Holography in de Sitter Space

More or less satisfying formulations of holography and, more generally, quantum gravity in Minkowski and Anti-de Sitter space have been obtained. Having thus addressed flat and negatively curved spaces, a natural next step would be to consider positively curved space, or de Sitter space. This case, however, has proven to be particularly difficult and a full description of quantum gravity in de Sitter space is still lacking. With recent observational data [98,106,113] indicating that the universe may be entering a de Sitter phase, this issue has taken on added significance.

Naturally, this has led to a lot of interest in the subject and in this chapter we will review some recent developments, focusing on the formulation of the holographic principle in de Sitter space. Of particular importance is the issue of finding a de Sitter solution to string theory. This would be an important step forward, but attempts in this direction have failed so far. Nonetheless, analogous to the well established AdS/CFT correspondence discussed in Section 2.2, progress has been made in formulating a similar correspondence in the de Sitter case.

We begin this chapter, in Section 4.1, by discussing the classical geometry of de Sitter space. Next, in Section 4.2, we discuss the notions of energy and entropy in a de Sitter context. In Section 4.3, we consider the recently proposed dS/CFT correspondence. Finally, in Section 4.4, we consider an interesting scenario in which the evolution of the universe corresponds to an RG-flow in the dual CFT. We consider a \( c \)-theorem in this context and its relation to the size of the apparent horizon in de Sitter space. Detailed accounts of the classical properties of this space can be found in [67,118]; observational constraints and theoretical approaches to the cosmological constant are reviewed in [31,141].
4. Holography in de Sitter Space

4.1 Classical Geometry of de Sitter Space

Pure de Sitter space is the unique vacuum solution to the Einstein equations with maximal symmetry and constant positive curvature. In $D = n + 1$ space-time dimensions, it is locally characterized by

$$R_{\mu\nu} = \frac{D-1}{R^2} g_{\mu\nu},$$

(4.1)

where $R$ is the radius of curvature of de Sitter space, and by the vanishing of the Weyl tensor. The cosmological constant, $\Lambda$, is given as a function of $R$ by

$$\Lambda = \frac{(D-1)(D-2)}{2R^2}.$$  

(4.2)

It is convenient to think of de Sitter space as a hypersurface embedded in $D+1$-dimensional flat Minkowski space. The embedding equation is

$$-X_0^2 + X_1^2 + \ldots + X_D^2 = R^2$$

(4.3)

and the resulting timelike hyperboloid is depicted in Figure 4.1. The embedding equation (4.3) makes manifest the $O(1, D)$ isometry group of de Sitter space. The de Sitter metric is the induced metric from the flat Minkowski metric on the embedding space. In this way several coordinate systems can be obtained.

Frequently used are the so called global coordinates, in terms of which the metric takes the form

$$ds^2 = -dT^2 + R^2 \cosh^2 \frac{T}{r} d\Omega^2_n.$$  

(4.4)
The time coordinate $T$ takes values $-\infty < T < \infty$. In these coordinates, which cover all of the hyperboloid shown in Figure 4.1, de Sitter space starts out as an infinitely large $n$-sphere at $T = -\infty$. Subsequently, it shrinks and reaches its minimal radius $R$ at $T = 0$, after which it re-expands. Note that the global time coordinate $T$ does not define a Killing vector. The appearance of a cosmological horizon is not manifest in these coordinates. Before discussing other useful coordinate systems for de Sitter space, we first consider its Penrose diagram.

4.1.1  **Penrose Diagram of de Sitter Space**

To elucidate the causal structure of de Sitter space, it is useful to construct the corresponding Penrose diagram. Through the coordinate transformation

$$\cosh \frac{T}{R} = \frac{1}{\cos \frac{\tau}{R}}, \quad (4.5)$$

put the metric (4.4) in the form

$$ds^2 = \frac{1}{\cos^2 \frac{\tau}{R}} (-d\tau^2 + R^2 d\Omega_n^2). \quad (4.6)$$

The range of the new time coordinate $\tau$ is $-\pi/2 < \tau/R < \pi/2$. Without affecting the causal structure, we can perform a conformal transformation, bringing the metric to the form

$$d\hat{s}^2 = \cos^2 \frac{\tau}{R} ds^2 = -d\tau^2 + R^2 d\Omega_n^2. \quad (4.7)$$

This can also be written as

$$d\hat{s}^2 = -d\tau^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\Omega_{n-1}^2 \right), \quad (4.8)$$

where $\theta$ is a polar angle with range $0 \leq \theta \leq \pi$. This leads to the square Penrose diagram depicted in Figure 4.2, where every point is a $n-1$-dimensional sphere with radius $R \sin \theta$. The equal time slices in global coordinates correspond simply to horizontal lines in the Penrose diagram. From the diagram it is clear that no observer can ever see all of de Sitter space. An observer on the south pole can see, as he approaches $I^+$, all events that happened in regions I and III. Similarly, he can send signals to events in regions I and IV. In the intersection of these, *i.e.* in region I, he can send signals to as well as receive signals from all events; this we call his causal diamond, or static patch. Region II, the causal diamond of an observer on the north pole, remains completely inaccessible to the observer on the south pole.
Another way to see that the Penrose diagram must be square is the following. Consider where a light ray that originates from the south pole at $\mathcal{I}^-$ ends up on $\mathcal{I}^+$. Choosing the ray such that it only moves in the $\theta$-direction selected in (4.8), we have from (4.4) that

$$dT^2 = R^2 \cosh^2 \frac{T}{R} d\theta^2.$$  \hfill (4.9)

The distance the light ray will travel in the $\theta$-direction before it reaches $\mathcal{I}^+$ is therefore given by

$$\Delta \theta = \int_{-\infty}^{\infty} dT \frac{d\theta}{dT} = \int_{-\infty}^{\infty} dT \frac{1}{R \cosh \frac{T}{R}} = \pi.$$

We see that it will exactly reach the north pole, independent of the value of $R$, i.e., independent of the cosmological constant $\Lambda$. It is interesting to consider what happens when one perturbs de Sitter space by adding some energy density to it. Like in Chapter 3, let us consider a radiation dominated FRW cosmology with radiation density $\rho$, but now with a positive cosmological constant. The Friedmann equation (3.3) becomes

$$H^2 = \frac{16\pi G_N}{n(n-1)} \left( \frac{1}{a^2} + \frac{2\Lambda}{n(n-1)} \right),$$

while the second Friedmann equation (3.4) remains unchanged. The metric
4.1 Classical geometry of de Sitter space

takes the general form

$$ds^2 = -dt^2 + a^2(t) d\Omega_n^2.$$  (4.12)

For the spacetime to be of the de Sitter form (4.4) at late times, $\rho$ cannot be larger than a certain critical value. For larger values of $\rho$ the spacetime will collapse to form a black hole instead of re-expanding. We will give a general definition of the notion ‘asymptotically de Sitter’ in Section 4.3.1. Precisely at the critical radiation density, the space will asymptote to its smallest radius and subsequently remain static instead of re-expanding. This final state is of course unstable as small perturbations will cause the space to either collapse to a black hole or to start expanding. The critical value for $\rho$ and the minimal radius $a_c$ are given by

$$\rho_c = \frac{2\Lambda}{8\pi G_N} \left( \frac{R_c}{a} \right)^{n+1},$$  (4.13)

$$a_c = \sqrt{\frac{n(n-1)^2}{2\Lambda(n+1)}}.$$  (4.14)

Now take the radiation density to be a fraction of this critical density,

$$\rho = \alpha \rho_c,$$  (4.15)

where $0 \leq \alpha \leq 1$. Solving (4.11) for $a(t)$ gives (henceforth setting $n = 3$)

$$a^2(t) = \frac{R^2}{2} \left( 1 + \sqrt{1 - 2\alpha \cosh 2t/R} \right).$$  (4.16)

Using the identity $\cosh 2x = 2 \cosh^2 x - 1$, we see that for $\alpha = 0$ this reduces to the pure de Sitter case (4.4), as expected. On the other hand, the critical case $\alpha = 1$ does not behave as described above. Instead, that behaviour corresponds to the particular solution

$$a^2(t) = \frac{R^2}{2} \left( 1 + \frac{1}{3} e^{-\frac{2t}{R}} \right).$$  (4.17)

For completeness, we also note the general solution, valid for any value of $\alpha$,

$$a^2(t') = \frac{R^2}{2} \left[ 1 + \frac{1}{3} e^{-\frac{2t'}{R}} + \frac{3}{4}(1 - \alpha) e^{\frac{2t'}{R}} \right],$$  (4.18)

where the new time coordinate $t'$ is defined by

$$t' = t + \frac{R}{4} \ln \left( \frac{9}{4}(1 - \alpha) \right).$$  (4.19)
Using the solution (4.16), we can now repeat the calculation (4.10) for $\Delta \theta$. Since again the result is independent of $\Lambda$, we put $R \equiv 1$. This leads to the expression

$$\Delta \theta = \int_{-\infty}^{\infty} dt \left( \frac{2}{1 + \sqrt{1 - \alpha \cosh 2t}} \right)^{\frac{1}{2}}. \quad (4.20)$$

Expanding for small $\alpha$ gives

$$\lim_{\alpha \to 0} \Delta \theta = \int_{-\infty}^{\infty} dt \left[ \frac{1}{\cosh t} + \alpha \frac{\cosh 2t}{8 \cosh^3 t} + \mathcal{O}(\alpha^2) \right], \quad (4.21)$$

$$= \pi + \alpha \frac{3\pi}{16} + \mathcal{O}(\alpha^2). \quad (4.22)$$

We thus find that in this case the light ray overshoots the antipodal point by some amount proportional to the energy density. This causes the Penrose diagram to become rectangular in shape, as shown in Figure 4.3. It was shown by Gao and Wald [50] that the same thing happens for general perturbations of de Sitter space. These ‘tall’ de Sitter spaces were considered in [87]. It is important to realize that this elongation of the Penrose diagram has drastic consequences for the causal structure of the spacetime. For example, an observer can now receive information from everywhere on $I^-$ at a finite time, i.e., before he reaches $I^+$. These changes in the causal structure present a severe challenge to the antipodal identification considered in the next chapter. We will comment on this in Section 5.7.
4.1.2 Coordinate systems

A coordinate set that is especially well adapted to describe a specific observer are the static coordinates. In these coordinates, the metric takes the form

\[ ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega_{n-1}^2. \]  

(4.23)

The cosmological horizon is at \( r_c = R \) and \( \partial/\partial t \) now is a Killing vector, although not globally timelike. Constructing these coordinates for an observer at the south pole, \( r = 0 \) corresponds to the south pole and the corresponding causal diamond (region I in Figure 4.2) has \( 0 \leq r \leq R \). The Killing vector \( \partial/\partial t \) is timelike in region I, making it a suitable generator of time evolution for this observer. But in regions III and IV, this vector becomes spacelike. In region II, it is again timelike, but it is directed in the opposite direction from region I, i.e. towards the past. The lack of a globally timelike Killing vector in de Sitter space affects the notions of entropy and energy as well as complicates the quantization of fields on this background. The equal time slices in the northern and southern causal diamonds are shown in Figure 4.4, where also the direction of the Killing field \( \partial/\partial t \) is indicated.

Another coordinate set, which we will encounter often, covers only half of de Sitter space. These are the planar coordinates appropriate for a big bang (or big crunch) de Sitter model. The metric takes the form

\[ ds^2 = -dt^2 + e^{\pm 2} dx_i dx^i, \]  

(4.24)

and, when applied to the south pole and depending on the sign in the exponential, covers either regions I and III or I and IV. These coordinates derive their name from the fact that the equal time slices, as depicted in Figure 4.5 for the combined regions I and III, are flat \( n \)-planes.

For future purpose, we introduce the Schwarzschild-de Sitter solution. In \( D = n + 1 \) dimensions, the line element in static coordinates takes the form

\[ ds^2 = -F(r) \, dt^2 + F^{-1}(r) \, dr^2 + r^2 d\Omega_{n-1}^2, \]  

(4.25)

\[ F(r) = 1 - \frac{2M}{r^{d-2}} - \frac{r^2}{R^2}. \]  

(4.26)
De Sitter space cannot support arbitrarily large black holes. In fact, the size of any object in de Sitter space is limited by the cosmological horizon. Putting a black hole in de Sitter space gives rise to a second horizon, namely a black hole horizon besides the cosmological horizon. Their locations, denoted respectively by \( r_{\text{BH}} \) and \( r_{\text{C}} \), are the positive roots of the equation \( F(r) = 0 \). The bound on the size of the black hole can then be expressed as \( r_{\text{C}} \geq r_{\text{BH}} \). This limits the mass of the black hole to be within the range \( 0 \leq M \leq M_{\text{max}} \), where

\[
M_{\text{max}} = \frac{1}{D-1} \left( \frac{(D-3)(D-2)}{2\Lambda} \right)^{\frac{D-3}{2}}.
\]

When \( M = M_{\text{max}} \), the black hole horizon coincides with the cosmological horizon.

### 4.2 Energy and Entropy

In a gravitational background there is no local notion of energy. However, the total energy can generally be defined, using an asymptotically timelike Killing vector. Since there exists no globally timelike Killing vector in de Sitter space, one cannot define a Hamiltonian in this way. Due to the spatial compactness, de Sitter space cannot support conserved Noether charges like mass and angular momentum. The only asymptotic regions within de Sitter space are temporal past and future infinity. An alternative definition of mass, using the Brown-York stress tensor, is given in [8].

In the absence of a Hamiltonian, there is no global notion of time-evolution. Indeed, we have seen above that the most natural timelike Killing vector to define the evolution of an observer, points towards the past in part of de Sitter space. Thus, a particle moving forward in time with respect to one observer, would be conceived as moving backwards in time by a second observer; that is, if he were able to see it. Note, however, that these different regions are causally disconnected. These considerations will play an important role in the next chapter.

#### 4.2.1 Entropy

One of the most enigmatic features of de Sitter space is its entropy [10]. That de Sitter space has a finite entropy may be expected, based on the appearance of a horizon in pure de Sitter. This horizon is, however, qualitatively different from a black hole horizon. The position of the horizon is observer dependent and, because of this, it is not entirely clear which concepts about black holes carry over to de Sitter space. In fact, the de Sitter cosmological horizon looks in many ways like the Rindler horizon in Minkowski space.
It was, however, shown by Gibbons and Hawking [53] that entropy and temperature can be ascribed to the cosmological horizon just as in the black hole case. Indeed, an observer in de Sitter space will observe a background of thermal radiation coming from the cosmological horizon that surrounds him. The area of the horizon is then a measure of the information hidden by it from the observer. By absorbing this thermal radiation, the observer can presumably gain knowledge on what lies beyond the horizon; thus causing the horizon to shrink. Indeed, in pure – or empty – de Sitter space the horizon size is maximal.

The radius of the cosmological horizon in $D$-dimensional de Sitter space, as described by the metric (4.23), is given by

$$r_c = R = \sqrt{\frac{(D-1)(D-2)}{2\Lambda}} \quad \text{(D=4)} \quad = \sqrt{\frac{3}{\Lambda}}.$$ 

(4.28)

The horizon area then becomes

$$A_{C} = 2\pi^{\frac{D}{2}} \left( \frac{(D-1)(D-2)}{2\Lambda} \right)^{\frac{D-1}{2}} \quad \text{(D=4)} \quad = \frac{12\pi}{\Lambda},$$

(4.29)

where $\Gamma$ denotes the Euler Gamma function. The corresponding Bekenstein-Hawking entropy equals

$$S_{dS} = \frac{A_{C}}{4G_N},$$

(4.30)

as usual. From the metric one can also deduce the Hawking temperature of the black hole by demanding it to be regular across the horizon. This leads to

$$T_H = \frac{1}{4\pi} \sqrt{\frac{8\Lambda}{(D-1)(D-2)}} \quad \text{(D=4)} \quad = \frac{1}{\pi} \sqrt{\frac{\Lambda}{12}}.$$ 

(4.31)

The values for the 4-dimensional case agree with those given in [53].

It is natural to interpret the Bekenstein-Hawking entropy for de Sitter space as the logarithm of the number of quantum states necessary to describe such a universe [12]. This is similar to the interpretation of the Bekenstein-Hawking entropy for a black hole, as discussed in Section 2.1.1, and extends the analogy between the two cases. More precisely, the Hilbert space of quantum gravity in asymptotically de Sitter space has finite dimension $\mathcal{N}$, in terms of which the entropy is given by

$$S_{dS} = \ln \mathcal{N},$$

(4.32)

cf. (4.30). Since in the presence of gravity the metric fluctuates, we need to speak of asymptotically de Sitter space. Keep in mind, however, that the only asymptopia are in the past and future; there is no spatial infinity.

*Note the necessity of an observer dependent notion of particles.
It is the finiteness of the entropy (4.30), and hence of the dimensionality of the Hilbert space, that leads to conceptual problems. It appears to be irreconcilable, for example, with the fact that at early times de Sitter space has very large volume. One can impose boundary conditions that have entropy larger than (4.32), where $\mathcal{N}$ is fixed by the value of the cosmological constant. However, these initial conditions will not evolve into a spacetime that is again de Sitter in the future. Said differently, the Einstein equations ensure that the FSB bound, the generalized Fischler-Susskind bound discussed in Section 3.2.1, is not violated.

Another important implication of a finite dimensional Hilbert space is that the de Sitter symmetry group, $O(D,1)$, cannot act on it [142]. Indeed, $O(D,1)$ is non compact and does not have (nontrivial) finite-dimensional unitary representations. In fact, one should not expect the full de Sitter group to act on the Hilbert space. The symmetry generators can be expressed as surface terms at infinity and, as we have seen before, de Sitter space cannot support such conserved charges. Hence, the de Sitter generators are zero. Any quantum gravity theory on de Sitter space will thus not have the full de Sitter isometry group that is the symmetry group of classical de Sitter space. The elliptical modification of de Sitter space that is the subject of Chapter 5 suggests that the relevant group is actually the compact subgroup $SO(D-1)$ of the full isometry group. This allows for unitary representations related to the de Sitter entropy, as discussed in Section 5.5.4.

4.2.2 Bounds from the Cosmological Horizon

As noted above, the cosmological horizon in de Sitter space attains its maximal size for pure de Sitter. Thus, pure de Sitter constitutes a state of maximal entropy for universes with a positive cosmological constant that tend towards pure de Sitter in the future, as can be understood in the following way. The exponential expansion of space will cause anything present within a cosmological horizon to be swept out of it, eventually leaving pure de Sitter. From the second law of thermodynamics we infer that entropy must have increased in the process. The generic final stage of such evolution being pure de Sitter, this must constitute a state of maximal entropy. By Bousso’s [25] proof of the so called $N$-bound this is generalized to include any universe with positive cosmological constant, not necessarily asymptoting to pure de Sitter in the future; see also [8,89].

In Section 2.1.2, we derived entropy bounds from studying the dynamics of black hole horizons. Similarly, entropy bounds can be derived from the cosmological horizon in de Sitter space. Consider a matter system, including possibly black holes, with entropy $S_m$ in asymptotically de Sitter space. By above arguments, the system is surrounded by a cosmological horizon with
4.2 Energy and Entropy

where $A_C$ is the horizon area in pure de Sitter given by (4.29). The total entropy then equals

$$S_{\text{total}} = S_m + \frac{A_C'}{4}. \tag{4.34}$$

From the point of view of an observer, the entire matter system will at some point pass through the horizon, as the final state is pure de Sitter space. In this process, the matter entropy $S_m$ is lost, while the Bekenstein-Hawking entropy of the cosmological horizon increases by an amount

$$\Delta S_{\text{DS}} = \frac{1}{4} (A_C - A_C'). \tag{4.35}$$

The generalized second law (2.8), with black hole entropy generalized to cosmological horizons, implies that the total entropy must not decrease, i.e.

$$\Delta S_{\text{DS}} \geq S_m. \tag{4.36}$$

This gives a bound on the matter entropy,

$$S_m \leq \frac{1}{4} (A_C - A_C'), \tag{4.37}$$

called the D-bound [25] on matter systems in asymptotically de Sitter space. Its relation to the Bekenstein bound is considered in [26].

As an example, let us check the D-bound for a black hole in de Sitter, as described by the Schwarzschild-de Sitter solution (4.25). For concreteness, we take $D = 4$, so that the metric reduces to

$$ds^2 = -F(r) dt^2 + F^{-1}(r) dr^2 + r^2 d\Omega_2^2, \tag{4.38}$$

$$F(r) = 1 - \frac{2M}{r} - \frac{r^2}{R^2}, \tag{4.39}$$

and $M_{\text{max}} = 1/\sqrt{\Lambda}$. For $M < M_{\text{max}}$, there are two horizons, whose locations are given by the positive roots of $F(r)$ and depend on the mass. These are the cosmological horizon, $r_c$, and the black hole horizon, $r_{\text{BH}}$, where $r_c < r_{\text{BH}}$. For $M = 0$, corresponding to pure de Sitter, there is only a cosmological horizon, no black hole horizon. For $M = M_{\text{max}}$, the two horizons coincide. It is easy to see that in between these two values of the mass, the black hole radius increases monotonically, while the cosmological radius decreases monotonically. This confirms the relation (4.33) for the case of a black hole in de Sitter space. More precisely, the entropy of Schwarzschild-de Sitter is given by

$$S_{\text{DS}} = \pi (r_{\text{BH}}^2 + r_c^2) \tag{4.40}$$
and the D-bound, (4.37), states that

$$S_{\text{dS}} \leq \pi R^2.$$ (4.41)

Note, that the cosmological horizon for pure de Sitter equals the radius of curvature, \( r_c = R \). Solving the cubic equation \( F(r) = 0 \) for its positive roots, gives for small \( M \) [26]

$$S_{\text{dS}} = \pi R^2 \left( 1 - \frac{2M}{R} \right) + O(M^2).$$ (4.42)

This is a monotonically decreasing function of \( M \), as expected. For the maximum value of the mass, \( M = M_{\text{max}} \), one finds \( S_{\text{dS}} = \frac{2}{3} \pi R^2 \) showing that the D-bound, in the form (4.41), holds.

### 4.2.3 De Sitter Space in String Theory

A logical step towards deriving the entropy and thermodynamic properties of de Sitter space from a microscopic description, would be to embed the space as a solution of string theory. As mentioned in Section 2.1.1, this approach has been successfully employed in deriving a microscopic description of a certain class of black holes. So far, various attempts at finding a de Sitter solution of string theory have failed. One of the main conclusions of this chapter and the next, will be that, in searching for a string theory embedding, the global perspective on de Sitter space may not be the correct one to take. In Chapter 5, we will consider a partial view which may be more appropriate. We will now discuss several of the obstacles encountered.

First of all, there is the issue of the finiteness of the de Sitter entropy, (4.30). In four dimensions, the entropy equals \( S_{\text{dS}} = \frac{2}{3} \pi/\Lambda. \) The number of degrees of freedom necessary to describe such a universe is

$$\mathcal{N} = e^{\frac{2\pi}{\Lambda}},$$ (4.43)

which for the currently observed value of the cosmological constant is very large, but finite. Conversely, this means that a quantum theory of gravity with finite dimensional Hilbert space of dimension \( \mathcal{N} \), for consistency requires a cosmological constant \( \Lambda = \frac{3\pi}{\ln \mathcal{N}}. \) As argued by Banks [12], this means that the cosmological constant, \( \Lambda \), should not be viewed as a derived quantity, but instead as a fundamental input parameter. This leads to the proposal that universes with positive cosmological constant are described by a fundamental theory with a finite number of degrees of freedom. A consequence of the finiteness of \( \mathcal{N} \) is that string theory, having an infinite dimensional Hilbert space, does not seem appropriate to describe de Sitter space. Banks [12] also connects
the finiteness of $\mathcal{N}$ with the breaking of supersymmetry; an issue that we will discuss next.

It is easy to see that there cannot be unbroken supersymmetry in de Sitter space. This is simply because there exist no superalgebra extensions of the de Sitter symmetry algebra. In [99] a de Sitter superalgebra was constructed, but it results in a Lagrangian that leads to a gauge field with the wrong sign kinetic term. The absence of a superalgebra is related to the fact that there is no positive conserved energy in de Sitter space. If there were to be nonzero supercharges $Q$, which we can assume to be Hermitian, then these would anticommute to give the Hamiltonian [142],

$$\sum [Q, Q^*] = \mathcal{H}. \quad (4.44)$$

Lacking a Hamiltonian, this is clearly impossible. Instead, the supercharges of the de Sitter superalgebra constructed in [99] square to zero. Since superstring theory is naturally supersymmetric, the lack of supersymmetry in de Sitter space complicates the issue of finding a de Sitter string solution. Indeed, a 'no go' theorem was proven in [91], stating that de Sitter space cannot arise from a conventional compactification of string theory. Attempts to find de Sitter solutions in a non-standard way include [35, 74, 116].

Finally, consider the issue of observables [44, 69] and that of the construction of an S-matrix in de Sitter space. In quantum field theory, asymptotic incoming and outgoing states are properly defined only in the asymptotic regions of spacetime. But for de Sitter space these regions are spacelike and there is no single observer who can determine the states both at past infinity as well as at future infinity. Consequently, the matrix elements of S-like matrices in de Sitter space are not measurable quantities; they are mere meta-observables, rather than observables. When one considers quantum gravity in asymptotically de Sitter space, the situation becomes even more serious. As has been pointed out by Witten, the only available pairing between in-states and out-states, CPT, is used to obtain an inner product for the Hilbert space [142]. There then does not seem to be an additional pairing between in- and out-states that could be used to arrive at an S-matrix. As the conventional formulation of string theory is based on the existence of an S-matrix, the lack of an analogue in de Sitter space heightens the difficulty of finding a relevant string solution.

An approach analogous to that which was successfully employed in the AdS/CFT [7, 102] case is taken in [119]. Employing the only available asymptotic regions, the infinite past and future, standard S-matrix elements are constructed despite their unobservable character. These are then used to relate correlation functions on de Sitter space to those of a proposed dual CFT; this is the subject of the next section. After re-addressing the issue of defining an S-matrix for de Sitter space in Section 5.5.2, we come back to that of finding a string realization in Section 5.6.
4.3 THE dS/CFT CORRESPONDENCE

In line with the AdS/CFT correspondence, it is natural to look for a similar duality in the case of quantum gravity in de Sitter space. Indeed, one might expect such a relation to exist, since de Sitter space can be obtained from Anti-de Sitter by analytic continuation. However, unlike the AdS case, there is no known realization of quantum gravity in de Sitter space. Nonetheless, following the approach of Brown and Henneaux [29] in the AdS$_3$/CFT$_2$ case, Strominger [121] formulated a tentative holographic duality relating quantum gravity in de Sitter space to a Euclidean CFT on a sphere of one lower dimension. This approach requires no input from string theory.

The duality thus constructed for de Sitter space exhibits many similarities, but also many differences with the AdS/CFT correspondence [93]. An important difference is that the boundary of de Sitter space not only is spatial, it also consists of two disconnected parts. This complicates the question as to where the dual CFT would live. Strominger employs the fact that, at least for pure de Sitter, the two boundary parts are causally connected, to arrive at a single CFT on a single sphere.

In this section, we consider the proposed dS/CFT correspondence in three spacetime dimensions. Although the correspondence should apply in general dimension, the 3-dimensional case is especially rich because of the infinite dimensionality of the 2-dimensional conformal group. In a sense this makes the 3-dimensional case the most restrictive regarding the properties of the dual theory; allowing for the most definite statements. In this section we set the radius of curvature of de Sitter space equal to one, $R = 1$.

4.3.1 ASYMPOTIC SYMMETRIES

The asymptotic symmetry group of de Sitter constitutes those nontrivial\(^1\) diffeomorphisms that preserve the boundary conditions on the asymptotic metric, i.e., the metric at $\mathcal{I}^\pm$. So, we need to specify the appropriate boundary conditions for what we would call ‘asymptotically de Sitter space’. An elegant way to do this is by using the Brown-York stress tensor [30], defined as

$$T_{ij} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \gamma^{ij}}, \quad (4.45)$$

where $\gamma_{ij}$ is the induced metric on the boundary and $S_{cl}$ denotes the action evaluated at a classical solution. Though similar in definition to the usual stress-energy tensor, $T_{ij}$ characterizes the entire system, including contributions from

---

\(^1\)Nontrivial in the sense that the transformation does not annihilate physical states.
both the gravitational field and, if present, matter fields. The bulk diffeomorphisms are generated by appropriate moments of this tensor, which lives on the boundary of the spacetime. We can then take as definition of an asymptotically de Sitter spacetime one for which the Brown-York stress tensor, and hence all symmetry generators, is finite.

The Brown-York tensor, (4.45), evaluated for the boundary $\mathcal{I}^-$ of de Sitter space in general dimension, takes the form \[ T_{ij} = \frac{1}{4G_N} \left[ K_{ij} - (D-2)\gamma_{ij} - \frac{g_{ij}}{(D-3)} \right] , \] (4.46)

where $K_{ij}$ is the extrinsic curvature, as defined in (3.56), and $g_{ij}$ denotes the boundary Einstein tensor. For $D = 3$ this reduces to

\[ T_{ij} = \frac{1}{4G_N} [K_{ij} - (K + 1)\gamma_{ij}] , \] (4.47)

which is identically equal to zero in planar coordinates (4.24). However, calculating (4.47) for perturbed de Sitter, with metric

\[ ds^2 = g_{\mu\nu} + h_{\mu\nu} , \] (4.48)

gives [118]

\[ T_{zz} = \frac{1}{4G_N} \left( h_{zz} - \partial_t h_{tz} + \frac{1}{2} \partial_z h_{zz} \right) + \mathcal{O}(h^2) , \] (4.49)

\[ T_{\bar{z}\bar{z}} = \frac{1}{4G_N} \left[ \frac{1}{4} e^{-2t} h_{tt} - h_{\bar{z}\bar{z}} + \frac{1}{2} (\partial_{\bar{z}} h_{tz} + \partial_z h_{\bar{z}} - \partial_t h_{\bar{z}\bar{z}}) \right] + \mathcal{O}(h^2) . \] (4.50)

Imposing the condition that the Brown-York tensor remains finite, one obtains the boundary conditions

\[ g_{\bar{z}\bar{z}} = \frac{e^{-2t}}{2} + \mathcal{O}(1) , \] (4.51)

\[ g_{tt} = -1 + \mathcal{O}(e^{2t}) , \] (4.52)

\[ g_{zz} = \mathcal{O}(1) , \] (4.53)

\[ g_{tz} = \mathcal{O}(1) . \] (4.54)

These boundary conditions define what is meant by ‘asymptotic de Sitter’. The asymptotic symmetry group hence consists of those diffeomorphisms that leave (4.51)-(4.54) invariant. The most general form of a diffeomorphism that does so, is given by [118]

\[ \zeta = U \partial_t + \frac{1}{2} U' \partial_z + \mathcal{O}(e^{2t}) + \text{complex conjugate} , \] (4.55)
where $U(z)$ is holomorphic in $z$. We can identify what these transformations are and, at the same time, find the central charge of the boundary theory, by acting with the diffeomorphism (4.55) on the Brown-York tensor, 

$$\delta \zeta T_{zz} = - U \partial T_{zz} - 2U' T_{zz} - \frac{1}{8G_N} U''.$$  (4.56)

The first two terms are those one expects for a conformal field of weight two under a conformal transformation. The third term, identified as the anomalous Schwarzian derivative term, corresponds to a central charge (restoring R) \[121\]

$$c = \frac{3R}{2G_N}. \quad (4.57)$$

Here the subscript 2 refers to the dimensionality of the boundary theory, but note that both $G_N$ and $R$ apply to the 3-dimensional bulk. We thus identify (4.55) as a conformal transformation in two dimensions and, since $U(z)$ is general, the asymptotic symmetry group of dS$_3$ as the conformal group of the Euclidean plane. It is important to note that is not possible to construct a diffeomorphism analogous to (4.55) for the case of global coordinates. This is because evolution in the global time $T$ is not part of any of the de Sitter isometries. Hence, it is impossible to associate evolution in the global time coordinate with conformal transformations of the boundary. This strongly suggests that the global perspective is not the relevant one when considering holography – and thus quantum gravity – in de Sitter space.

### 4.3.2 The Correspondence

Having found that the symmetry group of the boundary $\mathcal{I}^-$ is the Euclidean conformal group, it is to be expected that gravity correlators restricted to the boundary transform as in a 2-dimensional Euclidean CFT. Indeed, this is what one finds for massive scalar fields with mass $m$. The solutions to the wave equation these fields obey, which near the boundary takes the form

$$m^2 \phi = \nabla^2 \phi \sim - \partial_t^2 \phi + 2 \partial_t \phi \quad (t \to -\infty),$$  (4.58)

asymptotically behave as

$$\phi \sim e^{h \pm t} \quad (t \to -\infty), \quad (4.59)$$

where

$$h_{\pm} = 1 \pm \sqrt{1 - m^2}. \quad (4.60)$$

For masses in the range

$$0 \leq m^2 \leq 1,$$  (4.61)
4.3 The dS/CFT correspondence

$h_\pm$ are real and positive, while $h_- \leq h_+$. Picking the minus sign in (4.59),\(^\dagger\)
the imposed boundary condition has the form

$$\lim_{t \to -\infty} \phi(z, \bar{z}, t) = e^{h_- t} \phi_-(z, \bar{z}).$$

(4.62)

Analogous to the AdS/CFT correspondence, the proposed dS/CFT correspondence associates to \(\phi_-\) a dual operator in the boundary CFT, \(O_{\phi}\), of dimension \(h_+\). The two point function of this dual operator has the form [121]

$$\langle O_{\phi}(z, \bar{z})O_{\phi}(v, \bar{v}) \rangle \propto \frac{1}{|z - v|^{2h_+}}.$$  (4.63)

Outside of the range (4.61), i.e. for \(m^2 > 1\), the conformal weights \(h_\pm\) are no longer real, signalling that the dual CFT is not unitary. Albeit there are no a priori reasons that the CFT needs to be unitary, this might mean that consistent theories of quantum gravity on de Sitter space have no (stable) scalars with masses greater than one.

So far, we have only considered the combined regions I and III of Figure 4.2 on page 66, with a single boundary \(I^-\). When taking into account the full space, one might expect two separate CFT’s, living on disconnected spaces, to be necessary to describe the complete bulk dynamics. However, the two boundaries are causally connected. As we showed in Section 4.1, a light ray that originates from a certain point on \(I^-\), will arrive at the antipodal point on the sphere at \(I^+\). A singularity of a correlator between a point on \(I^-\) and a point on \(I^+\) can only occur if the two points are null separated. Through the singularity structure of the Green’s function on de Sitter space, such a correlation is related to one between the point on \(I^-\) and the antipode of the point on \(I^+\), also on \(I^-\). The insertion of a gravity operator on \(I^+\) then effectively corresponds to inserting the dual CFT operator at the antipodal point on \(I^-\). We are thus left with a single CFT on a single sphere. The antipodal relation in de Sitter space will be the main theme of Chapter 5 and we will consider the singularity structure of the Green’s function in more detail in Section 5.2.1.

Everything in this section generalizes to higher dimensions. The proposed correspondence can then be summarized in terms of bulk and boundary correlation functions as

$$\langle \phi(x^-_1) \cdots \phi(x^-_{i-1}) \phi(y^+_i) \cdots \phi(y^+_{j}) \rangle_{dS_0} \leftrightarrow \langle O_{\phi}(x_1) \cdots O_{\phi}(x_i) O_{\phi}(y_1) \cdots O_{\phi}(y_j) \rangle_{\mathcal{S}^{D-1}}.$$  (4.64)

Here, \(x^-_i\) and \(y^+_i\) are points on \(I^-\) and \(I^+\), respectively. On the right hand side, all points are on the sphere on which the dual CFT lives, denoted by

\(^\dagger\)We only consider the leading behaviour, ignoring the subleading term proportional to \(e^{h_+ t}\).
4. Holography in de Sitter Space

$S^{D-1}$, and the bar denotes the antipode on that sphere. Note that the fields $\phi$ with argument on $I^-\beta$ in (4.64) obey the boundary condition (4.62), while those with argument on $I^+\beta$ obey the corresponding boundary condition appropriate in the $t \to \infty$ limit.

4.4 RG FLOW

Recall how the conformal transformations of the boundary arise from bulk diffeomorphisms. Looking at the diffeomorphism $\zeta$, (4.55), we see that the first term generates a holomorphic diffeomorphism of the plane. In terms of the metric in planar coordinates, (4.24), this can be compensated for by a shift in $t$, as generated by the second term of (4.55). On the plane, this shift corresponds to a Weyl transformation. Together these transformations constitute the conformal transformations on the plane and, hence, on the boundary. A special class of these are the simple time shifts corresponding to scalings of the boundary. This is reminiscent of the AdS case where radial shifts correspond to scalings in the boundary metric, as is clear from the metric (2.49). In that case, renormalization group (RG) flow corresponds to radial flow of the bulk solutions [6,11,39]. For de Sitter, one might then expect RG flow of the putative dual theory to correspond – perhaps more naturally – to time evolution in the bulk [8,87,122].

4.4.1 INFLATION

An interesting RG scenario in de Sitter space was recently suggested by Strominger [122]. As opposed to Section 4.3, where we considered three spacetime dimensions because it was the most explicit case, we now focus on four dimensions since the current model is aimed at describing the real world. RG flows similar to the one we will discuss here were considered in [8].

As we have mentioned, recent astronomical observations suggest that the universe is currently entering a de Sitter phase. It is a much older idea that there has also been a temporary de Sitter phase in the early universe, referred to as ‘inflation’ [4,63,88]. Such a period of rapid expansion solves important cosmological issues like the flatness and horizon problems; see [84] for a review of early universe physics. Following Strominger [122], we assume that the metric is well-approximated by the flat Robertson-Walker big bang form

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i,$$  \hspace{1cm} (4.65)

where $a(t)$ is the scale factor of the universe. In the early and late universe, the
scale factor behaves as
\[
\frac{\dot{a}}{a} \to H_\pm \quad (t \to \pm \infty),
\quad (4.66)
\]
so that the metric takes the 4-dimensional planar de Sitter form
\[
ds^2 = -dt^2 + e^{2H_\pm t} dx_i dx^i \quad (t \to \pm \infty).
\quad (4.67)
\]
Here, $H_\pm$ denotes the Hubble constant during the de Sitter phase either in the early or late universe. Comparing to (4.24), note that
\[
H_\pm = \frac{1}{R}
\quad (4.68)
\]
in terms of the curvature radius of the appropriate de Sitter space. During the inflationary period, the cosmological constant is conjectured to have been about one hundred orders of magnitude larger than that currently observed. In terms of the Hubble constant, we have $H_- \approx 10^{24}$ cm$^{-1}$ versus $H_+ \approx 10^{-28}$ cm$^{-1}$.

As noted at the beginning of this section, the de Sitter metric (4.67) is invariant under a time translation accompanied by an appropriate scaling of the plane; namely under the transformation
\[
t \to t + \lambda, \quad x^i \to e^{-\lambda H_\pm} x^i.
\quad (4.69)
\]
This expresses that time evolution in the bulk generates scale transformations in the boundary theory. Moreover, we conclude from the behaviour (4.69) that early times in the bulk correspond to the IR regime of the boundary theory, while late times correspond to the UV regime. This is the de Sitter variant of the UV/IR correspondence discussed in Section 2.2.7. Note that a similar relation is not to be expected when taking the global perspective on de Sitter space. In that case, there is no monotonic relation between time and evolution from UV to IR in the bulk. Indeed, both early and late times would correspond to IR, while intermediate times would correspond to UV. This gives yet another indication that the global perspective is not the correct one when considering quantum gravity on de Sitter space.

In between the two de Sitter phases, the universe as described by (4.65) will have looked very different from (4.67) and likely will not have exhibited invariance under (4.69). There is thus no reason to expect the bulk gravity theory to be dual to a conformal theory on the boundary during that stage. An elegant interpretation is suggested by a similar situation encountered in AdS [48,59]. Radial translations in the bulk of AdS correspond to RG flow in the boundary theory. Viewing the universe as a perturbed de Sitter space,
the conformal invariance of the boundary theory is broken by insertion of the operators dual to the bulk perturbations. This way, the boundary theory is perturbed away from its UV conformal fixed point and ends up, by the RG flow, in an IR fixed point. Note that in this interpretation the RG flow runs opposite to the bulk time, since the universe actually starts out in the IR.

The above model has been applied in [86] to cosmic microwave background anisotropies. A simple dictionary between the holographic approach of the model just discussed and the standard inflationary model is constructed. Despite the very different inputs the two models give similar predictions.

4.4.2 \( c \)-FUNCTION

Let us consider the boundary theory in the RG flow model discussed above in a bit more detail. On dimensional grounds, the generalization of the central charge (4.57) to higher dimensions takes the form

\[
c_{D-1} \propto \frac{1}{G_N H^{D-2}} = \frac{1}{G_N H^2},
\]

where we have used (4.68). Of course, the expression (4.70) only makes sense for the de Sitter phases of the evolution. From the relative sizes of the Hubble constants \( H_\pm \) we have that

\[
c_+^3 \gg c_-^3.
\]

Since the central charge is a measure of the number of degrees of freedom of the theory, this indicates that there are overwhelmingly more degrees of freedom in the late universe than in the early universe. Such an increase of the number of degrees of freedom seems to be at odds with unitary evolution.¶ In the current context, this can be understood in the following way, again inspired by the AdS analogy [48,59].

For a generic unitary (non-conformal) field theory in two dimensions, Zamolodchikov [146] has proven that there exists a function \( c_1 \) that decreases from the UV to the IR along RG flows; this is called the ‘\( c \)-theorem’. So far, attempts have failed to generalize this theorem to higher dimensions. Moreover, while the dual boundary theory may not be unitary for de Sitter, unitarity was crucial in the proof given by Zamolodchikov. Nevertheless, by using the bulk theory, Strominger proves that \( c_3 \), as defined by (4.70), is an increasing function of time and hence a decreasing function along the RG flow. Indeed, through Einstein’s equations, the time derivative of Hubble’s constant in asymptotically

¶Note that it is only the dual boundary theory that is possibly non-unitary; the bulk theory is unitary. It is important to realize that this is true even though the bulk theory can be reconstructed – at least in principle – from the boundary theory.
de Sitter space in four dimensions can be written as

\[ \dot{H} = -4\pi G_N (p + \rho). \]  

(4.72)

Imposing the null energy condition, which requires \((p + \rho) \geq 0\) while \(\rho > 0\), ensures that the right hand side is non-positive. Hence, \(H\) decreases with time and \(c_3\) increases with time. This is an example of how the bulk/boundary duality can be put to use. We will come back to this below.

With time evolution corresponding to inverse RG flow, this suggests the following interpretation [122]. As the universe expands, more and more degrees of freedom become available; they are ‘integrated in’ in the language of RG flows. Only in the asymptotic future does the universe come into full existence.

The \(c\)-function as discussed in [8,122] and reviewed here only applies to spatially flat slicings of de Sitter space. It has been generalized in [87] to include spherical and hyperbolic slicings. Using (4.2), (4.70) can be re-written in terms of the cosmological constant. In [87], this is subsequently generalized by replacing \(\Lambda\) by an effective \(\Lambda_e\), appropriate for the different slicings, defined by

\[ \Lambda_e \equiv G_{\mu\nu} n^\mu n^\nu. \]  

(4.73)

Here, \(G_{\mu\nu}\) denotes the Einstein tensor and \(n^\mu\) is the unit normal to the respective slices. Evaluated for flat \((k = 0)\), spherical \((k = 1)\) and hyperbolic \((k = -1)\) slicings, this becomes [87]

\[ \Lambda_e = \frac{(D - 1)(D - 2)}{2} \left[ \frac{a'}{a} \right]^2 + \frac{k}{a^2}. \]  

(4.74)

The generalized \(c\)-function has the form

\[ c_{D+1} \propto \frac{1}{G_N \Lambda_e^{D/2}}. \]  

(4.75)

and a \(c\)-theorem is subsequently proven in [87].

### 4.4.3 APPARENT HORIZON

A nice geometric interpretation of the \(c\)-functions defined above is given in [85]. The boundary \(c\)-function is identified in the bulk as the area of the apparent horizon, which is a naturally increasing function of time. First, we need to define the apparent horizon. At the end of Section 3.2.1, we briefly mentioned Bousso’s [23,24] generalization of the Fischler-Susskind bound to general spacetimes. It associates to a general surface at least two light sheets.
Consider, e.g., a $D-2$-dimensional spherical surface surrounding an observer in de Sitter space. There are four families of radial null rays orthogonal to this surface; corresponding to future/past directed outgoing/ingoing. Let $\lambda$ be an appropriately chosen affine parameter on one of the four families of null rays. A family is then referred to as a light sheet of the corresponding spherical surface if it has non-positive expansion, $\theta(\lambda) \leq 0$. Recall the definition of $\theta$ as the expansion of the cross sectional area, $A$, of neighbouring light rays,

$$\theta = \frac{1}{A} \frac{dA}{d\lambda}.$$  

(2.34)

There are at least two such light sheets, which extend until they reach a caustic where the expansion becomes positive. Bousso then identifies three situations. A surface which has both a future directed and a past directed ingoing (outgoing) lightsheet is called normal. Similarly, a surface with two future (past) directed lightsheets, but no past (future) directed ones, is called (anti-)trapped. Spherical surfaces outside the cosmological horizon of an observer in de Sitter space are anti-trapped.

We are now ready to define the apparent horizon as the boundary between normal and (anti-)trapped regions. It is characterized by the fact that at least two light sheets have vanishing expansion, $\theta = 0$, there. For pure de Sitter space, the apparent horizon coincides with the cosmological horizon, but this is no longer the case for general asymptotically de Sitter spaces. However, in a de Sitter phase, the apparent horizon will tend towards the cosmological horizon.

For the class of models with positive $\Lambda$ considered in [85], including big bang models that tend to de Sitter in the future, the area of the apparent horizon is

$$A_{ah}(t) = \left( \frac{2\Lambda}{(D-1)(D-2)} \right)^{D-2} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right]^{-\frac{D-2}{2}} A_C. \quad (4.76)$$

The time independent area of the cosmological horizon, $A_C$, is given in (4.29). Note that $A_{ah} \leq A_C$ for physical values of the scale factor $a$ and, applying the limit (4.66), we see that

$$A_{ah} \sim A_C \quad (t \to \infty), \quad (4.77)$$

as expected for such models.

Based on the observation that the central charge (4.70) is proportional to the area of the cosmological horizon (4.29), the authors of [85] propose the area of the apparent horizon (4.76) in Planck units as $c$-function,

$$c_{D;} \propto \frac{A_{ah}}{G_N}. \quad (4.78)$$
This function can be evaluated from local data on any constant time slice. As a first check, we have already shown that it tends to the cosmological horizon in a de Sitter phase. Actually, as observed in [85], the function (4.78) defines the same $c$-function as (4.75). Indeed, substituting (4.74) into (4.75) and (4.76) into (4.78) one easily sees that both are of the form

$$c_{D,1} \propto \frac{1}{G_N \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right]^{D-2}}.$$  \hspace{1cm} (4.79)

This provides a geometric interpretation of the boundary theory $c$-function in terms of the increase in area of the apparent horizon in a future de Sitter universe.