At first sight, quantum field theories and gravity seem to have little to do with each other. Indeed, it has proven impossible to quantize gravity following the usual perturbative techniques of field theory. On the other hand, string theory naturally provides a quantum theory of gravity. It turns out that these different kinds of theories are intimately related.

In the nineteen seventies, 't Hooft [128] showed that in the limit of large gauge group the diagrammatic expansion of gauge theories looks like that of a free string theory. Although a description in terms of a worldsheet action was not found, this provides a direct connection between gauge theories and string theories. More recently, it was realized that the strings arising from certain conformal gauge theories are exactly the type IIB strings, moving on a curved background which has a boundary at spatial infinity. In a different approach, it was conjectured by 't Hooft [131] and Susskind [125] that any theory of quantum gravity should be dual to a quantum field theory living on the boundary of spacetime. This is called the holographic principle.

A major step forward came in 1995 with the discovery of D-branes by Polchinski [100]. Dp-branes are $p+1$-dimensional hypersurfaces on which open strings can end. The low energy theory of open strings ending on a Dp-brane is U(1) gauge theory in $p+1$ dimensions. If one puts $N$ Dp-branes on top of each other, this generalizes to U(N) gauge theory. On the other hand, by worldsheet duality, the Dp-brane also acts as a source for closed strings, which contain gravitons in their massless spectrum. This dual nature of the D-branes is depicted in Figure 2.6 on page 22.

All of this finally led Maldacena [90] in 1997 to conjecture the celebrated AdS/CFT correspondence, providing an explicit example of a gravity theory on a curved background and its dual field theory. These are type IIB string theory on Anti-de Sitter (AdS) space times a sphere and $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory. This last theory is a conformal field theory (CFT) which lives on the boundary of the AdS space.
We begin in Section 2.1 by discussing the holographic principle and the entropy bounds that can be inferred from it. In Section 2.2 we consider an explicit realization of the holographic principle: the AdS/CFT correspondence. This correspondence is then used to derive a holographic entropy formula for the dual CFT.

2.1 THE HOLOGRAPHIC PRINCIPLE

In a quantum theory of gravity there exists a natural upper limit on the amount of energy that a region of space can contain. Consider, for simplicity, a spherical region. The maximal energy content is then given by the mass of a black hole that fills the region. The important point is that the mass of a black hole in four dimensions is proportional to its horizon radius,

$$M_{BH} = \frac{r_H}{2},$$

and not to its volume. Since the ratio volume over radius, $V/r \propto r^2$, grows rapidly with $r$, the bound becomes more stringent as one considers larger volumes. This bound on the mass translates to a bound on the number of degrees of freedom that a region can support. Based on this, 't Hooft [131] and Susskind [125] proposed the holographic principle, which asserts that the number of accessible degrees of freedom in a specified region of space is proportional to the area of its boundary measured in Plank units. This is a radical step away from local field theory, which has degrees of freedom at every scale. Indeed, even when a finite number of degrees of freedom per unit volume is obtained by imposing infra-red (IR) and ultra-violet (UV) cutoffs, the total number is proportional to the volume. This discrepancy lies at the heart of the problems in unifying quantum field theory and gravity.

From the holographic principle, bounds on the entropy in a specified region can be derived. We will discuss these 'holographic entropy bounds' in the remainder of this section. For an extensive review of the holographic principle and the resulting entropy bounds see [21,27].

2.1.1 BLACK HOLE THERMODYNAMICS

Consider the Schwarzschild black hole solution in $D = n + 1$ dimensions. The metric takes the form

$$ds^2 = -\left(1 - \frac{\omega_n M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{\omega_n M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2.$$  \hspace{1cm} (2.2)
where
\[ \omega_n \equiv \frac{16\pi G_N}{(n-1)\text{Vol}(S^{n-1})} , \] (2.3)

\( G_N \) is the \( D \)-dimensional Newton constant, \( M \) denotes the mass of the black hole and \( d\Omega_{n-1} \) is shorthand for the metric on the \( n-1 \)-dimensional unit sphere. Focusing on the case \( D = 4 \) and setting \( G_N = 1 \), the metric reduces to
\[ ds^2 = -V(r)\, dt^2 + V(r)^{-1}\, dr^2 + r^2 \, d\Omega_2^2 , \] (2.4)

where
\[ V(r) = 1 - \frac{2M}{r} . \] (2.5)

The black hole horizon is at \( r_H = 2M \) and the area of the horizon is
\[ A_H = 16\pi M^2 . \] (2.6)

Hawking’s area theorem [64] states that the area of a black hole event horizon never decreases with time: \( dA_H \geq 0 \). For example, if something drops into a black hole this increases its mass and consequently its horizon area increases as well. The area theorem bears resemblance to the second law of thermodynamics, which states that entropy never decreases with time: \( dS \geq 0 \). However superficial the similarity between the two laws may seem, it turns out to be of a fundamental nature.

There is also the ‘no hair theorem’ [32,76], which states that a black hole is completely characterized by three quantities: its mass, charge and angular momentum. Since this allows for only a single quantum state, it implies that black holes have zero entropy. Throwing an entropy carrying thermodynamical system into a black hole would then cause entropy to be lost; in violation of the second law of thermodynamics.

As a resolution, Bekenstein [15–17] suggested to associate an entropy to a black hole proportional to its horizon area,
\[ S_{\text{BH}} = \eta A , \] (2.7)

where \( \eta \) is the constant of proportionality. Bekenstein then generalized the second law of thermodynamics to include black hole entropy,
\[ dS_{\text{matter}} + dS_{\text{BH}} \geq 0 . \] (2.8)

Any loss of matter entropy from objects falling into a black hole is compensated for by an increase of the black hole entropy. If the Bekenstein entropy of a black hole is to be interpreted as a thermodynamical entropy, the first law of thermodynamics,
\[ dM = TdS , \] (2.9)
implies that black holes must have a temperature. Classically this is not possible, since a black hole cannot radiate by definition. Hawking [65] showed that black holes do in fact radiate through a quantum process and found that they emit a black body spectrum at a temperature which is a function of the black hole mass. This result establishes the analogy between the laws of thermodynamics and those of black hole dynamics as a true physical principle. It gives an indication that a statistical system perhaps underlies the theory of gravitation.

To calculate the Hawking temperature for a 4-dimensional Schwarzschild black hole, consider the Euclidean version of the metric (2.4),

\[ ds^2_E = V(r) \, d\tau^2 + V(r)^{-1} \, dr^2 + r^2 \, d\Omega_2^2. \] (2.10)

\[ V(r) = 1 - \frac{2M}{r}. \] (2.11)

Demanding that the metric be non singular at the horizon gives a periodicity condition for the Euclidean time coordinate \( \tau \). The Hawking temperature is then given by the inverse period. In terms of a coordinate \( r_* \) given by

\[ r_* = r + r_n \ln(r - r_n), \] (2.12)

the metric becomes

\[ ds^2 = V(r)(d\tau^2 + dr_*^2) + r^2 \, d\Omega_2^2. \] (2.13)

Introduce coordinates \( u_{\pm} \) defined by

\[ u_{\pm} = e^{\frac{2\pi}{\beta}(r_* \pm i \tau)}, \] (2.14)

where \( \beta \) is the periodicity of \( \tau \). In terms of these coordinates, the metric takes the form

\[ ds^2 = \frac{\beta^2}{4\pi^2} V(r) e^{-\frac{4\pi r_*}{\beta}} \, du_+ du_- \ . \] (2.15)

Expanding \( V(r) \) around \( r = r_n \),

\[ V(r) \approx (r - r_n) \cdot V'(r_n) \quad (r \approx r_n), \] (2.16)

we see that the metric is regular around \( r = r_n \) if

\[ e^{\frac{4\pi r_*}{\beta}} \propto (r - r_n), \] (2.17)

or, equivalently, if

\[ r_* \propto \frac{\beta}{4\pi} \ln(r - r_n). \] (2.18)
Comparing with (2.12), we see that this requires
\[ \beta = 4\pi r_H = 8\pi M. \] (2.19)

We conclude that the Hawking temperature equals
\[ T_H = \frac{1}{8\pi M}. \] (2.20)

By the first law of thermodynamics, this fixes the proportionality constant \( \eta \) in the Bekenstein entropy (2.7). The so called Bekenstein-Hawking entropy of a Schwarzschild black hole in four dimensions is given in Planck units by
\[ S_{BH} = \frac{A}{4}. \] (2.21)

Reinstating all dimensionfull parameters, this formula takes the form
\[ S_{BH} = \frac{k_B A_H c^3}{4G_N \hbar}, \] (2.22)

where \( k_B \) is the Boltzmann constant, \( c \) denotes the speed of light and \( \hbar \) is the Planck constant.

The laws of black hole thermodynamics are by now well established and it is expected that a quantum theory of gravity will associate \( N = e^{A/4} \) microstates with a black hole, as suggested by (2.21). In fact, for a special class of black holes, string theory has succeeded in doing just that [38,92,123]. We will see in Chapter 4 that the Bekenstein-Hawking formula (2.21) applies just as well to the cosmological horizon that is present in the context of de Sitter space. This is exemplary of the universality of this formula, as advocated in the Introduction.

### 2.1.2 Entropy Bounds from Black Holes

For a system that includes gravity, an entropy bound can be deduced from the Bekenstein-Hawking formula (2.21) together with the generalized second law (2.8). Consider a spherical region of space \( V \). The area of the boundary of \( V \) equals \( A \). Start with a thermodynamical system \( \mathcal{Q} \), with entropy \( S \), that is completely contained within \( V \). The total energy of the system \( \mathcal{Q} \) cannot exceed that of a black hole of area \( A \), since it would then not fit within \( V \).

By collapsing a spherical shell of matter with precisely the right energy onto the system \( \mathcal{Q} \), this system can be converted into a black hole that fills the volume \( V \). This process of converting the system into a black hole is called the Susskind process [125]. The entropy of the resulting system is simply that
2. Holography

of a black hole with horizon area $A$, as given by (2.21). From the generalized second law it follows that the entropy of the original system is bounded by this entropy,

$$ S \leq S_{\text{BH}}. $$

(2.23)

This is called the spherical entropy bound, and it is an example of a holographic entropy bound. The most stringent form of this bound is obtained by choosing $V$ to be the smallest spherical volume that contains the system $Q$.

Because of the limited validity of the spherical entropy bound, e.g., it only applies to spherical regions, one would like to try and generalize it. The most natural extension is simply to drop the assumptions under which it is derived. The resulting entropy bound, called the spacelike entropy bound, states that the entropy within any spatial region cannot exceed the area of that region’s boundary,

$$ S(V) \leq \frac{A(V)}{4}. $$

(2.24)

Besides the fact that many counterexamples to this bound have been found, a crucial difficulty with any entropy bound on a spacelike volume is that the concept of a region and its boundary is not covariant. This makes it impossible to say exactly what the region is on which the entropy is bounded by a given boundary surface. Since a natural covariant notion is that of a lightlike volume, one might try to formulate holographic entropy bounds on such volumes [125]. This leads to covariant ‘lightlike entropy bounds’. Before discussing lightlike bounds, in the next section we first discuss an interesting relation between the Einstein equation and the first law of thermodynamics. The derivation of this relation by Jacobson [77] involves an early formulation of the idea to implement the holographic principle via entropy flow through light-sheets.

2.1.3 THERMODYNAMICS AND THE EINSTEIN EQUATION

The laws of black hole mechanics can be derived from the classical Einstein equation [14]. The discovery of Hawking radiation established the link between these laws and those of thermodynamics. How then did classical general relativity know that horizon area is a sort of entropy?

Jacobson [77] answers this question by deriving the Einstein equation from the proportionality of entropy and horizon area together with the fundamental relation

$$ \delta Q = T dS $$

(2.25)

connecting heat, entropy and temperature. To illustrate the idea, consider a thermodynamical system. Assume that the entropy $S(E, V)$ of the system is given as a function of energy and volume. The first law of thermodynamics,
2.1 The holographic principle

Together with the relation (2.25), yields

\[ \delta Q = dE + p \, dV , \]

(2.26)

where \( p \) is the pressure within the system. Differentiating the entropy gives

\[ dS = \frac{\partial S}{\partial E} \, dE + \frac{\partial S}{\partial V} \, dV . \]

(2.27)

From combining these relations, we infer the equation of state

\[ p = T \frac{\partial S}{\partial V} . \]

(2.28)

The approach of [77] is to start from the holographic entropy relation and then derive an equation of state for spacetime along these lines. This equation takes the form of the Einstein equation.

In order to apply the relation (2.25) to spacetime dynamics, we need appropriate definitions of the appearing quantities. Heat will be defined as energy that flows across a causal horizon. A causal horizon is not necessarily a black hole horizon, it can be simply the boundary of the past of any set. The entropy of the system hidden by the horizon is assumed to be proportional to that horizon’s area. The final quantity that has to be identified is the temperature of the system into which the heat is flowing. Jacobson defines this to be the Unruh temperature [134] that is associated with an uniformly accelerated observer hovering just inside the horizon.

For equilibrium thermodynamics to be applicable, construct a system that is instantaneously stationary in the following way. Through any spacetime point \( p \), there exists a spacelike, 2-dimensional surface element \( \mathcal{P} \) whose past directed null normal congruence to one side has vanishing expansion and shear in a first order neighbourhood of \( p \). Call the past horizon of such a \( \mathcal{P} \) the ‘local Rindler horizon of \( \mathcal{P} \)’. The part of spacetime beyond the Rindler horizon is in local equilibrium at \( p \).

Following [77], we now demonstrate that from the relation (2.25), interpreted in terms of energy flux and area of local Rindler horizons, it follows that gravitational lensing by matter energy affects the causal structure of spacetime in just the right way so that the Einstein equation holds.

We need to make the definitions of heat and temperature more precise. In a neighbourhood of \( \mathcal{P} \) spacetime is approximately flat and exhibits the usual Poincaré symmetries. In particular, there exists an approximate Killing field \( \chi^a \) generating boosts orthogonal to \( \mathcal{P} \) and vanishing at \( \mathcal{P} \) itself. Like in the familiar Rindler case, the vacuum state is a thermal state with respect to the boost Hamiltonian at temperature

\[ T = \frac{\hbar \kappa}{2\pi} . \]

(2.29)
where \( \kappa \) is the acceleration of the Killing orbit on which \( \chi^a \) has unit norm. Assuming that all the energy passing through the horizon is carried by matter, the local heat flow is defined by \( T_{ab} \chi^a \), where \( T_{ab} \) is the matter energy-momentum tensor.

Referring to Figure 2.1, consider a local Rindler horizon \( \mathcal{H} \) through a spacetime point \( p \). The horizon is generated by the approximate local boost Killing field \( \chi^a \). The heat flux through the horizon is given by [77]

\[
\delta Q = \int_{\mathcal{H}} T_{ab} \chi^a d\Sigma^b.
\]  

(2.30)

We can write \( \chi^a = -\kappa \lambda k^a \), where \( k^a \) is the tangent vector to the horizon generators for an affine parameter \( \lambda \) that vanishes at \( \mathcal{P} \) and is negative to the past of \( \mathcal{P} \). Then also \( d\Sigma^a = k^a d\lambda dA \), where \( dA \) is the area element on a cross section of the horizon. Inserting these relations into (2.30) puts it in the form

\[
\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b d\lambda dA.
\]  

(2.31)

As mentioned above, it is assumed that the entropy is proportional to the horizon area. The entropy variation associated with a piece of the horizon is then proportional to the variation \( \delta A \) of the cross sectional area of neighbouring horizon generators,

\[
dS = \eta \delta A,
\]  

(2.32)

where \( \eta \) is the constant of proportionality and the area variation is given by

\[
\delta A = \int_{\mathcal{H}} \theta d\lambda dA.
\]  

(2.33)
Here, $\theta$ denotes the expansion of the horizon generators defined by

$$\theta \equiv \frac{1}{A} \frac{dA}{d\lambda}.$$  

(2.34)

The relation $\delta Q = T dS$ relates the energy flux to a change in the horizon area, i.e., to a focusing of the horizon generators. By definition, at $\mathcal{P}$ the local Rindler horizon has vanishing expansion. It follows that the focusing to the past of $\mathcal{P}$ must cause the expansion to vanish there. Moreover, the rate of focusing must be so that the increase of a portion of the horizon will be proportional to the energy flux across it. This translates to a condition on the curvature of spacetime in the following way.

The Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - \mathcal{R}_{ab} k^a k^b$$  

(2.35)

relates the change in expansion along the generators parametrized by $\lambda$ to the shear $\sigma_{ab}$, the twist $\omega_{ab}$ and, through the Ricci tensor $\mathcal{R}_{ab}$, the curvature of spacetime. By stationarity of the horizon, $\theta$, $\sigma$ and $\omega$ vanish at $\mathcal{P}$. When integrating (2.35) to find $\theta$ near $\mathcal{P}$, the first three terms on the right hand side can thus be neglected as they are are higher order contributions. For sufficiently small $\lambda$, this integration yields $\theta = -\lambda \mathcal{R}_{ab} k^a k^b$ and by substituting this into (2.33) one obtains

$$\delta A = -\int_{\mathcal{P}} \lambda \mathcal{R}_{ab} k^a k^b d\lambda dA.$$  

(2.36)

From (2.31) and (2.36) we see that

$$\delta Q = T dS = \frac{\hbar \kappa}{2\pi} \delta A$$  

(2.37)

can only hold if

$$T_{ab} k^a k^b = \frac{\hbar \eta}{2\pi} \mathcal{R}_{ab} k^a k^b$$  

(2.38)

for all null vectors $k^a$. This condition implies that

$$\frac{2\pi}{\hbar \eta} T_{ab} = \mathcal{R}_{ab} + fg_{ab}$$  

(2.39)

for some function $f$. The stress-energy tensor $T_{ab}$ is divergence free by local conservation of energy and momentum. It follows by the contracted Bianchi identity that $f = -\mathcal{R}/2 + \Lambda$ for some constant $\Lambda$. This leads to the conclusion that the Einstein equations,

$$\mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} g_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar \eta} T_{ab},$$  

(2.40)

hold automatically [77].
2.1.4 Covariant Entropy Bounds

The most important aspect in which covariant entropy bounds differ from the spherical and spacelike bounds discussed before is that they bound the entropy not in a spatial volume but on a null hypersurface; a so called light-sheet. This formulation via light-sheets is what provides the covariance of the prescription. The most general formulation of a covariant entropy bound is given by Bousso [23,24]. Bousso provides a detailed discussion of the notion of a light-sheet, and how they can be constructed.

To illustrate the idea, we will discuss an early example of a covariant entropy bound, as constructed by Susskind [125]. Consider a 4-dimensional, asymptotically flat spacetime. Asymptotically, we can define Minkowski light cone coordinates $X^+, X^-, x^i$ ($i = 1, 2$), where $X^+$ is the light cone time coordinate. Define a light-sheet to be the set of light rays which, in the limit $X^- \to \infty$, have equal $X^+$. These light rays fill a 3-dimensional lightlike volume and are asymptotically parallel. The complete set of light-sheets, for all $X^+$ values, fills the entire spacetime (except for points inside black hole horizons).

Assign to a spacetime point $p$ ‘holographic coordinates’ $X^+, x^i$, according to the asymptotic coordinate values of the light ray that passes through $p$. In this way, all the points along a light ray are assigned the same holographic coordinates. The value of $X^-$ is thus projected out and for every value of the time coordinate $X^+$, the $x^i$ parameterize a 2-dimensional surface called a holographic screen. In this way the 3+1-dimensional theory is mapped onto 2+1-dimensional screens, as depicted in Figure 2.2.

This mapping defines an entropy density $\sigma(x^i)$ on the screen. The entropy of systems that are swept out by a light-sheet is mapped to part of the screen.
2.1 The holographic principle

In the following we will show that no distribution of energy will ever lead to an entropy density on the screen that exceeds the bound [125]

$$\sigma(x^i) \leq \frac{1}{4}. \quad (2.41)$$

To start with, consider a black hole. The entropy of a black hole is given in terms of its horizon area by (2.21). We can thus assign an entropy density of $\frac{1}{4}$ to the black hole horizon. By the mapping defined above, the horizon is projected onto a certain area on the holographic screen, as shown in Figure 2.3. To show that the proposed bound (2.41) holds in this case, we must proof that the horizon area is smaller than its image area on the screen. This can be done by applying the Raychaudhuri equation (2.35) in the following way. For a null vector $k^a$, we have from the Einstein equations that

$$R_{ab} k^a k^b = 8\pi T_{ab} k^a k^b \quad (g_{ab} k^a k^b = 0). \quad (2.42)$$

The Raychaudhuri equation (2.35) for the expansion $\theta$ of the cross sectional area of neighbouring light rays can then be written as

$$\frac{d\theta}{d\lambda} = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - 8\pi T_{ab} k^a k^b, \quad (2.43)$$

where $\lambda$ is an affine parameter along the light rays. For a surface-orthogonal family of light rays, such as a light-sheet, the twist $\omega$ vanishes. Moreover, by the null energy condition, the final term is non positive [67]. We thus see that the right hand side of (2.43) is manifestly non positive. It follows that the expansion $\theta$ never increases along the light rays that constitute a light-sheet,

$$\frac{d\theta}{d\lambda} \leq 0. \quad (2.44)$$
The physical content of this focusing theorem is to say that light is always focused by matter, never diverged.

Since asymptotically, in the flat region, the light rays are parallel, the expansion approaches zero there,

$$\theta \sim 0 \quad (\lambda \to \infty).$$

(2.45)

Hence, $\theta$ must be positive along the light rays that map the horizon onto the screen. This implies that the area on the holographic screen to which the horizon is mapped, is larger than the horizon area. The bound (2.41) on the entropy density follows.

One can try to increase the entropy density on the screen by adding matter to the light-sheet between the black hole and the screen. For example, we can add another black hole. However, as depicted in Figure 2.4, the bending of light by matter, together with the focusing theorem (2.44), ensures that the entropy density on the screen does not exceed the bound. These considerations lead to the conjecture [125] that for any entropy carrying system, when mapped to the screen, the entropy density obeys the bound (2.41). This conjecture was generalized by Fischler and Susskind [45] to include more general spacetimes. We will consider this generalization in detail when we discuss cosmological entropy bounds in Section 3.2.

2.2 AdS/CFT

If the holographic principle turns out to be of a fundamental nature, one would expect it to be manifest in a quantum theory of gravity. It is not yet clear...
whether this is true for string theory, but backgrounds that provide explicit realizations of holography have recently been found. In this section we consider the most prominent example to date: the AdS/CFT correspondence. The conjecture is that type IIB string theory on backgrounds of the form $\text{AdS}_n \times \mathbb{X}^{10-n}$ (where $\mathbb{X}$ is a compact manifold) is dual to a superconformal field theory that lives on the boundary of the Anti-de Sitter space. We will focus our attention on the case $\text{AdS}_5 \times S^5$, where the dual field theory is $\mathcal{N} = 4 \text{ SU}(N)$ super Yang-Mills.

The correspondence considered in this section is a realization of the idea that gauge theories might have a dual description in terms of a string theory. This idea originates from 't Hooft's large $N$ limit, which we will review below. The particular relation we consider was motivated by studies of D-branes and black holes in string theory. Before presenting the exact formulation of the correspondence, we will discuss these objects in some detail. We also consider the way the correspondence implements holography. Finally, a holographic entropy formula for the dual CFT is derived in Section 2.2.8.

The correspondence was conjectured by Maldacena [90] and subsequently made precise by Gubser, Klebanov and Polyakov [61] and independently by Witten [144]. For a review, please refer to [2,82].

### 2.2.1 Classical Geometry of AdS

For easy reference in the remainder of the text, let us begin by discussing a few geometrical aspects of Anti-de Sitter space and gathering some useful metrics. For a detailed account of the classical properties of this space please refer to [67].

AdS is the unique vacuum solution to the Einstein equations with maximal symmetry and constant negative curvature. In $D$ spacetime dimensions, it is locally characterized by

$$R_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu}. \quad (2.46)$$

where $L$ is the radius of curvature of AdS, and by the vanishing of the Weyl tensor. The cosmological constant, $\Lambda$, is given as a function of $L$ by

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}. \quad (2.47)$$

The whole space is covered by global coordinates, for which the metric becomes

$$ds^2 = L^2 \left( -\cosh^2 r \, dt^2 + dr^2 + \sinh^2 r \, d\Omega_{D-2}^2 \right). \quad (2.48)$$
A different coordinate system is defined by the so-called Poincaré coordinates. In terms of these coordinates, the metric takes the form

\[ ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu. \]  

(2.49)

A third form of the metric that we will encounter is

\[ ds^2 = L^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dz^2}{z^2}, \]  

(2.50)

where the boundary of the space is at \( z = 0 \).

2.2.2 'T Hooft Large \( N \) Limit

Gauge theories in four dimensions have no dimensionless parameters which can be used as perturbation parameters. \( SU(N) \) Yang-Mills theories have, however, an extra parameter: the rank \( N \) of the gauge group. It was suggested by 't Hooft [128] that these theories might simplify at large \( N \) and have a perturbation expansion in terms of \( 1/N \). We need to specify how the gauge coupling \( g_{YM} \) scales as we take \( N \) large. Of particular interest is the limit where \( \lambda \equiv g_{YM}^2 N \) is kept fixed while one takes \( N \to \infty \). This is called the 't Hooft large \( N \) limit. The observation made by 't Hooft is that, identifying \( 1/N \) with the string coupling constant, the Feynman diagram expansion in this limit takes a form similar to that of perturbative closed string theory. Associated with a field theory diagram with \( V \) vertices, \( P \) propagators and \( L \) loops is a coefficient proportional to

\[ g_{YM}^{2(P-V)N^L} = (g_{YM}^2 N)^{P-V} N^X = \lambda^{P-V} N^X, \]  

(2.51)

where \( \chi = L - P + V \) is the Euler character of the surface corresponding to the diagram. The perturbative expansion of a diagram in the field theory can thus be written as

\[ \sum_{\chi, P, V} c_{\chi, P, V} N^X \lambda^{P-V} = \sum_{\chi} N^X f_{\chi}(\lambda), \]  

(2.52)

where \( f_{\chi} \) is some polynomial in \( \lambda \). In terms of the genus \( g \) of a closed oriented surface, \( \chi = 2 - 2g \). Thus, each diagram is weighted by a factor \( N^{2-2g} \). In the large \( N \) limit, the first order diagrams in this expansion are those of lowest genus. When written in the 't Hooft double line notation, these are the diagrams with the topology of a plane, called planar diagrams, see Figure 2.5. These planar diagrams are in one-to-one correspondence with the lowest order string diagrams and similarly for higher orders. Notice that while the gauge
theory may be strongly coupled, $g_{\text{YM}} \gtrsim 1$, the string theory will be weakly coupled when $N$ is large. 't Hooft first suggested the large $N$ limit in the context of QCD, which is a very successful SU(3) gauge theory for the strong interactions. Being asymptotically free, the gauge theory is useful in describing the high energy regime. On the other hand, at low energies, where interesting phenomena like confinement occur, the theory is strongly coupled and it is difficult to extract information. Since the dual string theory will be weakly coupled (the coupling constant is $1/N$, or $1/3$ for QCD), the hope was that that theory would give a usable description of the low energy regime. However, formulation of the string theory dual to QCD in terms of a 2-dimensional worldsheet action remains an open problem.

For two dimensional gauge theories dual string theories have been constructed [57,58,95]. The case of four dimensional gauge theories is more complicated. Recently, however, there has been progress and, for a certain class of gauge theories, dual string theories have been constructed. These are large $N$ superconformal gauge theories. QCD on the other hand is neither conformal nor supersymmetric and it is unclear how to break these symmetries.

2.2.3 D-BRANES

D$p$-branes are solitonic solutions to string theory. They are defined as $p+1$-dimensional hypersurfaces on which open strings can end. String theory on a background with D-branes has two types of perturbative excitations: closed and open strings. Closed strings, propagating in the bulk, describe perturbations around the background metric, as they include a graviton mode in their massless spectrum. Open strings, which have their endpoints confined to the branes, describe excitations of the branes. The D-branes are topological defects in the sense that a closed string, when it hits a D-brane, can open up and become an open string living on the brane. Reversely, they must act as a closed
string source, as depicted in Figure 2.6. Through the open strings, the D-branes realize gauge theories on their worldvolume. Indeed, the massless spectrum of open strings living on a $Dp$-brane is that of a maximal supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. The spectrum contains $9-p$ massless scalars, which are associated with the transverse oscillations of the brane. Putting $N$ $Dp$-branes on top of each other, the gauge theory generalizes to a $U(N)$ theory. There are then $N^2$ different kinds of open strings, since the strings can begin and end on any of the branes; see Figure 2.7. The expectation values of the scalars determine the relative separations of the branes in the $9-p$ transverse directions. Turning on all of these expectation values breaks the gauge group to $U(1)^N$. In the current context, we are interested in the case of many coincident D-branes.

Before the discovery of D-branes by Polchinski [100], their low energy description in terms of black $p$-branes was already known. The $p$-branes are classical solutions to type I Dsupergravity, which is the low energy limit of string theory. The $p$-brane description provides a second, dual description of D-branes, besides the gauge theoretical description discussed above. The comparison of these two descriptions led to the discovery of the AdS/CFT correspondence.

A stack of $N$ $p$-branes is a heavy macroscopic object that curves spacetime. It can be described by a classical metric and other background fields, such as the Ramond-Ramond $p+1$ form potential. In the following we will focus on the 3-brane. In this case, the metric takes the form [71]

\[
\begin{align*}
    ds^2 &= \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) \\
    &\quad + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right), 
\end{align*}
\]

(2.53)
where $L^4 = 4\pi g_s l_s^4$ with $l_s$ the characteristic string length and the dilaton is constant. Notice that this metric is everywhere non-singular. The horizon is at $r = 0$. The metric for $N$ 3-branes on top of each other only differs in that

$$L^4 = 4\pi g_s l_s^4 N.$$ (2.54)

For the classical supergravity description to be valid, the curvature of the geometry of the $p$-brane has to be small compared to the string scale. This ensures that string corrections are negligible. To suppress string loop corrections, the effective string coupling also needs to be kept small. These requirements can be expressed as

$$1 \ll g_s N < N.$$ (2.55)

On the other hand, the D-brane description uses the string worldsheet and is thus a good description in string perturbation theory. In the case where there are $N$ D-branes on top of each other, every open string boundary loop ending on the D-branes comes with a factor $N$ times the string coupling. The D-brane description is thus valid in exactly the regime complementary to (2.55), namely when

$$g_s N \ll 1.$$ (2.56)

### 2.2.4 Low Energy Limit

The system of string theory on a background of D3-branes can be described by an action of the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}.$$ (2.57)

Here $S_{\text{bulk}}$ describes the 10-dimensional type IIB string theory in the bulk, $S_{\text{brane}}$ the 3 + 1-dimensional open string gauge theory on the branes and $S_{\text{int}}$ the interactions between these two theories. If we consider the system at low energies, i.e., at energies below the string scale $1/l_s$, only the massless modes
can be excited. In this limit the theory on the branes reduces to the low energy effective theory, which is pure $\mathcal{N} = 4 \text{ U}(N)$ SYM [143] and the bulk theory becomes free supergravity on a Minkowski background. Moreover, $S_{\text{int}}$ vanishes and the theory on the branes decouples from the theory in the bulk.

We can also consider the low energy limit in the dual supergravity $p$-brane description. From the perspective of an observer at infinity, there are two types of low energy excitations. There are the massless modes propagating in the bulk region. Since the metric (2.53) becomes flat at large $r$, these describe supergravity in flat space. On the other hand, since $g_{tt}$ in (2.53) depends on $r$, objects close to the horizon appear red-shifted by a factor

$$
\left(1 + \frac{L^4}{r^4}\right)^{-1/4} \sim \frac{r}{L} \quad (r \to 0)
$$

(2.58)
to an observer at infinity. Thus, any excitation becomes a low energy excitation as it is brought close to the horizon. The metric in the near-horizon regime takes the form

$$
ds^2 = \frac{r^2}{L^2} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + L^2 d\Omega_5^2.
$$

(2.59)

Comparing to (2.49) we see that this is the metric of $\text{AdS}_5 \times S^5$, where both components have equal radius $L$. The metric (2.53) thus interpolates between flat space and $\text{AdS}_5 \times S^5$. Figure 2.8 illustrates this separation into two regions. It depicts how the radius of the 5-sphere becomes constant as $r$ becomes small. Both geometries, which are thought to be exact string theory vacua, are separated by an infinitely long “throat”. The two types of excitations

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2_8}
\caption{D3-brane geometry as interpolation between flat 10-dimensional Minkowski space and $\text{AdS}_5 \times S^5$.}
\end{figure}
decouple, because the low energy absorption cross section is proportional to \( \omega^3 L^8 \) [62,81], where \( \omega \) is the energy. More intuitively, this happens because the wavelength of low energy bulk excitations becomes much larger than the typical size of the brane and, reversely, because it becomes increasingly difficult for the excitations near \( r = 0 \) to escape the gravitational well and propagate into the asymptotic region. We conclude that the low energy theory from the \( p \)-brane perspective consists of two decoupled regimes: free supergravity in 10-dimensional Minkowski space and string theory on the near horizon geometry of \( \text{AdS}_5 \times S^5 \).

2.2.5 Maldacena’s Conjecture

In this section we formulate the precise AdS/CFT conjecture, as proposed by Maldacena, and present some initial motivation for it. In both dual descriptions discussed above, the theory far from the branes is string theory (or supergravity) on Minkowski space. The theory near the branes is, however, not the same in both cases: it is a gauge theory in one case, string theory on \( \text{AdS}_5 \times S^5 \) in the other. This led Maldacena [90] to conjecture that \( \mathcal{N}=4 \ U(N) \) SYM is dual to type IIB superstring theory on \( \text{AdS}_5 \times S^5 \).

As motivation for the conjecture, note that the near horizon limit is equivalent to the low energy limit. Indeed, in taking the low energy limit, \( l_s \to 0 \), it is natural to keep fixed the energy of an open string stretched between a stack of D-branes and a probe brane, as in Figure 2.9. The energy of such a string is \( r / l_s^2 \). In order to keep this fixed, one needs to take \( r \to 0 \), which is the near horizon limit. Additionally, consider a massless particle incident from the asymptotic region. In the D-brane description, this will be absorbed by the D-branes and cause an excitation of the gauge theory. In the geometric description, it will tunnel into the throat region and cause an excitation there. The fact that the absorption cross sections for these dual processes are equal [81] provides strong
motivation for the conjecture.

As final motivation, consider the symmetry groups. The YM theory is conformal. Its symmetry group is the conformal group in four dimensions, $\text{SO}(2,4)$, times the R-symmetry group $\text{SU}(4)$. This group is isomorphic to the symmetry group of $\text{AdS}_5 \times S^5$, which is $\text{SO}(2,4)$ for $\text{AdS}_5$ and $\text{SO}(6)$ for $S^5$. Taking into account the supersymmetry, the isometry group of $\text{AdS}_5 \times S^5$ is $\text{SU}(2,2|4)$, which is exactly the superconformal group in four dimensions.

It is important to note that, because of the redshift (2.58), the entire spectrum of string excitations becomes low energy in the near horizon region. Indeed, we have taken the low energy limit in such a way, keeping $r/l_s^2$ fixed, that the energies of objects in the throat remain fixed in string units. The near horizon low energy spectrum is thus that of the full type IIB string theory.

2.2.6 THE CORRESPONDENCE BETWEEN FIELDS AND OPERATORS

Shortly after the AdS/CFT conjecture was made, a precise correspondence between gauge theory observables and those of supergravity was proposed in [61,144]. The basic idea is to identify the correlation functions in the conformal field theory with the dependence of the supergravity action on the asymptotic behaviour near the boundary of AdS.

Because of scale invariance, there is no notion of asymptotic states or of an S-matrix in conformal field theory. The natural objects to consider are then operators. Consider an operator $\mathcal{O}$ in $\mathcal{N} = 4$ super Yang-Mills which changes the value of the coupling constant $g_{\text{YM}}$. Since the gauge coupling is related to the string coupling by [42,100]

$$4\pi g_s = g_{\text{YM}}^2,$$  \hspace{1cm} (2.60)

this causes the string coupling to change. In turn, the string coupling is related to the expectation value of the dilaton $\phi$. This expectation value depends on the boundary value for the dilaton. We see that changing the gauge coupling is related to changing the boundary value of the dilaton. By adding a term $\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})$ to the Lagrangian, we can change the boundary condition on the dilaton to $\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})$, in the coordinates (2.50). It then seems natural to identify [61,144]

$$\left\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \right\rangle_{\text{CFT}} = Z_S[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})].$$  \hspace{1cm} (2.61)

The left hand side of this equation denotes the generating functional of correlation functions in the CFT and the right hand side the supergravity (or string) partition function with the boundary condition that the field $\phi$ approaches the
value $\phi_0$ on the boundary. In this way, the operators of the CFT are in one to one correspondence with the string theory fields. This is true not only for the scalar fields but for any field, including tensor and fermion fields. For example, the stress tensor of the boundary field theory corresponds to the graviton in the bulk.

The AdS/CFT correspondence is a strong/weak coupling duality in that it relates the strong coupling regime of one theory to the weak coupling regime of the dual theory. This complicates direct tests of the conjecture in which a certain quantity is calculated in both theories and compared. Nevertheless the correspondence has been tested extensively, e.g., through comparison of certain correlation functions that do not depend on the coupling \cite{34,49,70}. Based on these tests, it is fair to say that the AdS/CFT conjecture is by now well established and is beyond being a mere conjecture.

### 2.2.7 AdS/CFT and Holography

The AdS/CFT correspondence provides an explicit realization of holography. It allows one to describe the 5-dimensional bulk physics in terms of a 4-dimensional conformal field theory, which can be thought of as living on the boundary* of the AdS space. There is, however, an important aspect of the holographic principle that we have not yet addressed. Namely, the holographic theory should contain a finite number of degrees of freedom per Planck area. In the case at hand, the holographic theory, being a conformal theory, in fact has degrees of freedom at arbitrarily small scales. Also, the area of the boundary of AdS space is infinite. In order to perform a sensible counting of degrees of freedom, we need to regulate both the boundary area and the UV degrees of freedom of the gauge theory. In the following we will see that we can do both with a single regulator.

The number of degrees of freedom can be regulated by imposing an ultraviolet, or short distance cutoff $\delta$. Let us consider the bulk interpretation of such a cutoff. Close to the boundary of AdS we can use the metric (2.50), in which the boundary is at $z = 0$. The AdS isometry (in Euclidean coordinates) corresponding to a rescaling of the boundary is then given by

\begin{equation}
\begin{aligned}
x^i &\rightarrow \lambda x^i, \\
z &\rightarrow \lambda z.
\end{aligned}
\end{equation}

*It may seem that the boundary of AdS$_5 \times S^5$ is not 4-dimensional, but 9-dimensional instead. However, from the metric (2.59) on this space, we see that in approaching the boundary (at $r = \infty$) four of the dimensions blow up, while the $S^5$ remains constant. In order to obtain a finite metric on the boundary we will need to multiply by a factor that goes to zero at the boundary. Effectively, we are then left with a 4-dimensional boundary.
2. Holography

The scale size of objects in the boundary CFT thus corresponds to the radial coordinate, $z$, of AdS. This can also be seen directly in the AdS space. Consider a volume $V$ near the center of AdS. Using an AdS isometry, map the volume to a coordinate distance $\epsilon$ from the boundary. The volume will have scaled to a coordinate size $\epsilon V$. In this way, the scale size becomes a spatial dimension. This implies that a UV cutoff in the boundary at a length scale $\delta$ corresponds in the bulk theory to a IR cutoff at

$$z \leftrightarrow \delta.$$  \hspace{1cm} (2.63)

This relation between the bulk and boundary theories is called the UV/IR connection [127]. In Chapter 4 we will see that it is not a peculiarity of the AdS/CFT correspondence and also appears in a conjectured holographic duality of de Sitter space.

We can now proceed and calculate the entropy of the boundary theory. Assume that each independent quantum field has one degree of freedom per cutoff volume $\delta^3$. Write the metric of AdS as

$$ds^2 = L^2 \left[ -\left( \frac{1 + r^2}{1 - r^2} \right)^2 dt^2 + \frac{4}{(1 - r^2)^2} (dr^2 + r^2 d\Omega_1^2) \right].$$  \hspace{1cm} (2.64)

In these coordinates the radial coordinate is denoted by $r$ and the boundary is at $r = 1$. The regulated boundary is at $r = 1 - \delta$, where $\delta \ll 1$. Consider $\mathcal{N} = 4$ SYM on a three-sphere with unit radius. The number of degrees of freedom of a U($N$) theory is of order $N^2$. Since the volume of the three-sphere in terms of the cutoff $\delta$ is of order $\delta^{-3}$, the total number of degrees of freedom is

$$N_{\text{d.o.f.}} = \frac{N^2}{\delta^3}. \hspace{1cm} (2.65)$$

The area of the regulated sphere is $A \approx L^2/\delta^3$. Using (2.54), we can write

$$N_{\text{d.o.f.}} = \frac{AL^5}{l_s^8 G_5^2}. \hspace{1cm} (2.66)$$

In terms of the 5-dimensional Newton constant, $G_5 = l_s^8 G_5^2 L^{-5}$, this takes the form [127]

$$N_{\text{d.o.f.}} = \frac{A}{G_5}. \hspace{1cm} (2.67)$$

We see that, after suitable regularization, the SYM boundary theory provides a holographic theory including the information density bound.
Since in AdS space the volume and area of any region scale in the same way,\footnote{Note that \( L \), the curvature radius of the AdS space, is a constant.}
\[ \frac{A}{V} \sim \frac{1}{L} \quad (V \to \infty), \]  
(2.68)

one might ask how strong a statement it is to say that gravity in AdS is holographic. Indeed, for any field theory in AdS it holds true that the number of degrees of freedom is proportional to the boundary area. Compare this to the case of flat space where
\[ \frac{A}{V} \sim 0 \quad (V \to \infty). \]  
(2.69)

However, in the case at hand there is another parameter: the AdS length scale \( L \). In the boundary theory it corresponds to the rank of the gauge group, \( N \). We can then consider AdS spaces with different radii and observe whether the number of degrees of freedom goes like the volume or the area. This is relevant, since the volume and area depend differently on \( L \). Combining (2.66) and (2.68), which for AdS \( 5 \times S^5 \) takes the form
\[ V = L^6, \]  
(2.70)

combining (2.68) and (2.69), one obtains
\[ \frac{N_{\text{d.o.f}}}{V} = \frac{1}{L^4 g_s^2}. \]  
(2.70)

From this we see that, as \( L \) becomes large, the number of degrees of freedom per unit volume goes to zero. In this sense, then, the holographic bound is similarly restrictive as in flat space.

2.2.8 THE CARDY-VERLINDE FORMULA

The AdS/CFT correspondence can be applied to the situation where there is a black hole present in Anti-de Sitter space. Like black holes in asymptotically flat space, these solutions have thermodynamic properties including a characteristic temperature and an entropy equal to one quarter of the area of the event horizon in Planck units [68]. It was argued by Witten [145] that this temperature and entropy, as well as the mass of the black hole can be identified with the temperature, entropy and energy of a CFT at high temperatures.

Consider an Anti-de Sitter Schwarzschild black hole in \( D+1 \) dimensions, the metric is given by
\[ ds^2 = -\left(1 + \frac{r^2}{L^2} - \frac{\omega_D M}{r^{D-2}}\right) dt^2 + \left(1 + \frac{r^2}{L^2} - \frac{\omega_D M}{r^{D-2}}\right)^{-1} dr^2 + r^2 d\Omega_{D-1}. \]  
(2.71)
where
\[ \omega_D = \frac{16\pi G_N}{(D-1)\text{Vol}(S^{D-1})}, \]  
(2.72)

cf. (2.2). Here, \( G_N \) denotes the \( D+1 \)-dimensional Newton constant and \( L \) is the radius of curvature of the AdS spacetime. The radius of the black hole horizon is \( r_H \), with \( r_H \) the largest solution of the equation
\[ 1 + \frac{r^2}{L^2} - \frac{\omega_D M}{r^{D-2}} = 0. \]  
(2.73)
The dual CFT lives on the \( D = n + 1 \) dimensional boundary of the AdS Schwarzschild spacetime, with topology \( \mathbb{R} \times S^n \).

As shown in [145], the energy and entropy of the black hole (2.71) are given by
\[ E = M = \frac{(D-1)(r_n L^{-2} + r_n^{-1}) V}{16\pi G_N}, \]  
(2.74)
and
\[ S = \frac{V}{4G_N}, \]  
(2.75)
where \( V \) is the horizon volume,
\[ V = r_{\text{H}}^{D-1} \text{Vol}(S^{D-1}). \]  
(2.76)

For future purpose, rescale these formulas so that the CFT lives on a sphere with radius equal to that of the black hole horizon. Moreover, eliminate the \( D+1 \)-dimensional Newton constant using its relation with the central charge, \( c \), of the CFT,
\[ \frac{1}{4G_N} = \frac{c}{12} \frac{1}{L^n}. \]  
(2.77)
Since the entropy is dimensionless it does not scale. Using the substitution for \( G_N \) we can write it as
\[ S = \frac{c}{12} \frac{V}{L^n}. \]  
(2.78)
The energy does scale and this introduces a factor \( L/r_n \) as compared to (2.74), which can then be written as
\[ E = \frac{c}{12} \frac{n}{4\pi L} \left( 1 + \frac{L^2}{r_n^2} \right) \frac{V}{L^n}. \]  
(2.79)
The temperature of the black hole can be deduced from the metric (2.71) in the same way as for the asymptotically flat case, cf. (2.10)-(2.20). The temperature of the CFT living on a sphere of radius \( r_n \) then follows after the appropriate
rescaling. Alternatively, it follows from the first law of thermodynamics and is given by

\[ T = \frac{1}{4\pi L} \left[ (n + 1) + (n - 1) \frac{L^2}{r_n^2} \right]. \tag{2.80} \]

Based on the above formulas for the entropy and energy of the dual CFT, Verlinde \cite{137} constructed a formula for the entropy reminiscent of the Cardy formula for the entropy of a 1+1-dimensional CFT. Note that the energy (2.79) exhibits an extensive, but also a sub-extensive term,

\[ E = E_E + \frac{1}{2} E_C, \tag{2.81} \]

where \( E_E \) and \( E_C \) denote the extensive part and the sub-extensive part, respectively. The factor \( 1/2 \) is introduced for later convenience. As we will derive in more detail in the next chapter, the sub-extensive term is given by \cite{137}

\[ E_C = \frac{c}{12} \frac{n}{2\pi r_n} \frac{V}{L^{n-1} r_n}; \tag{2.82} \]

and is called the Casimir energy. Although the Casimir effect is usually discussed at zero temperature \cite{33}, a similar effect occurs at finite temperature. It results from finite size effects in the quantum fluctuations of the CFT and disappears in the infinite volume limit. Substituting for \( c \) and \( L \) leads to a unique expression for the entropy \cite{137},

\[ S = \frac{2\pi r_n}{n} \sqrt{E_C(2E - E_C)}. \tag{2.83} \]

This formula has become known as the Cardy-Verlinde formula. Indeed, substituting

\[ E r_n = L_0, \tag{2.84} \]

\[ E_C r_n = \frac{c}{12}, \]

the formula (2.83) reduces to the Cardy formula. It is rather surprising that the Cardy formula can so easily be generalized to higher dimensions, since the standard derivation, based on modular invariance, only works in 1+1 dimensions. These formulas will play an important role in Chapter 3 and the required substitutions (2.84) will be clarified there. In the derivation above it is assumed that \( r_n \gg L \). For future purpose we note that within this parameter range the Casimir energy \( E_C \) is smaller than the total energy \( E \).