Maximal supergravity in nine dimensions is known [57–60], but it has not been studied very extensively. The theory can be obtained by dimensional reduction of IIA or IIB supergravity in ten dimensions. In this chapter we study an extension of the nine-dimensional maximal supergravity theory by coupling various kinds of BPS multiplets to the supergravity multiplet.

In principle, the extended theory can be constructed purely in nine dimensions, but one can also use a compactification of a higher-dimensional theory as a guideline. We will follow the latter approach in this chapter. There are two kinds of 1/2-BPS multiplets that we couple to the massless theory. They arise in the compactification of eleven-dimensional supergravity theory on a two-torus, and of ten-dimensional type IIB supergravity theory on a circle. A supergravity theory coupled to BPS multiplets is called a BPS extended supergravity theory [61].

The reason for choosing the nine-dimensional theory is that it provides a reasonably simple example of such a BPS extended supergravity. Namely, the non-linearly realized global symmetry group $G = \text{GL}(2)$ is small and one is able to handle calculations much better than e.g. in four dimensions, where the hidden symmetry group is $E_{7(7)}$. Additionally the four-dimensional theory exhibits electro-magnetic duality, which would complicate the discussion. Nevertheless, it proves to be rather involved to actually write down a Lagrangian even for this simple BPS-extended supergravity theory in nine dimensions.

We will argue that the BPS-extended supergravity theory in nine dimensions goes beyond the standard eleven-dimensional supergravity theory. The new theory contains eleven-dimensional supergravity theory as a limiting case, but it also contain the ten-dimensional IIB supergravity theory in a different limit.

1. Supersymmetry algebra

We start the discussion of supersymmetry in nine dimensions by studying the supersymmetry algebra. This allows us to identify the various multiplets that can appear in the $\mathcal{N} = 2$ supergravity theory. We discuss the higher-dimensional origin and the interpretation of these BPS multiplets.
The supersymmetry algebra of maximal supergravity in nine dimensions can be obtained from the supersymmetry algebra in eleven dimensions (1.9) by straight-forward dimensional reduction on the two-torus $T^2$. The gamma matrices in eleven dimension, $\gamma^M$, are decomposed into nine-dimensional gamma matrices $\gamma^\mu$ and two-dimensional gamma matrices $\Gamma^m$.

$$\hat{\gamma}^\mu = \gamma^\mu \otimes \tilde{\gamma}, \quad \hat{\Gamma}^m = 1 \otimes \Gamma^m,$$

where $\tilde{\gamma} = -i \Gamma_9 \Gamma_{10}$. A Majorana spinor in nine dimensions has 16 components, i.e. half the number as in eleven dimensions, and consequently the supersymmetry algebra consists of two Majorana spinors $\tilde{Q}^i$, i.e. $\mathcal{N} = 2$. In the first place, we are only interested in massless fields. Therefore we simply neglect the central charges of the supersymmetry algebra, and we only consider states with vanishing momenta in the directions of the internal torus. The anti-commutator of two supersymmetry transformations reads\(^1\)

$$\{ \tilde{Q}^i, \tilde{Q}^j \} = -i \delta^{ij} P_\mu \gamma^\mu.$$

The $\mathcal{N} = 2$ supersymmetry algebra in nine dimensions can be realized on the supergravity multiplet consisting of 128 massless bosonic degrees of freedom and the same number of massless fermionic degrees of freedom.

Turning to the massive states, we can augment the supersymmetry algebra (3.2) by central charge terms. Again, we could directly construct the terms in nine-dimensions, but we prefer to deduce them from the eleven-dimensional supersymmetry algebra (1.9). In order to obtain only point-like central charges in nine dimensions, we assume that the two-form central charge $Z_{MN}$ only takes values in the ninth and tenth dimension, and we set the five-form central charge $Z_{MNPQR}$ to zero. The supersymmetry algebra in nine dimensions then takes the form

$$\{ \tilde{Q}^i, \tilde{Q}^j \} = -i \delta^{ij} P_\mu \gamma^\mu + Z^{ij},$$

where the central charge\(^2\) is given by

$$Z^{ij} = Z_{910} \delta^{ij} - (P_9 \tau_3 - P_{10} \tau_1)^{ij},$$

$$= M \left( a \left( \cos \theta \tau_3 + \sin \theta \tau_1 \right)^{ij} + b \delta^{ij} \right).$$

In this way, the central charge (3.4) decomposes into a singlet of SO(2), which is proportional to $\delta^{ij}$, and a doublet, which is a linear combination of $\tau_1$ and $\tau_3$. In this reduction from eleven dimensions, the doublet is formed by the momenta of the supergravity fields in the two internal directions. The singlet originates from a solitonic state of the supergravity theory, the so-called M2-brane, that is wrapped around the two-torus. The mass of a BPS state in its rest

---

\(^1\)Dirac-conjugated spinors in eleven and in nine dimensions are related by $\tilde{\psi}_1 = i \psi^1 \Gamma_0 = i \psi^1 \gamma_0 \tilde{\gamma} = \tilde{\psi}_9 \tilde{\gamma}$.

\(^2\)We have taken $\Gamma_9 = \tau_1$, $\Gamma_{10} = \tau_3$, which implies $\tilde{\gamma} = \tau_2$. Here, $\tau_i$ are the Pauli matrices.
frame is given by
\[ M = \sqrt{P_9^2 + P_{10}^2} + |Z_{9,10}|. \] (3.5)

From the perspective of the eleven-dimensional supermembrane [62, 63], the mass formula (3.5) can be rewritten as
\[ M = \frac{1}{4\tau_2} |q_1 + \tau q_2| + T_m A |p|, \] (3.6)
where \( p \) is the number of times that the membrane wraps around the torus with modular parameter \( \tau \). Here, \( q_1, q_2 \) are the momenta along the torus directions and \( T_m \) denotes the tension of the supermembrane.

We can also deduce the supersymmetry algebra (3.3) from the IIB supersymmetry algebra in ten dimensions [64], which is given by
\[ \{ Q^i, \bar{Q}^j \} = -i \delta^{ij} (P^M \Gamma^M) P_M + (\mathcal{P}^M \Gamma^M) Z_{ij}^{(M)}. \]
where \( \mathcal{P} = (1 + \Gamma^{10})/2 \) projects onto states with positive chirality. Upon compactification on a circle, the supersymmetry algebra decomposes as
\[ \{ Q^i, \bar{Q}^j \} = -i \delta^{ij} \rho \nu + Z^{ij}, \]
with the central charge
\[ Z^{ij} = -P_0 \delta^{ij} + (Z^F \tau_3 - Z^D \tau_1)^{ij}. \]
The origin of the singlet and the doublet charges in the reduction of the IIB supergravity theory is different from their origin in the reduction of the eleven-dimensional theory. Here, the singlet originates from the momentum of the supergravity fields on the internal circle, and the doublet is related to the winding of the fundamental string and the D1-string around the internal direction.

When one diagonalizes the anti-commutator (3.3), the right-hand side decomposes into four eight-dimensional blocks of unit matrices with coefficients equal to \( M \) times \((1 + a + b), (1 - a - b), (1 - a + b)\) and \((1 + a - b)\). Whenever one or more of these coefficients vanish, the algebra can be realized on a much smaller number of states and the corresponding states are BPS-states. This is the phenomenon of multiplet shortening, that we have already discussed in chapter 1 and in chapter 2.

We can distinguish a number of cases. For \( a = \pm 1 \) and \( b = 0 \), half of the components of the supersymmetry charge in (3.3) are zero on the states, which means that we are dealing with 1/2-BPS states. The multiplet contains the momentum states of eleven-dimensional supergravity (and consequently of the IIA supergravity theory in ten dimensions), cf. (3.4). This multiplet is the so-called KKA multiplet. Setting \( a = 0 \) and \( b = \pm 1 \), we obtain a different kind of 1/2-BPS multiplet which comprises the momentum modes of the type IIB supergravity theory in ten dimensions. This multiplet is the so-called KKB multiplet. Finally, the cases \( \pm a \pm b = \pm 1 \) lead to 1/4-BPS states, i.e. to states that are annihilated under one quarter of the original supersymmetry charges.
Supergravity in nine dimensions

These multiplets correspond to string theory states that carry both momentum and winding.

We will discuss the field content of the KKA and KKB multiplets in more detail in chapter 4. In particular we write down Lagrangians that describe the massive fields and their interaction with the massless fields and we discuss the relation between the two BPS multiplets in more depth.

2. Maximal supergravity

Let us first describe the theory of the massless fields, i.e. maximal $\mathcal{N} = 2$ supergravity in nine dimensions and its relation to IIA and IIB supergravity in ten dimensions. Maximal supergravity in nine dimensions coincides with the dimensionally reduced version of both the IIA and IIB supergravities in ten dimensions. The field content of the supergravity multiplet follows directly from the reduction of the supergravity multiplet in eleven dimensions. Alternatively, the field content can also be obtained by reducing the IIB supergravity theory. We use both approaches below, but we will mainly focus on the former.

Not only the field content but also some of the quantum numbers of the fields can be directly inferred from the supergravity multiplet in eleven dimensions by studying the decompositions of the various symmetries: the Lorentz symmetry in eleven dimensions is broken to $SO(1, 8) \times SO(2) \subset SO(1, 10)$, where the $SO(2)$ plays the role of the R-symmetry group in nine dimensions. Similarly, representations of the $SO(9)$ helicity group in eleven dimensions decompose into representations of the $SO(7)$ helicity group in nine dimensions. The diffeomorphism invariance of the torus gives rise to a global $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times SO(1, 1)$ symmetry. The group $SL(2, \mathbb{R})$ corresponds to transformations of the modular parameter of the internal torus, and the group $SO(1, 1)$ describes rescalings of the torus. When we include the BPS multiplets into the theory later in this chapter, the global $GL(2, \mathbb{R})$ is broken to the arithmetic subgroup $SL(2, \mathbb{Z})$.

On the IIB side, the quantum numbers arise in a somewhat different way. The $SO(1, 9)$ Lorentz symmetry in ten dimensions is broken to $SO(1, 8)$ in nine dimensions. The helicity group in ten dimensions is $SO(8)$, and it reduces to $SO(7)$ in nine dimensions. The $SL(2, \mathbb{R})$ symmetry does not originate from symmetries of the internal manifold as in the reduction of the eleven-dimensional theory, the symmetry already exists in ten dimensions as a strong-weak coupling self-duality symmetry. The group $SO(1, 1)$ corresponds to rescalings of the compactification circle.

Let us now discuss the decomposition of the eleven-dimensional fields. The graviton $g_{MN}$ in eleven dimensions transforms in the $44$ representation of the $SO(9)$ helicity group, and it splits up into the following $SO(7)$ helicity representations and their associated massless fields in nine dimensions: a graviton
2. Maximal supergravity

Table 2. The metric in $D$ dimensions decomposes into massless fields in $d$ dimensions, describing states with the given helicities.

<table>
<thead>
<tr>
<th>field</th>
<th>multiplicity</th>
<th>dimension of helicity representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{MN}$</td>
<td>1</td>
<td>$\frac{1}{2}D(D - 3)$</td>
</tr>
<tr>
<td>$g_{\mu \nu}$</td>
<td>1</td>
<td>$\frac{1}{2}d(d - 3)$</td>
</tr>
<tr>
<td>$A_\mu^m$</td>
<td>$D - d$</td>
<td>$d - 2$</td>
</tr>
<tr>
<td>$e_\mu^a$</td>
<td>$\frac{1}{2}(D - d)(D - d + 1)$</td>
<td>1</td>
</tr>
</tbody>
</table>

$g_{\mu \nu}$, two vector fields $A_\mu^m$ and three scalars $\phi^m$ and $\sigma$. The SO(9) helicity representations of the corresponding states decompose into SO(7) representations as follows,

$$
\begin{align*}
    g_{MN} & \rightarrow g_{\mu \nu} + A_\mu^m + \phi^m + \sigma \\
    44 & \rightarrow 27 + 7 + 7 + 1 + 1 + 1,
\end{align*}
$$

(3.7)

where $m = 1, 2$ is an SL(2, $\mathbb{R}$) index. The three scalar fields are proportional to the metric in the internal dimensions: the scalar field $\sigma$ is related to the determinant of the torus metric $g_{mn}$. The two scalar fields $\phi^m$ take values in the coset space SL(2, $\mathbb{R}$)/SO(2), and they are described by a nonlinear sigma model. The graviphotons $A_\mu^m$ form a doublet of SL(2, $\mathbb{R}$).

These results are easily generalized to a reduction from $D$ dimensions to $d$ dimensions. The helicity group in the unreduced space-time is SO($D - 2$), whereas it is SO($d - 2$) in the reduced space-time. The metric decomposes as in (3.7), but the representations are of course different. A list of the representations and their multiplicities can be found in table 2.

The eleven-dimensional three-form field $A_{MNP}$ describes states in the 84 representation of the SO(9) helicity group, which decomposes in the following way into SO(7) helicity representations in nine dimensions,

$$
\begin{align*}
    A_{MNP} & \rightarrow A_{\mu \nu \rho} + A_{\mu \nu}^m + B_\mu \\
    84 & \rightarrow 35 + 21 + 21 + 7.
\end{align*}
$$

The nine-dimensional fields are a three-form field $A_{\mu \nu \rho}$, two two-form fields $A_{\mu \nu}^m$, and one vector field $B_\mu$. Note that the vector field $B_\mu$, which is a singlet under SL(2, $\mathbb{R}$) originates from the reduction of the three-form.

The gravitino in eleven dimensions, $\psi_M$ splits into two gravitini $\psi_\mu^a$ and four fermions $\chi^a_m$.

$$
\begin{align*}
    \psi_M & \rightarrow \psi_\mu^a + \chi^a_m \\
    128 & \rightarrow 48 + 48 + 8 + 8 + 8.
\end{align*}
$$
Supergravity in nine dimensions

\[ D = 11 \quad D = 9 \quad \text{IIB weight} \]

\[
\begin{array}{cccc}
G_{\mu \nu} & g_{\mu \nu} & G_{\mu \nu} & 0 \\
A_{\mu 9^{10}} & B_\mu & G_{\mu 9} & -4 \\
G_{\mu 9}, G_{\mu 10} & A_\mu^m & A_{\mu 9^m} & 3 \\
A_{\mu \nu 9}, A_{\mu \nu 10} & A_{\mu \nu} & A_{\mu \nu 9} & -1 \\
A_{\mu \nu \rho} & A_{\mu \nu \rho 9} & 2 \\
G_{99}, G_{9^{10}}, G_{10^{10}} & \phi^m & \phi^m & 0 \\
\end{array}
\]

Table 3. The bosonic fields of eleven-dimensional, nine-dimensional, and type IIB supergravity. We have also included the SO(1, 1) scaling weight of the various fields in the Einstein frame.

where \(a = 1, 2\) is the SO(2) R-symmetry index. The doubling of the gravitinos in nine dimensions is due to the fact that a Majorana spinor in nine dimensions contains 16 components, whereas it contains 32 components in eleven dimension.

Dimensional reduction of the IIB theory gives rise to exactly the same field content, but the origin of the fields is of course different. The IIB origin of the massless fields is listed in detail in table 3.

The three abelian gauge fields \(A_\mu^m\) and \(B_\mu\) play an important role in the interpretation of the BPS multiplets which we will be discussing in the next section. From an eleven-dimensional point of view, the SL(2) doublet \(A_\mu^m\) is derived from the reduction of the metric. It therefore couples to all the massive Kaluza-Klein states, i.e. the massive Kaluza-Klein states carry two charges with respect to the two abelian gauge fields \(A_\mu^m\). From a IIB point of view the role of the \(A_\mu^m\) is rather different. The fields originate from the doublet of tensor fields in ten dimension, which means that they couple to the winding states of the fundamental string and of the D1-string in the IIB theory on the circle.

A similar analysis holds for the gauge field \(B_\mu\), which is an SL(2) singlet. In the reduction of the IIB theory it derives from the metric and therefore couples to the massive Kaluza-Klein states, i.e. the massive IIB Kaluza-Klein states are charged under \(B_\mu\). In the compactification from eleven dimension, the field \(B_\mu\) originates from the three-index tensor field, and consequently it couples to the winding modes of the M2-brane on the torus.
3. BPS multiplets

As described in section 1, there are two inequivalent 1/2-BPS multiplets in nine dimensions, the KKA and the KKB multiplet, which we study in the following. We derive their field content from the eleven-dimensional supergravity theory and the ten-dimensional IIB supergravity theory, respectively.

In order to identify the massive physical fields in nine dimensions, it is necessary to impose certain gauge choices on the fields. This gauge fixing is very similar to employing the unitary gauge in spontaneously broken theories. Since we are only interested in the classical theory in this section, we do not have to worry about Faddeev-Popov ghosts. We write down the gauge choices for the various fields in chapter 4, where we also discuss the precise relation between the lower-dimensional fields and the original higher-dimensional fields.

3.1. The KKA multiplet

The massive Kaluza-Klein modes of eleven-dimensional supergravity compactified on a torus make up the KKA multiplet, which contains 128 bosonic and 128 fermionic degrees of freedom. All of the KKA modes are charged with respect to the two graviphotons \( A_\mu \), and the charges \( q^m \) of the fields are equal to their mass \( m \), i.e. \( q^2 = m^2 \). The charges \( q^m \) form a two-dimensional lattice, which breaks the global symmetry group \( SL(2, \mathbb{R}) \) of the massless theory to the arithmetic subgroup \( SL(2, \mathbb{Z}) \). Under this \( SL(2, \mathbb{Z}) \), the charge-lattice of the KKA multiplets is mapped onto itself. To be precise, a multiplet with charges \( (p, q) \) is mapped onto a multiplet with charges \( (p', q') \) as follows,

\[
\begin{pmatrix}
p' \\
q'
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
p \\
q
\end{pmatrix},
\]

with integers \( a, b, c, d \) which are subject to \( ad - bc = 1 \). A theory of massless fields coupled to a “lattice” of massive fields is therefore invariant under a global \( SL(2, \mathbb{Z}) \) symmetry. From the perspective of IIB supergravity theory, the two vector fields \( A_\mu \) originate from the \( SL(2, \mathbb{Z}) \)-invariant tensor fields, as we have mentioned above. They couple to the winding states of the fundamental string and the D1-string.

The massless fields of the eleven-dimensional theory transform in representations of the helicity group \( SO(9) \). In nine dimensions, the massive fields transform under the spin-group \( SO(8) \), and the \( SO(9) \) representations split into \( SO(8) \) representations as follows,

\[
\begin{align*}
44 & \rightarrow 1 + 8_v + 35_v \\
84 & \rightarrow 28 + 56_v \\
128 & \rightarrow 8_v + 8_c + 56_v + 56_c .
\end{align*}
\]

Each of these \( SO(8) \) representations corresponds to a state in the BPS multiplet, and we will identify the corresponding fields below. The KKA multiplet
Table 4. The massless metric in $D$ dimensions splits up into massive fields in $d$ dimensions. The states described by the $D$-dimensional metric transforms in a representation of the helicity group $\text{SO}(D-2)$, whereas the states described by the $d$-dimensional massive fields transform in a representation of the spin group $\text{SO}(d-1)$.

<table>
<thead>
<tr>
<th>field</th>
<th>multiplicity</th>
<th>representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{MN}$</td>
<td>1</td>
<td>$\frac{1}{2}(D-2)(D-1)-1$</td>
</tr>
<tr>
<td>$g_{\mu\nu}$</td>
<td>1</td>
<td>$\frac{1}{2}d(d-1)-1$</td>
</tr>
<tr>
<td>$A_{\mu}^m$</td>
<td>$D-d-1$</td>
<td>$d-1$</td>
</tr>
<tr>
<td>$e_m{}^a$</td>
<td>$\frac{1}{2}(D-d)(D-d-1)$</td>
<td>1</td>
</tr>
</tbody>
</table>

therefore consists of the following $\text{SO}(8)$ representations,

$$1 + 8_v + 28 + 35_v + 56_v + 8_s + 8_c + 56_c .$$

Note that the KKA multiplet contains the same representations as the supergravity multiplet of the IIA theory. This can easily be understood by noting that the massless fields of the IIA theory transform under the helicity group $\text{SO}(8)$, which is also the spin group of the massive nine-dimensional fields. The associated fields are different, though, because the states describe massive and massless fields, respectively. Let us now discuss the decomposition of the individual fields.

§ The elfbein $E_M^A$ in eleven dimensions, which transforms in the $44$ of the helicity group $\text{SO}(9)$ splits up into the following representations of the spin-group $\text{SO}(8)$ in nine dimension,

$$E_M^A \longrightarrow e_\mu{}^a + A_\mu + \sigma$$

$$44 \longrightarrow 35_v + 8_c + 1 .$$

The $\text{SO}(8)$ representations of the states are attributed to the various massive fields as follows: the representation $35_v$ corresponds to a symmetric and traceless two-index field, which we identify as the massive neunbein $e_\mu{}^a$, or equivalently the massive graviton $g_{\mu\nu}$. The representation $8_c$ corresponds to a massive vector field $A_\mu$, and the representation $1$, finally, corresponds to a scalar field in nine dimensions. For a general compactifications table 4 shows how the representation of the state associated with metric splits up into representations and the associated fields in the lower dimension.
§ The three-index tensor field $A_{MN\rho}$ splits up into a massive three-form field $A_{\mu\nu\rho}$ and a massive two-form field $A_{\mu\nu}$,

$$A_{MN\rho} \rightarrow A_{\mu\nu\rho} + A_{\mu\nu}$$

$$84 \rightarrow 56_v + 28.$$  

The representation $56_v$ is associated with a massive anti-symmetric three-index tensor field $A_{\mu\nu\rho}$, and $28$ corresponds to a massive anti-symmetric two-index tensor field $A_{\mu\nu}$ in nine dimensions.

§ The gravitino $\psi_M$ in eleven dimensions splits up into two Rarita-Schwinger fields $\psi_\mu$ and two spinor fields $\chi$ in nine dimensions. The decomposition goes as follows,

$$\psi_M \rightarrow \psi_\mu + \chi^a$$

$$128 \rightarrow 56_s + 56_c + 8_s + 8_c.$$  

As we have mentioned earlier, the doubling of the number of spinors in nine dimensions is due to the split of a 32-component spinor in eleven dimensions to two 16-component spinors in nine dimensions.

### 3.2. The KKB multiplet

The KKB multiplet consists of the massive Kaluza-Klein modes of type IIB supergravity theory. The fields are charged with respect to the graviphoton $B_\mu$, and the charges form a one-dimensional lattice. The multiplet consists of states in the following representations of the SO(8) spin group,

$$1 + 1 + 28 + 28 + 35_u + 35_c + 8_s + 8_c + 56_s + 56_c$$

These are exactly the same representations as are contained in the massless IIB supergravity multiplet. The reason for this is obvious: in ten dimensions, the helicity group for massless fields is SO(8), and in nine dimensions the spin group for massive fields is also SO(8). In the process of the compactification, the states themselves do not change, only the fields describing the states. The degrees of freedom corresponding to the internal coordinate are absorbed by the other degrees of freedom in order to make up a massive field in nine dimensions. We will discuss this effect for the individual fields below.

The IIB supergravity theory in ten dimensions is invariant under $SL(2, \mathbb{R})$, which describes the strong-weak coupling self-duality of the theory. If we include the solitonic string modes into the theory, the symmetry is broken to the arithmetic subgroup, $SL(2, \mathbb{Z})$. Again, this global symmetry can be found back in nine dimensions. For the KKB multiplets, the $SL(2, \mathbb{Z})$ acts on the fields within one multiplet, i.e. it does not mix multiplets of different Kaluza-Klein charges. There is also an SO(1, 1) symmetry in nine dimensions, which corresponds to rescalings of the compact coordinate, and which is broken in
the coupled theory. Let us now discuss the decomposition of the individual fields in the IIB theory.

§ The zehnbein $E_M^A$ in the IIB supergravity theory reduces to a massive neunbein $e_\mu^a$ in nine dimensions,

\[
E_M^A \rightarrow e_\mu^a \\
35_v \rightarrow 35_e .
\]

As alluded to above, we can observe a mechanism that holds for the compactification of all fields: the SO(8) helicity representation $35_v$ of the ten-dimensional state is re-interpreted as an SO(8) spin representation of the nine-dimensional massive state. The massive field associated with $35_v$ is a symmetric traceless two-index field, the massive graviton.

§ The two two-index tensor fields $A_{MN}^m$ in ten dimensions, which transform as a doublet under SL(2, $\mathbb{Z}$), reduce to two massive two form-fields $A_{\mu\nu}^m$ in nine dimensions,

\[
A_{MN}^m \rightarrow A_{\mu\nu}^m \\
28 + 28 \rightarrow 28 + 28 .
\]

§ The two scalars $\phi^m$ in ten dimensions reduce to two massive scalar fields $\phi^m$ in nine dimensions,

\[
\phi^m \rightarrow \phi^m \\
1 + 1 \rightarrow 1 + 1 .
\]

Note that the scalar fields both in ten dimensions and nine dimensions transform as a doublet under the SL(2, $\mathbb{Z}$) symmetry.

§ The four-form field $A_{MNPQ}^c$ in ten dimensions with self-dual field strength reduces to a massive four-form field $A_{\mu\nu\rho\sigma}$ in nine dimensions, which is subject to a self-duality constraint,

\[
A_{MNPQ}^c \rightarrow A_{\mu\nu\rho\sigma} \\
35_c \rightarrow 35_c .
\]

While it is clear that we are dealing with a massive four-index tensor field in nine dimensions, it is a priori not obvious how the self-duality condition that holds in ten dimensions is interpreted in nine dimensions. We will comment on this in more detail in chapter 2.5.

§ The two gravitinos $\psi_M^a$ in ten dimensions reduce to two massive Rarita-Schwinger fields $\psi_\mu^a$ in nine dimensions,

\[
\psi_M^a \rightarrow \psi_\mu^a \\
56_v + 56_v \rightarrow 56_e + 56_e .
\]
§ The fermions $\lambda^a$ in ten dimensions reduce to two massive spinor fields $\lambda^a$ in nine dimensions,

$$
\begin{align*}
\lambda^a &\longrightarrow \lambda^a \\
8_s + 8_s &\longrightarrow 8 + 8.
\end{align*}
$$

Let us summarize the analysis of the BPS multiplets in nine dimensions before we proceed to the next chapter. The discussion of the multiplets in this chapter was based solely on group theoretical considerations. We studied how the helicity representations of the higher-dimensional theory decompose into representations of the lower-dimensional helicity group and spin group. This allowed us to identify the massless and massive fields in nine dimensions. In chapter 4, we are going to construct a theory describing the massless and massive fields in nine dimensions.

There is a difference in the structure of the BPS multiplets when compactifying one dimension (e.g. IIB supergravity) or two and more dimensions (e.g. eleven-dimensional supergravity). In the former case the helicity group in ten dimensions is identical to the spin group in nine dimensions, and the representations of the states do not change in the process of the compactification. In the latter case the spin group in nine dimensions is a subgroup of the helicity group in eleven dimensions, and the representations of the states are decomposed accordingly.