ (\sigma^\mu)_{\alpha\beta'} (\sigma_{\mu})_{\gamma\alpha'} = -2 \varepsilon_{\alpha\gamma} \varepsilon_{\beta'\alpha'} \]
\[ (\bar{\sigma}^{\dot{\mu}})^{\dot{\beta}\alpha'} (\sigma_{\mu})_{\gamma\alpha'} = -2 \delta_{\gamma}^{\dot{\beta}} \delta_{\alpha'}^\alpha \]
\[ (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta'} (\sigma_{\nu})_{\gamma\alpha'} = -\varepsilon_{\gamma}^{(\beta} \sigma^\mu_{\alpha')} . \] (B.2)

Finally, the gravitini and the graviphoton are rotated as follows
\[ \psi_{0i} = i \psi_{4i} \quad \bar{\psi}_{0i} = i \bar{\psi}_{4i} \quad A_0 = i A_4 . \]

For more information on the conventions we used, see Wess and Bagger [133].