

INDEPENDENT LOOP INVARIANTS FOR 2+1 GRAVITY

R. Loll¹

*Center for Gravitational Physics and Geometry
Physics Department, Pennsylvania State University
University Park, PA 16802, U.S.A.*

Abstract

We identify an explicit set of complete and independent Wilson loop invariants for 2+1 gravity on a three-manifold $M = \mathbb{R} \times \Sigma^g$, with Σ^g a compact oriented Riemann surface of arbitrary genus g . In the derivation we make use of a global cross section of the $PSU(1,1)$ -principal bundle over Teichmüller space given in terms of Fenchel-Nielsen coordinates.

¹ address after Aug 94: Sezione INFN di Firenze, Largo E. Fermi 2, Arcetri, I-50125 Firenze, Italy

1 Introduction

In recent years, Wilson loop variables have found manifold application in the investigation of fundamental physical theories. This holds in particular for the canonical quantization of gravity, and for quantum chromodynamics, the latter both in the continuum and the regularized lattice version (see [1] for a review of loop methods). Still many aspects of these loop formulations remain poorly understood. On the one hand they enjoy explicit gauge invariance, but on the other a vast overcompleteness is introduced by considering Wilson loops corresponding to arbitrary closed curves in space(-time). This obscures the physical content of the theory, which becomes particularly obvious in explicit calculations as for instance those occurring in lattice gauge theory. There the overcompleteness of the Wilson loop variables constitutes the biggest obstacle to making efficient numerical computations [2].

There are in general topological obstructions to finding sets of loop variables which are both independent and complete on the quotient space \mathcal{A}/\mathcal{G} of connection one-forms modulo gauge transformations. Although \mathcal{A} possesses an affine, i.e. an “almost linear” structure, this is in general not any more true for the corresponding space of \mathcal{G} -orbits in \mathcal{A} . This is the case for both $SU(N)$ -gauge theory and for 3+1 gravity in the Ashtekar formulation, where the gauge group is $SL(2, \mathbb{C})$.

One notable exception to this state of affairs is the theory of 2+1 gravity. This theory has been studied both for its mathematical beauty and as a toy model for the (3+1)-dimensional theory, in particular, to gain more understanding of quantum loop representations for gravitational theories [3,4]. In the pure (2+1)-dimensional theory, without matter coupling, the absence of local field degrees of freedom may be “compensated” for by allowing the underlying space-time manifold M to have a non-trivial topology. We will be concerned with the case $M = \mathbb{R} \times \Sigma^g$, where Σ^g is a compact orientable Riemann surface of genus $g \geq 2$. Using a space of connections as the basic variables of the theory, it could be demonstrated that its physical (i.e. classically reduced) configuration space is both finite-dimensional and contractible, and isomorphic to the $(6g-6)$ -dimensional Teichmüller space [5]. Thus it is possible in principle to find a set of loop variables that can serve as globally well-defined coordinates on the reduced configuration space.

Although there have been extensive investigations of both the classical and quantum theory of the genus-1 case (which from a mathematical point of view is somewhat degenerate), not a great deal is known about the explicit physical dynamics for higher genus. From the point of view of the loop quantization approach, it is important to understand which features of the genus-1 case generalize to higher g . One of the first steps in achieving this is to understand how Wilson loop variables describe the physical configuration (and phase) space. This problem can be phrased as follows. The reduced configuration space may be described

as (a sector of) the space \mathcal{A}^F of flat $SO(2,1)$ -connections $A(x)$ on Σ^g modulo the group \mathcal{G} of $SO(2,1)$ -gauge transformations. Because of their gauge invariance, the Wilson loop variables

$$T(\gamma) := \text{Tr } U_\gamma = \text{Tr } \text{P exp} \oint_\gamma A \quad (1.1)$$

are functions on the quotient space $\mathcal{A}^F/\mathcal{G}$. Since the connections are flat, the Wilson loop (1.1) depends only on the homotopy equivalence class of the closed curve γ . However, Wilson loop variables corresponding to arbitrary elements γ of the homotopy group $\pi_1(\Sigma^g)$ are not all independent but subject to i) identities among the traces of products of $SO(2,1)$ -representation matrices, so-called Mandelstam constraints, and ii) identities coming from the fundamental relation

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = 1 \quad (1.2)$$

between the generators $\alpha_i, \beta_i, i = 1 \dots g$ of the homotopy group (s. Fig.1). (Recall that a general homotopy group element γ is a “word” in terms of the α_i and β_i , i.e. a finite ordered product of the generators and their inverses.) The difficulty now lies in identifying a relevant set of such identities and solving them to obtain $6g - 6$ independent trace invariants. For $g = 2$, an explicit solution has been given by Nelson and Regge [6]. The computations are rather involved and their result so far has not been extended to genus $g > 2$.

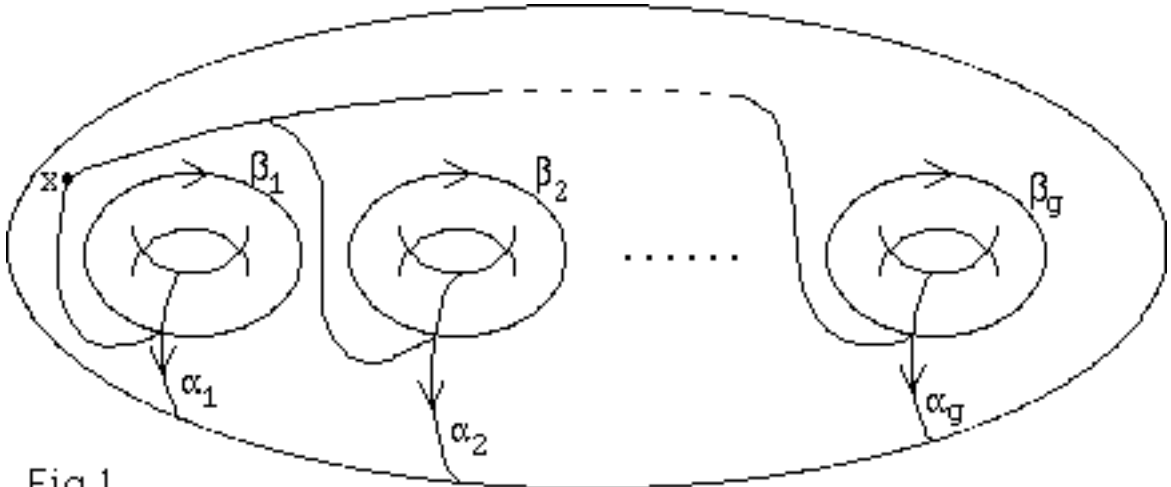


Fig.1

In this paper, we will present a solution for the general genus case. However, instead of tackling the algebra of the Wilson loop constraints directly, we will make use of the explicit

parametrization of the Teichmüller spaces in terms of Fenchel-Nielsen coordinates and of a result by Okai, namely, an explicit global section of the principal $PSU(1,1)$ -bundle over Teichmüller space [7]. This enables us to express arbitrary Wilson loops as functions of the Fenchel-Nielsen coordinates. It then remains to find an algebraically independent set of such Wilson loops. We give one solution, i.e. $6g - 6$ independent (linear combinations of) Wilson loops for any genus g , which have a particularly simple algebraic dependence on the Fenchel-Nielsen coordinates.

2 The loop invariants

We will now briefly review the necessary ingredients for deriving our result. The Riemann-Hilbert action for three-dimensional Lorentzian gravity on a space-time manifold $M = \mathbb{R} \times \Sigma^g$ may be substituted by an action on a space of connections, which requires the introduction of internal $SO(2,1)$ -degrees of freedom. After a Legendre transformation one obtains a first-class constrained system, which remarkably can be solved explicitly (for more details on the general theory, see [5,8]). As already mentioned in the introduction, the reduced configuration space may be described as a quotient space $\mathcal{A}^F/\mathcal{G}$ (and the corresponding physical phase space is the cotangent bundle over $\mathcal{A}^F/\mathcal{G}$). Alternatively, one may consider the set of holonomies U_{α_i} , U_{β_i} around the $2g$ generators of $\pi_1(\Sigma^g)$ modulo gauge transformations at the common base point $x \in \Sigma^g$ and subject to

$$U_{\alpha_1} U_{\beta_1} U_{\alpha_1}^{-1} U_{\beta_1}^{-1} \dots U_{\alpha_g} U_{\beta_g} U_{\alpha_g}^{-1} U_{\beta_g}^{-1} = \mathbb{1} \quad (2.1)$$

which is a direct consequence of relation (1.2). Furthermore, the holonomies must all lie in the sector of $SO(2,1)$ consisting of boosts around spacelike axes. For computational simplification we will work in the two-dimensional representation of $SU(1,1)$, identifying opposite points. The gauge group is therefore to be identified with $PSU(1,1) = SU(1,1)/\mathbb{Z}_2$, where we have divided out the normal subgroup. This form enables us to apply directly a result obtained by Okai [7], who constructed a global section of the trivial principal bundle

$$\begin{array}{c} \text{Hom}(\pi_1(\Sigma^g), PSU(1,1))^{e=2g-2} \\ \downarrow \\ \text{Hom}(\pi_1(\Sigma^g), PSU(1,1))^{e=2g-2} / PSU(1,1) = \mathcal{T}_g. \end{array} \quad (2.2)$$

The space of homomorphisms $\text{Hom}(\pi_1(\Sigma^g), PSU(1,1))^{e=2g-2}$ is the same as the space of the holonomies U described above, before factoring out by the gauge group action. The

superscript $e = 2g - 2$ denotes the connected component consisting of representations whose associated $\mathbb{R}P^1$ -bundle over Σ^g has Euler number $2g - 2$ [9]. This condition selects exactly the sector of holonomies we are interested in. The bundle fibre is given by the group $PSU(1, 1)$, acting adjointly at the base point x of the homotopy generators. The base space of the bundle is naturally isomorphic to the Teichmüller space \mathcal{T}_g of Σ^g .

The Teichmüller space \mathcal{T}_g is diffeomorphic to \mathbb{R}^{6g-6} and may be parametrized globally by the so-called Fenchel-Nielsen coordinates [10]. These are a set of length and angle coordinates associated with a pants decomposition of the genus- g surface. The surface is cut along $3g - 3$ simple geodesic curves (geodesic with respect to a constant negative-curvature metric of value -1) into $2g - 2$ “pants” P_i . As indicated in Figs.1 and 2, the geodesics will be labelled by α_i , $i = 1 \dots g$, γ_i , $i = 1 \dots g - 1$, and δ_i , $i = 2 \dots g - 1$. In terms of the homotopy group generators, one has $\gamma_i = \beta_i \alpha_i^{-1} \beta_i^{-1} \alpha_{i+1}$ and $\delta_i = \gamma_i \dots \gamma_{g-1} \beta_g \alpha_g^{-1} \beta_g^{-1}$. With each of these geodesics, we associate a pair of numbers $(l_j, \tau_j) \in \mathbb{R}^+ \times \mathbb{R}$ which measure the length of the geodesic and the relative twist with which the two pants meeting along the geodesic cut may be glued together. Following Okai’s notation, we associate the index j with the $3g - 3$ geodesics as follows: α_1 : $j = -\infty$; γ_1 : $j = 0$; δ_i : $j = 3i - 5$, α_i : $j = 3i - 4$, γ_i : $j = 3i - 3$ ($2 \leq i \leq g - 1$); α_g : $j = \infty$.

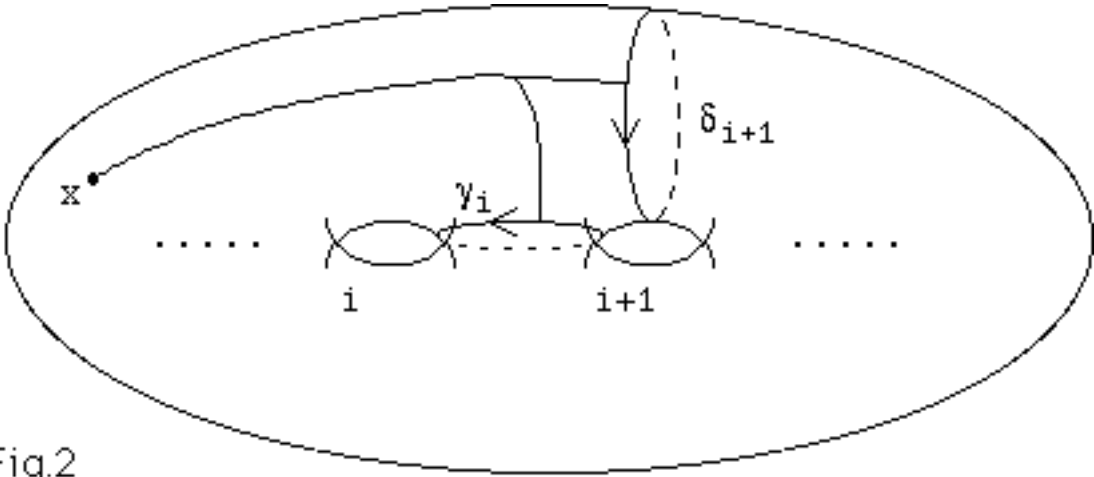


Fig.2

One proceeds by subdividing each P_i along three geodesic arcs (connecting pairs of its boundary components) into two right-angled hexagons. The lengths of these geodesic arcs are not independent but depend on the l_i through identities coming from hyperbolic geometry. One then associates $PSU(1, 1)$ -matrices to both the geodesic arcs and the boundary components, depending on their geodesic lengths. To determine the $PSU(1, 1)$ -holonomy matrix associated with a given element γ of $\pi_1(\Sigma^g)$, one chooses a representative that is homotopi-

cally equivalent to γ and is made up exclusively of such geodesic lines. Starting at the base point x , one multiplies together the corresponding $PSU(1,1)$ -matrices. Crossing over from one pair of pants to a neighbouring one contributes a $PSU(1,1)$ -matrix depending on a τ_i . This is described in detail in reference [7], where it is also proven that this leads to a global cross section of the bundle (2.2). Taking traces, one obtains arbitrary Wilson loop variables (1.1) as functions of the (l_i, τ_i) .

Our task is thus reduced to finding an independent, but complete set of trace invariants that can serve as parameters on \mathcal{T}_g . Obviously it does not suffice to consider just the Wilson loops of the homotopy generators, or those along the simple geodesic curves, because this does not lead to the desired number of $6g - 6$ degrees of freedom. One also has to take care that once a set of “basic loops” has been found, the corresponding Wilson loops are indeed good *global* coordinates on Teichmüller space. For example, the six invariants for $g = 2$ given in [4], based on a set of “simple-looking” loops, do not have this property, although they are locally independent.

Since the traces of holonomies essentially measure the lengths of closed geodesics, it is not hard to extract information about the $3g - 3$ length parameters l_j . One just takes the trace of the holonomy along the geodesic cut α_i , γ_i or δ_i . For example, for α_1 one obtains

$$T(\alpha_1) = 2 \cosh \frac{l_{-\infty}}{2}, \quad (2.3)$$

and similarly for the Wilson loops of the remaining geodesic cuts. The dependence on the twist variables τ_j is more difficult to extract. One possible solution will be given below after (2.6). However, for the sake of illustration, we will first give the set of complete and independent Wilson loop invariants for $g = 2$ and 3, before writing down the expressions for the general genus case. Using the abbreviation

$$s(l_i, l_j, l_k) := \cosh^2 \frac{l_i}{2} + \cosh^2 \frac{l_j}{2} + \cosh^2 \frac{l_k}{2} + 2 \cosh \frac{l_i}{2} \cosh \frac{l_j}{2} \cosh \frac{l_k}{2} - 1 \quad (2.4)$$

for the strictly positive function depending on the length parameters alone, our solution for $g = 2$ is

$$\begin{aligned}
T(\alpha_1) &= 2 \cosh \frac{l_{-\infty}}{2} \\
T(\beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2) &= 2 \cosh \frac{l_0}{2} \\
T(\alpha_2) &= 2 \cosh \frac{l_{\infty}}{2} \\
T(\beta_2 \alpha_2 \beta_2^{-1} \beta_1^{-1} \alpha_2^{-1} \beta_1 \alpha_1) - T(\beta_2 \alpha_2 \beta_2^{-1} \alpha_1 \beta_1^{-1} \alpha_2^{-1} \beta_1) &= 4 \frac{1}{\sinh \frac{l_{-\infty}}{2}} s(l_{-\infty}, l_0, l_{\infty}) \sinh \tau_{-\infty} \\
T(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_1 \alpha_1 \beta_1^{-1}) - T(\alpha_1^{-1} \alpha_2) &= 4 \frac{1}{\sinh \frac{l_0}{2}} s(l_{-\infty}, l_0, l_{\infty}) \sinh \tau_0 \\
T(\beta_1 \alpha_1 \beta_1^{-1} \alpha_2^{-1} \beta_2^{-1} \alpha_1^{-1} \beta_2) - T(\beta_1 \alpha_1 \beta_1^{-1} \beta_2^{-1} \alpha_1^{-1} \beta_2 \alpha_2^{-1}) &= 4 \frac{1}{\sinh \frac{l_{\infty}}{2}} s(l_{-\infty}, l_0, l_{\infty}) \sinh \tau_{\infty},
\end{aligned} \tag{2.5}$$

and the one for $g = 3$

$$\begin{aligned}
T(\alpha_1) &= 2 \cosh \frac{l_{-\infty}}{2} \\
T(\beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2) &= 2 \cosh \frac{l_0}{2} \\
T(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2) &= 2 \cosh \frac{l_1}{2} \\
T(\alpha_2) &= 2 \cosh \frac{l_2}{2} \\
T(\beta_2 \alpha_2^{-1} \beta_2^{-1} \alpha_3) &= 2 \cosh \frac{l_3}{2} \\
T(\alpha_3) &= 2 \cosh \frac{l_{\infty}}{2} \\
T(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_1^{-1} \alpha_2^{-1} \beta_1 \alpha_1) - T(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \alpha_1 \beta_1^{-1} \alpha_2^{-1} \beta_1) &= \\
&= 4 \frac{1}{\sinh \frac{l_{-\infty}}{2}} \sqrt{s(l_{-\infty}, l_0, l_1) s(l_{-\infty}, l_0, l_2)} \sinh \tau_{-\infty} \\
T(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_1 \alpha_1 \beta_1^{-1}) - T(\alpha_1^{-1} \alpha_2) &= \\
&= 4 \frac{1}{\sinh \frac{l_0}{2}} \sqrt{s(l_{-\infty}, l_0, l_1) s(l_{-\infty}, l_0, l_2)} \sinh \tau_0 \\
T(\alpha_1^{-1} \beta_3 \alpha_3 \beta_3^{-1} \alpha_2^{-1} \beta_1 \alpha_1 \beta_1^{-1} \alpha_1^{-1}) - T(\alpha_1^{-1} \beta_2 \alpha_2^{-1} \beta_2^{-1} \alpha_3) &= \\
&= 4 \frac{1}{\sinh \frac{l_1}{2}} \sqrt{s(l_{-\infty}, l_0, l_1) s(l_1, l_3, l_{\infty})} \sinh \tau_1 \\
T(\alpha_1 \beta_1^{-1} \beta_2^{-1} \alpha_3^{-1} \beta_2 \alpha_2 \beta_1) - T(\alpha_1 \beta_1^{-1} \alpha_2 \beta_2^{-1} \alpha_3^{-1} \beta_2 \beta_1) &= \\
&= 4 \frac{1}{\sinh \frac{l_2}{2}} \sqrt{s(l_{-\infty}, l_0, l_2) s(l_2, l_3, l_{\infty})} \sinh \tau_2
\end{aligned}$$

$$\begin{aligned}
& T(\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3) - T(\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_3) = \\
& \quad 4\frac{1}{\sinh \frac{l_3}{2}} \sqrt{s(l_1, l_3, l_\infty)s(l_2, l_3, l_\infty)} \sinh \tau_3 \\
& T(\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\beta_2\alpha_2\beta_2^{-1}\alpha_3^{-1}\beta_3^{-1}) - T(\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3\alpha_3^{-1}\alpha_3^{-1}\beta_2\alpha_2\beta_2^{-1}\beta_3^{-1}) = \\
& \quad 4\frac{1}{\sinh \frac{l_\infty}{2}} \sqrt{s(l_1, l_3, l_\infty)s(l_2, l_3, l_\infty)} \sinh \tau_\infty.
\end{aligned} \tag{2.6}$$

The verification of these formulae involves the multiplication of large numbers of 2×2 -matrices, which was done with the help of the algebraic program Mathematica. One can read off from (2.6) the algebraic form of the general trace invariant depending on τ_j . If the two pants meeting at the j 'th geodesic cut have boundary components labelled by (j, k, l) and (j, m, n) , say, the corresponding Wilson loop invariant is given by $4\sqrt{s(l_j, l_k, l_l)}\sqrt{s(l_j, l_m, l_n)} \sinh \tau_j / \sinh \frac{l_j}{2}$. No square roots occur in (2.5) because of symmetries special to the genus-2 case.

One verifies by inspection that these invariants are indeed complete and global sets of parameters on Teichmüller space. The only degeneracies occur in the singular limit when one or more of the l_j vanish, i.e. part of the genus- g surface “pinches off”.

The above generalizes straightforwardly to higher genus. The general expressions for the corresponding τ -dependent Wilson loops in terms of the generators α_i and β_i are given by

a) for $\tau_{-\infty}$:

$$T(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_1^{-1}\alpha_2^{-1}\beta_1\alpha_1) - T(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\alpha_1\beta_1^{-1}\alpha_2^{-1}\beta_1)$$

b) for τ_{3i-5} ($i = 2 \dots g-1$):

$$\begin{aligned}
& T(\beta_{i-1}\alpha_{i-1}^{-1}\beta_{i-1}^{-1}\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}\alpha_{i+1}\beta_i\alpha_i^{-1} \dots \alpha_g^{-1}\beta_g^{-1}) - \\
& T(\beta_i\alpha_i^{-1}\beta_i^{-1}\alpha_{i+1}\beta_{i-1}\alpha_{i-1}^{-1}\beta_{i-1}^{-1}\alpha_i\beta_i\alpha_i^{-1} \dots \alpha_g^{-1}\beta_g^{-1})
\end{aligned}$$

c) for τ_{3i-4} ($i = 2 \dots g-1$):

$$T(\beta_{i-1}\alpha_{i-1}\beta_{i-1}^{-1}\beta_i^{-1}\alpha_{i+1}^{-1}\beta_i\alpha_i) - T(\beta_{i-1}\alpha_{i-1}\beta_{i-1}^{-1}\alpha_i\beta_i^{-1}\alpha_{i+1}^{-1}\beta_i)$$

d) for τ_{3i-3} ($i = 1 \dots g-1$):

$$T(\beta_i\alpha_i^{-1}\beta_i^{-1}\alpha_{i+1}\alpha_{i+1}\beta_{i+1}\alpha_{i+1}^{-1} \dots \alpha_g^{-1}\beta_g^{-1}) - T(\alpha_{i+1}\beta_i\alpha_i^{-1}\beta_i^{-1}\alpha_{i+1}\beta_{i+1}\alpha_{i+1}^{-1} \dots \alpha_g^{-1}\beta_g^{-1})$$

e) for τ_∞ :

$$T(\beta_{g-1}\alpha_{g-1}^{-1}\beta_{g-1}^{-1}\alpha_g\beta_g\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_g^{-1}\beta_g^{-1}) - \\ T(\beta_{g-1}\alpha_{g-1}^{-1}\beta_{g-1}^{-1}\alpha_g\beta_g\alpha_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\beta_g^{-1}).$$

The dots stand for a product of generators as they occur in the fundamental relation (1.2). Using this relation, it can easily be verified that (2.5) and (2.6) are special cases of these expressions.

Given the explicit form for the independent Wilson loops, one may now translate back and forth between the classical Fenchel-Nielsen and the loop description. From the point of view of the loop space quantization of 2+1 gravity [4], it is important to know the explicit expression for the natural volume form on the Teichmüller space \mathcal{T}_g , which comes from the Weil-Petersson symplectic form [11]. Using the self-explanatory notation $T[l_j]$, $T[\tau_j]$ for the loop invariants given above, the volume form is

$$\prod_{j=1}^{3g-3} dl_j d\tau_j = \frac{1}{\left| \frac{\partial(T[l], T[\tau])}{\partial(l, \tau)} \right|} \prod_{j=1}^{3g-3} dT[l_j] dT[\tau_j]. \quad (2.7)$$

Abbreviating

$$S[l_i, l_j, l_k] := T[l_i]^2 + T[l_j]^2 + T[l_k]^2 + T[l_i] T[l_j] T[l_k] - 4 \quad (2.8)$$

one finds for the Jacobian of $g = 2$

$$\left| \frac{\partial(T[l], T[\tau])}{\partial(l, \tau)} \right| = \sqrt{S[l_{-\infty}, l_0, l_\infty]^2 + T[\tau_{-\infty}]^2 (T[l_{-\infty}]^2/4 - 1)} \times \\ \sqrt{S[l_{-\infty}, l_0, l_\infty]^2 + T[\tau_0]^2 (T[l_0]^2/4 - 1)} \sqrt{S[l_{-\infty}, l_0, l_\infty]^2 + T[\tau_\infty]^2 (T[l_\infty]^2/4 - 1)}, \quad (2.9)$$

for $g = 3$

$$\begin{aligned}
\left| \frac{\partial(T[l], T[\tau])}{\partial(l, \tau)} \right| = & \\
& \sqrt{S[l_{-\infty}, l_0, l_1]S[l_{-\infty}, l_0, l_2] + T[\tau_{-\infty}]^2(T[l_{-\infty}]^2/4 - 1)} \times \\
& \sqrt{S[l_{-\infty}, l_0, l_1]S[l_{-\infty}, l_0, l_2] + T[\tau_0]^2(T[l_0]^2/4 - 1)} \times \\
& \sqrt{S[l_{-\infty}, l_0, l_1]S[l_1, l_3, l_{\infty}] + T[\tau_1]^2(T[l_1]^2/4 - 1)} \times \\
& \sqrt{S[l_{-\infty}, l_0, l_2]S[l_2, l_3, l_{\infty}] + T[\tau_2]^2(T[l_2]^2/4 - 1)} \times \\
& \sqrt{S[l_1, l_3, l_{\infty}]S[l_2, l_3, l_{\infty}] + T[\tau_3]^2(T[l_3]^2/4 - 1)} \times \\
& \sqrt{S[l_1, l_3, l_{\infty}]S[l_2, l_3, l_{\infty}] + T[\tau_{\infty}]^2(T[l_{\infty}]^2/4 - 1)},
\end{aligned} \tag{2.10}$$

and similarly for higher genus. We observe that the measure, unlike in its form in terms of Fenchel-Nielsen coordinates, does not factorize completely. Given the explicit form of these measures, one may now go ahead and solve a problem posed in [4], namely that of introducing damping factors to make the loop transform well-defined. We will not pursue this line of investigation any further in the present paper.

3 Conclusions

We have presented above a complete and independent set of (linear combinations of) Wilson loop variables for 2+1 gravity on a compact spatial manifold Σ^g of arbitrary genus g , and have therefore identified the true physical degrees of freedom in terms of loop invariants. In the derivation, crucial use was made of an explicit cross section of the bundle (2.2) over Teichmüller space. The existence of such global loop invariants is possible because of the topologically trivial nature of the physical configuration space, and therefore does not generalize to more complicated theories of connections like for example the $SU(2)$ -Yang-Mills theory. Also, our final solution is simple in that the independent loop variables are not subject to any inequalities which a priori might have occurred [12].

The elements of the homotopy group $\pi_1(\Sigma^g)$ going into the construction of the independent trace invariants are sufficiently complicated to make it plausible that it would be difficult to obtain similar solutions by solving the trace identities directly. The algebraic form of our final solution reiterates the well-known fact that it is usually quite involved to transform back and forth between the loop and the connection formulation, and also that much of the original geometric simplicity of the loop formulation gets lost when solving for the physical degrees of freedom.

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