

# GENERALIZED COORDINATES ON THE PHASE SPACE OF YANG-MILLS THEORY

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## Abstract

We study the suitability of complex Wilson loop variables as (generalized) coordinates on the physical phase space of  $SU(2)$ -Yang-Mills theory. To this end, we construct a natural one-to-one map from the physical phase space of the Yang-Mills theory with compact gauge group  $G$  to a subspace of the physical configuration space of the complex  $G^{\mathbb{C}}$ -Yang-Mills theory. Together with a recent result by Ashtekar and Lewandowski this implies that the complex Wilson loop variables form a complete set of generalized coordinates on the physical phase space of  $SU(2)$ -Yang-Mills theory. They also form a generalized canonical loop algebra. Implications for both general relativity and gauge theory are discussed.

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# 1. Introduction

The  $SL(2, \mathbb{C})$ -Ashtekar connection introduced in [1] has led to important progress in the Hamiltonian formulation of general relativity [2-4]. In particular a wide class of solutions to the quantum gravitational equations has been found with the help of the Wilson loop variables constructed from this connection [3,4].

The situation we encounter in the gravitational application is similar to the gauge theory case to be discussed below. Namely, one has the  $SL(2, \mathbb{C})$ -Ashtekar connection  $A_{\text{grav}}^{\mathbb{C}}$  on a spatial manifold  $\Sigma$ ,  $\dim(\Sigma) = 3$ ,

$$A_{\text{grava}}^{\mathbb{C}j}(x) = \Gamma_a^j(x) - i K_a^j(x), \quad (1.1)$$

together with its complex conjugate, to be thought of as complex coordinates on the real phase space of general relativity. In (1.1),  $\Gamma_a^i$  is the spin connection determined by the triad  $E_i^a = \frac{1}{\sqrt{h}} \tilde{E}_i^a$ , and  $K_a^i$  is related to the extrinsic curvature  $K_{ab}$  on  $\Sigma$  by  $K_a^i = \frac{1}{\sqrt{h}} K_{ab} E^{bi}$  [1,2]. Both  $\Gamma$  and  $K$  are coordinates on phase space, but unlike in the Yang-Mills case we will consider, they are not canonically conjugate to each other.

The Wilson loop variables constructed from the connections (1.1) too are functions on the phase space of general relativity, invariant with respect to  $SL(2, \mathbb{C})$ -gauge transformations. Since physically one requires only invariance under gauge transformations corresponding to a  $SU(2)$ -subgroup of  $SL(2, \mathbb{C})$  [2], it may happen that the complex,  $SL(2, \mathbb{C})$ -invariant Wilson loop variables do not separate points in the “physical” phase space. (Here we mean the phase space obtained after enforcing the reality conditions and the Gauss law constraint but prior to imposing the diffeomorphism and hamiltonian constraints.) Such pathologies may occur in the course of going to the quotient space with respect to the  $SL(2, \mathbb{C})$ -transformations, although, as was shown in [5], the Wilson loops separate all (separable) points in the space of  $SL(2, \mathbb{C})$ -connections modulo  $SL(2, \mathbb{C})$ -gauge transformations.

One finds an analogous situation, but with a much simpler symplectic structure, in  $SU(2)$  (or in general  $G$ )-Yang-Mills theories in a  $d + 1$ -dimensional space-time  $M = \mathbb{R} \times \Sigma$ . In the present paper we prove that, for any compact gauge group  $G$ , the physical phase space  $\mathcal{P}_{\text{phys}}$  of the  $G$ -Yang-Mills theory can be naturally identified with a subset of  $\mathcal{Q}_{\text{phys}}^{\mathbb{C}}$ , the physical *configuration* space of the  $G^{\mathbb{C}}$ -Yang-Mills theory (with  $G^{\mathbb{C}}$  denoting the complexification of the Lie group  $G$ ). This is the case although the group  $\mathcal{G}^{\mathbb{C}}$  of complex  $G^{\mathbb{C}}$ -gauge transformations acting on the big phase space  $\mathcal{P}$  is “twice as large” as the group  $\mathcal{G}$  of real  $G$ -gauge transformations. Locally, the identification between the space  $\mathcal{P}_{\text{phys}}$  and (a subspace of)  $\mathcal{Q}_{\text{phys}}^{\mathbb{C}}$

is made possible by the fact that the imaginary  $G^{\mathbb{C}}$ -gauge transformations are transverse in the phase space  $\mathcal{P}$  to the constraint surface  $\mathcal{C}$ , defined by the vanishing of the non-linear Gauss law constraints,

$$\mathcal{C} = \{(A_a, \tilde{E}^a) \in \mathcal{P} : D[A]_a \tilde{E}^a = 0\}, \quad (1.2)$$

where  $A_a$  is a  $G$ -gauge field,  $\tilde{E}^a$  is a one-density electric field and  $D[A]_a$  is the covariant derivative  $D[A]_a \equiv D_a = \partial_a + [A_a, \cdot]$ . Our proof however will be of a global nature.

For any closed piecewise smooth spatial curve  $\alpha$ , consider the complex Wilson loop variable

$$T_\alpha(A_a + i\lambda E_a) := \frac{1}{N} \text{Tr} \text{P exp} \left( \int_\alpha (A_a + i\lambda E_a) dx^a \right), \quad (1.3)$$

where  $A_a \in \mathcal{A}^G$  is a  $G$ -connection and  $E_a(x)$  is defined by  $E_a(x) = h_{ab}(x) \frac{1}{\sqrt{h}} \tilde{E}^b(x)$ . The normalization factor on the right-hand side is the dimension  $N$  of the linear representation of  $G$ ,  $h_{ab}$  is a fixed Riemannian metric on the spatial manifold  $\Sigma$ , and  $\lambda > 0$  a positive real constant with dimension of length. By virtue of their local  $G^{\mathbb{C}}$ -gauge invariance, variables of the form (1.3) project down to the physical configuration space  $\mathcal{Q}_{\text{phys}}^{\mathbb{C}} = \mathcal{A}^{G^{\mathbb{C}}} / \mathcal{G}^{G^{\mathbb{C}}}$  of  $G^{\mathbb{C}}$ -Yang-Mills theory.

Moreover, it was shown in [5] that for  $G^{\mathbb{C}} = SL(2, \mathbb{C})$  they constitute a complete set of generalized coordinates on  $\mathcal{Q}_{\text{phys}}^{\mathbb{C}}$  (i.e. they separate all points that are separable with the help of continuous functions). Our result, for the particular case of  $G = SU(2)$ , and the result of [5] imply that the complex variables  $T_\alpha$  can be used as global generalized coordinates on the physical phase space  $\mathcal{P}_{\text{phys}}$  of the  $SU(2)$ -Yang-Mills theory (see comment after (2.9)). By “generalized” we mean to indicate that, due to the non-linearity of the spaces  $\mathcal{A}/\mathcal{G}$ , the Wilson loop variables are always overcomplete, subject to a set of identities, the so-called Mandelstam constraints [6,7].

## 2. Physical phase space of real Yang-Mills theory versus physical configuration space of complex Yang-Mills theory

Consider the Yang-Mills theory with compact gauge group  $G$  in a  $d + 1$ -dimensional space-time  $\Sigma \times \mathbb{R}$ . The canonical pairs

$$(A_a(x), \tilde{E}^a(x)),$$

are a natural set of coordinates on phase space, with Poisson brackets given by

$$\{A_a(x), \tilde{E}^b(y)\} = \delta_a^b \delta^d(x, y). \quad (2.1)$$

Note that the parametrization of the phase space with  $(A, \tilde{E})$  is valid globally whenever  $\dim(\Sigma) = 3$  and  $G = SU(2)$ , because then the underlying principal fibre bundle is trivial. The action of the group of gauge transformations  $\mathcal{G}$ ,

$$\begin{aligned} A_a(x) &\mapsto g^{-1}(x)A_a(x)g(x) + g^{-1}(x)\partial_a g(x) \\ \tilde{E}^a(x) &\mapsto g^{-1}(x)\tilde{E}^a(x)g(x), \end{aligned} \quad (2.2)$$

can be naturally extended to an action of the complexified group  $\mathcal{G}^{\mathbb{C}}$  if we identify the *big phase space*  $\mathcal{P}$  of the  $G$ -Yang-Mills theory with the *big configuration space*  $\mathcal{Q}^{\mathbb{C}}$  of the  $G^{\mathbb{C}}$ -Yang-Mills theory through the ( $\lambda$ -dependent) map

$$\begin{aligned} \phi_\lambda &: \mathcal{P} \rightarrow \mathcal{Q}^{\mathbb{C}} \\ \phi_\lambda &: \left( A_a(x), \tilde{E}^a(x) \right) \mapsto A_a^{\mathbb{C}}(x) = A_a(x) + i\lambda E_a(x). \end{aligned} \quad (2.3)$$

The diffeomorphism  $\phi_\lambda$  allows us to take across group actions from the right- to the left-hand side of (2.3) and vice versa. The  $G$ -gauge transformations (2.2) acting on the right-hand side of (2.3) have the form

$$A_a^{\mathbb{C}}(x) \mapsto g^{-1}(x)A_a^{\mathbb{C}}(x)g(x) + g^{-1}(x)\partial_a g(x) \quad (2.4)$$

with  $g(x) \in G$ . Consider the Lie algebra  $Lie(G^{\mathbb{C}}) = Lie(G)^{\mathbb{C}} = Lie(G) + i Lie(G)$  of  $G^{\mathbb{C}}$ , where

$$G^{\mathbb{C}} = G e^{i Lie(G)} \quad (2.5)$$

is the (unique for a compact connected Lie group  $G$ ) universal complexification of the group  $G$  (see, for example, [8]). In particular for  $G = SU(N)$ , its complexification  $SU(N)^{\mathbb{C}}$  is (group theoretically) isomorphic to  $SL(N, \mathbb{C})$ ,  $N \geq 2$ . Since  $A_a^{\mathbb{C}}(x)$  is a one-form with values in  $Lie(G)^{\mathbb{C}}$ , we may regard it as a  $G^{\mathbb{C}}$ -connection and define on it  $G^{\mathbb{C}}$ -transformations given by the following obvious extension of (2.4),

$$A_a^{\mathbb{C}}(x) \mapsto A_a^{\mathbb{C}'}(x) = g_{\mathbb{C}}^{-1}(x)A_a^{\mathbb{C}}(x)g_{\mathbb{C}}(x) + g_{\mathbb{C}}^{-1}(x)\partial_a g_{\mathbb{C}}(x), \quad (2.6)$$

where  $g_{\mathbb{C}}(x) \in G^{\mathbb{C}}$ . The  $\lambda$ -dependent action induced on the left-hand side of (2.3) reads

$$\begin{aligned} A_a(x) &\mapsto Re(A_a^{\mathbb{C}'}(x)) \\ \tilde{E}^a(x) &\mapsto \frac{\sqrt{\hbar}h^{ab}(x)}{\lambda} Im(A_b^{\mathbb{C}'}(x)) \end{aligned} \quad (2.7)$$

or, under an infinitesimal change generated by the algebra element  $\xi(x) + i\eta(x) \in Lie(G)^{\mathbb{C}}$ ,

$$\begin{aligned} \delta A_a(x) &= D_a \xi(x) - \lambda [E_a(x), \eta(x)] \\ \lambda \delta \tilde{E}^a(x) &= \lambda [\tilde{E}^a(x), \xi(x)] + \tilde{D}^a \eta(x). \end{aligned} \quad (2.8)$$

Hence the identification of the big configuration space of the complex  $G^{\mathbb{C}}$ -Yang-Mills theory with the big phase space of the real  $G$ -Yang-Mills theory via (2.3) allows us to introduce in the latter an action of the group of local  $G^{\mathbb{C}}$ -transformations (one action for each choice of  $\lambda$ ,  $\lambda \in \mathbb{R}$ ). Note that the action (2.7) on  $\mathcal{P}$  is not symplectic, only the subgroup of “real” gauge transformations is.

By introducing a larger group  $\mathcal{G}^{\mathbb{C}}$  as a group of gauge transformation on these spaces one may a priori worry that requiring invariance with respect to the  $\mathcal{G}^{\mathbb{C}}$ -action leads to a loss of relevant physical observables. The result we obtain in the following together with the result of [5] will show that, at least for  $SU(2)$ -Yang-Mills theory, this is not the case.

Recall that physical observables in gauge theory are defined as the  $\mathcal{G}$ -invariant functions on the constraint surface  $\mathcal{C} \subset \mathcal{P}$ , and not on all of  $\mathcal{P}$ . We prove that for every complex  $\mathcal{G}^{\mathbb{C}}$ -orbit  $\mathcal{O}^{\mathbb{C}} \subset \mathcal{P}$  that intersects the constraint surface  $\mathcal{C}$ , we have

$$\mathcal{O}^{\mathbb{C}} \cap \mathcal{C} = \mathcal{O}, \quad (2.9)$$

where  $\mathcal{O}$  is a single real  $\mathcal{G}$ -orbit, and therefore the intersection contains only one real orbit (corresponding to a unique physical configuration). Hence the extra invariance conditions imposed by the  $G^{\mathbb{C}}$ -gauge transformations correspond merely to identifying unphysical configurations outside the constraint surface  $\mathcal{C}$ , along orbits transverse to  $\mathcal{C}$ . This result, together with the known fact that the  $SU(2)^{\mathbb{C}}$ -Wilson loop variables (1.3) separate (closed)  $SL(2, \mathbb{C})$ -gauge orbits in  $\mathcal{P}$  [5], implies that these variables form a good set of generalized coordinates on the physical phase space of  $SU(2)$ -Yang-Mills theory, i.e. they separate all points in  $\mathcal{P}_{phys}$  (with the possible exception of singular points for which the holonomy group is a subgroup of the group of null rotations [5]).

We now prove our result by contradiction. Consider the Gauss constraint surface  $\mathcal{C}$  in the big phase space  $\mathcal{P} = T^*\mathcal{A}$  of the  $G$ -Yang-Mills theory,

$$\mathcal{C} = \{(A_a, \tilde{E}^a) \in \mathcal{P} \mid D_a \tilde{E}^a = 0\}. \quad (2.10)$$

Assume that two different  $\mathcal{G}$ -orbits  $\mathcal{O}^{(0)}$ ,  $\mathcal{O}^{(1)}$  in  $\mathcal{C}$  are contained in the same  $\mathcal{G}^{\mathbb{C}}$ -orbit and let

$$p_j = (A_a^{(j)}, \tilde{E}^{(j)a}); \quad j = 0, 1 \quad (2.11)$$

denote two points in these orbits. Then by assumption there exists a complex gauge transformation connecting them, which according to (2.5) is of the form

$$g_0(x) e^{i\xi_0(x)},$$

where  $g_0(x) \in \mathcal{G}$ ,  $\xi_0(x) \in Lie(\mathcal{G})$ . Without loss of generality we can assume that  $g_0(x) = e$ . Indeed let  $g_0(x)$  be nontrivial. Then the point  $p'_1 = e^{i\xi_0(x)} p_0$  is in the same  $\mathcal{G}$  orbit as  $p_1$  ( $p_1 = g_0(x) e^{i\xi_0(x)} p_0 = g_0(x) p'_1$ ) and we can prove our result for the pair  $(p_0, p'_1)$ .

Consider the curve

$$q(s) = (A_a^{(s)}, \tilde{E}^{(s)a}) = e^{is\xi_0} p_0$$

in phase space, with  $q(0) = p_0$  and  $q(1) = p_1$ . We will show that necessarily  $p_0 = p_1$ , which contradicts the initial assumption that these points belong to different real orbits.

Consider the following real function on the unit interval: <sup>1</sup>

$$r(s) = \int_{\Sigma} dx Tr \left\{ \left( \partial_a \tilde{E}^{(s)a}(x) + \left[ A_a^{(s)}(x), \tilde{E}^{(s)a}(x) \right] \right) \xi_0(x) \right\}. \quad (2.12)$$

for which we have

$$r(0) = r(1) = 0. \quad (2.13)$$

For the derivative of  $r(s)$  with respect to the curve parameter  $s$  we obtain (using (2.8)) the following non-negative expression

$$\begin{aligned} \dot{r}(s) = - \int_{\Sigma} \sqrt{h} Tr & \left( \frac{h^{ab}}{\lambda} \left( D[A^{(s)}]_a \xi_0 \right) (x) \left( D[A^{(s)}]_b \xi_0 \right) (x) \right. \\ & \left. + \frac{\lambda}{\sqrt{h}} h_{ab} \left[ \tilde{E}^{(s)a}(x), \xi_0(x) \right] \left[ \tilde{E}^{(s)b}(x), \xi_0(x) \right] \right) \geq 0, \end{aligned} \quad (2.14)$$

where we have assumed appropriate boundary conditions on the fields to ensure the vanishing of boundary terms. Note that we are using a normalization for the  $G$ -generators  $\tau^i$  for which  $Tr \tau^i \tau^j = -\frac{1}{2} \delta^{ij}$ . Combining (2.13) with (2.14) we conclude that

$$r(s) \equiv \dot{r}(s) \equiv 0, \quad s \in [0, 1].$$

(Note that the same argument would not be valid for non-compact gauge groups  $G$ , since in that case the inner product on the Lie algebra used in (2.14) would have a different signature.) This implies that the vector

$$\left( \delta_{\xi_0} A_a^{(s)}, \delta_{\xi_0} \tilde{E}^{(s)a} \right) = \left( (D[A^{(s)}]_a \xi_0), [\tilde{E}^{(s)a}, \xi_0] \right) \quad (2.15)$$

and therefore also the vector

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<sup>1</sup> Our proof is the extension to infinite-dimensional Yang-Mills systems of analogous proofs used in finite-dimensional systems. The geometric interpretation of our construction and finite-dimensional examples are discussed elsewhere [9].

$$\left(\delta_{i\xi_o} A_a^{(s)}, \delta_{i\xi_o} \tilde{E}^{(s)a}\right) = \left(-\lambda[E_a^{(s)}, \xi_0], \frac{1}{\lambda} D[\widetilde{A^{(s)}}]^a \xi_0\right) \quad (2.16)$$

vanish at any point  $q(s) = (A_a^{(s)}, \tilde{E}^{(s)a})$ ,  $s \in [0, 1]$ . Since (2.16) are the infinitesimal “imaginary” transformations generated by  $i\xi_0$  we conclude that

$$(A_a^{(0)}, \tilde{E}^{(0)a}) = (A_a^{(1)}, \tilde{E}^{(1)a}). \quad (2.17)$$

We have therefore proven our claim that no additional conditions are imposed by requiring invariance under the  $\mathcal{G}^{\mathbb{C}}$ -action on  $\mathcal{P}$ . The following comment is in order:

Equations (2.12) and (2.14) show that locally the imaginary gauge transformations are transverse to the constraint surface. This is in accordance with the Moncrief decomposition of the tangent space at every point of the constraint surface on the phase space of Yang-Mills systems [10].

### 3. The loop algebra

It follows from the previous section that the complex  $SU(2, \mathbb{C})$ -Wilson loop variables (1.3) may serve as an alternative to the usually employed set of generalized coordinates

$$T_{\alpha}^{(n) a_1 a_2 \dots a_n}(A, \tilde{E}) = Tr \tilde{E}^{a_1}(x_1) U_{\alpha}(x_1, x_2) \tilde{E}^{a_2}(x_2) U_{\alpha}(x_2, x_3) \dots \tilde{E}^{a_n}(x_n) U_{\alpha}(x_n, x_1) \quad (3.1)$$

on the  $SU(2)$ -Yang-Mills phase space [3, 11]. In (3.1),  $U_{\alpha}(x_i, x_j) = U_{\alpha}(A)(x_i, x_j)$  denotes the holonomy along  $\alpha$ , taken between the two points  $x_i$  and  $x_j$ ,  $U_{\alpha}(x_i, x_j) = \text{P exp} \int_{x_i}^{x_j} A_a dx^a$ . We may expand the complex Wilson loop (1.3) as a power series in  $\lambda$ ,

$$T_{\alpha}(A + i\lambda E) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} T^n(\alpha, A, E), \quad (3.2)$$

where

$$T^n(\alpha, A, E) = Tr \int_0^1 dt_1 \int_{t_1}^1 dt_2 \dots \int_{t_{n-1}}^1 dt_n U(\alpha(0), \alpha(t_1)) E_{a_1}(\alpha(t_1)) \dot{\alpha}^{a_1}(t_1) U(\alpha(t_1), \alpha(t_2)) \\ E_{a_2}(\alpha(t_2)) \dot{\alpha}^{a_2}(t_2) \dots E_{a_n}(\alpha(t_n)) \dot{\alpha}^{a_n}(t_n) U(\alpha(t_n), \alpha(1)), \quad (3.3)$$

are nothing but integrated versions of the  $T^{(n)}$  in (3.1). Hence the  $T_\alpha(A + i\lambda E)$  may be regarded as generating functions for the “higher Wilson loop momenta”  $T^n$ . (A similar expansion is used in the zero-curvature formulation of reduced gravitational models [12].) We may now explicitly calculate the Poisson bracket between two complex Wilson loops, using the fundamental relation (2.1):

$$\{T_\alpha(A + i\lambda E), T_{\alpha'}(A + i\lambda' E)\} = \\ i(\lambda' - \lambda) S[\alpha, \alpha'] \sum_{l=1}^3 Tr \left[ U_\alpha(A + i\lambda E)(\alpha(t), \alpha(t)) \tau^l \right] Tr \left[ U_{\alpha'}(A + i\lambda' E)(\alpha'(t'), \alpha'(t')) \tau^l \right]. \quad (3.4)$$

The path-dependent, distributional (for  $d > 2$ ) structure constants are defined by

$$S[\alpha, \alpha'] = \frac{1}{\sqrt{h}} \int_0^1 dt \int_0^1 dt' \delta^d(\alpha(t), \alpha'(t')) h_{ab}(\alpha(t)) \dot{\alpha}^a(t) \dot{\alpha}'^b(t'), \quad (3.5)$$

and vanish for perpendicular tangent vectors  $\dot{\alpha}$ ,  $\dot{\alpha}'$ . The right-hand side of (3.4) may be evaluated further by using the identities for the  $\tau$ -matrix generators of the  $su(2)$ -algebra.

The cases of special interest are those where  $\lambda' = \lambda$ , for which the Poisson brackets vanish, and  $\lambda' = -\lambda$ , for which the connection arguments are canonically conjugate. For the latter case, the right-hand side of (3.4) may again be expanded as a power series in  $\lambda$ . However, it is not true that the right-hand side is expressible as (a sum of terms)  $T_\gamma(A \pm i\lambda E)$  for some loop(s)  $\gamma$ . Rather, one obtains Wilson loop variables with a “colouring” for pieces of loops between intersections, which keeps track of whether the holonomy for that piece comes from an integration of the connection  $A + i\lambda E$  or of its conjugate  $A - i\lambda E$ .

The idea of using the complex Wilson loops  $T_\alpha(A + i\lambda E)$  as generating functions for Wilson loop momenta may now profitably be used to define natural momenta  $T^n(\alpha, A, E)$  for the case when  $\alpha$  possesses self-intersections. For simple (i.e. non-selfintersecting) loops we keep the definition (3.3). Then, by taking Poisson brackets as in (3.4), with both  $\alpha$  and

$\alpha'$  simple, we may define  $T^n(\alpha, \alpha', A, E)$  as the coefficient at order  $\lambda^{n+1}$ , which is a finite sum of terms. For loop configurations with more complicated intersections, we can continue to take Poisson brackets of the resulting quantities. This is a bona fide procedure from the point of view of the real gauge theory, since the Poisson bracket of two  $SL(2, \mathbb{C})$ -invariant quantities is always invariant under  $SU(2)$ - (though generally not under  $SL(2, \mathbb{C})$ -)gauge transformations. The reason for this is that the  $SU(2)$ -gauge transformations are canonical (i.e. Poisson bracket preserving) transformations while the  $SL(2, \mathbb{C})$  gauge transformations are not.

## 4. Conclusions

We have shown that the complex Wilson loops  $T_\alpha(A \pm i\lambda E)$  form a good set of generalized coordinates on the phase space of the real  $SU(2)$ -Yang-Mills theory. If the result of Ashtekar and Lewandowski [5] extends to any compact gauge group  $G$ , also our result immediately generalizes. This shows that there is a sense in which the identification  $(A, E) \leftrightarrow A + i\lambda E$  between  $T^*\mathcal{A}$  and  $\mathcal{A}^{\mathbb{C}}$  continues to hold at the level of the corresponding physical spaces  $T^*(\mathcal{A}/\mathcal{G})$  and  $\mathcal{A}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}$ .

This is a non-trivial result because of the non-linear character of the quotient spaces involved. Note that we have not shown that each point in  $\mathcal{A}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}$  does in fact correspond to a physical phase space point. There are finite-dimensional gauge model systems for which one can prove a one-to-one correspondence between a dense subset of  $\mathcal{A}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}$  and  $T^*(\mathcal{A}/\mathcal{G})$  [9], but there is as yet no proof for the infinite-dimensional (non-abelian) gauge theory case.

It remains to be investigated how our alternative Hamiltonian description for Yang-Mills theory intertwines with the dynamical evolution, and whether it leads to any simplification in an explicitly gauge-invariant description, for example, in a regularized lattice formulation. More generally, since our result is of a kinematical nature, it may be applied to any theory whose configuration space is a space of connections, for example, a Chern-Simons theory [13].

As already mentioned in the introduction, the big phase space of general relativity in the Ashtekar formulation is  $\mathcal{A}^{SL(2, \mathbb{C})}$ , but with a symplectic structure significantly more complicated than the one of  $SU(2)$ -Yang-Mills theory (see [2]). Notice that the fact that  $A_{grav}^{\mathbb{C}}$  is a (holomorphic) coordinate on the phase space of general relativity is clear from (1.1) or from the reality conditions  $\overline{A_{grav}^{\mathbb{C}}} = -A_{grav}^{\mathbb{C}} + 2\Gamma(E)$ . We hope that techniques similar to the ones employed in the present paper will be useful in establishing an analogous rigorous result in general relativity.

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## References

- [1] A. Ashtekar, *Phys. Rev. Lett.* **57** (1986) 2244; *Phys. Rev. D* **36** (1987) 1587
- [2] A. Ashtekar, *Non-perturbative canonical quantum gravity* (notes prepared in collaboration with R.S. Tate), World Scientific, Singapore, 1991
- [3] C. Rovelli and L. Smolin, *Nucl. Phys.* **B331** (1990) 80
- [4] B. Brügmann, R. Gambini and J. Pullin, *Phys. Rev. Lett.* **68** (1992) 431
- [5] A. Ashtekar and J. Lewandowski, *Class. Quant. Grav.* **10** (1993) L69
- [6] S. Mandelstam, *Phys. Rev.* **175** (1968) 1580
- [7] R. Loll, *Nucl. Phys.* **B368** (1992) 121; **B400** (1993) 126
- [8] G. Hochschild, *The structure of Lie groups*, Halden-Day, San Francisco, 1965
- [9] R. Loll, J.M. Mourão and J.N. Tavares, *Complexification of gauge theories*, to appear in *J. Geom. Phys.*
- [10] V. Moncrief, *J. Math. Phys.* **16**(1975) 1556
- [11] R. Gambini and A. Trias, *Nucl. Phys.* **B278** (1986) 436
- [12] V. Husain, *Phys. Rev. D* **55** (1994) 6207
- [13] R. Loll, J.M. Mourão and J.N. Tavares, in *Constraint theory and quantization methods*, ed. F. Colomo, L. Lusanna and G. Marmo, World Scientific, Singapore, 1994, 291