Supersymmetry in anti-de Sitter space

Instead of discussing supersymmetry in flat Minkowski space, we now present an introduction to supersymmetry in anti-de Sitter space, which is an Einstein space of constant negative curvature. In the framework of this thesis, this chapter serves mainly two purposes: first to deepen our general understanding of supersymmetry applied to field theories and gravity theories, and to point out some features of supersymmetry that are generally ignored when considering supersymmetry in flat space-time. Second, this chapter prepares the reader for the study of supergravity in three dimensions in chapter 6, where we will see that anti-de Sitter space is often found as a ground-state geometry.

Field theory in anti-de Sitter space is not a new subject. Already in the thirties of the twentieth century Dirac considered wave equations that are invariant under the anti-de Sitter group [18]. Later, in 1963, he discovered the ‘remarkable representation’ which is now known as the singleton [19]. Shortly afterwards there was a series of papers by Fronsdal and collaborators discussing the representations of the anti-de Sitter group [20–23]. Quantum field theory in anti-de Sitter space was studied for instance in [24, 25]. Many new developments were inspired by the discovery that gauged supergravity theories have ground states corresponding to anti-de Sitter space-times [26–36]. This led to a study of the stability of these ground states with respect to fluctuations of the scalar fields [37] as well as to an extended discussion of supermultiplets in anti-de Sitter space [37–42]. In recent years, field theory in anti-de Sitter space has attracted a lot of interest because of the so-called AdS/CFT correspondence [43–46]. This conjecture states that certain supergravity theories on anti-de Sitter space-times are in some sense dual to field theories on the boundary of the anti-de Sitter space. For example, ten-dimensional IIB supergravity theory on a space-time of the form $\text{AdS}_5 \times S^5$ is conjectured to be dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions with gauge group $U(n)$.

In the following, we are able to cover only a few of these topics. We restrict ourselves to an introduction to supersymmetry in anti-de Sitter space and discuss the presence of the so-called masslike terms in wave equations for various fields in anti-de Sitter space. Then we will analyze the various irreducible representations of the anti-de Sitter isometry group, and at the end we
will consider the consequences for supermultiplets. We emphasize the issue of multiplet shortening for both multiplets of given spin and for supermultiplets. Throughout the whole chapter we make contact with supersymmetry in flat space, i.e. we show that Poincaré supersymmetry can be obtained by taking a certain limit in anti-de Sitter supersymmetry.

This chapter is based on [47].

1. Supersymmetry and anti-de Sitter space

In this section, we discuss some properties of anti-de Sitter space and consider the supersymmetry algebra in anti-de Sitter space. We will also give an example of a simple supergravity theory in anti-de Sitter space.

1.1. Properties of anti-de Sitter space

Anti-de Sitter space is a maximally symmetric space with constant negative curvature. It has \( d \) \((d+1)/2\) isometries which constitute the group \( \text{SO}(d-1, 2) \). Anti-de Sitter space can be described as a hypersurface embedded into a \((d+1)\)-dimensional embedding space. Denoting the extra coordinate of the embedding space by \( Y \), so that we have coordinates \( Y^A \) with \( A = -, 0, 1, 2, \ldots, d-1 \), this hypersurface is defined by

\[
-(Y^-)^2 - (Y^0)^2 + Y^2 = \eta_{AB} Y^A Y^B = -g^{-2}.
\]

The parameter \( g \) is the inverse radius of the anti-de Sitter space. In the limit of \( g \to 0 \) one recovers \( d \)-dimensional Minkowski space. The hypersurface defined by (2.1) is invariant under linear transformations that leave the metric \( \eta_{AB} = \text{diag} (-, -, +, +, \ldots, +) \) invariant. These transformations constitute the isometry group \( \text{SO}(d-1, 2) \). The \((d+1)/2\) generators of the group \( \text{SO}(d-1, 2) \), denoted by \( M_{AB} \), act on the embedding coordinates by

\[
M_{AB} = Y_A \frac{\partial}{\partial Y^B} - Y_B \frac{\partial}{\partial Y^A},
\]

where we lower and raise indices by contracting with \( \eta_{AB} \) and its inverse \( \eta^{AB} \).

Anti-de Sitter space is a homogeneous space, which means that any two points on it can be related via an isometry. It has the topology of \( S^1 \) [time] \( \times \mathbb{R}^{d-1} \). When unwrapping \( S^1 \) one finds the universal covering space denoted by \( \text{CAdS} \), which has the topology of \( \mathbb{R}^d \). There are many ways to coordinatize anti-de Sitter space; however, we will try to avoid using specific coordinates.

1.2. The supersymmetry algebra

The generators \( M_{AB} \) of the isometries \( \text{SO}(d-1, 2) \) form an algebra. The commutation relations for two generators \( M_{AB} \) are given by

\[
[M_{AB}, M_{CD}] = \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}.
\]
On spinors, the anti-de Sitter algebra can be realized by the following combination of gamma matrices,

\[ M_{AB} = \frac{1}{2} \Gamma_{AB} = \begin{cases} \frac{1}{2} \Gamma_{ab} & \text{for } A, B = a, b, \\ \frac{1}{2} \Gamma_a & \text{for } A = -, B = a, \end{cases} \]

with \( a, b = 0, 1, \ldots, d - 1 \). Our gamma matrices satisfy the Clifford property \( \{ \Gamma^a, \Gamma^b \} = 2 \eta^{ab} \mathbb{I} \), where \( \eta^{ab} = \text{diag} (-, +, \ldots, +) \).

The supersymmetric extension of the algebra (2.2) is called the anti-de Sitter superalgebra. It contains the following additional (anti-)commutation relations,

\[ \{ Q_a, \tilde{Q}_b \} = -\frac{1}{2} (\Gamma_{AB})_{ab} M^{AB}, \quad (2.3a) \]

\[ [M_{AB}, \tilde{Q}_a] = \frac{1}{2} (\tilde{Q} \Gamma_{AB})_a. \quad (2.3b) \]

In order to recognize the relation between the anti-de Sitter superalgebra and the Poincaré superalgebra one rescales the fields as follows,

\[ Q \rightarrow \frac{1}{\sqrt{g}} \tilde{Q}, \]

\[ M_{-a} \rightarrow \frac{1}{g} M_{-a}. \]

One then takes the limit \( g \rightarrow 0 \), which corresponds to taking the radius of anti-de Sitter space to infinity and reproduces flat Minkowski space. This singular transformation is a so-called Wigner-Inönü contraction [48], and the resulting algebra is the Poincaré superalgebra. Important relations in the Poincaré superalgebra are

\[ \{ Q_a, \tilde{Q} \} = -\frac{1}{2} \Gamma^a P_a, \quad (2.4a) \]

\[ [M_{ab}, \tilde{Q}] = \frac{1}{2} (\tilde{Q} \Gamma_{ab})_a. \quad (2.4b) \]

where we have defined the momentum operator \( P_a = M_{-a} \). One important difference between the Poincaré superalgebra and the anti-de Sitter superalgebra is that in the former, \( P^2 \) commutes with all elements of the algebra, and \( P^2 \) is therefore a Casimir operator. In the anti-de Sitter superalgebra \( (M_{-a})^2 \), which corresponds to \( P^2 \) in the Poincaré algebra, is not a Casimir operator. Instead, the quadratic Casimir operator for the anti-de Sitter algebra is given by

\[ C_2 = \frac{1}{2} M^{AB} M_{AB}. \]

In general, i.e. for \( D > 3 \), there are also higher order Casimir operators for the anti-de Sitter algebra, but we will not be concerned with those. The quadratic
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The Casimir operator is related to the covariantized d'Alambertian operator $\Box_{\text{AdS}}$ in anti-de Sitter space. It can be shown \cite{16} that

$$C_2 = \Box_{\text{AdS}} + C_L^2,$$

where $C_L^2$ is the quadratic Casimir operator for the spin-$s$ representation of the Lorentz group \cite{49}. As we will see at the end of section 2, $C_L^2$ vanishes for scalar fields, and the d'Alambertian equals the quadratic Casimir operator.

The structure of the anti-de Sitter algebra changes drastically for dimensions $d > 7$, see \cite{50} and references cited therein. For $d \leq 7$ the bosonic subalgebra coincides with the anti-de Sitter algebra. As we have seen in section 1 of chapter 1, there are $\mathcal{N}$-extended supersymmetry generators, each transforming as a spinor under the anti-de Sitter group. These $\mathcal{N}$ generators transform under a compact group, whose generators appear as central charges in the $\{Q, \tilde{Q}\}$ anticommutator. For $d > 7$ the bosonic subalgebra can no longer be restricted to the anti-de Sitter algebra and the algebra corresponding to a compact group, but one needs extra bosonic generators that transform as high-rank antisymmetric tensors under the Lorentz group. In contrast to this, there exists an ($\mathcal{N}$-extended) super-Poincaré algebra associated with flat Minkowski space of any dimension, whose bosonic generators correspond to the Poincaré group, possibly augmented with the generators of a compact group associated with rotations of the supercharges.

In this chapter, we are mainly dealing with the case $\mathcal{N} = 1$ and we always assume that $d \leq 7$.

1.3. Supergravity

Let us now study a simple supergravity theory in an unspecified number of space-time dimensions. In section 4 of chapter 1, we have already encountered the standard Einstein-Hilbert Lagrangian of general relativity and the Rarita-Schwinger Lagrangian for the gravitino field(s), cf. (1.10). We now add a cosmological term to the Lagrangian as well as a suitably chosen masslike term for the gravitino field,

$$\mathcal{L} = -\frac{1}{2} e R(\omega) - \frac{1}{2} e \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu (\omega) \psi_\rho + \frac{1}{4} g (d - 2) e \bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu + \frac{1}{2} g^2 (d - 1)(d - 2) e + \cdots,$$

where the covariant derivative on a spinor $\psi$ reads

$$D_\mu (\omega) \psi = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{\ ab} \Gamma_{ab} \right) \psi.$$

Here, $\omega_\mu^{\ ab}$ is the spin-connection field defined such that the torsion tensor (or a supercovariant version thereof) vanishes. As we will see shortly, the
1. Supersymmetry and anti-de Sitter space

...cosmological constant term in (2.6) can give rise to an anti-de Sitter spacetime. The masslike term for the gravitino is required if one demands that the ground-state preserves supersymmetry.

...It turns out that the action corresponding to (2.6) is locally supersymmetric, up to terms that are cubic in the gravitino field, and the supersymmetry transformation rules are given by,

\[ \delta e_{\mu}^a = \frac{1}{2} \xi \Gamma^a \psi_\mu , \]

\[ \delta \psi_\mu = \left( D_\mu (\omega) + \frac{1}{2} g \Gamma_\mu \right) \epsilon . \]

Note that the second term in the transformation rule for the gravitino is induced by the cosmological constant term in the Lagrangian and the masslike term for the gravitino.

The above demonstrates that, a priori, supersymmetry does not forbid a cosmological term, but it must be of definite sign—at least, if the ground state is to preserve supersymmetry.\(^1\) To construct a fully supersymmetric field theory is difficult and there are strong restrictions on the number of space-time dimensions. The Lagrangian (2.6) was first written down in [53] in four space-time dimensions and the correct interpretation of the masslike term was given in [54].

The Einstein-Hilbert equation corresponding to (2.6) reads (suppressing the gravitino field),

\[ R_{\mu \nu} - \frac{1}{2} R_{\mu \nu} g + \frac{1}{2} g^2 (d - 1)(d - 2) g_{\mu \nu} = 0 , \]

which implies that

\[ R_{\mu \nu} = g^2 (d - 1) g_{\mu \nu} , \quad R = g^2 d (d - 1) . \]

Hence we are dealing with a \( d \)-dimensional Einstein space. The maximally symmetric solution of this equation is an anti-de Sitter space, whose Riemann curvature equals

\[ R_{\mu \nu}^{ab} = 2 g^2 e_{\mu [a} e_{\nu b]} . \]

This solution leaves all the supersymmetries intact just as flat Minkowski space does. One can verify this directly by considering the supersymmetry variation of the gravitino field and by requiring that it vanishes in the bosonic background. This happens for spinors \( \epsilon (x) \) satisfying

\[ \left( D_\mu (\omega) + \frac{1}{2} g \Gamma_\mu \right) \epsilon = 0 . \quad (2.7) \]

\(^1\)For a discussion see [51, 52] and references therein.
Spinors satisfying this equation are called Killing spinors. As a direct consequence of (2.7) also \( (D_\mu (\omega) + g \Gamma_\mu / 2) (D_\nu (\omega) + g \Gamma_\nu / 2) \epsilon \) must vanish. Antisymmetrizing this expression in \( \mu \) and \( \nu \) then yields the integrability condition
\[
\left( -\frac{1}{4} R^{ab}_{\mu \nu} + \frac{1}{2} g^2 \Gamma_{\mu \nu} \right) \epsilon = 0 ,
\]
which is precisely satisfied in anti-de Sitter space.

As we have seen, anti-de Sitter space is consistent with supersymmetry. This is just as for flat Minkowski space, which has the same number of isometries but now corresponding to the Poincaré group, and which is also consistent with supersymmetry. We have already mentioned that the two cases are related; namely, the flat space results are obtained in the limit \( g \to 0 \).

The commutator of two supersymmetry transformations yields an infinitesimal general-coordinate transformation and a tangent-space Lorentz transformation. For example, we obtain for the vielbein,
\[
[\delta_1, \delta_2] e^a_\mu = \frac{1}{2} \bar{\epsilon}_2 \Gamma^a \delta_1 \psi_\mu - \frac{1}{2} \bar{\epsilon}_1 \Gamma^a \delta_2 \psi_\mu
\]
\[
= \frac{1}{2} D_\mu (\bar{\epsilon}_2 \Gamma^a \epsilon_1) + \frac{1}{2} g (\bar{\epsilon}_2 \Gamma^a \epsilon_1) e_\mu ,
\]
where \( \delta g_{\mu \nu} = D_\mu \bar{\epsilon}_\nu + D_\nu \bar{\epsilon}_\mu = 0 , \)
which is precisely satisfied in anti-de Sitter space.

We remind the reader of the fact that we are dealing with an incomplete theory. For a complete theory the above result should hold uniformly on all the fields (possibly modulo field equations). As before we have ignored terms proportional to the gravitino field. In the anti-de Sitter background the vielbein is left invariant by the combination of symmetries on the right-hand side. Consequently the metric is invariant under these coordinate transformations and we have the so-called Killing equation,
\[
\delta g_{\mu \nu} = D_\mu \bar{\epsilon}_\nu + D_\nu \bar{\epsilon}_\mu = 0 ,
\]
where \( \delta g_{\mu \nu} = (\bar{\epsilon}_2 \Gamma^a \epsilon_1) / 2 \) is a Killing vector and where \( \epsilon_1, 2 \) are Killing spinors. Since \( D_\mu \bar{\epsilon}_\nu = (g \bar{\epsilon}_2 \Gamma^a \epsilon_1) / 2 \), the right-hand side of (2.8) vanishes for this choice of supersymmetry parameters, and \( \epsilon^\mu \) satisfies the Killing equation (2.9). As for all Killing vectors, higher derivatives can be decomposed into the Killing vector and its first derivative, e.g. \( D_\mu (g \bar{\epsilon}_2 \Gamma^a \epsilon_1) = - g^2 \bar{g}_{\mu \rho} \bar{\epsilon}_1 \). The Killing vector can be decomposed into the \( d(d+1)/2 \) Killing vectors of the anti-de Sitter space.

2. Anti-de Sitter supersymmetry and masslike terms

Readers familiar with supersymmetry in flat space will remember that in Minkowski space all fields belonging to a supermultiplet are subject to field equations with the same mass. This follows from the fact that the momentum operator \( P_\mu \) commutes with the supersymmetry charges, so that \( P^2 \) is a Casimir operator. As we have seen in the previous section, \( P^2 \) is not a Casimir operator.
in anti-de Sitter space, but $M^2_{AB}$ plays that role instead. Therefore masslike terms are not necessarily the same for different fields belonging to the same anti-de Sitter supermultiplet. In the following we illustrate this phenomenon for the example of a chiral supermultiplet in four space-time dimensions. Further facts about the anti-de Sitter superalgebra will be given in section 4.

A chiral supermultiplet in four space-time dimensions consists of a scalar field $A$, a pseudoscalar field $B$ and a Majorana spinor field $\psi$. In anti-de Sitter space the supersymmetry transformations of the fields are proportional to a spinor parameter $x_i$, which is a Killing spinor in the anti-de Sitter space, i.e. $x_i$ must satisfy the Killing spinor equation (2.7). We allow for two constants $a$ and $b$ in the supersymmetry transformations, which we parameterize as follows,

$$
\delta A = \frac{1}{4} \bar{\epsilon} \psi ,
$$

$$
\delta B = \frac{1}{4} i \bar{\epsilon} \gamma_5 \psi ,
$$

$$
\delta \psi = \bar{\psi} (A + i \gamma_5 B) \epsilon - (a A + i b \gamma_5 B) \epsilon .
$$

The coefficient of the first term in $\delta \psi$ has been chosen such as to ensure that $[\delta_1, \delta_2]$ yields the correct coordinate transformation $\xi^\mu D_{\mu}$ on the fields $A$ and $B$. To determine the constants $a$ and $b$ and the field equations of the chiral multiplet, we consider the closure of the supersymmetry algebra on the spinor field. After some Fierz reordering we find

$$
[\delta_1, \delta_2] \psi = \xi^\mu D_{\mu} \psi + \frac{1}{16} (a - b) (\bar{\epsilon}_2 \gamma^{ab} \epsilon_1 \gamma_{ab} \psi
$$

$$
- \frac{1}{2} \bar{\xi} \mu \nu \left( \bar{\psi} + \frac{1}{2} (a + b) \psi \right) .
$$

We point out that derivatives acting on $\epsilon(x)$ occur in this calculation at an intermediate stage and should not be suppressed in view of (2.7). However, they produce terms proportional to $g$ which turn out to cancel in the above commutator. Now we note that the right-hand side should constitute a coordinate transformation and a Lorentz transformation, possibly up to a field equation. Obviously, the coordinate transformation coincides with (2.8) but the correct Lorentz transformation is only reproduced provided that $a - b = 2g$. If we now denote the mass of the fermion by $m = (a + b)/2$, so that the last term is just the Dirac equation with mass $m$, then we find

$$
a = m + g , \quad b = m - g .
$$

Consequently, the supersymmetry transformation of the $\psi$ equals

$$
\delta \psi = \bar{\psi} (A + i \gamma_5 B) \epsilon - m (A + i \gamma_5 B) \epsilon - g (A - i \gamma_5 B) \epsilon ,
$$

(2.10)
and the fermionic field equation equals $\mathcal{D} \psi = 0$. The second term in (2.10), which is proportional to $m$, can be accounted for by adding an auxiliary field to the supermultiplet. The third term, which is proportional to $g$, can be understood as a compensating S-supersymmetry transformation associated with auxiliary fields in the supergravity sector, see e.g. [55]. In order to construct the corresponding field equations for $A$ and $B$, we consider the variation of the fermionic field equation. Again we have to take into account that derivatives on the supersymmetry parameter are not equal to zero. This yields the following second-order differential equations,

\begin{align}
(\Box_{\text{AdS}} + 2g^2 - m(m - g)) A &= 0, \\
(\Box_{\text{AdS}} + 2g^2 - m(m + g)) B &= 0, \\
(\Box_{\text{AdS}} + 3g^2 - m^2) \psi &= 0.
\end{align}

The last equation follows from the Dirac equation. Namely, one evaluates $D m / D C m$, which gives rise to the wave operator $\Box_{\text{AdS}} + \mathcal{D} / 2 - m^2$. The commutator yields the Riemann curvature of the anti-de Sitter space. In an anti-de Sitter space of arbitrary dimension $d$ this equation then reads,

\begin{align}
(\Box_{\text{AdS}} + \frac{1}{4} d(d - 1) g^2 - m^2) \psi &= 0,
\end{align}

which, for $d = 4$ agrees with (2.11c). A striking feature of the above result is that the field equations (2.11) all have different mass terms, in spite of the fact that they belong to the same supermultiplet. Consequently, the role of mass is quite different in anti-de Sitter space as compared to flat Minkowski space. This will be elucidated later.

The $g^2$ term in the field equations for the scalar fields can be understood from the observation that the scalar d’Alambertian can be extended to a conformally invariant operator, see e.g. [55],

\begin{align}
\Box + \frac{1}{4} d(d - 2) g^2 = \Box + \frac{1}{4} d(d - 2) g^2,
\end{align}

which seems the obvious candidate for a massless wave operator for scalar fields. Indeed, for $d = 4$, we do reproduce the $g^2$ dependence in the first two equations (2.11). Observe that the Dirac operator $\mathcal{D}$ is also conformally invariant.

The quadratic Casimir operator for the Lorentz group in four dimensions, $C_L^2$ takes the values 0 and 3/2 for scalars and spinors, respectively. Comparing these values with the field equations (2.11) and with (2.5) yields the following values for the Casimir operators of the scalar field and the spinor,

\begin{align}
C_L^2(\text{scalar}) &= -2 + m^2, \\
C_L^2(\text{spinor}) &= -\frac{3}{2} + m^2.
\end{align}
This shows very clearly that for scalar fields and for spinor fields, the coefficient $m^2$ is not the mass term in the equation of motion.

3. Unitary representations of the anti-de Sitter algebra

In this section we discuss unitary representations of the anti-de Sitter algebra. For definiteness we mainly look at the case of four space-time dimensions. We will be able to construct massless and massive multiplets, and we will encounter the phenomenon of multiplet shortening for certain massive multiplets. In this context, a special representation that is not known from the Poincaré algebra is discussed, the so-called singleton representation. We refer to [20–23] for some of the original work, and to [39, 40] where some of this work was reviewed.

In order to underline the general features we start in $d$ space-time dimensions. Obviously, the group $\text{SO}(d - 1, 2)$ is non-compact. This implies that unitary representations will be infinitely-dimensional. The generators are then all anti-hermitian,

$$M_{AB}^\dagger = -M_{BA}.$$ 

Note that the covering group of $\text{SO}(d - 1, 2)$ has the generators $\Gamma^{\mu\nu}/2$ and $\Gamma^\mu/2$. They act on spinors, which are finite-dimensional objects. These generators, however, have different hermiticity properties from the ones above.

The compact subgroup of the anti-de Sitter group is $\text{SO}(2) \times \text{SO}(d - 1)$ corresponding to rotations of the compact anti-de Sitter time and spatial rotations. It is convenient to decompose the $d(d + 1)/2$ generators as follows. We have seen in section 1.2 that the generator $M_{-0}$ is related to the energy operator when the radius of the anti-de Sitter space is taken to infinity. The eigenvalues of this generator, which is associated with motions along the circle, are quantized in integer units in order to have single-valued functions, unless one goes to the covering space CAdS. So we define the energy operator $H$ by

$$H = -i M_{-0}.$$ 

Obviously the generators of the spatial rotations are the operators $M_{ab}$ with $a, b = 1, \ldots, d - 1$. Note that we have changed notation: here and henceforth the indices $a, b, \ldots$ refer only to space-like indices. The remaining $2(d - 1)$ generators $M_{-a}$ and $M_{0a}$ are combined into pairs of mutually conjugate operators,

$$M_{a}^\pm = -i M_{0a} \pm M_{-a}.$$
and we have \((M^+_a)^\dagger = M^-_a\). The anti-de Sitter commutation relations now read

\[
\begin{align*}
[H, M^\pm_a] &= \pm M^\pm_a, \tag{2.13a} \\
[M^\pm_a, M^\pm_b] &= 0, \tag{2.13b} \\
[M^+_a, M^-_b] &= -2(H \delta_{ab} + M_{ab}). \tag{2.13c}
\end{align*}
\]

Obviously, the operators \(M^\pm_a\) play the role of raising and lowering operators: when applied to an eigenstate of \(H\) with eigenvalue \(E\), application of \(M^+_a\) yields a state with eigenvalue \(E + 1\).

In this section we restrict ourselves to the bosonic case. Nevertheless, let us briefly indicate how some of the other (anti-)commutators of the anti-de Sitter superalgebra decompose, cf. (2.3),

\[
\begin{align*}
\{ Q_a, Q_b^\dagger \} &= H \delta_{\alpha\beta} - \frac{1}{2} i M_{ab} \left( \Gamma^a \Gamma^b \Gamma^0 \right)_{\alpha\beta} \tag{2.14a} \\
&\quad + \frac{1}{2} \left( M^+_a \Gamma^a (1 + i \Gamma^0) + M^-_a \Gamma^a (1 - i \Gamma^0) \right)_{\alpha\beta}, \\
[H, Q_a] &= -\frac{1}{2} i (\Gamma^0 Q)_a, \tag{2.14b} \\
[M^+_a, Q_a] &= \mp \frac{1}{2} \left( \Gamma_a (1 \mp i \Gamma^0) Q \right)_a. \tag{2.14c}
\end{align*}
\]

For the anti-de Sitter superalgebra, all the bosonic operators can be expressed as bilinears of the supercharges, so that in principle one could restrict oneself to fermionic operators only and employ the projections \((1 \pm i \Gamma^0) Q\) as the basic lowering and raising operators. However, this is not quite what we will be doing later in section 4.

Let us now assume that the spectrum of \(H\) is bounded from below,

\[H \geq E_0,\]

so that in mathematical terms we are considering lowest-weight irreducible unitary representations. The lowest eigenvalue \(E_0\) is realized on states that we denote by \(|E_0, s\rangle\), where \(E_0\) is the eigenvalue of \(H\) and \(s\) indicates the value of the total angular momentum operator. Of course there are more quantum numbers, e.g. associated with the angular momentum operator directed along some axis (in \(D = 4\) there are thus \(2s + 1\) degenerate states), but this is not important for the construction and these quantum numbers are suppressed. Since states with \(E < E_0\) should not appear, ground states are characterized by the condition,

\[M^-_a |E_0, s\rangle = 0.\]

The representation can now be constructed by acting with the raising operators on the vacuum state \(|E_0, s\rangle\). To be precise, all states of energy \(E = E_0 + n\)
are constructed by an \( n \)-fold product of creation operators \( M_a^+ \). In this way one obtains states of higher eigenvalues \( E \) with higher spin. The simplest case is the one where the vacuum has no spin (\( s = 0 \)). For given eigenvalue \( E \), the highest spin state is given by the traceless symmetric product of \( E - E_0 \) operators \( M_a^+ \) on the ground state. These states are shown in figure 1.

Henceforth we specialize to the case \( d = 4 \) in order to keep the aspects related to spin simple. To obtain spin-1/2 is trivial; one simply takes the direct product with a spin-1/2 state. That implies that every point with spin \( j \) in figure 1 generates two points with spin \( j \pm 1/2 \), with the exception of points associated with \( j = 0 \), which will simply move to \( j = 1/2 \). The result of this is shown in figure 2 on the next page.

Likewise one can take the direct product with a spin-1 state, but now the situation is more complicated as the resulting multiplet is not always irreducible. In principle, each point with spin \( j \) now generates three points, associated with \( j \) and \( j \pm 1 \), again with the exception of the \( j = 0 \) points, which simply move to \( j = 1 \). The result of this procedure is shown in figure 3 on page 31.
Figure 2. States of the spinor representation in terms of the energy eigenvalues $E$ and the angular momentum; the half-integer values for $j = l + 1/2$ denote a symmetric tensor-spinor of rank $l$. The small circles denote the original spinless multiplet from which the spinor multiplet has been constructed by a direct product with a spinor.

Let us now turn to the quadratic Casimir operator, which for four space-time dimensions can be written as

$$
C_2 = -\frac{1}{2} M^{AB} M_{AB} \\
= H^2 - \frac{1}{2} \{ M_a^+ , M_a^- \} - \frac{1}{2} (M_{ab})^2 \\
= H (H - 3) - \frac{1}{2} (M_{ab})^2 - M_a^+ M_a^- .
$$

Applying the last expression on the ground state $|E_0, s\rangle$ we derive

$$
C_2 = E_0 (E_0 - 3) + s(s + 1) , \quad (2.15)
$$

and, since $C_2$ is a Casimir operator, this result holds for any state belonging to the corresponding irreducible representation. Note, that the angular momentum operator is given by $J^2 = -(M_{ab})^2/2$. Assuming that $E_0$ only takes real
Figure 3. States of the spin-1 representation in terms of the energy eigenvalues $E$ and the angular momentum $j$. Observe that there are now points with double occupancy, indicated by the circle superimposed on the dots and states transforming as mixed tensors (with rank $l = j$) denoted by a star. The double-occupancy points exhibit the structure of a spin-0 multiplet with ground state energy $E_0 + 1$. This multiplet becomes reducible and can be dropped when $E_0 = 2$, as is explained in the text. The remaining points then constitute a massless spin-1 multiplet, shown in figure 4 on the next page.

values, (2.15) imposes a lower bound on $C_2$,

$$C_2 \geq s(s + 1) - \frac{9}{4}.$$ 

We can apply this result to an excited state (which is generically present in the spectrum) with $E = E_0 + 1$ and $j = s - 1$. Here, we assume that the ground state has $s \geq 1$. In that case we find

$$C_2 = (E_0 + 1)(E_0 - 2) + s(s - 1) - |M_a^-|E_0 + 1, s - 1|^2$$

$$= E_0(E_0 - 3) + s(s + 1),$$

so that

$$E_0 - s - 1 = \frac{1}{2}|M_a^-|E_0 + 1, s - 1|^2.$$
Supersymmetry in anti-de Sitter space

Figure 4. States of the massless $s = 1$ representation in terms of the energy eigenvalues $E$ and the angular momentum $j$. Now $E_0$ is no longer arbitrary but it is fixed to $E_0 = 2$.

This shows that $E_0 \geq s + 1$ in order to have a unitary multiplet. When $E_0 = s + 1$, however, the state $|E_0 + 1, s - 1\rangle$ is itself a ground state, which decouples from the original multiplet, together with its corresponding excited states. This can be interpreted as the result of a gauge symmetry and therefore we call these multiplets massless. Hence massless multiplets with $s \geq 1$ are characterized by

$$E_0 = s + 1, \quad \text{for } s \geq 1. $$

For these particular values the quadratic Casimir operator is

$$C_2 = 2(s^2 - 1). \quad (2.16)$$

Although this result is only derived for $s \geq 1$, it also applies to massless $s = 0$ and $s = 1/2$ representations, as we shall see later. Massless $s = 0$ multiplets have either $E_0 = 1$ or $E_0 = 2$, while massless $s = 1/2$ multiplets have $E_0 = 3/2$.

One can try and use the same argument again to see if there is a possibility that even more states decouple. Consider for instance a state with the same spin as the ground state, with energy $E$. In that case

$$E(E - 3) = E_0(E_0 - 3) + |M_n^+|E, s|^2. \quad (2.17)$$
3. Unitary representations of the anti-de Sitter algebra

For spin $s \geq 1$, this condition is always satisfied in view of the bound $E_0 \geq s+1$. But for $s = 0$, one can apply (2.17) for the first excited $s = 0$ state which has $E = E_0 + 2$. In that case one derives

$$2(E_0 - 1) = |M^+_a [E_0 + 2, s = 0]|^2,$$

so that

$$E_0 \geq \frac{1}{2}.$$ 

For $E_0 = 1/2$ we have the so-called singleton representation, where we have only one state for a given value of the spin. A similar result can be derived for $s = 1/2$, where one can consider the first excited state with $s = 1/2$, which has $E = E_0 + 1$. One then derives

$$2(E_0 - 1) = |M^+_a [E_0 + 1, s = 1/2]|^2,$$

so that

$$E_0 \geq 1.$$ 

For $E_0 = 1$ we have the spin-1/2 singleton representation, where again we are left with just one state for every spin value. The existence of these singleton representations was first noted by Dirac [19]. They are shown in figure 5 on the following page. Both singletons have the same value of the Casimir operator,

$$C_2 = -\frac{5}{4}.$$ 

In four dimensions, the spin-0 singleton and the spin-1/2 singleton are the only singleton representations. For dimensions higher than four, there are infinitely many singleton representations, which is related to the fact that the rotation group is of higher rank, so that there is a large variety of representations. Singleton representations do not have a flat space limit, and they therefore have no analogue in the Poincaré superalgebra. In order to understand this phenomenon, note that Poincaré representations correspond to plane waves, which can be decomposed into an infinite number of spherical harmonics. Therefore, for any given spin, one is dealing with an infinite tower of modes. The spectrum of the singleton, on the other hand, is different because a state has a single energy eigenvalue for any given value of the spin, as shown in figure 5 on the next page.

In the above treatment, we have come across the phenomenon of multiplet shortening for specific values of the energy and the spin of the representation. In fact, this is very similar to the multiplet shortening of massive multiplets in flat space to BPS multiplets, which occurs when the mass and the central charge obey the BPS relation (1.2).

From the above it is clear that we are dealing with the phenomenon of multiplet shortening for specific values of the energy and spin of the ground state. This can be understood more generally from the fact that the $[M^+_a, M^-_b]$
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Figure 5. The spin-0 and spin-1/2 singleton representations. The solid dots indicate the states of the spin-0 singleton, the circles the states of the spin-1/2 singleton. It is obvious that singletons contain fewer degrees of freedom than a generic local field. The value of $E_0$, which denotes the spin-0 ground state energy, is equal to $E_0 = 1/2$. The spin-1/2 singleton ground state has an energy which is one half unit higher, as is explained in the text.

commutator acquires zero or negative eigenvalues for certain values of $E_0$ and $s$. We will return to this phenomenon in section 4 in the context of the anti-de Sitter superalgebra.

4. The $\mathcal{N} = 1$ anti-de Sitter superalgebra

In this section we return to the anti-de Sitter superalgebra. We start from the (anti-)commutation relations already established in (2.13) and (2.14). For definiteness we discuss the case of four space-time dimensions with a Majorana supercharge $Q$. This allows us to make contact with the material discussed in section 2. These anti-de Sitter multiplets were discussed in [37–40].

We choose conventions where the gamma matrices are given by

$$
\Gamma^0 = \begin{pmatrix} -i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix}, \quad \Gamma^a = \begin{pmatrix} 0 & -i\sigma^a \\ i\sigma^a & 0 \end{pmatrix},
$$
where $a = 1, 2, 3$, and write the Majorana spinor $Q$ in the form

$$Q = \left( \begin{array}{c} q_\alpha \\ \epsilon_{a\beta} q^\beta \end{array} \right),$$

where $q^\alpha \equiv q_\alpha$ and the indices $\alpha, \beta, \ldots$ are two-component spinor indices. We substitute these definitions into (2.14) and obtain

$$\begin{align*}
\left[ H, q_\alpha \right] &= -\frac{1}{2} q_\alpha, \\
\left[ H, q^\alpha \right] &= \frac{1}{2} q^\alpha, \\
\left\{ q_\alpha, q_\beta \right\} &= (H 1 + J \cdot \sigma)_\alpha^\beta, \\
\left\{ q_\alpha, q^\beta \right\} &= M^-_\alpha (\sigma^\alpha \sigma^\beta)_\alpha^\beta, \\
\left\{ q^\alpha, q^\beta \right\} &= M^+_\alpha (\sigma^2 \sigma^\alpha)^\alpha^\beta,
\end{align*}$$

where we have defined the angular momentum operator $J_a = -i \epsilon_{abc} M^{bc}/2$. We see that the operators $q_\alpha$ and $q^\alpha$ are lowering and raising operators, respectively. They change the energy of a state by half a unit.

In analogy to the bosonic case, we study unitary irreducible representations of the $\text{OSp}(1|4)$ superalgebra. We assume that there exists a lowest-weight state $|E_0, s\rangle$, characterized by the fact that it is annihilated by the lowering operators $q_\alpha$.

$$q_\alpha |E_0, s\rangle = 0.$$

In principle we can now choose a ground state and build the whole representation upon it by applying products of raising operators $q^\alpha$. However, we only have to study the antisymmetrized products of the $q^\alpha$, because the symmetric ones just yield products of the operators $M^+_a$ by virtue of (2.18). Products of the $M^+_a$ simply lead to the higher-energy states in the anti-de Sitter representations of given spin that we considered in section 3. By restricting ourselves to the antisymmetrized products of the $q^\alpha$ we thus restrict ourselves to the ground states upon which the full anti-de Sitter representations are build. These ground states are $|E_0, s\rangle$, $q^\alpha |E_0, s\rangle$ and $q^\alpha q^\beta |E_0, s\rangle$. Let us briefly discuss these representations for different $s$.

The $s = 0$ case is special since it contains fewer anti-de Sitter representations than the generic case. It includes the spinless states $|E_0, 0\rangle$ and $q^\alpha q^\beta |E_0, 0\rangle$ with ground-state energies $E_0$ and $E_0 + 1$, respectively. There is one spin-$1/2$ pair of ground states $q^\alpha |E_0, 0\rangle$, with energy $E_0 + 1/2$. As we will see below, these states correspond exactly to the scalar field $A$, the pseudo-scalar field $B$ and the spinor field $\psi$ of the chiral supermultiplet, that we studied in section 2.
For $s \geq 1/2$ we are in the generic situation. We obtain the ground states $|E_0, s \rangle$ and $q^\alpha q^\beta |E_0, s \rangle$ which have both spin $s$ and which have energies $E_0$ and $E_0 + 1$, respectively. There are two more (degenerate) ground states, $q^\alpha |E_0, s \rangle$, both with energy $E_0 + 1/2$, which decompose into the ground states with spin $j = s - 1/2$ and $j = s + 1/2$.

As in the purely bosonic case of section 3, there can be situations in which states decouple so that we are dealing with multiplet shortening associated with gauge invariance in the corresponding field theory. The corresponding multiplets are then again called massless. We now discuss this in a general way analogous to the way in which one discusses BPS multiplets in flat space. Namely, we consider the matrix elements of the operator $q^\alpha q^\beta$ between the ground states $E_0, s \rangle$,

$$
\langle E_0, s | q^\alpha q^\beta | E_0, s \rangle = \langle E_0, s | \{q^\alpha, q^\beta\} | E_0, s \rangle = \langle E_0, s | (E_0 \mathbb{1} + J \cdot \sigma)_{\alpha\beta} | E_0, s \rangle.
$$

This expression constitutes an hermitian matrix in both the quantum numbers of the degenerate ground-state and in the indices $\alpha$ and $\beta$, so that it is $2s$-by-$2s$. Because we assume that the representation is unitary, this matrix must be positive definite, as one can verify by inserting a complete set of intermediate states between the operators $q^\alpha$ and $q^\beta$ in the matrix element on the left-hand side. Obviously, the right-hand side is manifestly hermitian as well, but in order to be positive definite the eigenvalue $E_0$ of $H$ must be big enough to compensate for possible negative eigenvalues of $J \cdot \sigma$, where the latter is again regarded as a $(4s + 2)$-by-$(4s + 2)$ matrix. To determine its eigenvalues, we note that $J \cdot \sigma$ satisfies the following identity,

$$(J \cdot \sigma)^2 + (J \cdot \sigma) = s(s + 1) \mathbb{1},$$

as follows by straightforward calculation. This shows that $J \cdot \sigma$ has only two (degenerate) eigenvalues (assuming $s \neq 0$, so that the above equation is not trivially satisfied), namely $s$ and $-(s + 1)$. Hence in order for (2.19) to be positive definite, $E_0$ must satisfy the inequality

$$E_0 \geq s + 1, \quad \text{for } s \geq \frac{1}{2}.$$

If the bound is saturated, i.e. if $E_0 = s + 1$, the expression on the right-hand side of (2.19) has zero eigenvalues so that there are zero-norm states in the multiplet which decouple. In that case we must be dealing with a massless multiplet. As an example we mention the case $s = 1/2$, $E_0 = 3/2$, which corresponds to the massless vector supermultiplet in four space-time dimensions. Observe that we have multiplet shortening here without the presence of central charges.

Armed with these results we return to the masslike terms of section 2 for the chiral supermultiplet. The ground-state energy for anti-de Sitter multiplets corresponding to the scalar field $A$, the pseudo-scalar field $B$ and the Majorana
spinor field $\psi$, are equal to $E_0$, $E_0 + 1$ and $E_0 + 1/2$, respectively. The Casimir operator therefore takes the values

$$C_2(A) = E_0(E_0 - 3), \quad (2.20a)$$
$$C_2(B) = (E_0 + 1)(E_0 - 2), \quad (2.20b)$$
$$C_2(\psi) = \left( E_0 + \frac{1}{2} \right) \left( E_0 - \frac{5}{2} \right) + \frac{3}{4}. \quad (2.20c)$$

For massless anti-de Sitter multiplets, we know that the quadratic Casimir operator is given by (2.16), so we present the value for $C_2 - 2(s^2 - 1)$ for the three multiplets, i.e.

$$C_2(A) + 2 = (E_0 - 1)(E_0 - 2),$$
$$C_2(B) + 2 = E_0(E_0 - 1),$$
$$C_2(\psi) + \frac{3}{2} = (E_0 - 1)^2.$$

The terms on the right-hand side are not present for massless fields and we should therefore identify them somehow with the common mass parameter. Comparison with the field equations (2.11) shows for $g = 1$ that we obtain the correct contributions provided we make the identification $E_0 = m + 1$. Observe that we could have made a slightly different identification here; the above result remains the same under the interchange of $A$ and $B$ combined with a change of sign in $m$ (the latter is accompanied by a chiral redefinition of $\psi$).

Outside the context of supersymmetry, we could simply assign independent mass terms with a mass parameter $\mu$ for each of the fields, by equating $C_2 - 2(s^2 - 1)$ to $\mu^2$. In this way we obtain

$$E_0(E_0 - 3) - (s + 1)(s - 2) = \mu^2,$$

which leads to

$$E_0 = \frac{3}{2} \pm \sqrt{\left( s - \frac{1}{2} \right)^2 + \mu^2}.$$

(2.21)

For $s \geq 1/2$ we must choose the plus sign in (2.21) in order to satisfy the unitarity bound $E_0 \geq s + 1$. For $s = 0$ both signs are acceptable as long as $\mu^2 \leq 3/4$. Observe, however, that $\mu^2$ can be negative but remains subject to the condition

$$\mu^2 \geq -\left( s - \frac{1}{2} \right)^2$$

in order that $E_0$ remains real. For $s = 0$, this is precisely the bound of Breitenlohner and Freedman for the stability of the anti-de Sitter background against small fluctuations of the scalar fields [37]. We can also compare $C_2 - 2(s^2 - 1)$
to the conformal wave operator for the corresponding spin. This shows that (again with unit anti-de Sitter radius), $c_2 = \Box_{\text{AdS}} + c_2^L$.

In the case of $N$-extended supersymmetry the supercharges transform under an $\text{SO}(N)$ group and we are dealing with the so-called $\text{OSp}(N|4)$ algebras. Their representations can be constructed by elaborate methods, e.g. by the oscillator method [56], that we have not discussed in this chapter because it lies outside the scope of this thesis. However, the generators of $\text{SO}(N)$ will now also appear on the right-hand side of the anticommutator of the two supercharges, thus leading to new possibilities for multiplet shortening. For an explicit discussion of this we refer the reader to [39].

Most of our discussion of the irreducible representations of the anti-de Sitter algebra and its superextension was restricted to four space-time dimensions, but in principle the same methods can be used for anti-de Sitter space-times of arbitrary dimension. For higher-extended supergravity, the only way to generate a cosmological constant is by elevating a subgroup of the rigid invariances that act on the gravitini to a local group, i.e. to gauge the supergravity theory. This then leads to a cosmological constant, or to a potential with possibly a variety of extrema, and corresponding masslike terms which are quadratic and linear in the gauge coupling constant, respectively. So the relative strength of the anti-de Sitter and the gauge group generators on the right-hand side of the $[Q, \bar{Q}]$ anticommutator is not arbitrary and because of that maximal multiplet shortening can take place so that the theory can realize a supermultiplet of massless states that contains the graviton and the gravitini. Of course, this is all under the assumption that the ground state is supersymmetric.