Quantum Gravity and the Holographic Principle

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Quantum Gravity and the Holographic Principle

Quantumgravitatie en het holografische beginsel

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Universiteit Utrecht, op gezag van de Rector Magnificus, Prof. dr. W.H. Gispen, ingevolge het besluit van het College voor Promoties in het openbaar te verdedigen op maandag 18 juni 2001 des namiddags te 16.15 uur

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Politics is for the moment. An equation is for eternity
Albert Einstein

Voetbal is simpel. Het is echter moeilijk om simpel te voetballen
Johan Cruijff

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Chapter 1

Introduction

Ever since in 1974 Hawking discovered that black holes emit radiation [65], there has been great controversy about the fact that black holes can evaporate, and about their fate after have they have done so. Indeed, as is well-known, pairs of particles and anti-particles can form in vacuum. These particles however tend to recombine and, under usual circumstances, they will annihilate each other after a very short time. However, when these pairs form in the vicinity of a black hole, there is a small chance for the particle to have just enough energy to escape to infinity, whereas its partner with negative energy is doomed to fall into the black hole. Obviously this is a small effect, as the probability for a Hawking particle to have enough energy to escape to infinity, where we can measure it, is extremely small. Yet the mere idea that such a process is possible is a great challenge for theoretical physics, for it raises the question what would happen if we were able to isolate a black hole (in our minds) so that nothing falls in but it only can emit particles and hence evaporate. In fact, small black holes will evaporate very fast, as the temperature of Hawking radiation is inversely proportional to the mass:

$$T = \frac{\hbar c^3}{8\pi k G_{\rm N} M} = 6 \cdot 10^{-8} (M_\odot / M)\,{\rm K}, \tag{1.1}$$

where $M_\odot$ is the solar mass. For a black hole as heavy as the sun this is a very tiny effect, but small black holes, like the ones that were formed in the early universe, may have a mass small enough to emit strong radiation.

The controversy we alluded to above is not so much concerned with the fact that black holes emit radiation, but rather with the nature of the radiation: it is purely thermal. Its spectrum is that of black-body radiation, which means that it contains little information about the initial state of the black hole. Take for example a page of this thesis and burn it (this is just a thought experiment). After the paper is completely burned, all the precious information that was in it is lost. A close look at the few ashes that are left behind or an analysis of the radiation that is emitted will not help us puzzling out what was written on the paper. We can recover a great deal of the information about it — its chemical composition, etc.—, but not the detailed information about how molecules were precisely arranged on the surface of the sheet.

With black holes the situation is very similar. The Hawking radiation that is emitted is coarse-grained, it does not contain precise information about for example how the black hole exactly formed and all its past history. This is why information is lost in the process of evaporation.

Yet for the burned page we know this is not completely true: if we were able to keep track of each single molecule after the page is burned, applying the laws of physics (and chemistry) we would be able to give their precise configuration when the thesis was still intact, and so we would succeed at recovering the lost information. This is nothing else than the statement that thermodynamics can be derived from microscopic physics by a coarse-graining procedure. In quantum mechanical terms, if the final state is pure, the initial state must be pure as well unless there is a violation of quantum mechanics. Now if Hawking's argument is correct, black holes violate quantum mechanics, as their final state is mixed and not pure.

One can hardly overestimate the importance of Hawking's paradox for our understanding of nature. If true, it points to a fundamental discrepancy between general relativity and quantum mechanics, and so it is extremely important to find out whether there is a mistake in the formulation of Hawking's argument, or whether we have to change the fundamental laws of physics by allowing quantum mechanics to be even less "classical". Indeed, conventional quantum mechanics already leads to conceptual difficulties, but a theory where transitions between pure and mixed states are allowed would be even less transparent and, what is worse, it would be very unlikely to respect basic principles of physics like energy conservation.

A few years before Hawking radiation was discovered, Jacob Bekenstein developed the laws of black hole thermodynamics, based on the analogy between black hole mechanics and thermodynamics [19]. He argued that, up to a constant, the entropy of a black hole must be proportional to its area:

$$S_{\text{BH}} = \frac{kc^3 A}{4G\hbar}, \tag{1.2}$$

and the precise proportionality factor of 1/4 was only determined when Hawking radiation was discovered. Based on the analogy with statistical mechanics, this suggests that the area of the black hole is a measure for the number of microscopical states that give rise to the same macroscopic black hole of mass M, charge Q and angular momentum J.

In string theory, several microscopic countings have been made that confirm the area-law (1.2) with the right proportionality coefficient. Though performed for so-called extremal and near-extremal black holes, which are presumably not of much astrophysical relevance, these countings give, for the first time, a microscopical explanation of the black hole entropy formula.

Motivated by the above relation between entropy and area, in 1993 't Hooft conjectured that at Planckian energies our world is not three-, but two-dimensional [112]. The argument, in simplified form, was as follows. Consider a closed region of space-time of volume $V \sim R^3$ and energy E and ask how many physical states there are in this region. For the states to be physical and thus measurable for an outside observer, we must require that the radial size of the region we consider is larger than the size of its Schwarzschild radius. Otherwise the surface would lie within its own horizon and would be hidden to the observer outside. Since the Schwarzschild radius is given by the energy inside, we get the bound

$$2E < R \tag{1.3}$$

i.e. the Schwarzschild radius should always be smaller than the actual radius, and so the energy density inside the volume is not allowed to be too large.

Given ordinary quantum field theory, the most probable state would be a gas at some temperature T. Its energy would be given by Boltzmann's law,

$$E \sim VT^4. \tag{1.4}$$

In what follows we suppress all multiplicative constants of order 1. The total entropy is

$$S \sim VT^3, \tag{1.5}$$

and so combining (1.3) with (1.4) one gets a bound on the temperature. This gives, for the entropy,

$$S < V^{\frac{1}{2}} \sim A^{\frac{3}{4}}, \tag{1.6}$$

which for large area does not exceed the entropy of a black hole of the same size. Thus, black holes have the largest entropy ordinary matter can possibly have. In fact, they have a larger entropy than what is suggested by the stronger bound (1.6). This is not surprising, as any form of matter will form a black hole if we increase its energy density more and more.

What is surprising is that the limit on the entropy is set by the area, (1.2), and not by the volume. 't Hooft's explanation was that most of the states of field theory are not physical, for their energy is so large that they are confined inside their own Schwarzschild radius. So, the expectation is that gravitational physics reduces the number of physical degrees of freedom: states with energy corresponding to a Schwarzschild size larger than the size of the physical system are not physical and so should be disregarded, hence the number of states grows exponentially with the area instead of the volume. It was then conjectured that quantum gravity should be described by a topological field theory, in the sense that all its degrees of freedom live on the boundary. This is called the *holographic hypothesis*.

There have been various generalisations of the holographic principle which we will not go in detail into, as in this thesis we will only consider the, from a geometrical point of view, most simple cases. In general, one has to define the boundary of a certain region, and its inside and outside. This can be done by looking at the propagation of light rays from a certain region [22].

Much progress in the understanding of the holographic principle came from very different considerations when in 1997 Maldacena conjectured the so-called AdS/CFT correspondence [87]. The AdS/CFT correspondence goes back to the long-ago conjectured relationship between gauge theories and strings [118]. It relates string and gravity theories in a certain back-ground (so-called “anti-de Sitter”, AdS for short) to certain field theories which do not contain gravity (CFT stands for “conformal field theory”). AdS space is a space with a timelike boundary, and in this sense it can be compared with a "box" (one can think of it as a cylinder of circular base). The field theory is defined at the boundary of the space, which corresponds to the wall of the cylinder. Thus, the field theory lives in a space of one dimension less. The AdS/CFT correspondence thus gives a simple realisation of the holographic principle: the gravitational degrees of freedom in the bulk can be arranged in such a way that they describe a non-gravitational theory living on the boundary of the space.

The holographic principle is not only a statement about the number of microstates of the theory. It also implicitly assumes that these degrees of freedom reorganise on the boundary in a somehow physically meaningful way. This implies that the boundary theory should at least respect causality. The AdS/CFT correspondence is a nice arena to perform tests of causality, and in fact some non-trivial tests have been performed with black holes and collisions between massless particles. Although some bizarre behaviour has been found [98, 108, 85] from the boundary point of view, so far no contradictions have been perceived with the causality principles of quantum field theory. Perhaps even more surprising than the fact that the theory lives on the boundary is the fact that the AdS/CFT correspondence relates bulk gravity to one of the field theories that were already known.

In this thesis we are mainly concerned with two different approaches to holography. The first one is an analysis of the eikonal regime of quantum gravity, where the theory reduces to a topological field theory. This is the regime where particles interact at high energies but with small momentum transfer. We also consider quantum gravity away from the extreme eikonal limit and find indications that the theory remains topological. The second approach we pursue is the AdS/CFT correspondence, where one can ask very precise questions about the way the geometry of the bulk and the matter fields are encoded in the boundary theory. We also study warped compactifications, where our d-dimensional world is regarded as a slice of a $d+1$-dimensional space-time, and analyse in detail the question as to where the d-dimensional observer can find the information about the extra dimension. Much of what we do does not assume string theory directly, although most of our results can be embedded in string theory, and in fact we think string theory is probably the best way to understand and think about our results. In particular, the discussion of the AdS/CFT correspondence does assume string theory. Even though in this thesis we investigate two apparently very different approaches, our aim is in fact to apply them to situations where both can be used. In this way we are naturally led to considering Planckian scattering in AdS. This will be studied in chapter 3, where we make a few preliminary remarks about the relation between both.

The thesis is organised as follows. The first chapter is introductory: we first explain the sorts of problems related to black holes and Hawking radiation which motivate this work. We explain why the assumption of the holographic principle can be a way to solve them. Then we review the features of quantum gravity in the eikonal regime, string theory and the AdS/CFT correspondence. Particular emphasis is laid on how holography arises in the context of quantum gravity and of the AdS/CFT correspondence. In chapter 2 we study in detail high-energy scattering between massless particles: classical and quantum mechanical features of gravitational scattering, and how to go beyond the eikonal approximation. In chapter 3 we generalise some of these results to spaces with a cosmological constant (positive or negative) and find the corresponding dual theories. A particularly interesting case is that of a positive cosmological constant. We believe our results are relevant to the discussions in [22, 67, 45] on the possibility of describing holographic duals of de Sitter space. In chapter 4 we study the reconstruction of space-time and of space-time fields from the CFT. We do this perturbatively in the distance to the boundary. We develop a systematic method to regularise and renormalise the bulk action, and interpret our results from the CFT point of view. In chapter 5 we reinterpret the counter-terms of the gravitational action as generating the dynamics from the point of view of an observer living on a brane of codimension 1. We analyse the cases of asymptotically AdS, dS and flat space-time.

1.1 Holography in Quantum Gravity

The most clear and astonishing example of a holographic map between a gravitational and a non-gravitational theory is perhaps the AdS/CFT correspondence. It remains very mysterious, however, how holography may work if the bulk space-time is not AdS but asymptotically flat. In particular, a satisfactory description of the four-dimensional Schwarzschild black hole is still lacking.

As a matter of fact there exists a holographic description, if not of an evaporating Schwarzschild black hole, of a Rindler space-based model that is to mimic the most important features of the near-horizon region of the four-dimensional black hole. This is the S-matrix description discussed by 't Hooft [113], which we are going to examine in detail in this thesis. However, even if this is truly a holographic model, the quantum mechanical properties of the model are not well understood beyond the eikonal approximation, and no entropy formula has been derived. Nevertheless, it is quite remarkable that this model does exhibit explicitly how the information that falls into the black hole is stored into the outgoing radiation without violating any no-quantum-copying-machine principle. In particular, one can compute an approximated S-matrix. It furthermore has a striking similarity with string theories and non-commutative geometry. It also gives interesting insights in the non-perturbative regime of quantum gravity in the eikonal approximation. For these reasons, we think that the model is worth studying, the more because it is applicable in the context of AdS where we also have a dual CFT description. It would be extremely interesting if one could "compare" both holographic duals, and we will make a few preliminary remarks in that direction. It is clear that a cross-fertilisation between the S-matrix model, where the issue of unitarity is exhibited explicitly, and the CFT description, for which there exists an extraordinarily precise dictionary, is most desirable (for a discussion of the issue, see, e.g., [85, 106, 97]).

The next sections are an introduction to some aspects of the eikonal regime of quantum gravity, first in the specific context of point-like particles on a fixed background, and later in a more general set-up. We review in particular how holography arises in the context of quantum gravity. There are many other relevant papers on the subject (see, e.g., [73, 74]), but for the purpose of this thesis we restrict ourselves to the ones that will be used in later sections.

1.1.1 Quantum Gravity in the Eikonal Regime

The main ingredient of the S-matrix Ansatz is the gravitational interactions between in-going particles and out-coming radiation on a black hole horizon. These interactions are not taken care of in the derivation of Hawking radiation, and because of the extreme high frequencies of the in-falling modes these interactions cannot be neglected.

If quantum gravity would be non-predictable in the way originally discussed by Hawking, we would have to enlarge the uncertainty in quantum mechanics to allow for an uncertainty in the state of the wave-function: on top of the statistical description of observables postulated by quantum mechanics, there would be an uncertainty in the quantum state [66]. However, there are strong reasons to believe that gravity can be reconciled with quantum mechanics without giving up unitarity. String theory, and in particular the AdS/CFT correspondence, supports such a view. Nevertheless it is important for the understanding of quantum gravity to be able to point to a loophole in the original argument. Although the contents of this section have already been discussed at length in [113] and other publications by 't Hooft, we will review the S-matrix Ansatz once more because it is the starting point for other considerations in the next chapters.

The basic idea is to take into account the fact that in-going and out-coming particles interact gravitationally at the horizon. If the black hole was formed by some in-falling matter configuration, there will be traces of its initial state on the geometry near the horizon, and so, when Hawking radiation is emitted, it will be scattered off that non-trivial surrounding geometry.

Consider a Schwarzschild black hole in a typical state, say a superposition of in-going and out-going particles. States for the Schwarzschild (Rindler) observer are related to the Kruskal (Minkowski) vacuum by the well-known Bogolyubov transformation,

$$a_{\tilde{k},\omega} \quad = \quad \frac{1}{\sqrt{1-e^{-2\pi\omega}}}\left[b_{\tilde{k},\omega}+e^{-\pi\omega}b^{\dagger}_{-\tilde{k},-\omega}\right]. \tag{1.7}$$

The operator a annihilates a particle of energy ω and momentum $\tilde{k}$ in Rindler space, whereas b is directly related to the annihilation operator in Minkowski space. One can easily check that this mixing between creation and annihilation operators gives rise to the following relation between states:

$$|0\rangle_{\mathrm{M}} = \prod_{\tilde{k},\omega}\sqrt{1-e^{-2\pi\omega}}\sum_{n=0}^{\infty}e^{-n\pi\omega}|n,n\rangle_{\tilde{k},\omega}. \tag{1.8}$$

Consider now the process of "purifying" such a state by removing first one particle and subsequently all the others, until we are left with the vacuum. For the Kruskal observer, more and more particles are being added to his state, with such tremendous energies that they will interact gravitationally, eventually forming small and even big black holes. It is clear that in such a situation the Bologyubov transformation (1.7) will not be correct, as gravitational interactions were neglected in its derivation. So, for the Kruskal observer, the vacuum of the Rindler observer is not at all a vacuum state nor a thermal bath of particles, but it will rather be a highly complicated, gravitationally interacting state. If we had a way of adding or removing particles from our state, keeping track of correlations with other particles, we could then reach any state in Fock space if only we had one reference state.

To realise this in practise goes beyond present knowledge, but we can give an approximated picture. In the next section we will consider an arbitrary state of out-going particles and add one in-going particle to see how the state changes. Repeating this procedure many times, we can compute the S-matrix of the whole process, up to an unknown phase which is the transition element between those reference in- and out-states. Notice that in this context it is not possible to compute this phase because in the eikonal approximation which we will be considering there is no black hole formation. To describe the creation of small black holes one has to consider the full transfer of momentum. We will not discuss black hole creation, but we will discuss how to go beyond the eikonal approximation. Black hole formation is a very important issue which has been considered in a simplified 2+1-dimensional set-up in [89]. Important related discussions in the context of the AdS/CFT correspondence and string theory can be found in [10, 82].

The natural objects to have falling into a black hole are massless objects, since any massive object that is falling into the black hole will be boosted to the speed of light with a tremendous energy. Therefore, we concentrate on massless point particles. The momenta of in-falling particles grow exponentially with Schwarzschild time, whereas momenta of out-coming particles decrease exponentially. A time lapse $\delta t = 4M\gamma$ in Rindler co-ordinates corresponds to a Lorentz-boost in Kruskal co-ordinates,

$$\begin{aligned} u &\rightarrow e^{\gamma}\, u \\ v &\rightarrow e^{-\gamma}\, v \\ p_u &\rightarrow e^{-\gamma}\, p_u \\ p_v &\rightarrow e^{\gamma}\, p_v, \end{aligned} \tag{1.9}$$

in co-ordinates where the future horizon is at $v = 0$, and the past horizon at $u = 0$. So the momentum of in-falling particles grows exponentially as they approach the horizon.

We anticipate that the gravitational effect of such a massless particle on the trajectories of the out-going Hawking particles takes the form of a shift,

$$u \rightarrow u + p_v^{\rm in}(\theta', \phi')\, f(\theta, \phi, \theta', \phi'), \tag{1.10}$$

so also the horizon shifts and out-coming particles come out at time $u = p_v^{\rm in} f$. This means that the size of the black hole has become larger. Notice that, according to (1.9), this shift grows larger and larger as Schwarzschild time goes by, and so at some point it will not be negligible. The point of view we advocate in this thesis is that this is a relevant effect that should be taken into account in the unitarity argument. Indeed, as explained in [113], there seems to be a hidden assumption in the derivation of the Hawking spectrum. This derivation performs a co-ordinate transformation from Minkowski to Rindler co-ordinates in the asymptotic region, where the energy of particles is rather low, and so this transformation seems a good approximation, at least as long as one computes macroscopic properties like the intensity of the emitted flux. However, when it comes to microscopic correlations between the radiation and the in-going particles, this approximation fails because it does not take into account the fact that particles collided at very high energy near the horizon and so they remain correlated afterwards.

In this thesis we will concentrate on the effect of the shift (1.10). Since the shift only takes into account the in-going momentum and not other possible charges of the particle, this is only an approximation to the real problem. However, notice that in a world with no other charges momentum would be enough to recover the information about the particle that was sent in. In realistic models this is also a good approximation because at those energies gravity is the dominant interaction. Electromagnetic interactions are subdominant and they can be easily incorporated in this model, but other charges are more difficult to account for. For a discussion of this issue we refer to [113, 76].

Several objections have been raised against the existence of an S-matrix with such properties. The strongest one seems to be the no-quantum-copying-machine principle [107], which can be formulated as follows. Imagine sending some pure state into a black hole, and assume there is some linear operator X copying this information on an outgoing state. Since the Hilbert space decomposes into an in- and an out-component, $\mathcal{H} = \mathcal{H}_{\rm in} \otimes \mathcal{H}_{\rm out}$, the operator X acts as

$$X\left(|\psi\rangle_{\rm in} \otimes |\phi\rangle_{\rm out}\right) = |\psi\rangle_{\rm in} \otimes |\psi\rangle_{\rm out}. \tag{1.11}$$

However, by letting X act on a superposition $|\psi\rangle_{\rm in} = |\alpha\rangle_{\rm in} + |\beta\rangle_{\rm in}$ one easily sees that an operator defined as above would not be linear and so would violate one of the basic principles of quantum mechanics, the linear evolution of states. Therefore, according to this argument there is no such a thing as a quantum copying machine.

This argument assumes that Hilbert space can be separated into an in- and an out-component, but that turns out not to be true in the S-matrix Ansatz. Actually both Hilbert spaces are complementary just like position and momentum space are in quantum mechanics. So we are forced to describe physics in either one Hilbert space or in the other, but not in both at the same time. The operator that will do the job of "copying" the information of the in-going waves to out-coming radiation will be $X = e^{ip_{\rm in} f p_{\rm out}}$, where p is the momentum of the in- or out-going waves, and f is a function of the impact parameter. This operator certainly acts linearly on wave-functions, and it is actually directly related to the S-matrix. The fact that the in- and out-Hilbert spaces are complementary means that we cannot do measurements on outgoing waves without influencing the outcomes of measurements done on in-going waves: if we choose to measure certain observables outside the black hole, this will imply an uncertainty for the outcomes of measurements inside the black hole.

It is easy to see how the shift (1.10) comes about. An in-falling massless particle with momentum $p_{\rm in}$ is described by the following shock-wave metric [2]

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v - p_{\rm in}\delta(u)\, f\, \mathrm{d}u) + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{1.12}$$

The geodesics of massless test particles in this metric are easy to compute and give

$$\begin{aligned} v(u) &= v_0 + p_{\rm in}\,\theta(u)\left(f + u\,\frac{\partial x^i}{\partial u}\,\partial_i f\right) \\ x^i(u) &= x^i_0 - \frac{1}{2}\,p_{\rm in}\,\partial_i f\, u\,\theta(u) \end{aligned} \tag{1.13}$$

where u parametrises the null geodesic. The function f is given by

$$f = -4G_{\rm N}\,\log(x^2 + y^2). \tag{1.14}$$

The impact parameter is the transverse distance between both particles, $b = \sqrt{x^2+y^2}$ (in co-ordinates where one of the particles is at the origin). Therefore, for large transverse separations as compared to the Planck length, the derivatives of f, $\partial_i f \sim \frac{1}{b}$, can be neglected. More precisely, we have the following small dimensionless parameter: $\varepsilon = G_{\text{N}} p_{\text{in}}/b$. The approximation where this parameter is taken to be small is called the eikonal approximation. In that approximation, we see from the above formulae that v is modified purely by a shift as the test particle crosses the world-line of the in-going particle, whereas the transverse co-ordinates remain unchanged. So, after the collision the particle continues along the same straight line, and the only effect of the collision is a time delay. This in turn means that the momentum transfer during the collision is negligible.

Next we briefly summarise the considerations leading to the black hole S-matrix [113]. Take some reference state $|p_{\text{in}}\rangle$ of particles falling into a black hole, distributed over the horizon as $p_{\text{in}} = p_{\text{in}}(\Omega)$. Then assume that we have an element of the S-matrix that describes the formation and evaporation of the black hole,

$$\mathcal{N} = \langle \text{in}_0 | \text{out}_0 \rangle = \langle p_{\text{in},0}(\Omega) | p_{\text{out},0}(\Omega) \rangle. \tag{1.15}$$

If we perturb the in-going state by adding some momentum δp_{in}, $p_{\text{in}} \to p_{\text{in}} + \delta p_{\text{in}}$, out-going particles will be shifted according to (1.13):

$$\delta v = f \delta p_{\text{in}}. \tag{1.16}$$

So the out-state is modified by:

$$|p'_{\text{out}}\rangle = e^{i\delta v\, \hat{p}_{\text{out}}} |p_{\text{out},0}\rangle, \tag{1.17}$$

the caret meaning that we are generating a shift. So we get a new S-matrix element

$$\langle p'_{\text{out}} | p_{\text{in}} \rangle = \mathcal{N}\, e^{-i\delta p_{\text{in}}\, p_{\text{out}} f}. \tag{1.18}$$

This way we can reach any state $|p_{\text{out}}\rangle$ from a known state $|\text{in}_0\rangle$ by the successive addition of infinitesimal amounts of momentum, and so we get

$$\langle p_{\text{out}} | p_{\text{in}} \rangle = \mathcal{N}' e^{-i p_{\text{in}} p_{\text{out}} f} \tag{1.19}$$

where we filled in the expression for the shift. The magnitude of $\mathcal{N}'$ is fixed by unitarity, but its phase is arbitrary and may depend on the details of the formation of the black hole. We refer to [113] for further details.

When computing the scattering amplitude from (1.19), one finds [114] the Veneziano amplitude for scattering between strings, with an imaginary string constant related to Newton's constant.

A Fourier transform of the above gives

$$\begin{aligned} \langle p_{\text{out}}(\Omega) | p_{\text{in}}(\Omega) \rangle \quad = \quad & \int \mathcal{D}u_{\text{in}} \mathcal{D}u_{\text{out}} \, \exp\Big[i \int \mathrm{d}^2\Omega \, (\partial u_{\text{in}} \partial u_{\text{out}} + \\ & + p_{\text{in}} u_{\text{in}} - p_{\text{out}} u_{\text{out}} + u_{\text{in}} u_{\text{out}}) \Big] \end{aligned} \tag{1.20}$$

which resembles very much the path integral over the world-sheet action of a string. Notice that there is a mass term that breaks conformal invariance. This term, however, is absent if instead of a black hole we consider a Minkowski background[1].

The fields $u_{\rm in}$ and $u_{\rm out}$ are introduced as the Fourier transforms of $p_{\rm in}$ and $p_{\rm out}$, and so at the quantum level we have

$$\begin{aligned} [u_{\rm in}(\Omega), p_{\rm in}(\Omega')] &= i\delta(\Omega-\Omega') \\ [u_{\rm in}(\Omega), p_{\rm in}(\Omega')] &= i\delta(\Omega-\Omega') \\ [u_{\rm in}(\Omega), u_{\rm out}(\Omega')] &= if(\Omega-\Omega'). \end{aligned} \tag{1.21}$$

We see that gravity drastically changes the structure of space-time as seen by massless particles. Co-ordinates between particles become mutually non-commuting operators.

This has far-reaching consequences for the interpretation of Minkowski space as the near-horizon region of Kruskal space. The positions of particles that fall into a black hole are correlated with the positions of the emitted particles, and so Hilbert space does not reduce to a direct product of in and out Hilbert spaces. In other words, modifying the state of in-falling particles does modify the state of the Hawking radiation that is sent out. This obviously reduces the dimensionality of Hilbert space drastically, although we haste to add that every state u still depends on a continuous variable, the angular variable Ω, and so a transverse cutoff is still needed in this crude approximation.

It should now be clear why it is claimed that high-energy scattering presents holographic features. The theory that one gets is the sigma model (1.20), whose fields are defined on a two-dimensional surface, the two-sphere for the case of a Kruskal background. This can be best understood in the context of the results of [121], which we will review in the next section.

1.1.2 Quantum Gravity as a Topological Field Theory

In the previous section we saw that collisions of massless particles at high energies exhibit great similarity with strings, the reason being the extended nature of the gravitational shock-wave. One can wonder whether this is a specific feature of the shock-wave solution, or a general property of gravity at high energies.

In references [73, 74, 121] it was shown that most of the features of the S-matrix model can be understood as specific properties of the eikonal limit of quantum gravity. Indeed, in this regime quantum gravity can be shown to have zero bulk degrees of freedom, all the degrees of freedom living purely on the boundary. So in that regime quantum gravity reduces to a topological field theory. The boundary here is the usual asymptotic null boundary of Minkowski space if we are talking about asymptotically flat spaces, but in chapter 3 we will see that it can also be the boundary of dS and AdS space. This result is at first extremely puzzling, as in general one would expect gravity to have a nonzero number of degrees of freedom in the bulk.

The derivation by Verlinde and Verlinde also sheds light on the validity regime of the S-matrix ansatz. As we will see, the eikonal regime is a perturbative regime as far as transverse processes are concerned, but is non-perturbative in the longitudinal length

[1]When talking about a *background* in the context of shock-wave solutions, we mean a shock-wave on some background space-time.

scale. We will review this argument in some detail here, as it will be the starting point of our generalisation in chapter 3.

The basic argument involves dimensional analysis of the different length scales in the problem. This is a usual argument used in field theory to derive the perturbation expansion. We can set all the dependence on dimensionful quantities into the metric by a rescaling of co-ordinates. Imagine that the typical length scale of the problem is given by some quantity ℓ, then the metric scales like

$$G_{\mu\nu} = \ell^2 \hat{G}_{\mu\nu}, \tag{1.22}$$

where $\hat{G}_{\mu\nu}$ is dimensionless. In four dimensions, the Einstein-Hilbert action scales like

$$S_{\text{EH}}[\ell^2 \hat{G}] = \ell^2 S_{\text{EH}}[\hat{G}]. \tag{1.23}$$

Now although in the path integral one integrates over all metrics, one expects that the dominant contribution will be given by those configurations whose size is that of the physical system, and so it seems reasonable to expect that $\hat{G}_{\mu\nu}$ is typically of order 1. With this assumption, the coupling constant multiplying the action is

$$g = \frac{\ell_{\text{Pl}}}{\ell}. \tag{1.24}$$

This argument is commonly used to argue that when energies are of the Planck size, the theory is strongly coupled and so one needs the full quantum gravity theory to make sensible predictions.

Consider, however, a process where particles collide with Planckian energies but almost head-on. In such a collision, the longitudinal variables $x^\alpha = (t, x)$ fluctuate rapidly, whereas fluctuations in the transverse plane $y^i = (y, z)$ are much slower. In such a situation we have not one but rather two relevant length scales, namely, the longitudinal and the transverse scales. Therefore we can form two dimensionless ratios:

$$\begin{aligned} g_{\parallel} &= \frac{\ell_{\text{Pl}}}{\ell_{\parallel}} \sim 1 \\ g_{\perp} &= \frac{\ell_{\text{Pl}}}{\ell_{\perp}} \ll 1. \end{aligned} \tag{1.25}$$

From now on, the first few Greek characters $\alpha, \beta, \ldots$ refer to the longitudinal space, and middle Latin letters $i, j, \ldots$ refer to the transverse plane.

Taking $\ell_{\parallel}$ to be of order ℓ_{Pl}, we are left with one dimensionless coupling:

$$\kappa = \frac{\ell_{\text{Pl}}}{\ell_{\perp}} \ll 1. \tag{1.26}$$

Performing the rescaling in the action explicitly, in four dimensions the Einstein-Hilbert action splits into three terms:

$$S[G]_{\text{EH}} = \frac{1}{8\pi G_{\text{N}}} \int \mathrm{d}^4 x \sqrt{-G}\, R[G] \;=\; \frac{1}{\kappa^2} S_0[\hat{G}] + \frac{1}{\kappa} S_1[\hat{G}] + S_2[\hat{G}]. \tag{1.27}$$

Thus, part of the action is strongly coupled, whereas S_0 is weakly coupled. The important conclusion is that for the weakly coupled piece we can use the saddle-point approximation. As far as this part of the action is concerned, the leading contribution is given by the classical configurations. Therefore, in the limit of low-momentum transfer, high-energy amplitudes can be computed using semi-classical techniques.

Considering perturbations around a classical background,

$$g_{\mu\nu} = g^{\rm cl}_{\mu\nu}(x) + \kappa\, h_{\mu\nu}, \tag{1.28}$$

the authors of [121] found that the action reduces to

$$S_{\rm EH} = \int \sqrt{-g_{\rm cl}}\, [h^i_i K^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4}\epsilon^{ik}\epsilon^{jl}\nabla_\alpha h_{ij}\nabla^\alpha h_{kl} - \frac{1}{2}(R_i + \epsilon^{\alpha\beta}\partial_\alpha h_{i\beta})^2] + \text{tot. der.} \tag{1.29}$$

where $R_i = \epsilon^{\alpha\beta}\partial_\alpha\partial_i X^a \partial_\beta X_a$ and $K^{\alpha\beta} = \nabla^\alpha\nabla^\beta - g^{\alpha\beta}_{\rm cl}\nabla^2$, and the fields X^a are to be defined below.

The field equations for the metric $g^{\rm cl}_{\mu\nu}$ are determined by the term of the action linear in the perturbation, $h_{\mu\nu}$. The lowest order term S_0 vanishes identically for solutions of the equations of motion. Verlinde and Verlinde found the following solutions to the equations of motion:

$$\begin{aligned} g^{\rm cl}_{\alpha\beta} &= \eta_{ab}\,\partial_\alpha X^a \partial_\beta X^b \\ g^{\rm cl}_{ij} &= g_{ij}(y) \\ g^{\rm cl}_{i\alpha} &= 0, \end{aligned} \tag{1.30}$$

and one also has $R_i = 0$. Notice that the action (1.29) contains no y-derivatives, and so it is like a dimensionally reduced 2-dimensional action. The X-fields then represent y-dependent displacements of the longitudinal plane into itself.

It is now convenient to define a vector field V^α_i with the following properties

$$\begin{aligned} \partial_i X^a &= V^\alpha_i \partial_\alpha X^a, \\ \partial_i g_{\alpha\beta} &= \nabla_\alpha V_{i\beta} + \nabla_\beta V_{i\alpha}. \end{aligned} \tag{1.31}$$

This vector field describes the flow of the X^a-fields in the y-direction. The action then reduces to

$$S_{\rm EH} = \int \sqrt{g_\| g_\perp} \left(R[g_\perp] - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\nabla^\alpha V^\beta_i \nabla^\gamma V^{i\delta} - \frac{1}{2}(\epsilon^{\alpha\beta}\partial_\alpha V_{i\beta})^2 \right). \tag{1.32}$$

This theory is topological: the first two terms can be written as a total derivative, and the last term is set to zero by the constraint $R_i = 0$.

Therefore, the action (1.32) reduces to the following boundary term:

$$S_{\rm EH} = S_{\partial M}[\bar X] = \int_{\partial M} \mathrm{d}x^\alpha \int \sqrt{g_\perp}\,\epsilon_{ab}\left(R[g_\perp]\bar X^a \partial_\alpha \bar X^b + \partial_i \bar X^a \partial_\alpha \partial^i \bar X^b\right) \tag{1.33}$$

where $\bar X$ are the boundary values of X. The boundary here corresponds to the four asymptotic null regions of the 2-dimensional Minkowski plane.

For full details, we refer to [121]. After including point particles, it turns out that the X^a's couple to the longitudinal momenta of the particles. The S-matrix computed from (1.32) with point particles gives exctly the amplitude computed by 't Hooft. In fact, quantisation of the model gives rise to the following commutator:

$$[X^a(y), X^b(y')] = i\epsilon^{ab} f(y, y') \tag{1.34}$$

where f is the Green's function.

$$(\triangle_h - \frac{1}{2}R[h])\, f(y, y') = \delta^{(2)}(y - y') \tag{1.35}$$

This is obviously 't Hooft's result (1.21).

The important conclusion of [121] is that, in the eikonal regime, quantum gravity is a topological field theory: its degrees of freedom live on the boundary, and its only physical perturbations are the global variations of the fields X^a. When coupled to point particles, the saddle-point of these variations correspond to shock waves. Indeed, after inserting the solutions (1.30), the full four-dimensional metric is :

$$\mathrm{d}s^2 \quad = \quad \eta_{ab}\,\partial_\alpha X^a \partial_\beta X^b\,\mathrm{d}x^\alpha \mathrm{d}x^\beta + g_{ij}(y)\,\mathrm{d}y^i \mathrm{d}y^j, \tag{1.36}$$

with

$$\begin{aligned} X^- &= x^- + p^- \theta(x^+) f(y), \\ X^+ &= x^+ - p^+ \theta(x^-) f(y), \end{aligned} \tag{1.37}$$

and this is obviously a generalisation of the Aichelburg-Sexl metric (1.12) for the case of two shock-waves[2].

In chapter 3 we will perform a systematic study of the eikonal regime, valid for spaces with a cosmological constant and of any dimension.

1.2 String Theory

Although still unsolved, Hawking's information paradox has proven to be a very useful scenario to obtain new insights that can help us construct a consistent theory of quantum gravity. Discussions about black holes have led to the discovery of several guiding principles that should be present in quantum gravity. Holography, complementarity, some sort of extendedness beyond the point particle approximation, and non-commutativity, seem to be some of the features that quantum gravity should meet. All of these are present in the eikonal regime of quantum gravity which we studied in the previous sections. In general, however, quantum gravity as a theory of point particles is quite intractable and one may need to make some additional assumption like the assumption that particles have a string-like extension. This leads us to string theory.

There are several reasons to think that making such an assumption is a good idea. Suffice it to say that string theory seems to have built in some of the above principles, in particular the principle of holography, as we will discuss in the next section.

[2]In four dimensions, there are no exact two-particle solutions known. Equation (1.36) is only valid at the linearised level.

The action of a point particle is simply given by the invariant length of its world line:

$$S = -m \int_\gamma \mathrm{d}s \sqrt{-G_{\mu\nu}(z)\, \dot{z}^\mu \dot{z}^\nu} \tag{1.38}$$

where γ is the world line of the particle and z^μ its trajectory along this world line. It is, however, more convenient to have a quadratic action. This can be done by introducing an auxiliary field:

$$S = \frac{1}{2} \int \mathrm{d}s\, (\frac{1}{e}\, G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu - e\, m^2). \tag{1.39}$$

For a quantum mechanical particle, one integrates over all possible trajectories and also over the auxiliary field:

$$\int \mathcal{D}z \mathcal{D}e\, e^{iS[z,e]}. \tag{1.40}$$

The saddle point approximation to the path integral selects the classical trajectory with minimal length.

For strings the situation is analogous to the point particle case. The path integral now contains the exponentiated area of the string,

$$\int \mathcal{D}X \mathcal{D}h\, e^{iS[X,h]} \tag{1.41}$$

where $X(\tau, \sigma)$ denotes the embedding of the string into target space and h_{ij} is an auxiliary field representing the metric on the string. The action is given by

$$\begin{aligned} S \;=\; & -\frac{T}{2} \int \mathrm{d}^2\sigma\, [\sqrt{h} h^{ij} G_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu + \epsilon^{ij} B_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu] \\ & + \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h}\, \phi(X)\, R[h] \end{aligned} \tag{1.42}$$

where we are allowing for additional background fields apart from the metric: the dilaton $\phi(X)$ and an antisymmetric tensor field $B_{\mu\nu}(X)$.

It is well known that at low energies string theory reproduces gravity. The vanishing of the β-functions of the sigma-model (1.42) imposes, at first order in α' [26]:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\alpha\beta} H_{\nu\alpha\beta} + \nabla_\mu \nabla_\nu \phi &= 0 \\ \nabla^\alpha H_{\alpha\mu\nu} - 2\nabla^\alpha \phi\, H_{\alpha\mu\nu} &= 0 \\ 4(\nabla\phi)^2 - 4\Box\phi - R + \frac{1}{12} H^2 &= 0, \end{aligned} \tag{1.43}$$

where $H_{\mu\nu\alpha}$ is the field-strentgh constructed from $B_{\mu\nu}$. The expansion parameter α' is proportional to the string length and is inversely proportional to the string tension T. The β-function equations at lowest order determine the space-time dimension, $D = 26$ for the bosonic string, and $D = 10$ for the superstring.

These equations can be integrated to the following effective action:

$$S = -\frac{1}{\kappa^2}\int \mathrm{d}^D x\,\sqrt{G}\,e^{-2\phi}[R + 4(\nabla\phi)^2 - \frac{1}{12}\,H^2]. \tag{1.44}$$

By a field redefinition of the metric one can bring the action to the Einstein frame. It is clear that higher order terms in the expansion of the β-functions (1.43) will show up as α'-corrections in the effective action (1.44). These are typically of order R^2 and higher, and they predict specific corrections to Einstein's theory.

The action (1.44) does not contain all of the massless supergravity fields. Let us for example concentrate on type IIB string theory. In this case there are additional terms one can add to the effective action. One of these is a self-dual 5-form F_5, which then gives rise to an extremal 3-brane solution of the following form:

$$\begin{aligned} \mathrm{d}s^2 &= f^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}_3^2) + f^{1/2}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2) \\ f &= h + \frac{R^4}{r^4} \end{aligned} \tag{1.45}$$

and $h = 1$. This solution has a constant dilaton, a covariantly constant 5-form flux along the S^5 and $H = 0$. The strentgh of the flux and the value of the dilaton are absorbed in the definition of R. However, 3-branes can also be viewed from a different point of view: they are the hyperplanes on a 10-dimensional flat space on which open (and closed) strings can end and they are called Dirichlet branes. From this point of view, one can effectively describe the physics by the effective action on the D3-brane by describing its collective modes, which are the excitations of the open strings. The effective action in the case of N D-branes placed on top of each other is the Dirac-Born-Infeld action, which generalises the world-volume action of a single D-brane and accounts for the strings being stretched between the branes.

One can also consider the above solution for $h = 0$. The space-time is then $\mathrm{AdS}_5 \times S^5$. The D3-brane and the AdS metrics agree at $r/R \ll 1$, which is precisely the near-horizon limit considered in the AdS/CFT correspondence. In other words, near the horizon of the D3-brane the space looks locally like $\mathrm{AdS}_5 \times S^5$, just as the near-horizon geometry of the Schwarzschild black hole is Rindler space times a two-sphere of constant radius.

$\mathrm{AdS}_5 \times S^5$ is an exact solution of string theory, but the above extremal D3-brane metric is not. α'-corrections to the effective above action (1.44) become important as the energy increases. Here we again concentrate on the case of type IIB, which is where these corrections are best known. Keeping only the 5-form in the RR sector, the action at next order in α' is given by [61, 47, 63]:

$$S = \int \mathrm{d}^{10}x\sqrt{g}\,[e^{-2\phi}(R + 4(\partial\phi)^2 + \gamma W) - \frac{1}{2\cdot 5!}\,F_5^2], \tag{1.46}$$

where γ is a number of order α'^3. The self-duality of F_5 ensures that there are no higher order corrections in F. W is a sum of certain contractions of four Weyl tensors, $W \sim C^4$. Terms of order R^2 and R^3 are removed by a field redefinition. The Einstein frame is reached by a redefinition $g \to e^{\phi/2}g$, and the action becomes:

$$S = \int \mathrm{d}^{10}x\sqrt{g}\,[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 5!}\,F_5^2 + \gamma e^{-\frac{3}{2}\phi}W]. \tag{1.47}$$

Branes play an essential role in the arguments leading to the AdS/CFT correspondence. Since the D3-brane is not an exact solution of the β-function equations, it would be very interesting to analyse α'-corrections to the metric (1.45). The analysis of these corrections will be presented elsewhere [34].

1.3 The AdS/CFT Correspondence

The AdS/CFT correspondence is the most concrete example of a holographic duality. It states that *string theory in an AdS space-time is equivalent to a certain conformal field theory formulated on the boundary of AdS.* So, for example, if the bulk is AdS_5, the dual CFT on the boundary is $\mathcal{N} = 4$ SYM on the boundary of AdS_5 which can be thought of as a cylinder. Of course, as it stands the formulation of this duality is still too vague. Later on we will give more details about the correspondence.

One of the surprising things about the Maldacena or AdS/CFT conjecture is that string theory contains gravity, whereas the field theory does not. This suggests that gravitational theories have redundant degrees of freedom [117] or, at least, they can be reorganised in a more economic way. This gives rise to a theory that is non-gravitational and, furthermore, is defined on a manifold with one space dimension less. In other words, gravity does not contain as many degrees of freedom as one would naively think.

The relationship between gauge theories and theories containing gravity like string theory is long standing [118]. However, a precise connection exists only since the discovery of the AdS/CFT correspondence [87, 125, 64]. There is a large literature on checks of the correspondence between supergravity in AdS and the large N limit of conformal field theories. In this thesis we will concentrate on rather generic but precise questions concerning the holographic map between both theories. Indeed, it is important to have a precise understanding of how quantities in the bulk and on the boundary are mapped into each other in order to understand how holography works.

As said, the focus will be on generic questions concerning the duality. Mostly we will not specify the details of the CFT that we are studying but assume that it exists and require minimal knowledge about it, like which sources are turned on. Then we try to reconstruct the bulk theory as far as we can with this information, until new information from the CFT is required. That we are interested in generic properties of the holographic map is due to the fact that we would like to understand holography in general, i.e. also for other backgrounds than AdS. Hopefully this will give more insight in why the duality works. In the case of the AdS/CFT correspondence, the duality between open and closed strings lies at the heart of the holographic relation [79].

Among the many phrases that can be found in the holographic dictionary, a very important notion is that of the UV/IR connection [109, 94, 13], i.e. the duality between high and low energies on both sides of the duality. More precisely, the renormalisation group scale in the gauge theory is interpreted as the compactification radius of the gravity theory. Radial evolution is then related to the renormalisation group equations [30, 119, 120].

Another, related aspect one would like to understand precisely is the geometry. How is the information about the geometry of the bulk precisely encoded in the boundary theory? More precisely, we can ask: given a certain boundary theory, how does one reconstruct the classical bulk space-time and the fields on this space-time? This an other questions will be addressed in chapter 4. Some of those results will be extended in chapter 5 to asymptotically flat and asymptotically de Sitter space-times: the information about the bulk geometry is encoded in certain specific "holographic" stress tensors.

For a review of the arguments motivating this duality, see [1]. One important issue that one has to address with any duality is its limits of validity. Indeed, string theory in AdS is usually too complicated to be dealt with in detail. One of the tractable limits is the supergravity limit where $\ell_{\rm Pl} < l_s \ll R$, which implies $1 \ll g_s N < N$. The condition $R \gg \ell_{\rm Pl}$ is needed in order for higher curvature corrections to be small. $\ell_{\rm Pl} < l_s$ is equivalent to $g_s < 1$ which is needed in order to avoid string loop corrections in the string coupling e^{ϕ}, which are not well defined in supergravity which is a non-renormalisable theory. On the SYM side this corresponds to the large N, strong 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ limit of the theory. There are of course other intersting limits that one can look at but we will not consider those here.

Let us now discuss how to make the AdS/CFT correspondence more precise. In particular there is an important issue about boundary conditions at infinity that needs to be considered [64, 125]. AdS has a timelike boundary at infinity. Therefore, fields on this space can propagate to the boundary and so one has to supplement them with certain boundary conditions. There is a precise 1-1 correspondence between the boundary values of fields on AdS and operators on the CFT. We collectively denote bulk fields by Φ, and their boundary values by $\phi_{(0)}$. The string partition function is then a functional of the boundary values of the fields:

$$Z_{\rm string}[\phi_{(0)}] = \int_{\phi_{(0)}} \mathcal{D}\Phi \, \exp(-S[\Phi]). \qquad (1.48)$$

According to the proposal in [125], this should be equal to the generating functional of correlation functions in the CFT,

$$Z_{\rm string}[\phi_{(0)}] = Z_{\rm CFT}[\phi_{(0)}] = \langle \exp[\int_{\partial X} \mathrm{d}^d x \sqrt{g}\, \phi_{(0)}(x) O(x)] \rangle_{\rm CFT} \qquad (1.49)$$

where $O(x)$ is a specific composite operator in the CFT and ∂M is the boundary of the manifold M. Thus, the boundary values of the fields act as sources for computing correlation functions of operators in the CFT.

The partition function (1.48) is an intractable object to deal with, so one has to consider some limit like for example the supergravity limit. In this limit, one of the fields that will be integrated over in (1.48) is the metric. Thus, we are strictly speaking not considering AdS space, but any Einstein manifold with fixed metric at infinity. Therefore, the metric in the bulk is allowed to fluctuate as long as it preserves the boundary conditions.

Obviously, at low energies we are interested in the supergravity limit of (1.48) where the dominant contribution to the path integral is given by the saddle-point approximation. The partition function then reduces to:

$$Z_{\rm sugra}[\phi_{(0)}] = \exp(-S[\Phi_{\rm cl}(\phi_{(0)})]), \qquad (1.50)$$

where $\Phi_{\rm cl}$ are now fields that satisfy the low-energy equations of motion with fixed boundary values $\Phi(r,x)|_{r=0} = \phi_{(0)}(x)$.

In general, massive scalar fields that solve the equations of motion behave differently from $\Phi(r,x) \to \phi_{(0)}(x)$ as they approach the boundary. They can either decay more rapidly or develop singularities. A more detailed analysis gives:

$$\Phi(r,x) = r^{d-\Delta}\phi_{(0)}(x) + \cdots, \tag{1.51}$$

where Δ satisfies

$$\Delta(\Delta - d) = m^2 \tag{1.52}$$

and m is the mass of the scalar field. So Δ has two possible values, $\Delta = d/2 \pm \sqrt{d^2/4 + m^2}$ which satisfy $\Delta_+ + \Delta_- = d$. This means that the expansion in general has the following asymptotic form:

$$\Phi(r,x) = r^{d-\Delta}(\phi_{(0)} + \mathcal{O}(r^2)) + r^{\Delta}(\varphi(x) + \mathcal{O}(r^2)). \tag{1.53}$$

where $\phi_{(0)}$ and φ are two independent modes. The unitarity bound on the mass implies $\Delta > (d-2)/2$. The existence of two independent solutions to the equations of motion reflects the fact that usually one needs to impose two boundary conditions on the fields: initial conditions for the positions and the momenta[3]. In AdS, usually one of these two modes will vanish if we also impose some regularity condition in the centre of AdS or some global condition like the vanishing of the Weyl tensor.

The correspondence between the gravity and the CFT computations has been tested for 2-, 3- and 4-point functions of several operators [46, 38, 1].

Klebanov and Witten have argued [81] that for $-d^2/4 < m^2 < -d^2/4 + 1$ the existence of two independent modes for fields in this mass range implies the existence of two conformal field theories dual to the same bulk metric. These are called the Δ_+ and the Δ_--theory. In the Δ_+-theory, the lowest-order mode[4] $\phi_{(0)}$ has the usual interpretation as an external source that couples to an operator $O(x)$ of conformal dimension Δ_+, whereas $\varphi(x)$ (which appears at order Δ_-) is related to the expectation value of $O(x)$. In the Δ_--theory, on the other hand, $\phi_{(0)}$ is interpreted as an expectation value whereas φ is the source. Both theories are related by a Legendre transformation. The case $\Delta_+ = \Delta_-$ is special and corresponds to the tachyon of minimal mass.

As it stands, the correspondence (1.50) is meaningless as both sides suffer from divergences. These, however, can be regularised and renormalised by adding appropriate counter-terms [70, 11, 83, 35]. It has been shown [109, 70] that the IR divergences on the gravitational side correspond to UV divergences on the gauge theory side. In chapter 4 we will develop a systematic method to regularise and renormalise the on-shell supergravity action. Although from the gravity point of view the divergences are purely classical and related to the infinite volume of AdS, it is essential to remove them in order for gravity solutions to have a sensible interpretation in terms of mass, entropy, etc. [25, 11, 83].

In chapter 4 we will study these issues in detail for scalar fields, for the metric and for the coupled gravity-matter system.

[3] However, quantisation in AdS is subtle due to the fact that there is no complete Cauchy surface [9, 50].

[4] Lowest order in r. This mode is the first mode to appear in a perturbative expansion in terms of r.

Chapter 2

Holography in High-Energy Scattering

In this chapter we consider collisions between massless particles at very high energies. We do this perturbatively in the eikonal approximation, where collisions are almost head-on and the impact parameter is large. In this regime, gravity reduces to a topological field theory with zero bulk degrees of freedom. We discuss how to go beyond the extreme eikonal regime as well as first and second quantisation of gravitationally interacting particles.

The contents of this chapter are based mainly on [32] and [33]. The chapter is organised as follows. The first section reviews massless particle solutions of Einstein's equations for various back-ground geometries. In section 2.2 we discuss classical scattering between these particles at very high energies, and in section 2.3 we give a covariant generalisation (in transverse space) of 't Hooft's S-matrix, discussed in section 1.1.1 of the introduction. A first step towards the restoration of covariance in the longitudinal plane is taken in section 2.4 where we compute the transfer of momentum during collitions at high energies. In the next section, section 2.5, we discuss the quantum theory and find a closed algebra between momenta and a gravitational correction to Heisenberg's uncertainty which is nothing but an expression of this momentum transfer. In section 2.6 a precise link is proven between transfer of momentum and covariance. In section 2.7 we discuss second quantisation of gravitationally interacting particles and find that they satisfy an exchange algebra that is very much reminiscent of the Moyal product. We close the chapter with a discussion and some conclusions in section 2.8.

2.1 Pointlike massless particles in Einstein's theory

When energies are so high that gravity becomes the dominant force and particles start interacting gravitationally, one needs to take into account the back-reaction of particles on the back-ground geometry. That is, one cannot trust the free Einstein equations, but one has to couple them to the matter fields of the particles. Our main focus will be massless particles, as these are the relevant excitations when we discuss scattering in the neighbourhood of a black hole.

Massless particles are included in Einstein's theory as follows. The gravitational action is given by

$$S = S_{\rm EH} + S_{\rm matter}, \tag{2.1}$$

where

$$S_{\rm EH} \;=\; \frac{1}{16\pi G_{\rm N}} \int_X \mathrm{d}^d x \sqrt{-G}\,(R + 2\Lambda) \tag{2.2}$$

and $S_{\rm matter}$ is the matter action belonging to the massless particle. In spaces with a boundary, as is the case when the cosmological constant is negative, the action (5.2) may be supplemented with additional boundary terms to ensure a well-defined variational problem. This point will be discussed in detail in later chapters.

As is well-known, the matter action for a massless particle includes an auxiliary field e:

$$S_{\rm matter} = -\frac{1}{2}\int \mathrm{d}s\, e(s)\, G_{\mu\nu}\, \dot{z}^\mu \dot{z}^\nu. \tag{2.3}$$

Making use of the gauge invariance of the action, the equations of motion of the matter fields z^μ and of the auxiliary field give the usual geodesic equation together with the constraint that the particle is massless. Einstein's equations then take the following form:

$$R_{\mu\nu} - \frac{1}{2}\, G_{\mu\nu}\, R - \Lambda\, G_{\mu\nu} = -8\pi G_{\rm N}\, T_{\mu\nu}, \tag{2.4}$$

where the stress-energy tensor is given by:

$$T^{\mu\nu}(x) = -\frac{p}{\sqrt{-G(x)}} \int_\gamma \mathrm{d}s\, \delta^{(d)}(x - z(s))\, \dot{z}^\mu \dot{z}^\nu, \tag{2.5}$$

and γ is the world-line of the particle, parametrised by s, $z^\mu(s)$ the trajectory of the particle along the world-line, p the momentum of the particle along the light-cone, and d the space-time dimension. To find solutions describing massless particles, one solves (2.4) coupled to the geodesic equation and the constraint.

A useful technique to obtain solutions describing massless particles from existent vacuum solutions is Penrose's cut-and-paste method [95]. As explained in the introduction, massless particles can be seen as space-time defects of dimension 1. Penrose's method provides a space-time with a delta-function singularity with support on the null line along the particle's trajectory, given two flat pieces of Minkowski space. As shown by Dray and 't Hooft [40], this kind of gravitational solution generalises to a much larger class of asymptotically flat space-times. It can be further generalised to spaces with either positive or negative cosmological constant.

Let us start with the following rather general class of metrics:

$$\mathrm{d}s^2 = G_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = 2A(u,v)\, \mathrm{d}u\mathrm{d}v + g(u,v)\, h_{ij}(x^k)\mathrm{d}x^i \mathrm{d}x^j, \tag{2.6}$$

in co-ordinates $x^\mu = (u, v, x^i)$. Let us assume that this metric is a vacuum solution of Einstein's equations. In this space-time, the stress tensor takes the following form:

$$T_{vv} = 4pA^2\delta(v)\delta^{(d-2)}(x) \tag{2.7}$$

for a particle travelling along the null line $v = 0$, $x^i = 0$. The cut-and-paste method suggests the following ansatz for the metric:

$$\mathrm{d}s^2 = 2A(u,v)\,\mathrm{d}v(\mathrm{d}u - f(x^k)\delta(v)\mathrm{d}v) + g(u,v)\,h_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j, \tag{2.8}$$

and indeed by direct computation (see Appendix A.2) one finds that Einstein's equations reduce to the following equations at $v = 0$:

$$\triangle_h f - \frac{1}{A}\,\partial_u\partial_v g\, f \quad = \quad \frac{gA}{\sqrt{-G}}\,\delta^{(d-2)}(x), \tag{2.9}$$

$$\partial_u A = \partial_u g \quad = \quad 0. \tag{2.10}$$

The first equation is a junction condition for gluing together both parts of the metric along the null line $v = 0$. The second equation is the requirement that the metric has a Killing vector along the null trajectory of the particle. So, two regions of space-time can be glued together only along a direction with a Killing vector.

The metric (2.8) is singular at $v = 0$. However, this singularity can be removed by a discontinuous co-ordinate transformation

$$\tilde{u} = u - f(x^k)\,\theta(v). \tag{2.11}$$

In these co-ordinates, the metric is finite but not continuous. Geodesics are continuous but not differentiable. With a further (continuous but not differentiable) co-ordinate transformation one can make the metric continuous.

One first remark is that the d-dimensional Einstein equations reduce to the equation of motion of a massive scalar, coupled to a source, on the space-time defect. In this case the space-time defect is a null surface of dimension 1. This result underlies the S-matrix Ansatz. In later chapters we will see that when the defect is timelike and of codimension 1, the induced equations are Einstein's equations coupled to certain stress-tensors. It would be interesting to perform a similar analysis for other types of defects, like e.g. null defects of codimension 1.

The solutions to (2.9) and (2.10) are easy to find if the background is Minkowski space. In four dimensions we find the logarithmic solution given in the introduction, equation (1.14). In other dimensions the solution generically goes like $f \sim \frac{1}{|x|^{d-4}}$. For more general backgrounds, like the Schwazschild black hole, some solutions are given in [40].

It is not difficult to extend the above analysis to space-times including a cosmological constant [72]. Let us just give the solution for pure AdS space with a massless particle travelling from the boundary to the bulk. We write pure AdS$_d$ in the following co-ordinates (see Appendix A.1 for the transformation to Poincare co-ordinates):

$$\mathrm{d}s^2 = \frac{4}{(1 - y^2/\ell^2)^2}\,\eta_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu, \tag{2.12}$$

where ℓ is the AdS radius and $y^2 = \eta_{\mu\nu} y^\mu y^\nu$. The stress tensor of a massless particle can straightforwardly be computed and gives $T_{uu} = -p\,\delta(u)\delta(\rho)$, where ρ is the radial co-ordinate $\rho = \sum_{i=1}^{d-2} y_i^2$.

This metric is not of the class considered above. However, it gives the following solution of Einstein's equations with a massless particle:

$$\mathrm{d}s^2 = \frac{4}{(1 - y^2/\ell^2)^2} \left(\eta_{\mu\nu} \mathrm{d}y^\mu \mathrm{d}y^\nu + 8\pi G_{\mathrm{N}}\, p_u \delta(u)(1 - \rho^2/\ell^2) f(\rho) \mathrm{d}u^2 \right) \tag{2.13}$$

provided

$$\triangle_h f - 4\,\frac{d-2}{\ell^2}\, f = \delta(\rho). \tag{2.14}$$

$\triangle_h$ is the Laplacian on the transverse hyperbolic space,

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_{d-3}^2}{(1 - \rho^2/\ell^2)^2}. \tag{2.15}$$

The solutions to (2.14) are given in chapter 3 and they of course reduce to the Minkowski solutions $f \sim \frac{1}{|x|^{d-4}}$ in the limit when the AdS radius goes to infinity, $\ell \to \infty$.

This metric can also be obtained with Penrose's method because the conformal factor has no dependence on the longitudinal co-ordinates at the locus of the shock-wave. In fact, the condition (2.14) is very similar to (2.10) and it is very likely that one can easily generalise the construction of Dray and 't Hooft to space-times with a cosmological constant that have the Horowitz-Itzhaki shock-wave as a special case. For this case one needs to introduce a dependence on the transverse length ρ in the Ansatz (2.6). Note that the effect on outgoing massless particles takes the form of a shift also in this case (see Appendix A). This can be shown either by direct computation or by using the fact that massless geodesics are invariant under Weyl rescalings of the metric. The latter fact relates the trajectories in AdS to trajectories in flat space.

It is interesting to note that shock-wave solutions are exact solutions of string theory. Indeed, in [5] it has been shown that shock-wave backgrounds are solutions to all orders in the sigma-model perturbation theory. In [72], it was shown that also the AdS shock-wave does not receive any α'-corrections from a geometrical argument used in [77, 71]. The argument uses the fact that all scalar combinations that can be formed from the contribution to the Riemann tensor due to the shock-wave vanish. Thus, corrections to the supergravity action can only come from the AdS part of the metric, but these are known to be equally zero. Thus, shock-waves are among the few known examples of exact backgrounds of string theory. Another interesting fact is that the amplitude computed by 't Hooft agrees, at large distances, with the amplitude of a free string in the shock-wave background generated by another string. The latter also agrees with the (infinite genus) amplitude of two interacting strings in a flat background. So, the shock wave can be regarded as a non-perturbative effect coming from the resummation of flat-metric string contributions [3, 4]. At small distances, however, the string amplitudes do not exhibit the singular behaviour of the point particle case. Let us however point of that to our knowledge no amplitude valid beyond the eikonal regime has been computed so far for the point particle case, and so there is not much one can conclude from the discrepancy.

2.2 Classical scattering at Planckian energies

Next we compute the effect of shock-waves on the trajectories of test particles. This is a straightforward computation if one is careful [33], although there are mathematical subtleties on has to take into account [104, 84]. We illustrate this for the case of a Minkowski background, but the computation generalises straightforwardly to other spaces. Take the metric

$$\mathrm{d}s^2 = 2\mathrm{d}v\,(\mathrm{d}u - \mathsf{f}_v(\tilde{x})\,\delta(v)\mathrm{d}v) + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{2.16}$$

where the shift function is $\mathsf{f}_v(\tilde{x}) \equiv -\frac{1}{T}\int \mathrm{d}^2\tilde{x}' P_v(\tilde{x}')\, f(\tilde{x}-\tilde{x}')$. This is a straightforward generalisation for the case that the total momentum is not concentrated at one point, but is a distribution over the shock-wave. This allows to describe an arbitrary amount of left-movers (see Figure 2.1) all sitting on a plane of constant v with total momentum distribution P_v. The in-going momentum distribution $P_v(\tilde{x})$ is typically equal to

$$P_v(\tilde{x}) = \sum_{i=1}^{N} p_v^i \delta(\tilde{x} - \tilde{x}^i), \tag{2.17}$$

if there are N particles with transverse positions x^i on the plane of the shock-wave. The right-moving particles have initial momentum p_u^0. All particles satisfy the mass-shell condition $p_\mu^2 = 0$.

The first geodesic equation in the metric (2.16) gives

$$\ddot{v} = 0, \tag{2.18}$$

where the dot denotes the derivative with respect to the affine parameter λ along the geodesic. This equation allows us to use v as a time co-ordinate. The other equations are solved as follows:

$$\begin{aligned} u(v) &= u(0) - \frac{1}{2T}\,\mathrm{sgn}\,(v)\int \mathrm{d}^2\tilde{x}'\,P_v(\tilde{x}')\left(f(\tilde{x}_0-\tilde{x}') + v\frac{\partial x^i}{\partial v}(0)\,\partial_i f(\tilde{x}_0-\tilde{x}')\right) \\ x^i(v) &= x^i(0) + p_0^i v + \frac{1}{2T}\,v\,\mathrm{sgn}\,(v)\int \mathrm{d}^2\tilde{x}'\,P_v(\tilde{x}')\,\partial_i f(\tilde{x}_0-\tilde{x}'), \end{aligned} \tag{2.19}$$

where $\tilde{x}_0 \equiv \tilde{x}(0)$. As a boundary condition, we have chosen that the initial momentum in the u-direction is zero, and in the transverse i-direction[1] it is p^i.

If we now concentrate on the $y - v$ plane, differentiating (2.19) yields

$$\frac{\partial y}{\partial v} = \frac{1}{2T}\,\mathrm{sgn}\,(v)\int \mathrm{d}^2\tilde{x}' P_v(\tilde{x}')\,\partial_y f(\tilde{x}_0-\tilde{x}'). \tag{2.20}$$

This agrees with a standard computation by Dray and 't Hooft [40] where massless geodesics are obtained from massive ones by boosting a black hole to the speed of light while sending its mass to zero.

[1] The latter will be set to zero in the following.

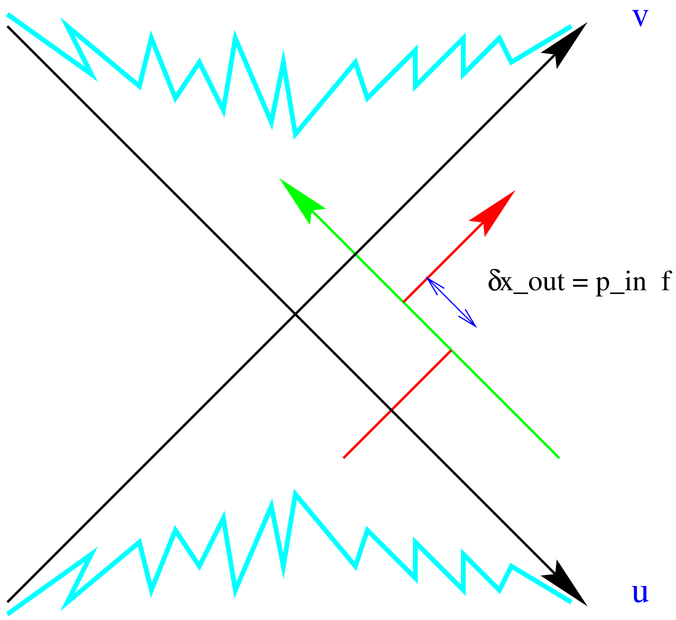

Figure 2.1: Effect of a shock-wave in the lightcone directions

As mentioned in the introduction, we will be working in the first few orders in the eikonal approximation. We introduce the expansion parameter $\varepsilon \equiv G\, p_{\rm in} b$, where $p_{\rm in}$ is the in-going momentum and b the impact parameter, given by the transverse separation between the colliding particles. ε can be taken to be small in the eikonal regime, and it will control our perturbative expansion. Notice that, since f is logarithmic in the transverse distance, $\partial_i f \sim \frac{1}{b}$.

The first of (2.19) gives us the shift (5.2) in the longitudinal co-ordinate u as a consequence of the in-going particle plus a correction that is $\mathcal{O}(\varepsilon^2)$ and can be neglected as long as the in-going transverse momentum is small, $p_\perp \ll p_\parallel$. The second of (2.19) can be represented by a kink in the trajectory of the out-coming particle, see Figure 2.2. This is a higher-order effect.

As mentioned, these are also the trajectories in AdS with a shock-wave, (2.13), with f replaced by the corresponding shift (A.8).

So far we have discussed how the trajectories of out-coming particles are modified by the shock-waves of in-going particles. Next we will consider the momentum transfer involved.

In Figure 2.2 the trajectories in the $y-v$ plane are shown. These follow from (2.19). We learn from the figure that

$$\tan\gamma = \frac{p_y}{p_u}, \tag{2.21}$$

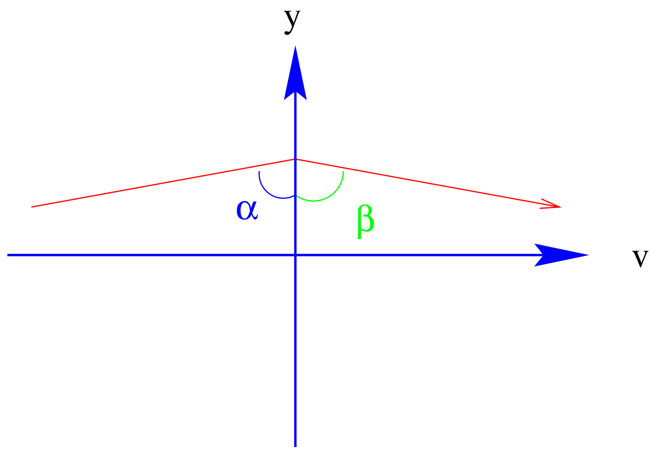

Figure 2.2: Effect of the shock-wave in the transverse direction

where the angle γ is defined by $\gamma = \pi - \alpha - \beta$, and α and β are defined as in the figure. p_y and p_u are the momentum of the out-coming particle in the y and v-directions, respectively, *after* it passes the shock wave. These quantities are different from the momenta before the interaction, which we denote by p^0_μ. We do not explicitly write the superscripts in or out, as it should be clear from the context whether the momentum refers to the in-going or out-coming particle[2].

We now take the initial transverse momentum to be zero, $p^0_y = 0$. This means that $\alpha = \pi/2$ and hence, from (2.20),

$$\cot\alpha + \cot\beta = \tan\gamma = \frac{1}{T}\, p_v\, \partial_y f(y_0). \tag{2.22}$$

One can easily check that the exchange of momentum in the v-direction, to first order in ε, is equal to zero and hence $p_u \simeq p^0_u$. Therefore, we have

$$p_y = \frac{1}{T}\, p_u p_v \partial_y f. \tag{2.23}$$

Since $p_y \sim \frac{\partial y}{\partial v}$, this can also be directly deduced from (2.20).

If the initial transverse momentum is nonzero, differentiating (2.19) once yields, at $v > 0$,

$$\frac{\partial u}{\partial v} = -\frac{1}{2T}\,\frac{\partial x^i}{\partial v}\, p_v \partial_i f(\tilde{x}_0), \tag{2.24}$$

so for the out-coming particle we have

$$p_v^{\rm out} = -\frac{1}{T}\, p_i^{0,\rm out} p_v^{\rm in}\, \partial^i f(\tilde{x}_0). \tag{2.25}$$

[2]In the remainder of this section we assume there is only one particle coming in.

The same relation is obtained from the mass-shell condition $p_\mu p^\mu = 0$.

From (2.23) and (2.25) we find that, roughly speaking, $\delta p_\perp \sim p_\parallel\, \varepsilon$ and $\delta p_\parallel \sim p_\perp\, \varepsilon$, and so if $p_\perp \ll p_\parallel$, the transfer of momentum in the transverse plane is much larger than in the longitudinal plane.

Furthermore, as the transfer of momentum in both the longitudinal and the transverse plane are $\mathcal{O}(\varepsilon)$, they are negligible for large transverse separations (compared to the Planck length). That is the regime where the eikonal approximation is valid.

2.3 The S-matrix

The classical trajectories found in section 2.2 are enough to obtain the scattering amplitude of two particles in the eikonal approximation. In this approximation, the net effect of the presence of a shock-wave on another particle is a shift of the corresponding wave-function. Naively one would think that since the whole effect is only a shift, it can be gauged away with a suitable choice of co-ordinates. However, as argued in section 1.1.2, although locally on both sides of the shock-wave there is no effect, there is an important global effect which is the shift. This shift cannot be removed by a co-ordinate transformation, despite the suggestive form of the metric (1.36). This can be more easily understood in analogy with the electromagnetic case [113, 76]. When a charged particle is boosted towards the speed of light, the electromagnetic field A_μ of the particle is pure gauge outside the light-cone of the particle, i.e. $A_\mu = \frac{1}{2}\partial_\mu \Lambda$ and so has no net physical effect there. However, the gauge field is discontinuous along the world line of the particle, and so the transformations needed to gauge it away are different on the future and past light-cones, $A^\pm_\mu = \pm\frac{1}{2}\,\partial_\mu\Lambda$ with $\Lambda = Q/2\pi \log|x|$, Q being the charge of the particle. Therefore, the total effect is physical.

As explained in the introduction, the scattering amplitude computed from the shift (1.16) is the Veneziano amplitude. The effective action that one finds after a Fourier transformation of the amplitude is

$$S = \int \mathrm{d}^2\tilde{x}\left(-T\partial_i u \partial^i v + P_u u - P_v v\right). \tag{2.26}$$

This is nothing but a rewriting of (1.20) for a flat background. As remarked in the introduction, this is the action of a non-linear sigma model with a coupling to an external source P_μ. The coupling constant is $T = \frac{1}{8\pi G_\mathrm{N}}$. The equations of motion following from this action directly lead to the geodesic equation in the eikonal approximation:

$$\begin{aligned}\partial_i^2 u(\tilde{x}) &= \frac{1}{T}\, P_v(\tilde{x}) \\ \partial_i^2 v(\tilde{x}) &= -\frac{1}{T}\, P_u(\tilde{x}),\end{aligned} \tag{2.27}$$

which are solved by

$$\begin{aligned}u(\tilde{x}) &= u_0 - \frac{1}{T}\int \mathrm{d}^2\tilde{x}'\, P_v(\tilde{x}')\, f(\tilde{x}-\tilde{x}') \\ v(\tilde{x}) &= v_0 + \frac{1}{T}\int \mathrm{d}^2\tilde{x}'\, P_u(\tilde{x}')\, f(\tilde{x}-\tilde{x}').\end{aligned} \tag{2.28}$$

We first write these equations according to a 2+2-splitting of space-time. This is easy to do in the longitudinal plane. We find:

$$X^a(\sigma) \quad = \quad x^a - \frac{1}{T}\,\epsilon^{ab}\int \mathrm{d}^2\sigma' P_b(\tilde\sigma)\,f(\tilde\sigma - \tilde\sigma'). \tag{2.29}$$

The quantisation of this model has been discussed in (1.21). We get:

$$[X^a(\sigma), X^b(\sigma')] = -\frac{1}{T}\,\epsilon^{ab} f(\tilde\sigma - \tilde\sigma'). \tag{2.30}$$

Notice that the minus sign difference in (2.28) is crucial to obtain the epsilon tensor. Indeed, had we guessed a relation of the type $\partial_i^2 x^a \sim p^a$, then the right-hand side of (2.30) would not have been antisymmetric and the model would have been inconsistent at the quantum level[3]. Indeed, due to the complete symmetry between u and v, one's naive guess would have been a geodesic equation where both terms in (2.27) have the same sign. However, the minus sign is directly linked to causality: one of the particles is in-going, whereas the other is out-going. We will comment some more on this in the conclusion.

The presence of an epsilon-tensor is also proven in [121] from the manipulations of the Einstein-Hilbert action coupled to massless particles in the eikonal limit, as reviewed in the introduction.

Recall that the equation of motion (2.27) is only valid in Minkowski space. Indeed, when the manifold is curved the shift function f gets a mass term as shown in (2.9), and this is the equation one has to take as a starting point for more general backgrounds. It can be expressed in terms of the induced metric h_{ij} if one considers that the second term on the left-hand side of (2.9) is a relic of the two-dimensional Ricci-tensor. From the computation outlined in Appendix A.2, we find that the Ricci tensor of the vacuum metric (2.6) equals

$$R_{ij}[G] = R_{ij}[h] - \frac{\partial_u\partial_v g}{A}\,h_{ij}, \tag{2.31}$$

where $R_{ij}[G]$ is the transverse part of Ricci tensor obtained from the full four-dimensional Riemann tensor, see (A.13), and $R_{ij}[h]$ is the Ricci tensor corresponding to the two-dimensional metric h_{ij}. Since $R_{ij}[G]$ satisfies the vacuum Einstein equations, the constraint reduces to

$$R[h] = \frac{2}{A}\,\partial_u\partial_v g \tag{2.32}$$

for a two-dimensional metric h. This obviously gives the metric on the sphere if $g = r^2$ and $A = 1$. We can write equation (2.9) as

$$\left(\triangle_h - \frac{1}{2}\,R[h]\right) f \quad = \quad \frac{1}{\sqrt{h}}\,\delta^{(2)}(\tilde x - \tilde x_0). \tag{2.33}$$

[3]Of course, the sign can be reabsorbed in the definition of momentum, but this leads to non-standard commutation relations and is therefore not very useful for the discussion of covariant generalisations.

It is now obvious how to include this extra term in (2.27):

$$(\triangle_h - \frac{1}{2}\,R[h])\,X^a = \frac{1}{2T}\,\epsilon^{ab}\,P_b. \tag{2.34}$$

This equation is reminiscent of the focusing theorem. It is solved exactly as before,

$$X^a(\sigma) = x^a + \frac{1}{2T}\int \mathrm{d}^2\tilde{\sigma}\,\sqrt{h}\,\epsilon^{ab}\,P_b(\sigma')f(\tilde{\sigma}-\tilde{\sigma}'), \tag{2.35}$$

where f is now the solution of the generalised Green equation (2.33).

It is now straightforward to find a "covariant" generalisation of the action (2.26) in the eikonal limit:

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\,\sqrt{h}\,[h^{ij}\partial_i X^a\partial_j X_a + \frac{1}{2}\,R[h]X^aX_a + \frac{1}{T}\,\epsilon^{ab}X_aP_b]. \tag{2.36}$$

We put the word "covariant" between quotation marks because the fields X^a are still two-dimensional as we are still in the eikonal limit. Thus, covariance here is only with respect to the transverse co-ordinates.

An alternative way to derive this equation is by performing a Fourier transformation of the amplitude (1.19) with a generalised shift that satisfies (2.33). In the case that the metric h is the metric on the unit sphere, we get the amplitude (1.19) computed for the Schwarzschild background.

Let us consider the symmetries of (2.36) for a moment. First of all there is the Lorentz symmetry which we just referred to. It is interesting to note that this symmetry is induced by time translations in Rindler time, as shown in (1.9). Thus, one can say that time translations in the bulk induce Lorentz boosts on the boundary. This is very reminiscent of the relation between radial translations in the bulk and conformal transformations on the boundary for the case of AdS, although the groups are obviously different. It will be interesting to investigate the symmetries of the boundary action in the case of AdS.

The original 't Hooft action (2.26) was invariant under Weyl rescalings of the boundary metric. In the general case we find that the term in (2.36) proportional to the curvature explicitly breaks this symmetry and we are left with a global symmetry only.

2.4 The eikonal limit and beyond

We have mentioned that shock-wave solutions are exact solutions of Einstein's equations, even if one includes any higher-curvature corrections, like for example the ones that appear in string theory. These come from the conformal invariance of the sigma-model. Furthermore one can compute the exact effect of the shock-wave on outgoing particles and the transfer of momentum. Therefore one can ask the question: what happens when one increases the energy up to the Planck scale and perhaps beyond? In other words, how does one go beyond the eikonal approximation? This is the question we are going to analyse in detail in this section.

’t Hooft has suggested [115, 113] that a covariant generalisation of the equations of motion (2.34) should automatically account for the transfer of momentum[4]. However, as we will explain later, it is extremely difficult to find a consistent generalisation of this formula in four dimensions.

Instead, we will choose another approach here. In (2.23) and (2.25) we found the exact momentum transfer. These formulae indeed hold without any approximations. So we will write these formulae in a manifestly covariant form, and will then discuss quantisation. The covariant expression will automatically account for the transfer of momentum. In this section we will study this formula in detail, and in later sections we show that it is consistent with quantisation. Let us first give the expression:

$$P^{\mu}_{\rm out}(\tilde\sigma) = (g^{\mu\nu} + A^{\mu\nu}) P^{0,{\rm out}}_{\nu}(\tilde\sigma), \tag{2.37}$$

where

$$A^{\mu\nu}(\tilde\sigma) = -\frac{1}{T}\,\epsilon^{\mu\nu\lambda\rho}\epsilon^{ij}\partial_i X_\lambda(\tilde\sigma)\int {\rm d}^2\tilde\sigma'\,P_{\rho,{\rm in}}(\tilde\sigma')\,\partial_j f(\tilde\sigma-\tilde\sigma'). \tag{2.38}$$

For the in-operators, one interchanges the labels in-out in the above expression. The quantities $P^0_{\rm out}$ and $P_{\rm out}$ are the momenta of the out-coming particle before and after the interaction, respectively.

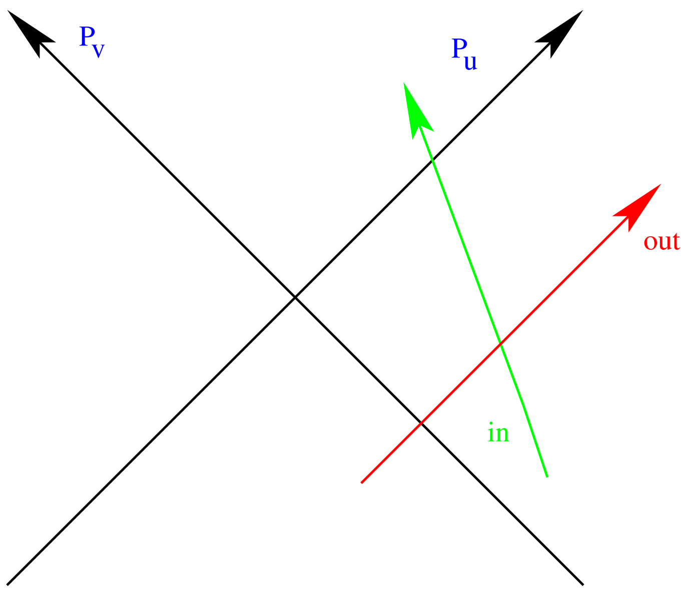

Figure 2.3: Collision at non-zero angle

[4]By covariant we really mean covariant with respect to 4-dimensional diffeomorphisms.

Let us now check that this covariant expression reproduces the momentum transfer computed before. To that end, we first have to set up some notation. We use the notation of reference [121], explained in section 1.1.2 of the introduction. Four-dimensional fields X^μ split into a longitudinal and a transverse component, $X^\mu = (X^a, Y^m)$. The internal co-ordinates x^α and $y^i \equiv \sigma^i$ are (in the usual gauge) the zero modes of X^a and Y^m, respectively, roughly: $X^a = x^\alpha + \cdots$ and $Y^m = \sigma^i + \cdots$. In view of this, and since we will be making a distinction between (X, Y) and (x, y), the indices a and α can be identified, and also m and i (but notice that $X^\alpha \neq x^\alpha, Y^i \neq \sigma^i$). We analyse the case when the background is Minkowski. Both for the out- and the in-particles we have $P_\parallel = P_\alpha = (P_u, P_v)$, $P_\perp = P_i$. We will consider the change of momentum for the out-coming particles produced by the in-going particles, but the expressions for the in-going particles are trivially obtained by exchanging the labels "in" and "out". The kinematics is illustrated in Figure 2.3.

There is still a point in using this (2+2)-splitting of space-time even if the transverse momentum is not zero, because the longitudinal and transverse momenta behave differently in the first few orders in the eikonal approximation. We get:

$$\begin{aligned} P_i^{\rm out}(\tilde\sigma) &= P_i^{0,\rm out}(\tilde\sigma) + \frac{1}{T}\, P_v^{0,\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_u^{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ &- \frac{1}{T}\, P_u^{0,\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_v^{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') + \mathcal{O}(\varepsilon^2) \\ &= P_i^{0,\rm out}(\tilde\sigma) + \frac{1}{T}\, \epsilon^{ab} P_a^{0,\rm out} \int \mathrm{d}^2\tilde\sigma'\, P_b^{\rm in}\, \partial_i f(\tilde\sigma - \tilde\sigma'). \end{aligned} \tag{2.39}$$

Notice that if the operator on the left-hand side of (2.39) carries an out-label, then the operator on the right-hand side of (2.41) which is evaluated at $\tilde\sigma$ corresponds to the same out-particle, whereas the operators which are integrated over give the contributions from the in-particles. The same is true if one reverses the labels.

Note that even if the initial transverse momentum is zero, like in head-on collisions, it will be non-vanishing after the interaction. The two particles will spin around each other for a short time. This agrees with equations (2.23) and (2.25), which were obtained from kinematical considerations. If the momentum of the out-going particle satisfies $p_\parallel^{\rm out} \gg p_\perp^{\rm out}$, using the equation for the shift

$$X^a(\tilde\sigma) = x^a + \frac{1}{T} \int \mathrm{d}^2\sigma'\, \epsilon^{ab} P_b(\sigma')\, f(\sigma - \sigma') \tag{2.40}$$

we find from (2.39)

$$P_i(\tilde\sigma) = P_i^0(\tilde\sigma) + P_a^0 \partial_i X^a, \tag{2.41}$$

to first order in ε. This expression was found in [113] from the consideration that the transverse momentum is not an independent variable, together with the requirement that the transverse momentum generates transverse translations. Here we see that it straightforwardly follows from the transfer of momentum during the collision.

The transverse momentum (2.39) can also be written as

$$P_i(\tilde\sigma) = P_i^0(\tilde\sigma) + \frac{\epsilon}{T}\, P_a^0(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_0^a(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma'), \tag{2.42}$$

where $\epsilon = 1$ if P_i is an operator corresponding to the in-going particles and $\epsilon = -1$ for the out-operators. This is the usual sign convention, where all in-going momenta are defined to be positive, and out-coming momenta to be negative [113]. Indeed, if initially the in-going particles only have momentum P_v, and the out-coming ones only momentum P_u, (2.42) gives

$$\begin{aligned} P_i^{\rm in}(\tilde\sigma) &= P_i^{0,\rm in}(\tilde\sigma) + \frac{1}{T}\, P_{v,0}^{\rm in}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_u^{\rm out}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ P_i^{\rm out}(\tilde\sigma) &= P_i^{0,\rm out}(\tilde\sigma) - \frac{1}{T}\, P_{u,0}^{\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_v^{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma'). \end{aligned} \tag{2.43}$$

The next nontrivial check concerns the longitudinal momentum transfer. Using (2.37), we find

$$\begin{aligned} P_u(\tilde\sigma)^{\rm out} &= P_u^{0,\rm out}(\tilde\sigma) - \frac{1}{T}\, P_0^i \int \mathrm{d}^2\tilde\sigma'\, P_u\, \partial_i f + \frac{1}{T}\, P_u^0 \int \mathrm{d}^2\tilde\sigma'\, P^i\, \partial_i f, \\ P_v^{\rm out}(\tilde\sigma) &= P_v^{0,\rm out}(\tilde\sigma) + \frac{1}{T}\, P_0^i \int \mathrm{d}^2\tilde\sigma'\, P_v\, \partial_i f - \frac{1}{T}\, P_v^0 \int \mathrm{d}^2\tilde\sigma'\, P^i\, \partial_i f. \end{aligned} \tag{2.44}$$

In covariant (2+2)-notation,

$$\begin{aligned} P^a_{\rm out}(\tilde\sigma) &= P^a_{0,\rm out}(\tilde\sigma) + \frac{1}{T}\, \epsilon^{ab}\, P_b^{0,\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma' P^i_{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ &- \frac{1}{T}\, \epsilon^{ab}\, P^i_{0,\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma' P_b^{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ &= P^a_{0,\rm out}(\tilde\sigma) - P^i_{0,\rm out}(\tilde\sigma)\, \partial_i X^a(\tilde\sigma) \\ &+ \frac{1}{T}\, \epsilon^{ab} P^0_{b,\rm out}(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma' P^i_{\rm in}(\tilde\sigma') \partial_i f(\tilde\sigma - \tilde\sigma'). \end{aligned} \tag{2.45}$$

Again, it perfectly agrees with (2.25) in the corresponding limit, $P^i_{\rm in} = 0$. Notice that the transfer of longitudinal momentum is zero if the initial transverse momentum is zero. So, although for vanishing initial transverse momentum there is still a transverse momentum transfer, in the longitudinal plane this transfer is zero to first order in ε.

Notice that in the situation that is usually considered, $P^i_{\rm in} = 0$, equations (2.41) and (2.45) can be rewritten as

$$\begin{aligned} \delta P_i &= +W_i^a P_a^0 \\ \delta P^a &= -W_i^a P^{i,0}. \end{aligned} \tag{2.46}$$

where we defined $W_i^a = \partial_i X^a$. Since W_i^a is proportional to the vector field V_i^α in formula (1.31) by $W_i^a = V_i^\alpha \partial_\alpha X^a$, we see that the latter is responsible for the transfer of momentum. The analogy with fluid dynamics suggested in [121] becomes more transparent from this computation: this vector field accounts for the “flow” or “vorticity” of momentum during collisions at high energies.

2.5 Quantisation

A full quantum theory for this non-linear four-dimensional model is extremely difficult to write down away from the eikonal limit. To quantise the theory we have to give a complete set of observables and the way they act on states in Hilbert space. Now in this model the space-time co-ordinates are not independent, but are related by shift equations. Upon expanding the fields into eigenmodes of this equation of motion with the corresponding creation and annihilation operators, we will find that co-ordinates do not commute. This result has been known for a long time (see [113] and references therein). Quantisation in the eikonal limit is quite straightforward and gives rise to non-commuting co-ordinates (2.30). However, beyond the eikonal approximation we encounter non-linearities which are difficult to deal with. We anticipate that we will not be able to give an exhaustive list of commutation rules among all operators beyond the eikonal limit, for basically the same reason it was not found in earlier works [113, 33]. Instead, we will give a complete set of commutators between the momenta, P. These commutators of course satisfy the Jacobi identity. It is however not clear how to derive a well-defined commutator between the X's. One would think it can be obtained from a generalisation of (2.35) or (2.30), but this is not straightforward as these equations become highly non-linear at low impact parameter. Part of the problem also stems from the fact that the commutator is non-local and so does not transform properly under non-linear co-ordinate transformations of the world-sheet co-ordinates. In fact, it is not even clear whether X^μ is a good starting point to define a quantum theory that incorporates non-linear effects, as it does not transform as a vector in target space. 't Hooft has stressed [113] that a consistent quantisation scheme can perhaps be found if one introduces variables that are better behaved. In the next section we will see the derivative of X, ∂X, is a better physical observable. In this section we will concentrate on the operator P, which is also well behaved as it is a natural object of the tangent space.

As said, here we will assume the canonical commutator between position and momentum operators, and find commutation relations for the momenta from the momentum transfer equation (2.37). We will see that one does find a set of commutation rules that is consistent, where the momentum operator has the usual interpretation as the generator of translations. We will find that the commutators appearing in [115], postulated from the condition that momentum operators generate translations, automatically follow from our equations of motion. We also find new commutators which close the algebra of momenta.

As for the commutators between the X's, these should follow from the Jacobi identity. Indeed, in three dimensions we have solved the Jacobi identity and recovered the results which were found in [116] and [33] directly from the shock-wave equations of motion. However, we have not been able to integrate the equation in four dimensions.

We first consider the action of the operators $\hat{P}^\mu$ and $\hat{X}^\mu$ on state vectors $|P_0\rangle$ and $|X\rangle$. We obviously have

$$\begin{aligned}\hat{P}^\mu|P_0\rangle &= P_0^\mu|P_0\rangle;\\ \hat{X}^\mu|X\rangle &= X^\mu|X\rangle.\end{aligned} \tag{2.47}$$

Furthermore, the operators $\hat{P}$ and $\hat{X}$ satisfy the usual commutation relation

$$[\hat{X}^\mu(\tilde{\sigma}), \hat{P}^\nu(\tilde{\sigma}')] = ig^{\mu\nu}\,\delta(\tilde{\sigma}-\tilde{\sigma}'). \tag{2.48}$$

Indeed, at the level of the path integral and in the eikonal limit these operators were related by a Fourier transformation [32]. From now on we drop the carets on operator-valued quantities.

The quantum theory will however be an interacting theory, and has to take (2.37) into account. Therefore, just as in the eikonal limit (2.29) was promoted to an operator identity, leading to (2.30), our assumption will be that (2.37) is also a relation between a free operator P_0 and the interacting field P. We get the following modified commutator:

$$[X^\mu(\tilde\sigma), P^\nu(\tilde\sigma')] = i\,(g^{\mu\nu} + A^{\mu\nu})\,\delta(\tilde\sigma - \tilde\sigma'). \tag{2.49}$$

This simply means that, due to the back-reaction, the number of independent measurements one can do simultaneously is reduced according to:

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \mathcal{O}\left(\frac{\ell_{\rm Pl}^2 p_{\rm in}}{b}\right). \tag{2.50}$$

The modification of the canonical commutation relation (2.48) in the presence of gravitational interactions has also been predicted (although in different contexts) by several authors [86, 78, 124].

The generalised commutator (2.49) has a simple interpretation if we go back to the underlying shock-wave picture. Before the interaction takes place, the different momenta are independent variables. However, *after* the interaction, they are coupled through the momentum transfer equation (2.37). Then the longitudinal momenta generate sideways displacements as well. So it is natural to identify the canonical momentum $P^\mu_{\rm can}$ with the momentum before the interaction, which we denote by P^μ_0, and P^μ with the momentum after the collision. The latter describes the momentum transfer, and can be seen to be a measure for the recoil of the particles. This holds both for the in-going and the out-coming particles. Although P^μ is not a canonical operator, when writing (2.49) out in components we will see that it generates translations in the sense of field theory. The situation here is similar to cases with background electromagnetic fields, take for example a particle in an electromagnetic field. In that case, the kinetical momentum, which is the operator that (by Ehrenfest's theorem) satisfies the classical equation of motion, is not the canonical momentum operator.

Let us now take a closer look at the commutation relation (2.49),

$$[X^\mu(\tilde\sigma), P^\nu(\tilde\sigma')] = iG^{\mu\nu}\,\delta(\tilde\sigma - \tilde\sigma'), \tag{2.51}$$

where the "generalised metric" is defined as

$$G^{\mu\nu} \equiv g^{\mu\nu} + A^{\mu\nu}. \tag{2.52}$$

Writing (2.51) out in components, we find

$$\begin{aligned}
[u(\tilde\sigma), p_i(\tilde\sigma')] &= i\partial_i u\,\delta(\tilde\sigma - \tilde\sigma');\\
[v(\tilde\sigma), p_i(\tilde\sigma')] &= i\partial_i v\,\delta(\tilde\sigma - \tilde\sigma');\\
[Y^m(\tilde\sigma), p_u(\tilde\sigma')] &= i\partial_u Y^m\,\delta(\tilde\sigma - \tilde\sigma');\\
[Y^m(\tilde\sigma), p_v(\tilde\sigma')] &= i\partial_v Y^m\,\delta(\tilde\sigma - \tilde\sigma').
\end{aligned} \tag{2.53}$$

In the 2+2 splitting, this can be reexpressed as

$$\begin{aligned}[X^a(\tilde\sigma), p_i(\tilde\sigma')] &= i\partial_i X^a(\tilde\sigma)\,\delta(\tilde\sigma-\tilde\sigma') \\ [Y^m(\tilde\sigma), p_\alpha(\sigma')] &= i\partial_\alpha Y^m(\tilde\sigma)\,\delta(\tilde\sigma-\tilde\sigma'). \end{aligned} \tag{2.54}$$

In the gauge where longitudinal indices α are along X^a, and transverse indices i are along Y^m, defining $\delta X^a = X^a - x^a$ and $\delta Y^m = Y^m - \sigma^m$, we have:

$$\partial_\alpha \delta Y_i + \partial_i \delta X_\alpha = 0. \tag{2.55}$$

Note that the operators P_μ are not usual translation operators. They rather generate translations of the fields X^μ along the internal directions.

In quantum mechanics, co-ordinates are independent of each other, and so the right-hand side of (2.51) reduces to the canonical commutator $ig^{\mu\nu}\,\delta(\tilde\sigma-\tilde\sigma')$. But in our case we have a two-dimensional field theory where the longitudinal and the transverse co-ordinates become mutually dependent fields. This renders (2.51) non-vanishing even if the indices μ and ν are different (notice that, for $\mu\neq\nu$, (2.51) is nonzero if one of the indices is transverse, say i, and the other one is a longitudinal index α). So p generates translations just as in field theory, as one directly sees from (2.53).

One can also get an algebra for the commutator of the p's among themselves. One finds (the operators referring all to the in- or all to the out-states)

$$\begin{aligned}[p_\alpha(\tilde\sigma), p_i(\tilde\sigma')] &= ip_\alpha(\tilde\sigma')\,\partial_i\delta(\tilde\sigma-\tilde\sigma'); \\ [p_i(\tilde\sigma), p_j(\tilde\sigma')] &= ip_i(\tilde\sigma')\partial_j\delta(\tilde\sigma-\tilde\sigma') + ip_j(\tilde\sigma)\partial_i\delta(\tilde\sigma-\tilde\sigma'), \end{aligned} \tag{2.56}$$

Now we can also obtain an algebra that relates the in- and the out-operators. Using (2.56), we get:

$$\begin{aligned}[p_v^{\rm in}(\tilde\sigma), p_i^{\rm out}(\tilde\sigma')] &= -iT\,\partial_i u(\tilde\sigma')\,f^{-1}(\tilde\sigma-\tilde\sigma'); \\ [p_u^{\rm out}(\tilde\sigma), p_i^{\rm in}(\tilde\sigma')] &= iT\,\partial_i v(\tilde\sigma')\,f^{-1}(\tilde\sigma-\tilde\sigma'); \\ [p_i^{\rm in}(\tilde\sigma), p_j^{\rm out}(\tilde\sigma')] &= -iT\,\partial_i v(\tilde\sigma)\,\partial_j u(\tilde\sigma')\,f^{-1}(\tilde\sigma-\tilde\sigma') \\ &+ \frac{i}{T}\,p_v^{\rm in}(\tilde\sigma)\,p_u^{\rm out}(\tilde\sigma')\,\partial_j f(\tilde\sigma-\tilde\sigma'). \end{aligned} \tag{2.57}$$

In reference [113] it was not possible to find correct expressions for the commutators between in- and out-operators. The expected expression for the last of (2.57) did not satisfy the Jacobi identity when combined with (2.56). One can check that the above expression does satisfy the Jacobi identity.

The algebra (2.57) is very non-local and, furthermore, non-linear. It, however, can be significantly simplified by defining the total momentum

$$P_\mu = \int \mathrm{d}^2\tilde\sigma\, p_\mu(\tilde\sigma). \tag{2.58}$$

This leads to the following local expressions:

$$\begin{aligned}[p_i^{\rm in}(\tilde\sigma), P_j^{\rm out}] &= i\partial_j p_i^{\rm in}(\tilde\sigma); \\ [p_i^{\rm out}(\tilde\sigma), P_j^{\rm in}] &= i\partial_j p_i^{\rm out}(\tilde\sigma); \\ [p_\alpha^{\rm in}(\tilde\sigma), P_i^{\rm out}] &= i\partial_i p_\alpha^{\rm in}(\tilde\sigma); \\ [p_\alpha^{\rm out}(\tilde\sigma), P_i^{\rm in}] &= i\partial_i p_\alpha^{\rm out}(\tilde\sigma), \end{aligned} \tag{2.59}$$

so the total transverse momentum generates translations.

One can check that the algebra between the transverse in-operators or the out-operators is similar to (2.59). However, we do not expect the theory to have two different generators of transverse translations. So we expect

$$\delta P^i_{\rm in} = \delta P^i_{\rm out}. \tag{2.60}$$

Integrating (2.39) we indeed see that this is the case. The same holds for the lightcone directions, as one sees from equation (2.44). Therefore, for the integrated momentum operators we get the constraint

$$\delta P^\mu_{\rm in} = \delta P^\mu_{\rm out}. \tag{2.61}$$

Recalling that these operators give the momentum transfer, this is nothing else than the expression of the conservation of momentum. As a constraint on Hilbert space, in our case it is also equivalent to the usual asymptotic completeness [48] of the in- and out-Hilbert spaces.

Equation (2.61) implies that momentum is a globaly conserved quantity. But locally it is not conserved, as one can see from the individual local expressions. Only after integrating over $\tilde{\sigma}$ the total momentum is conserved. This is also the usual expectation in field theory.

Recalling that we started off regarding the Minkowski plane as the near-horizon region of a Schwarzschild black hole, we have shown that one can go beyond the eikonal approximation and compute the momentum transfer, thereby respecting momentum conservation which is a minimal requirement for the unitarity of the S-matrix. The assumption that the S-matrix is unitary was the starting point of 't Hooft's considerations, as explained in the introduction. We now see that this assumption leads to a consistent algebra of momenta. In fact, it would be interesting to take the algebra (2.56)-(2.57) as the starting point of some field theory, the momentum being related to the stress-energy tensor in the usual way, and to study the Hilbert-space structure of this theory.

Since the results presented in this section are valid to the first non-trivial order in the eikonal approximation, it seems that the framework developed in [121] would be most appropriate to do an additional check of our results, and would perhaps provide some more conceptual insight in the near-eikonal regime of quantum gravity.

Some of the results in this section had already been found in [115] from general considerations. Here we learn that they straightforwardly follow when recoil effects are taken into account. Furthermore, we also get the additional equations (2.57) and (2.59), which close the algebra. In the next section we perform another check of (2.37).

2.6 Quantum gravity in 2+1 dimensions

When looking for a formulation of the S-matrix beyond the eikonal approximation, in four dimensions one encounters several problems [113, 33] that originate in the non-linearity of the equations. Indeed, as the dimension increases the equations become more and more non-linear [32]. However, when one reduces to 2+1 dimensions things simplify considerably as the algebra becomes linear.

In this section we compactify one of the space-time (and world-sheet) directions on a small circle of radius R and assume the three remaining fields X^μ to be independent of this internal dimension. We also assume that the momentum along this direction is zero and hence we only take the zero modes into account. For a complete theory one should of course also consider the excited modes.

There are several ways to set up the theory. In references [116, 33] it was chosen to find the commutator for the X fields from a covariant generalisation of the dimensionally-reduced system. Reference [116] wrote the covariant formula only after deriving the commutators, whereas in [33] the equation of motion for X was first covariantly generalised, and from there the commutator was found. Both approaches gave the same result. In reference [33] the commutator between the X's and P's was then found from the Jacobi identity, and it was checked that it agrees with the commutator one finds if one directly dimensionally reduces (2.49). It was concluded that the covariant generalisation of the algebra is directly related to the transfer of momentum. This also served as a check of the four-dimensional algebra, which, as stressed in [33], is not free of problems.

Here we choose an alternative route, which, as we will see, is equivalent to that of [33] and provides a nice check of our formulae in the previous section. We take the expression for the commutator between X and P, equation (2.49), as our starting point for the dimensional reduction. In four dimensions this expression comes from the transfer of momentum, (2.37). We then use the Jacobi identity to find the commutator between the X's.

As said, both methods give the same results. The advantage of the latter method is that it only assumes transfer of momentum and not knowledge of the commutator between the X's. Furthermore, in principle this method can be generalised to higher dimensions, where we can compute the transfer of momentum as in the previous sections, but we do not have a fully consistent equation of motion for X for the reasons explained in [33]. Solving the Jacobi identity should give the equation of motion for X. Nevertheless, we have not been able to find a solution to the Jacobi identity in four dimensions, although we do not see any reason why it should not have a solution.

We parametrise the compactified dimension by $\sigma_2 = y$, $0 \leq y \leq R_3$. We define $\sigma = \sigma_1$, $\partial = \frac{\partial}{\partial\sigma}$ and $\epsilon_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda y}$. Notice that the effective 2+1-dimensional Newton's constant is obtained from the 3+1-dimensional one by

$$G_3 = \frac{G_4}{R}. \tag{2.62}$$

Therefore we will find an effective coupling $T = RT_4$.

Since the momentum $p^\mu(\sigma_1, \sigma_2)$ is a momentum density, we have to integrate over the internal direction to obtain the observable momentum from the three-dimensional point of view: $P^\mu(\sigma_1) = \int \mathrm{d}\sigma_2\, p^\mu(\sigma_1, \sigma_2)$.

In 2+1 dimensions, (2.49) becomes

$$[X^\mu(\sigma), P^\nu(\sigma')] = i\left(g^{\mu\nu} - \frac{1}{T}\,\epsilon^{\mu\nu\lambda}\int \mathrm{d}\sigma''\, P_\lambda(\sigma'')\,\partial f(\sigma-\sigma'')\right)\delta(\sigma-\sigma') \tag{2.63}$$

where the shift function is now given by $f(\sigma-\sigma') = \frac{1}{2}|\sigma-\sigma'|$.

We can obtain the commutator between two X's from the Jacobi identity. As remarked in [33] and stressed in previous sections, it is better to consider its derivative,

∂X^μ, rather than X itself, because the former satisfies a local algebra. So we work out the following relation:

$$[[\partial X^\mu(\sigma), P^\nu(\sigma')], \partial X^\lambda(\sigma'')] + \text{cyclic} = 0. \tag{2.64}$$

We get the following solution:

$$[\partial X^\mu(\sigma), \partial X^\nu(\sigma')] = -\frac{i}{T}\,\epsilon^{\mu\nu\lambda} g_{\lambda\rho}\,\partial X^\rho(\sigma)\,\delta(\tilde\sigma - \tilde\sigma'). \tag{2.65}$$

This is the SO(2,1) algebra obtained in [116, 33].

Following [116], the presence of the delta-function in (2.65) suggests to define the following integrated variables:

$$x^\mu_A = \int_A \mathrm{d}\sigma\,\partial x^\mu = x^\mu(A_1) - x^\mu(A_0), \tag{2.66}$$

where A is an interval $A = [A_0, A_1]$ along the line σ.

These variables have the nice property:

$$[x^\mu_A, x^\nu_A] = -\frac{i}{T}\,\epsilon^{\mu\nu\lambda} g_{\lambda\rho}\,x^\rho_A. \tag{2.67}$$

As argued by 't Hooft, this gives rise to a time variable that is quantised in units of t_{Pl}/R.

Another useful quantity is the total momentum flowing through A,

$$p^\mu_A \equiv \int_A \mathrm{d}\sigma\,P^\mu(\sigma). \tag{2.68}$$

The commutator then becomes

$$[x^\mu_A, p^\nu_A] = iG^{\mu\nu}, \tag{2.69}$$

with the "generalised metric"

$$G^{\mu\nu} = g^{\mu\nu} - \frac{1}{T}\,\epsilon^{\mu\nu\lambda} g_{\lambda\rho}\,p^\rho_A. \tag{2.70}$$

The same results can be derived [33] from a covariant generalisation of the three-dimensional equation of motion (2.35):

$$\partial^2 x^\mu = \frac{1}{T}\,\epsilon^{\mu\nu\lambda} g_{\lambda\rho}\,\partial x^\rho\,p_\nu. \tag{2.71}$$

One can also work out the commutation relations in a way analogous to equations (2.53)-(2.54), finding that P again has an interpretation as the generator of translations:

$$\begin{aligned} [u_A, p_x^A] &= i(\partial u)_A \\ [v_A, p_x^A] &= i(\partial v)_A, \end{aligned} \tag{2.72}$$

etc., in a way analogous to the 3+1-dimensional case.

It is particularly beautiful that the link between a covariant algebra, where all coordinates are treated on the same footing, and the inclusion of the transverse gravitational force, can be made so precise in 2+1 dimensions: including the transverse gravitational force leads to an algebra that is invariant under the full three-dimensional Lorentz group, and viceversa: writing the algebra in a manifestly SO(2,1) invariant form automatically accounts for transverse effects.

2.7 Second quantisation of gravitationally interacting particles

Gravitational interactions at high energies lead to a non-commutative space-time. One can wonder what consequences this has for fields that live on this space-time. In reference [80], it was found that taking into account the back-reaction of particles on a black-hole horizon leads to quantised fields that satisfy a so-called exchange algebra. This exchange algebra exhibits great similarity with the Moyal product defined in non-commutative gauge theories.

The computation of [80] uses 't Hooft's results to model a forming black hole with a horizon that fluctuates in time. The formation of the future horizon depends on the time of arrival of in-coming particles, and thus it matters whether we first add in-going particles and then measure the positions of out-going particles, or viceversa.

In this section we show that this effect is not at all an exclusive feature of time-dependent black holes (although black-holes are the natural scenario where these effects become important). Gravitationally interacting fields in Minkowski space already obey such an exchange algebra if they interact gravitationally. All the considerations in this section are independent of the dimension, except for the details of the eikonal approximation. This section is based on [39].

Consider two massless particles in Minkowski space. Particle 1 is "hard" and carries a shock-wave with it, whereas particle 2 is "soft" and so its back-reaction can be neglected. Particle 1 is a left-mover with momentum k^- along x^-, and particle 2 is a right-mover with momentum k^+ along x^+. When particle 2 crosses the trajectory of particle 1 at $x^+ = 0$, it will get shifted:

$$\delta x^- = k^- f, \tag{2.73}$$

and the impact parameter is kept fixed.

Next we consider quantised fields in this Minkowski background. For the moment we restrict ourselves to fields with no transverse momentum. These fields fall apart into a + and a − component:

$$\phi(x^+, x^-) = \phi_+(x^+) + \phi_-(x^-). \tag{2.74}$$

Therefore, the Hilbert space decomposes into a left- and a right-moving part.

To have an S-matrix description, we must have some notion of asymptotic states. Because of (2.74), the Hilbert space of the in-states will fall apart into:

$$|\text{in}\rangle_-|\text{in}\rangle_+, \tag{2.75}$$

and likewise for the out-states. The S-matrix will relate both sets of states. In a momentum representation, if there are for example N in-going particles with momentum along the x^--direction, we have a state

$$|k_1^-, \cdots, k_N^-\rangle_{\text{in},-}. \tag{2.76}$$

We now consider creation and annihilation operators of particles at I_- and I_+. A creation operator $a_+^\dagger(k^+)$ that naturally acts on an in-state is defined by

$$a_+^\dagger(k^+)|0\rangle_{\text{in},+} = |k^+\rangle_{\text{in},+}, \tag{2.77}$$

and likewise for the x^--direction (see Figure 2.4). We require these operators to satisfy the usual commutation rules

$$[a_\alpha(k), a^\dagger_\beta(k')] = \delta(k-k')\,\delta_{\alpha\beta}, \tag{2.78}$$

where the Greek indices stand for $+$ or $-$[5]. For the out-states we have a similar definition, and the corresponding operators will be called $b_\alpha(k)$.

We now consider the commutation rules between in- and out-operators. In the absence of any interactions, the S-matrix is simply unity and so the Hilbert spaces $|\text{in}\rangle_+$ and $|\text{out}\rangle_+$ are identified. a_α and b_β then satisfy

$$\begin{aligned}[a_+(k), b^\dagger_+(k')] &= \delta(k-k') \\ [a_-(k), b^\dagger_-(k')] &= \delta(k-k'). \end{aligned} \tag{2.79}$$

The $+$ and the $-$-operators mutually commute in this case.

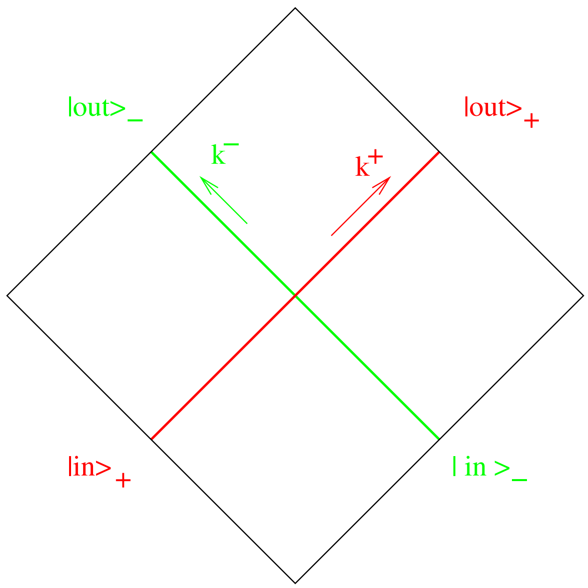

Figure 2.4: Asymptotic states in a two-particle collision

We now include the gravitational interaction (2.73). We assume that the operators

[5]This definition is slightly different from, but completely equivalent to, the usual Fock space. In usual Fock space, the states are characterised by the occupation numbers $|\{n_k\}\rangle$. This is a more economic arrangement of the state (2.76), but it is not useful for our purposes.

a_- and a_+, and b_- and b_+, will still commute[6]. Furthermore, since in the shock-wave approximation there are no self-interactions, (2.79) still holds. When transforming a state $|\text{in}\rangle_+$ into a state $|\text{out}\rangle_+$, the S-matrix element is still trivial.

The interaction (2.73) gives something non-trivial when one considers the commutators $[a_+, b_-]$ and $[a_-, b_+]$. In these cases, one has to take into account the shift (2.73).

The proposal is that in-going operators act also on the Hilbert space of out-going particles. When we add an in-going particle with momentum k^-, we are also shifting the trajectories of out-going particles with momentum along x^+ by the amount (2.73). So we define the operator $a_-(k)$ to act on out-states as follows:

$$\begin{aligned} a_-^\dagger(k)|0\rangle_{\text{out},-}|k_1^+,\cdots,k_{\text{N}}^+\rangle_{\text{out},+} &= \exp\left[-i\sum_{i=1}^{\text{N}} k_i^+ k^- f(\tilde{x}-\tilde{x}^i)\right]\times \\ &\quad \times|k^-\rangle_{\text{out},-}|k_1^+,\cdots,k_{\text{N}}^+\rangle_{\text{out},+}. \end{aligned} \tag{2.80}$$

So this operator translates out-coming particles by the corresponding shift. One can check that the states created by $a_-^\dagger$ form a natural set of states for the out-going Hilbert space. Indeed, solving the Klein-Gordon equation in a shock-wave geometry one finds that the complete set of wave-functions are not simply plane waves, but rather plane waves translated over the corresponding shift. Equation (2.80) also defines the S-matrix.

Notice that the shift f depends on the transverse position of each particle, but this has no meaning in a momentum representation, where we have taken $\tilde{k} \approx 0$, since in principle such a particle cannot be localised. However, one can neglect this effect as long as the transverse distances are large, so that quantum fluctuations are small. As soon as the transverse distance becomes small, one also has to take transverse momentum transfer into account, and the eikonal approximation (2.80) is no longer valid.

We are now in a position to compute the difference between the products $a_-^\dagger b_+^\dagger$, $b_+^\dagger a_-^\dagger$:

$$\begin{aligned} b_+^\dagger(k)\,a_-^+(p) \quad & |k^1,\cdots,k^{\text{N}}\rangle_{\text{out},+}|p^1,\cdots,p^{\text{M}}\rangle_{\text{out},-} = \\ & = \exp\left[-i\sum_{i=1}^{\text{N}} k_i^+ p^- f(\tilde{x}-\tilde{x}^i)\right]\times \\ & \times \quad |k,k^1,\cdots,k^{\text{N}}\rangle_{\text{out},+}|p,p^1,\cdots,p^{\text{M}}\rangle_{\text{out},-}; \\ a_-^+(p)\,b_+^\dagger(k) \quad & |k^1,\cdots,k^{\text{N}}\rangle_{\text{out},+}|p^1,\cdots,p^{\text{M}}\rangle_{\text{out},-} = \\ & = \exp\left[-i\sum_{i=1}^{\text{N}} k_i^+ p^- f(\tilde{x}-\tilde{x}^i) - ik^+p^-f\right]\times \\ & \times \quad |k,k^1,\cdots,k^{\text{N}}\rangle_{\text{out},+}|p,p^1,\cdots,p^{\text{M}}\rangle_{\text{out},-}. \end{aligned} \tag{2.81}$$

Since the states considered here are arbitrary, we conclude that

$$b_+^\dagger(k^+)a_-^\dagger(k^-) = \exp\left[-ik^+k^-f\right] a_-^\dagger(k^-)b_+^\dagger(k^+). \tag{2.82}$$

[6] This is actually different from the philosophy advocated by 't Hooft, who considered non-vanishing commutators for operators at spacelike separated distances, although still preserving causality. Notice, however, that even if two operators act at the same point of the light-cone x^+-x^-, they still can be separated by a spacelike distance since there is still a large transverse separation $\tilde{x}-\tilde{y}$. The extended nature of the shock-wave makes it impossible to avoid non-locality.

Of course, the commutation rules for the annihilation operators can be obtained by replacing $k \to -k$.

Notice that the exchange factor (2.82) generates shifts both in the + and in the $-$-direction, depending on the state it acts on. Therefore we require that b_- and a_+ obey the same algebra:

$$a_+^\dagger(k^+) b_-^\dagger(k^-) = \exp\left[-ik^+k^- f\right] b_-^\dagger(k^-) a_+^\dagger(k^+). \tag{2.83}$$

One can now define scalar fields $\phi(x^\pm)_{\text{in},\pm}$, $\phi(x^\pm)_{\text{out},\pm}$ in terms of these operators:

$$\begin{aligned} \phi_{\text{in},+}(x^+) &= \int \mathrm{d}k_+\, a_-(k_+)\, e^{ik_+x^+} \\ \phi_{\text{out},-}(x^-) &= \int \mathrm{d}k_-\, b_+(k_-)\, e^{ik_-x^-}, \end{aligned} \tag{2.84}$$

and analogously for the other two fields. Notice that, since we integrate over positive and negative frequencies, these fields are automatically real and contain both creation and annihilation modes. As remarked before, one can also check that they satisfy the Klein-Gordon equation in the shock-wave geometry:

$$\left[\partial_+\partial_- - p^- f(\tilde{x} - \tilde{x}')\, \delta(x^+) \partial_-^2\right] \phi(x) = 0, \tag{2.85}$$

and we have neglected transverse derivatives which give factors quadratic in $\tilde{k}$ and $\partial_{\tilde{x}} f$, which are assumed to be small. In this approximation, the solution to this equation is:

$$\phi(x) = \int \mathrm{d}k_- \mathrm{d}\tilde{k}\, F(k_-, \tilde{k}) \exp\left[ip^- k_- \theta(x^+) f(\tilde{x}) + ik_- x^- + ik_+ x^+ + i\tilde{k} \cdot \tilde{x}\right], \tag{2.86}$$

where $k_+ = -\tilde{k}^2/k_-$ and under the assumption that the main contribution to the integral comes from the region of small $\tilde{k}$. The function F is arbitrary, and has to be fixed by imposing some boundary conditions on the field and its derivative.

To simplify notation, we write (2.84) as

$$\begin{aligned} \phi_{\text{in}}(x^+) &= \int \mathrm{d}k_+\, a(k_+)\, e^{ik_+x^+} \\ \phi_{\text{out}}(x^-) &= \int \mathrm{d}k_-\, b(k_-)\, e^{ik_-x^-}. \end{aligned} \tag{2.87}$$

Now these fields satisfy the following exchange algebra:

$$\phi_{\text{out}}(x^-)\phi_{\text{in}}(x^+) = \exp\left[if\partial_+\partial_-\right] \phi_{\text{in}}(x^+)\phi_{\text{out}}(x^-), \tag{2.88}$$

which looks like the $M \to \infty$ limit of the algebra obtained in [80].

Ultimately we would like to consider not only zero modes but rather fields with transverse momentum. Indeed, the transverse distance has not properly been taken care of in (2.88). f depends on the transverse separation of ϕ_{in} and ϕ_{out}, so it is clear that the fields should depend on the transverse co-ordinates too. It is straightforward to include transverse momentum as long as we are in the eikonal regime. Consider fields like in (2.86),

$$\phi_{\rm in}(x) = \int {\rm d}k_+ {\rm d}\tilde{k}\, a(k_+, \tilde{k})\, e^{ik_+x^+ + ik_-x^- + i\tilde{k}\tilde{x}}, \tag{2.89}$$

where $k_- = -\tilde{k}^2/k_+$. This expression is valid as long as we consider large transverse separations between the fields. One gets:

$$\phi_{\rm out}(y)\phi_{\rm in}(x) = \exp\left[if^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right]\phi_{\rm in}(x)\phi_{\rm out}(y), \tag{2.90}$$

where $f^{\mu\nu} = \epsilon^{\mu\nu} f(\tilde{x} - \tilde{x}')$, the indices running over the light-cone directions only. The epsilon-tensor is due to the minus sign coming from the antisymmetry under interchange of the in- and out-labels, and x^+ and x^- in (2.90).

Notice that, by assumption, the main contribution to the integral over transverse momenta comes from the region of small $\tilde{k}$. This would seem to imply that the effect of the shift of the in-field in the x^+-direction is negligible, since its momentum $k^- = -\tilde{k}^2/k^+$ is small. However, this is not necessarily true as the factor appearing in the exponential is proportional to $-\tilde{k}^2 p^+ f(\tilde{x} - \tilde{y})/k^+$, so in 4-dimensional Minkowksi space, where $f \sim \log|\tilde{x} - \tilde{y}|$, this need not be small for large transverse separations and large momenta p_+. In other words, the eikonal approximation only requires the derivative of f to be small, but the shift itself can be large in Planck units.

The above expression is suspiciously similar to the Moyal product that one gets in non-commutative field theory. The obvious guess is that this is related to our original commutator

$$[x^\mu, y^\nu] = if^{\mu\nu}. \tag{2.91}$$

Notice, however, that the situation here is slightly different from that in non-commutative field theory in that our back-ground is commutative now. Non-commuting particle co-ordinates have been replaced by non-commuting fields.

2.8 Discussion and conclusions

The eikonal regime turns out to be a very interesting corner of the moduli space of quantum gravity. Things simplify so enormously in this regime that the theory becomes topological. Still it has non-trivial dynamics. Global variations of the fields correspond to massless particles in the bulk, similarly to the way massive particles in 2+1 dimensions correspond to topological defects. If the holographic principle is to be true, one should not be surprised by this conclusion but should rather wonder whether the same is true away from the eikonal regime.

It is also found that the theory is a non-commutative theory whose natural length scale is the Planck length. Furthermore, Heisenberg's relation is modified by a term proportional to Newton's constant. This has been proposed by other authors [86, 78, 124], but in the context of collisions between particles at high energies it appears to be a simple consequence of the entanglement between the particles after interactions.

Attempts to construct the S-matrix for a two-particle collision at arbitrary angles and arbitrary momentum transfer have failed so far. Note that for this it is not at all necessary to have a two-particle solution of Einstein's equations as long as the rest mass of the particles is small, as one can always go to a frame where the momentum of one of the particles is small. We have performed a somewhat indirect analysis. The momentum transfer between the particles was computed, and from this it is easy to obtain the commutation rules between momenta. At every stage conservation of momentum was explicit. However, we were not able to find an algebra between the co-ordinates, although in principle this can be found by integrating the Jacobi identity. In 2+1 dimensions, this is easy to do and we get an SO(2,1) algebra between position operators, in agreement with earlier works [116, 33]. In the derivation given here it is clear that these complicated quantum mechanical effects are again rooted in the entanglement between the particles produced by the momentum transfer.

The precise analysis of the momentum transfer also gives interesting insights in the decoupling of the longitudinal and transverse degrees of freedom. Transverse momentum transfer is of the order $\delta p_\perp \sim p_\| \, \varepsilon$, whereas longitudinal momentum transfer is of the order $\delta p_\| \sim p_\perp \, \varepsilon$. Thus, as long as $p_\perp \ll p_\|$, transverse physics is frozen and the transverse modes can be treated classically, whereas the longitudinal modes are still rapidly fluctuating. In fact, we have explicitly seen that when this condition is no longer valid, the transverse modes start fluctuating and become quantum mechanical operators as well.

A case of particular interest is AdS. Based on the AdS-shock-wave solution of by Horowitz and Itzhaki, we found that also in AdS interactions between massless particles are given in terms of shifts. In particular, for scalar fields the effect is a phase shift. In the next chapter we will find that, from the CFT point of view, the dual operator has a different expectation value inside the light-cone from its value outside.

Another interesting result concerns second quantisation of these gravitationally interacting particles. They satisfy an exchange algebra which is very similar to the Moyal product defined in non-commutative gauge theories, with the difference that in our case the θ-parameter is a function of the transverse co-ordinates. The non-commutativity of the algebra is rooted in the non-commutativity of the first-quantised space-time. However, despite the similarity the situation is different from that in non-commutative gauge theories as our algebra is not derived from an action on a non-commutative space. In our case the co-ordinates commute. The non-commutativity arises when we include gravitational interactions. This is true both for the first and the second quantised system: in our case, it is always the matter fields of particles that are non-commuting. These are either co-ordinates of particles, $X^\mu(\tilde{\sigma})$, or scalar fields, $\phi(x)$. The underlying space-time ($\tilde{\sigma}$ or x, respectively) is always commutative.

There are several interesting open questions which are left for future study.

One interesting problem is how to fully take into account the transverse effects in the non-commutative scalar field theory without having to restrict ourselves to the eikonal approximation. This could be done most easily in the 2+1-dimensional context where we have the full commutator between the x's, which satisfy the SO(2,1) algebra, and it may be easier to obtain the exact solutions to the Klein-Gordon equation.

Another crucial question in the context of holography is the interpretation of the commutators (2.91) and (2.90) in the context of the AdS/CFT duality. The computation of the trajectories in the AdS-shock-wave metric in Appendix A.1 reveals that once again the shift is proportional to the momentum, and so upon quantisation one expects co-ordinates to be non-commuting. However, one now has to take into account the additional problems with quantisation that arise in AdS. It is likely that the techniques developed in chapter 4 can help us understand the meaning of the commutator (2.91) in terms of sources or operators related to the bulk fields $z^\mu(s)$. Another, more straightforward approach, will be to directly study the algebra (2.90) from the point of view of the dual operators on the boundary. A previous step in this direction is taken in section 3.7 of the next chapter.

As remarked by 't Hooft [116], the epsilon-tensor in (2.67) is directly related to the position of the observer with respect to a black hole horizon. In turn, the appearance of such an epsilon tensor can be traced back to the minus sign difference in (2.27), which gives rise to an epsilon tensor in the longitudinal space in formula (2.30). At the level of the S-matrix, this sign difference comes from the fact that particles with momentum p_u are in-going, whereas those with momentum p_v are out-going, as one easily sees from (1.20). Thus, this epsilon tensor is indeed connected with the distinction between in-going and out-going and thus with causality. It is at first somewhat surprising that the same epsilon tensor appears in (1.33) and (1.34), but also here it has to do with the orientation with respect to the asymptotic boundary of the space-time, and thus again it is a global property closely related with causality.

Let us end with a somewhat speculative remark. Beyond the eikonal approximation, although there may still be some hidden redundancy in our formulae, we have seen that there are more than two fields X^a whose variations contain physical information. In four dimensions, there are four such fields, X^a and Y^m. One is therefore led to speculate that the path-integral approach in [121] at the next order in the eikonal approximation will still be topological, the physical fields now being the boundary values of X^a and Y^m. Of course, at some point one expects to encounter the usual non-renormalisable infinities in quantum gravity, and at that point one may need to invoke string theory.

Chapter 3

Boundary Description of High-Energy Scattering in Curved Space-times

We show that for an eikonal limit of gravity in a space-time of any dimension with a non-vanishing cosmological constant, the Einstein – Hilbert action reduces to a boundary action. This boundary action describes the interaction of shock-waves up to the point of evolution at which the forward light-cone of a collision meets the boundary of the space-time. The conclusions are quite general and in particular generalise the work of E. and H. Verlinde [122]. The role of the off-diagonal Einstein action in removing the bulk part of the action is emphasised. We discuss the sense in which our result is a particular example of holography and also the relation of our solutions in AdS to those of Horowitz and Itzhaki [72]. We also find a boundary action for the case of asymptotically de Sitter space. This is relevant to the discussions of holographic duals of de Sitter space in [22, 67, 45].

The contents of this chapter are based on [7].

3.1 Introduction

Although one could claim that high-energy scattering in gravity should be treated in string theory the philosophy adopted in this chapter, based on the holographic principle is that such collisions should be treatable in the context of quantum gravity. The holographic principle is taken to be the guiding feature behind quantum gravity, rather than the string principle. As such it implies a reduction in the true number of quantum gravity degrees of freedom in line with the counting of degrees of freedom in string theory. Thus implementing correctly the holographic principle [112, 113, 105, 1] in quantum gravity should result in a softening of amplitudes akin to that which occurs in string theory. From here on all discussions will take place in the context of gravity using the Einstein – Hilbert action including cosmological constant apart from some string-theory related comments in the final sections.

The role of high energy scattering has been emphasized by 't Hooft in the context of the black hole evaporation process. As is well known, the appearance of Hawking radiation can be attributed to the diverging red-shift of outgoing wave packets when propagated back to the region close to the horizon. Quantum gravitational effects are therefore expected to play a fundamental role and their inclusion is expected to restore the unitarity of the Hawking radiation. According to the picture of 't Hooft these gravitational interactions close to the horizon can be effectively described by shock wave configurations associated to the boosted particles. They have non-trivial backreaction effects, bringing about a shift in the geodesics of the outgoing particles and in the position of the horizon, as we have seen in the previous chapters. These correlations should in principle reduce the enormous degeneracy of states at the horizon of the black hole that one naively calculates using quantum field theory in the curved space-time of the near-horizon geometry. In this picture the horizon of the black hole becomes a sort of fluctuacting membrane due to incoming and outgoing particles and information of the bulk spacetime is projected holographically onto this surface.

In view of these developments it seems interesting to search for a more concrete relation between the general arguments of 't Hooft and Susskind and the AdS/CFT construction. In this chapter we discuss in general the eikonal limit of scattering in curved space-times and find that under certain rather general assumptions about the relevant classical backgrounds, the dynamics of gravity is described by a theory that lives only on the boundary of the space-time. We also find that, from the bulk point of view, some of the classical solutions of this boundary theory describe shock-waves moving from the boundary to the bulk in Einstein spaces. On the way to finding classical backgrounds for our quantum theory we need the general solution of a two-dimensional gravity model analysed in [14]. Our solutions include the shock-wave solution constructed by Horowitz and Itzhaki [72] and this will be discussed in some detail in section 3.7.

We will also find a boundary description of scattering in the case of asymptotically de Sitter space-times. This boundary action is defined on the past and future space-like boundaries of the de Sitter space and may be important for discussions of causality and locality of holographic duals of de Sitter space [22, 67, 45]. To our knowledge, this is the first explicit boundary descripition of the dynamics in de Sitter space.

This chapter is organised as follows. In section 3.2 we will describe the setup in which our analysis takes place in particular reviewing the basic idea of [121, 122] in which a rescaling is made of the Einstein − Hilbert action thus separating it into three pieces each scaling differently in the eikonal limit. In section 3.3 we discuss the solutions to the classical part of this action in various regimes. In section 3.4 we introduce shock-wave configurations and then in section 3.5 we show how the off-diagonal part of the Einstein equations will be implemented. In section 3.6 we discuss the derivation and details of the resulting boundary action and in section 3.7 we show how our analysis is related to and extends the construction of Horowitz and Itzhaki [72]. Finally in section 3.8 we make some comments on our results and some other concluding remarks.

3.2 The setup

We consider high-energy scattering in spacetimes with a non vanishing cosmological constant Λ. Our basic construction is a direct generalization of that used in [121, 122] and thus we will consider an almost forward scattering situation. One introduces two scales, $\ell_{\|}$ and $\ell_{\perp}$: the former is the typical longitudinal wavelength of the particles while the latter represents the impact parameter. Due to the presence of the cosmological constant we also have an additional scale ℓ – the radius of curvature $\Lambda \sim \frac{1}{\ell^2}$. For high-energy forward scattering $\ell_{\|}$ is typically of the order of the Planck length $\ell_{\rm Pl}$, $\ell_{\perp} \gg \ell_{\|}$. This set of length scales characterizes the eikonal limit of the scattering process which for gravity is a linearized regime. We will also deal with two different cases according to large or small values of the cosmological constant present in the problem. In general we then find that for $\ell_{\perp}$ small on the cosmological scale the scattering takes place in the locally flat space-time. On the other hand for impact parameters that are large on the cosmological scale, there are significant changes in the scattering process due to the curvature. The final result is conceptually the same however as we find that for shock-wave scattering the process can always be described by a lagrangian on the boundary at infinity of the space-time.

Our general strategy will be to choose dimensionless co-ordinates by extracting the natural length scale in the corresponding directions and therefore we will consider the Einstein – Hilbert action plus a non-vanishing cosmological constant and exterior curvature K,

$$S = \frac{1}{\ell_{\rm Pl}^{d-2}}\left[\int_M \mathrm{d}^d x \sqrt{-G}\,(R - 2\Lambda) + \int_{\partial M} \mathrm{d}^{d-1} x \sqrt{\gamma}\, 2K\right], \tag{3.1}$$

making a rescaling in the longitudinal x^α and transverse co-ordinates y^i according to the respective scales, as explained in section 1.1.2 of the introduction. Under a rescaling of the metric, the action rescales as

$$\epsilon^{d-4} S_E = \left(\frac{S_0}{\epsilon^2} + \frac{S_1}{\epsilon} + S_2\right) \tag{3.2}$$

where $\epsilon = \ell_{\rm Pl}/\ell_{\perp} \sim \ell_{\|}/\ell_{\perp}$ is a very small dimensionless parameter. $S_2 = S_{\|}$ therefore is the strongly coupled part of the action while $S_0 = S_{\perp}$ is the weakly coupled part. The former is non pertubative while the latter is essentially classical. The role of S_1 will be discussed in the following but as is clear it also contributes to the classical part of the action in the limit of small ϵ. Under the above rescaling the cosmological term scales as $\frac{\ell_{\perp}^2}{\epsilon^{d-4}}$ and thus becomes part of the classical $S_{\perp}$ or the "quantum" $S_{\|}$ depending on the size of $\ell_{\perp}$ in comparison to the cosmological scale, ℓ. We will consider both the case in which the cosmological constant is added to the classical part of the action – the "strongly curved regime" or the regime of large impact parameter – and the case when the cosmological constant is included in the strongly coupled part of the action – the "flat regime" or regime of small impact parameter.

3.2.1 Scaling and small fluctuations

We will actually consider a metric that at leading order is block diagonal – the blocks corresponding to the plane of the scattering and the plane transverse to the scattering. We will consider a rescaling of the metric (equivalent but more convenient than that of the co-ordinates discussed above) such that

$$G_{\mu\nu} = \begin{pmatrix} g_{\alpha\beta} & h_{\alpha i} \\ h_{i\alpha} & h_{ij} \end{pmatrix} \to \begin{pmatrix} \ell_{\parallel}^2 \, g_{\alpha\beta} & \ell_{\parallel}\ell_{\perp} \, h_{\alpha i} \\ \ell_{\parallel}\ell_{\perp} \, h_{i\alpha} & \ell_{\perp}^2 \, h_{ij} \end{pmatrix}. \tag{3.3}$$

We use a notation where Greek indices label longitudinal variables, and Latin indices label transverse ones, $x^\mu = (x^\alpha, y^i)$.

In addition to this rescaling of the energy scales, we will also make the assumption that the off-diagonal blocks of the metric are small. In the end then we will be making a double expansion of the action, in ϵ and in $h_{i\alpha}$.

In the limit that $\epsilon \to 0$ the leading terms in the action become classical and thus we need to derive and examine first the equations of motion arising from S_0 and S_1 given our choice of metric. S_0 always becomes a covariant $1+1$ dimensional action and has no terms linear in the small off-diagonal part of the metric $h_{i\alpha}$. S_1 starts at linear order in $h_{i\alpha}$ and the equation of motion here comes from the variation with respect to $h_{i\alpha}$ imposing the vanishing of the off-diagonal block of the Ricci tensor $R_{i\alpha}$. The remaining part S_2 of the action is the most interesting part as it is not removed in our limit and basically describes the dynamics of the eikonal limit of scattering at high-energy and large impact parameter. We will find that this action contains no bulk degrees of freedom and thus reduces to a boundary term. The details of the scaling of the curvature components are in Appendix B.1. At each order in ϵ the action gets contributions from different orders in the expansion of the Ricci tensor. The leading order ϵ^{-2} term in the action has contributions from $R_{\alpha\beta}$ at order ϵ^0 and from R_{ij} at order ϵ^{-2}; the subleading order at ϵ^{-1} in the action comes solely from the leading term in $R_{i\alpha}$; while the final term at order ϵ^0 has contributions from the remaining higher order terms in $R_{\alpha\beta}$ and R_{ij}. The resulting double expansion in ϵ and $h_{i\alpha}$ is:

$$\begin{aligned} \epsilon^{d-4} \, S &= \frac{1}{\epsilon^2} \int_{\mathcal{M}} \sqrt{-gh} \left(R_g + \frac{1}{4}(h^{ik}h^{lm} - h^{il}h^{km}) \partial_\alpha h_{ik} \partial_\beta h_{lm} g^{\alpha\beta} \right) \\ &- \frac{2}{\epsilon} \int_{\mathcal{M}} \sqrt{-gh} \, h^{i\alpha} R_{i\alpha} \\ &+ \int_{\mathcal{M}} \sqrt{-gh} \left(R_h + \frac{1}{4}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}) \partial_i g_{\alpha\beta} \partial_j g_{\gamma\delta} h^{ij} \right) \\ &+ \int_{\partial\mathcal{M}} \sqrt{\gamma} \, 2K - 2\ell_{\perp}^2 \int_{\mathcal{M}} \sqrt{-gh} \, \Lambda. \end{aligned} \tag{3.4}$$

Considering a path integral for this action we see that the first two terms become classical as $\epsilon \to 0$. The cosmological constant can be moved to different orders of ϵ depending on its scaling with $\ell_{\parallel}$ or with $\ell_{\perp}$. Physically the mobility of the cosmological constant corresponds to the relationship between the scale of curvature of the space-time and the impact parameter of the scattering process under consideration. In the regime for which the curvature of the space-time does not really enter into the dicussion we find that the analysis is similar to that in flat space, though with corrections to the boundary action coming from the cosmological constant. In the other regime for which the impact parameter is larger than the radius of curvature, the space-time in the plane of scattering is curved and the analysis more subtle. The result again is that the scattering process can be described by a now non-quadratic lagrangian that lives on the boundary of the space-time.

The contribution of the exterior curvature will follow the usual construction of the Einstein – Hilbert action. It will split under rescaling to give contributions to the boundary in such a way that these boundary terms have their usual effect. That is, at the leading "classical" orders they will simply cancel boundary terms that come from integrating by parts when varying the action to get the equations of motion. The rescaling of the co-ordinates acts on the exterior curvature part of the action in such a way that it only contributes to the action at order ϵ^{-2} and ϵ^{-1} and thus will not provide any addition to our final boundary action which is at order ϵ^0. The details of the scaling of the exterior curvature part of the action are given in Appendix B.2.

The general setup that is obtained via this rescaling of the action by the factor ϵ (which depends on the energy scales of the problem) is one in which we have an energy dependent action. This means that we are not considering a high energy process in a theory that is already defined, but rather we are using the high energy "eikonal" limit to define for us a new action that (hopefully) isolates the degrees of freedom that are important for the problem at hand. In particular, as we will see from the classical solutions that come from the small ϵ limit, the space-time splits into a $2 + (d - 2)$ configuration in which the two parts are coupled only through the constraint that the off-diagonal part of the curvature vanish. The interaction between the two parts of the space-time - that transversal and that longitudinal – is restricted by $R_{i\alpha} = 0$. Therefore in the case of large cosmological constant although one may be tempted to interpret this as a limit of small AdS_d it is not. It is more simply a case in which the separation of the shock-waves in the transverse part of the space-time is large, and the size of the AdS_2 in the longitudinal space corresponds to a large curvature. However this is not obviously the same as a scattering in say the context of string theory in an AdS_d with large curvature, though it does retain some of the important features.

In the next three sections we will discuss in turn each order in ϵ of this rescaled action.

3.3 The solutions

The geometry in the longitudinal plane of the scattering is determined by the saddle point of S_0. This is the classical part of the action which is of order $\frac{1}{\epsilon^2}$:

$$\begin{aligned} S_0[g,h] &= \int_{\mathcal{M}} \sqrt{gh}\,[R[g] - 2\Lambda + \frac{1}{4}\,g^{\alpha\beta}\partial_\alpha h_{ij}\partial_\beta h_{kl}(h^{ik}h^{jl} - h^{ij}h^{kl})] \\ &= \int_{\mathcal{M}} \sqrt{gh}\,[R[g] - 2\Lambda + \frac{1}{4}h^{ij}\Box h_{ij} - \frac{1}{2}(\nabla \log \sqrt{h})^2]. \end{aligned} \tag{3.5}$$

The form of the action after the last equality sign is particularly useful to derive the equations of motion with respect to the transverse metric. We use an index-free notation where all derivatives are with respect to longitudinal variables. The action indeed does not contain transverse derivatives, and so the dynamics in the transverse space is trivial.

The equations of motion derived from this action are:

$$\begin{aligned} g_{\alpha\beta}[-\Lambda + \frac{1}{8}\,\mathrm{Tr}\,(\nabla h)^2 - \frac{1}{8}\,(\mathrm{Tr}\,\nabla h)^2] - \frac{1}{4}\mathrm{Tr}\,(\nabla h)^2_{\alpha\beta} + \frac{1}{4}(\mathrm{Tr}\,\nabla h)^2_{\alpha\beta} + & \\ + \frac{1}{\sqrt{h}}(\nabla_\alpha \nabla_\beta - g_{\alpha\beta}\Box)\sqrt{h} &= 0 \\ \frac{1}{2}h^{ij}[R[g] - 2\Lambda + \frac{1}{4}h^{kl}\Box h_{kl}] - \frac{1}{4}h^{ik}h^{jl}\Box h_{kl} + \frac{1}{2}h^{ij}(\nabla \log \sqrt{h})^2 + & \\ + \frac{1}{2}\nabla \log \sqrt{h}\nabla h^{ij} + \frac{1}{4}\Box h^{ij} + \frac{3}{4}h^{ij}\Box \log \sqrt{h} &= 0. \end{aligned} \quad (3.6)$$

Note that taking the trace of the first equation we get:

$$(\Box_g + 2\Lambda)\sqrt{h} = 0. \quad (3.7)$$

So far we made no assumptions concerning the particular form of the solution. However, to solve the equations of motion we need some specific ansatz that is suitable for describing a forward scattering situation and allows among the various possibilities for the presence of the cosmological constant. In particular, the form of the metric used in [121],

$$h_{ij} = \tilde{h}_{ij}(y), \quad (3.8)$$

does not allow for non-trivial solutions in all cases that we will study. We then need to assume that in general the transverse metric depends on the longitudinal co-ordinates through a warp factor

$$h_{ij}(x^\mu) = e^{\chi(x,y)}\tilde{h}_{ij}(y). \quad (3.9)$$

This is of course not the most general ansatz but it is general enough so as to give non-trivial solutions of the equations of motion. This ansatz can also be used to study radial scattering situations provided one chooses a time and a radial co-ordinate in the longitudinal directions. We will treat the $d = 3$ case separately due to various inconvenient factors of $(d-3)$ in the following general analysis.

Substituting the ansatz (3.9) in the equations of motion (3.6) gives the same equations of motion that follow from the reduced action:

$$S_0 = S_\perp = \int_M \sqrt{-g\tilde{h}}\, e^{\frac{(d-2)\chi}{2}} \left(R[g] - 2\Lambda - \frac{(d-2)(d-3)}{4} g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \right) \quad (3.10)$$

Making the following field redefinition

$$\phi(x,y) = (\frac{d-3}{2(d-2)})^{1/2} \exp\left((\frac{d-2}{4})\chi(x,y) \right) \quad (3.11)$$

one gets

$$S_\perp = -8\int_M \sqrt{-g\tilde{h}}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{d-2}{4(d-3)}\phi^2(R[g]-2\Lambda)\right). \tag{3.12}$$

The eikonal limit restricts us to consider the extrema of (3.12). It is interesting to note that with the assumption (3.9) the problem is reduced to a general two-dimensional gravity plus scalar field as studied in [14]. More properly, since the transverse fluctuations are suppressed in the leading order (in $h_{i\alpha}$) term of the weakly coupled action, its explicit expression will not contain, as shown by the scaling arguments, transverse derivatives. Therefore the action still depends on all four co-ordinates but the dependence on the transverse directions is only parametric. The equations of motion for the metric and the scalar field ϕ are:

$$\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\,g_{\alpha\beta}g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi = \frac{d-2}{4(d-3)}(\Lambda\, g_{\alpha\beta}\,\phi^2 + (g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta)\phi^2) \tag{3.13}$$

$$\Box\phi + \frac{(d-2)}{4(d-3)}(R[g]-2\Lambda)\phi = 0 \tag{3.14}$$

As mentioned, these are the same equations of motion as (3.6) for our warped metric.

As proved in the paper [14], all classical solutions have a Killing vector that is perpendicular to the curves of constant scalar field. When $\Lambda < 0$, the solutions are static. Therefore we can in this case choose the longitudinal metric $g_{\alpha\beta}$ to be of the form

$$\mathrm{d}s^2 = -e(x)^2\mathrm{d}t^2 + g(x)^2\mathrm{d}x^2, \tag{3.15}$$

where also

$$\phi = \phi(x), \tag{3.16}$$

in co-ordinates where $x^\alpha = (x,t)$. These solutions have a boundary at spacelike infinity. More properly in our case, as we will see below, e and g depend on the transverse co-ordinates too, since we are considering the two dimensional longitudinal manifold times the transverse space.

When $\Lambda > 0$, the solutions are time-dependent [14]. They are de Sitter-type cosmological solutions with a spacelike boundary. These solutions can be obtained either by analytic continuation of the above solutions from negative to positive cosmological constant, or by choosing a time-dependent metric from the beginning:

$$\mathrm{d}s^2 = -g(t)^2\mathrm{d}t^2 + e(t)^2\mathrm{d}x^2, \tag{3.17}$$

where now

$$\phi = \phi(t). \tag{3.18}$$

These are in fact the type of solutions analysed in [14].

3.3.1 Large Curvature

For the case of a negative cosmological constant, the general solution to the equations (3.13)-(3.14) can easily (details in Appendix B.3) be found and is:

$$\begin{aligned} \phi(r) &= \psi(r)^{\gamma} \\ e(r) &= C\psi(r)^{\frac{\gamma}{4Q}}\dot{\psi}(r) \end{aligned} \tag{3.19}$$

where

$$\psi(r) = Ae^{\sqrt{\frac{\lambda}{4Q\gamma}}r} + Be^{-\sqrt{\frac{\lambda}{4Q\gamma}}r}, \tag{3.20}$$

$$\mathrm{d}r = g(x)\mathrm{d}x, \tag{3.21}$$

and

$$\gamma = \frac{4Q}{1+8Q} \tag{3.22}$$

and where $\lambda = -\frac{(d-2)}{2(d-3)}\Lambda$, $Q = \frac{(d-2)}{4(d-3)}$.

Note that A, B and C are constant with respect to the longitudinal co-ordinates. However they can have an arbitrary dependence on the transverse co-ordinates y^i. Their precise form is fixed by imposing opportune boundary conditions depending on the spacetime under consideration.

It is also interesting to notice that if one takes either of A or B to zero, this two-dimensional metric has constant curvature and is actually just the metric on AdS_2 – the entire space-time metric being AdS_2 times the $(d-2)$-dimensional transverse geometry plus a warp factor.

The fact that the transverse metric $\tilde{h}_{ij}(y)$ is not determined by the equations of motion means that it is an arbitrary classical back-ground. This is also the case for the small curvature case to be considered in the next subsection.

3.3.2 Small Curvature

In the small curvature regime, the cosmological constant term belongs to the strongly coupled part of the action, as discussed. The classical action that we then have to consider is therefore (3.10) with $\Lambda = 0$. However, putting the cosmological constant to zero in the solutions above is a singular limit. It is easy to directly solve for the metric in this case and one finds:

$$\begin{aligned} \phi(r) &= (Ar+B)^{\gamma} \\ e(r) &= C(Ar+B)^{\frac{\gamma}{4Q}}, \end{aligned} \tag{3.23}$$

Again, A, B and C are allowed to depend on the transverse co-ordinates. The curvature is

$$R[g] = \frac{16QA^2}{(1+8Q)^2(Ar+B)^2}. \tag{3.24}$$

Notice that it is always positive. In the limit $B \to \infty$ we recover flat space, which was not a solution of the equations of motion in the strong curvature regime. In the region $r \ll B$, the space has locally positive constant curvature.

Note that there is also a degenerate flat space solution for which ϕ is constant and

$$e = Ar + B. \tag{3.25}$$

Even though these solutions could not be obtained directly from those with Λ non-zero by setting Λ to zero they can be obtained as near-horizon limits of those geometries, and this exactly corresponds to first shifting the co-ordinate r and then taking the limit of small cosmological constant.

The second case, (3.25), is the near-horizon geometry for the solutions of the previous section, for $B/A > 0$. Indeed, a simple co-ordinate transformation brings the near-horizon metric into the form of the Rindler metric (see Appendix A.13). In the same way (3.23) is the near-horizon geometry in the case $B/A < 0$, and again a simple co-ordinate transformation brings it into the form of a Rindler type metric with singular horizon (again see Appendix B.1).

3.3.3 Three-dimensional space-time

The above formulae are not directly applicable for $d = 3$, although one can obtain the equations of motion by carefully setting $d = 3$ in the above equations. The action in three dimensions is:

$$S_\perp = \int \sqrt{-g\tilde{h}}\,\phi^2(R[g] - 2\Lambda), \tag{3.26}$$

where $\phi = e^{\chi/4}$. The equations of motion for the scalar field and the metric are:

$$\nabla_\alpha \nabla_\beta \phi^2 - g_{\alpha\beta}\nabla^2\phi^2 - g_{\alpha\beta}\Lambda\phi^2 = 0, \tag{3.27}$$

$$R[g] = 2\Lambda, \tag{3.28}$$

and so the space-time always has constant curvature. It is therefore not surprising that the only solution we find is AdS_2. These equations are totally symmetric under interchanges of ϕ and e, and under reflections $r \to -r$. Therefore, the general solution is

$$\begin{aligned} \phi(r) &= A\,e^{r/2\ell} \\ e(r) &= B\,e^{-r/\ell}, \end{aligned} \tag{3.29}$$

where ℓ is the AdS radius. This solution corresponds to pure AdS, as expected, with a scalar field ϕ that vanishes at the boundary and has a singularity at the horizon.

When $\Lambda > 0$, we obtain 3-dimensional de Sitter space.

3.4 Gravity at high energy and shock waves

This section is a necessary digression into the shock-wave solutions to the classical part of our action. We need to understand the form of these shock-waves as they will motivate our final choice for the metric that we will use in the remaining non-classical part of our action. The physics in the bulk that can be described classically via these shock-waves will then be the physics that is encoded in the boundary action.

It turns out that scattering at Planckian energies is dominated by the gravitational force. Therefore one should have a complete theory of quantum gravity to describe these processes. However already in the eikonal regime that we are considering one can use semiclassical methods to get useful information.

At leading order gravitational interactions can indeed be described by shock wave configurations – gravitational waves with a longitudinal impulsive profile. Essentially this is the gravitational field surrounding a particle whose mass is dominated by kinetic energy therefore representing a sort of massless regime of general relativity [2, 3, 33, 40, 73, 74, 95, 114].

Explicit solutions in general spacetimes and their physical effects have been described by Dray and 't Hooft [40], using the so called cut and paste method (see chapter 2).

For example, the gravitational field of a massless particle in flat spacetime can be described by a metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v - 4p\ln(|x^i|^2)\,\delta(u)\mathrm{d}u^2 + \mathrm{d}x^i\mathrm{d}x_i,$$

where p is the momentum of the massless particle. The physical effects of such configurations play a crucial role in 't Hooft's description of the evaporation of a black hole. We refer to [113] for a detailed account and references.

Choosing $x^\mu = (x^+, x^-, x^i)$ and placing the massless particle at $x^+ = x^i = 0$, a natural way to rewrite this metric is then

$$\mathrm{d}s^2 = \partial_\alpha X^- \mathrm{d}x^\alpha \mathrm{d}x^+ + \mathrm{d}x_i^2$$

with

$$X^- = x^- + p\,\theta(x^+)\ln(|x^i|^2)$$

This means that if we want to describe the scattering of two high energy particles before a collision takes place we must use the generalised shockwave configuration with the metric in the longitudinal plane being of the form:

$$\mathrm{d}s^2 = \partial_\alpha X^a \partial_\beta X^b \eta_{ab}\,\mathrm{d}x^\alpha \mathrm{d}x^\beta,$$

thus allowing a pair shock-waves of the above type in both x^+ and in x^-. Here the SO(1,1) X^a vectors can in principle depend on all space-time co-ordinates. These are the configurations studied in [122] and below we will generalise this construction to include the presence of curvature in the longitudinal plane.

3.5 The constraint and solution-ansatz

The second order in our expansion is quite simple. It is

$$-\frac{2}{\epsilon}\int\sqrt{-g}hh^{i\alpha}R_{i\alpha} \tag{3.30}$$

As this is order ϵ^{-1} we also need to implement the corresponding equation of motion (as we did for the leading order in the previous section). In this order basically the equation of motion appears as a constraint $R_{i\alpha}=0$ on the general solutions.

Before implementing this constraint we will go back to the construction of [122] where it is shown how to change variables in a way that simplifies the following analysis. The saddle-point of the transverse part of the action S_0 gives the dominant vacuum field configurations. In the absence of the cosmological constant there was only

$$\begin{aligned} R[g] &= 0 \\ h_{ij} &= h_{ij}(y). \end{aligned} \tag{3.31}$$

As recounted in the previous section for massless shock-wave configurations we will choose a parametrization of the metric via diffeomorphisms that represents these shock-waves,

$$g_{\alpha\beta}=\partial_\alpha X^a\partial_\beta X^b\eta_{ab} \tag{3.32}$$

where the $X^a(x,y)$ are diffeomorphims which relate $g_{\alpha\beta}$ to the flat metric. Note that they are maps of the two dimensional x^α plane onto itself being however allowed to vary in the transverse directions and therefore represent transverse co-ordinate dependent displacements in the longitudinal co-ordinates. These X^a fields have the appearance of diffeomorphisms in the world-volume of the two-dimensional sigma-model and as such would appear to not introduce any new degrees of freedom. However, in the d-dimensional theory this is no longer really true as we are not considering the full transformation of the higher dimensional metric under these transformations. Nevertheless, due to the constraint coming from the off-diagonal part of the Einstein action we will see that these fields do not contribute additional bulk degrees of freedom.

An intermediate and useful step required to derive the boundary action and used to great effect in [121] is to express the strongly coupled action in terms of fields V_i^α defined as

$$\partial_i X^a = V_i^\alpha\partial_\alpha X^a. \tag{3.33}$$

These fields were introduced and motivated physically in terms of fluid velocity in [121]. In the gravitational setup presented here they can be thought of as zweibeins (see also [77]) for a two-dimensional sigma-model describing the embedding of the scattering plane into the transverse space. They considerably simplify the action and help to conceptualize our configuration from the sigma-model point of view.

This definition could also have been motivated by the simple practical consideration that in order to rewrite the strongly coupled action as a boundary action one needs to remove derivatives in the transverse directions to give one an action that is covariant in the longitudinal directions. As a consequence one tries to express every derivative in the transverse directions in terms of a derivative in the longitudinal ones. This is precisely obtained utilizing this definition of the V_i^α fields. In this way the indices labelling transverse directions act as an internal symmetry of the sigma-model from the point of view of the longitudinal spacetime. This will become clear in the next section where we write the general explicit form for the boundary action for all $d \geq 3$ and in both the strong and weak curvature regimes.

This construction is basically identical for the more general metrics considered here. As we have seen in Section 4, the conditions (3.31) are too restrictive and one ends up in this general setup case with a family of solutions specified by $g_{\alpha\beta}$ and χ. A natural generalization of the above parametrization of $g_{\alpha\beta}$ is then,

$$g_{\alpha\beta} = e^{\sigma(X)} \partial_\alpha X^a \partial_\beta X^b \eta_{ab}. \tag{3.34}$$

thus allowing the presence of a warp factor in the $2 + (d-2)$ decomposition of the metric. In principle σ may also have some explicit y dependence, however this would correspond to a more complicated sigma model than the one we are presently considering. As stated several times, the introduction of X is simply a statement that the scattering configuration described by shock-waves is described simply via singular coordinate tranformations with support only along light-cones in the scattering plane and thus the classical solution $\sigma(x)$ after the shock wave ansatz becomes simply $\sigma(X)$ and similarly $\chi(x)$ becomes $\chi(X)$. As in the previous case [121], we define fields V_i^α by

$$\partial_i X^a = V_i^\alpha \partial_\alpha X^a, \tag{3.35}$$

which in turn gives when lowering the longitudinal index

$$V_{i\alpha} = e^{\sigma(X)} \partial_i X^a \partial_\alpha X_a. \tag{3.36}$$

With the use of the $V_{i\alpha}$ the longitudinal metric changes under reparametrisations of the transverse co-ordinates according to,

$$\partial_i g_{\alpha\beta} = \nabla_\alpha V_{\beta i} + \nabla_\beta V_{\alpha i}. \tag{3.37}$$

Finally we also will have

$$h_{ij} = e^{\chi(X)} \tilde{h}_{ij}(y). \tag{3.38}$$

Notice that this form of the solutions captures both the cases $\Lambda < 0$ and $\Lambda > 0$.

3.6 The Effective Boundary Theory

We now examine how our classical solution – ansatz leads us to the general result that in this setup the transverse action $S_\parallel$ always reduces to a boundary action.

Making the substitutions of our solution – ansatz in the leading order (ϵ^0) action,

$$\begin{aligned} S_\parallel[g,h] &= \int \sqrt{-gh} \left[R[h] - \frac{1}{4} h^{ij} \partial_i g_{\alpha\beta} \partial_j g_{\gamma\delta} \, \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \right] \\ &= \int \sqrt{-gh} \left[R[h] - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \, h^{ij} \nabla^\alpha V_i^\beta \nabla^\gamma V_j^\delta + \frac{1}{2} h^{ij} R_i R_j \right], \end{aligned} \tag{3.39}$$

where

$$R_i \;=\; \epsilon^{\alpha\beta}\nabla_\alpha V_{i\beta}. \tag{3.40}$$

3.6.1 Strong curvature regime

Filling in the solutions of the classical equations of motion,

$$\begin{aligned} g_{\alpha\beta} &= e^{\sigma(X)}\,\partial_\alpha X^a \partial_\beta X^b\,\eta_{ab} \\ h_{ij} &= e^{\chi(X)}\,\tilde{h}_{ij}, \end{aligned} \tag{3.41}$$

we get

$$\begin{aligned} S_{\|} &= \int \sqrt{-g\tilde{h}}\, e^{\frac{d-4}{2}\chi}\Big[R[\tilde{h}] - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\tilde{h}^{ij}\nabla^\alpha V_i^\beta \nabla^\gamma V_j^\delta + \frac{1}{2}R_i^2 + \\ &- (d-3)\Box_{\tilde{h}}\chi - \frac{1}{4}(d-3)(d-4)(\partial_i\chi)^2\Big], \end{aligned} \tag{3.42}$$

and from now on we raise and lower transverse indices by means of the rescaled metric $\tilde{h}_{ij}$.

The action splits into a bulk and a boundary term:

$$\begin{aligned} S_{\|} &= S_{\text{bulk}} + S_{\text{bdry}} \\ &= \int_{\partial M} \mathrm{d}x^\alpha \sqrt{\tilde{h}}\, e^{\frac{d-4}{2}\chi}\Big(R[\tilde{h}]\, e^\sigma\, X^0\partial_\alpha X^1 + e^\sigma \epsilon_{ab}\ \partial_i X^a \times \\ &\times\ [\partial^i\partial_\alpha X^b + \frac{1}{2}\nabla^\beta\sigma(\partial_\alpha X^b V^i_\beta - \partial_\beta X^b V^i_\alpha)] - \frac{1}{2}V_{i\alpha}R^i - \frac{d-3}{2}\epsilon_{\alpha\beta}V^{i\beta}\partial_i\chi\Big) + \\ &+ \int_M \sqrt{-g\tilde{h}}\, e^{\frac{d-4}{2}\chi}\, V^{i\alpha}R_{i\alpha}, \end{aligned} \tag{3.43}$$

where by X^a we mean the variation of X^a around its infinite value. Filling in the constraint

$$\begin{aligned} R_{i\alpha} &= \frac{1}{2}R[g]V_{i\alpha} + \frac{1}{2}\epsilon_{\alpha\beta}\nabla^\beta R_i + \frac{1}{2}\partial_\alpha\chi\nabla^\beta V_{i\beta} - \frac{d-3}{2}\partial_i\partial_\alpha\chi + \\ &+ \frac{d-4}{4}R_i\nabla^\beta\chi(\nabla_\alpha V_{i\beta} + \nabla_\beta V_{i\alpha}) = 0, \end{aligned} \tag{3.44}$$

it obviously reduces to a boundary action. Note that this action will generally consist of two disconnected pieces corresponding to the two boundaries of the longitudinal spacetime.

When $\Lambda < 0$, as we have been implicitly assuming in this section, the boundary is timelike. In the large curvature regime, the discussion for $\Lambda > 0$ is more intrincate. Formally, the above derived action is valid in de Sitter space as well, the boundary being now a spacelike boundary at the future and past infinities. This agrees with Bousso's considerations on de Sitter space in [22]. As in this case the boundary theory is defined on an Euclidean manifold ∂M, the physical interpretation in terms of causality and locality of a corresponding holographic map is somewhat more mysterious than in the AdS case, as discussed in [22, 67, 45].

3.6.2 Weak curvature regime

In this regime there are two types of solutions, curved (singular) and flat. The action is the same as in the strong curvature regime, apart from an additional term proportional to the cosmological constant,

$$\begin{aligned} S_{\|}[g,h] &= \int \sqrt{-gh}\left[R[h] - 2\Lambda - \frac{1}{4}\, h^{ij}\partial_i g_{\alpha\beta}\partial_j g_{\gamma\delta}\, \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\right] \\ &= \int \sqrt{-gh}\left[R[h] - 2\Lambda - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\, h^{ij}\nabla^{\alpha}V_i^{\beta}\nabla^{\gamma}V_j^{\delta} + \frac{1}{2}h^{ij}R_iR_j\right]. \quad (3.45) \end{aligned}$$

Again filling in the general solutions we get:

$$\begin{aligned} S_{\|} &= S_{\text{bulk}} + S_{\text{bdry}} \\ &= \int_{\partial M} \mathrm{d}x^{\alpha}\sqrt{\tilde{h}}\,\Big((R[\tilde{h}] - 2e^{\chi}\Lambda)\ \ X^0\partial_{\alpha}X^1 + e^{\frac{d-4}{2}\chi+\sigma}\epsilon_{ab}\,\partial_i X^a \times \\ &\times\ \ [\partial^i\partial_{\alpha}X^b + \frac{1}{2}\nabla^{\beta}\sigma(\partial_{\alpha}X^bV_{\beta}^i - \partial_{\beta}X^bV_{\alpha}^i)] - e^{\frac{d-4}{2}\chi}(\frac{1}{2}\,V_{i\alpha}R^i + \frac{d-3}{2}\,\epsilon_{\alpha\beta}V^{i\beta}\partial_i\chi)\Big) + \\ &+\ \ \int_M \sqrt{-g\tilde{h}}\, e^{\frac{d-4}{2}\chi}\, V^{i\alpha}R_{i\alpha} \quad (3.46) \end{aligned}$$

The most interesting case is the flat-space solution, where the action is simply quadratic:

$$\begin{aligned} S_{\|} &= S_{\text{bulk}} + S_{\text{bdry}} \\ &= \int_{\partial M} \mathrm{d}x^{\alpha}\sqrt{\tilde{h}}\,[\epsilon_{ab}\,\partial_{\alpha}X^b\,(\frac{1}{2}R[\tilde{h}] - \Lambda - \triangle_{\tilde{h}})\,X^a - \frac{1}{2}\,V_{i\alpha}R^i] \\ &+\ \int_M \sqrt{-g\tilde{h}}\,e^{\frac{d-4}{2}\chi}\,V^{i\alpha}R_{i\alpha} \quad (3.47) \end{aligned}$$

The constraint then reads

$$R_{i\alpha} = \frac{1}{2}\,\epsilon_{\alpha\beta}\nabla^{\beta}R_i = 0. \quad (3.48)$$

Note that unlike [122] this does not imply that $R_i = 0$ but that R_i is a function only of the transverse co-ordinates $R_i(y)$. In particular then even for the flat space we have found that the complete analysis of this limit actually implies that there can be an additional term in the boundary action. It would be interesting to understand the physical meaning of this extra piece.

The action can be rewritten as

$$\begin{aligned} S_{\|} &= S_{\text{bdry}} \\ &= \int_{\partial M} \mathrm{d}x^{\alpha}\sqrt{\tilde{h}}\,\Big(\epsilon_{ab}\,\partial_{\alpha}X^a(\triangle_{\tilde{h}} + \Lambda - \frac{1}{2}R[\tilde{h}])\,X^b + \frac{1}{2}\partial_{\alpha}X^aR^i(y)\partial_iX_a\Big) \quad (3.49) \end{aligned}$$

which will be convenient for the discussions in the next section. Needless to say that in this case the classical solutions are independent of the value of the cosmological constant, and therefore the action (3.49) allows any value of Λ. We thus find ourselves with a quadratic action like that of [121]. Correspondingly there will be a way to quantize this action, write down the S-matrix and to study the inevitable non-commutativity of the boundary co-ordinates. In the next section we will consider the relationship between our construction and the curved space-time shock-wave scattering considered in particular in a paper by Horowitz and Itzhaki [72].

3.7 Shock-waves from eikonal gravity and the AdS/CFT Correspondence

The boundary action found in the small curvature regime for $R_i(y) = 0$ is quadratic and therefore easy to deal with. In fact it is a straightforward generalisation of the boundary action found in [121, 122].

Let us briefly discuss its quantum mechanical properties when we couple it to point particles. In this regime, and restricting ourselves to the classical solutions of the equations of motion, the longitudinal space is basically flat. Therefore, the coupling to point particles in this case goes precisely along the lines of section 5.1 of [121]. For details about the stress-energy tensor of a pointlike particle we refer to Appendix A.1.

Following [121], we represent the stress-energy tensor in terms of a momentum flux $P_{a\alpha}$ as follows:

$$T_{\alpha\beta} = P_{a\alpha}\,\partial_\beta X^a. \tag{3.50}$$

As shown in [121], in a forward scattering situation where the stress-energy tensor is concentrated in the longitudinal plane, the matter part of the action depends only on the boundary values of X^a. Its variation equals:

$$\delta S_{\text{matter}} = 2\int_{\partial\mathcal{M}} \mathrm{d}x^\alpha \sqrt{h}\,\epsilon_\alpha{}^\beta\, P_{a\beta}\delta X^a. \tag{3.51}$$

Coupling this action to the gravitational part of the action, (3.49), gives the standard shift equation, see e.g. (2.29).

Quantisation is now straightforward and, as discussed in [121] and also in chapters 1 and 2 of this thesis, it leads to non-trivial commutators for the co-ordinates X^a:

$$[X^a(y), X^b(y')] = i\epsilon^{ab} f(y, y'), \tag{3.52}$$

where f now satisfies the Green's function equation

$$(\triangle_{\tilde{h}} + \Lambda - \frac{1}{2}\,R[\tilde{h}])\,f(y, y') = \delta^{(d-2)}(y - y'). \tag{3.53}$$

As we have already discussed, we expect shock-waves to be described by our boundary action also. Let us first briefly discuss how shock-waves can be implemented in AdS [72].

We write pure AdS in the following co-ordinates,

$$\mathrm{d}s^2 = \frac{4}{(1 - y^2/\ell^2)^2}\,\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu, \tag{3.54}$$

where $y^2 = \eta_{\mu\nu} y^\mu y^\nu$. The stress tensor of a massless particle is computed in Appendix A.1 and gives

$$T_{uu} = -p\,\delta(u)\delta(\rho), \tag{3.55}$$

where ρ is the radial co-ordinate $\rho = \sum_{i=1}^{d-2} y_i^2$.

Horowitz and Itzhaki found the following solution of Einstein's equations with a massless particle:

$$\mathrm{d}s^2 = \frac{4}{(1 - y^2/\ell^2)^2}\,\left(\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu + 8\pi G_{\mathrm{N}}\, p\delta(u)(1-\rho^2/\ell^2)f(\rho)\mathrm{d}u^2\right) \tag{3.56}$$

provided

$$\triangle_h f - 4\,\frac{d-2}{\ell^2} f = \delta(\rho). \tag{3.57}$$

$\triangle_h$ is the Laplacian on the transverse hyperbolic space,

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_{d-3}^2}{(1-\rho^2/\ell^2)^2}, \tag{3.58}$$

and therefore (3.57) takes the form

$$f'' + \frac{d-3+(d-5)\rho^2/\ell^2}{\rho(1-\rho^2/\ell^2)} f' - \frac{4(d-2)}{\ell^2(1-\rho^2/\ell^2)^2}\, f \quad = \quad \delta(\rho). \tag{3.59}$$

The solutions to (3.57) are given by:

$$\begin{aligned}
\mathrm{AdS}_3: \qquad f(\rho) &= \frac{\ell}{2}(C+\theta(\rho))\,\sinh\log\left(\frac{\ell+\rho}{\ell-\rho}\right) + \ell D\cosh\log\left(\frac{\ell+\rho}{\ell-\rho}\right)\\
\mathrm{AdS}_4: \qquad f(\rho) &= C\,\frac{1+\rho^2/\ell^2}{1-\rho^2/\ell^2}\,\log(\rho/D) + \frac{2C}{1-\rho^2/\ell^2}\\
\mathrm{AdS}_5: \qquad f(\rho) &= \frac{C}{1-\rho^2/\ell^2}\left(\frac{1}{\rho} + \frac{6\rho}{\ell^2} + \frac{\rho^3}{\ell^4}\right) + \frac{D}{\ell}\,\frac{1+\rho^2/\ell^2}{1-\rho^2/\ell^2},
\end{aligned} \tag{3.60}$$

D is an arbitrary constant to be determined by boundary conditions. C is a constant of order 1 that can be computed either by explicit computation or by matching with the Minkowski solutions. The shift function f of course behaves like the solutions for shock-wave in Minkowski space $f \sim \frac{1}{|x|^{d-4}}$ in the limit when the AdS radius divided by the impact parameter goes to infinity, $\ell/\rho \to \infty$. In fact the metric (3.56) was derived by boosting a black hole to the speed of light while sending its mass to zero and keeping its energy fixed.

Notice that for an Einstein space with negative curvature and curvature radius $\Lambda = -\frac{1}{2\ell^2}\,(d-1)(d-2)$, the above general equation for the shift function derived via our boundary action method (3.53) reduces to the condition (3.57) found by Horowitz and Itzhaki precisely when the transverse space is Euclidean AdS_{d-2}:

$$\mathrm{d}s^2 = 4\,\frac{\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_{d-3}^2}{(1-\frac{\rho^2}{\ell^2})^2}. \tag{3.61}$$

In this case, the transverse curvature is

$$R[\tilde{h}] = -\frac{1}{\ell^2}(d-2)(d-3). \tag{3.62}$$

The class of solutions to (3.53) is however much larger than only shock-waves in pure AdS. It allows for values of the transverse curvature that are postitive, negative or zero, and the cosmological constant is also allowed to be positive. In the limit $\Lambda \to 0$, all our results of course agree with the results found in [40].

It is not surprising that we find an approximate shock-wave from our boundary action only in the small curvature regime. These shock-waves have a smooth limit as $\Lambda \to 0$ which of course could not happen in the large curvature regime. Note that, as for the Dray-'t Hooft Ansatz (2.6), the transverse metric $\tilde{h}_{ij}(y)$ is not determined by Einstein's equations and is to be treated as a classical back-ground.

Horowitz and Itzhaki have argued that the CFT duals of shock-waves are "light-cone states" – states with their energy-momentum tensor localised on the boundary light-cone. It is tempting to argue that our boundary description should somehow be related to these light-cone states. Indeed, we have shown that our boundary theory describes bulk shock-waves in an approximate fashion. Hence one is led to speculate that our boundary action is somehow related to some sort of eikonal limit of a boundary CFT perturbed by the addition of light-cone states. Notice, however, that it is not at all clear how to prove such a relation. In particular it is not clear how light-cone states should be precisely described in field theory, although some attempts have been made in [98]. Related discussions can be found in [97, 106, 53] and, recently, in [56]. Furthermore if quantum gravity has a boundary description at all energies we have taken the eikonal limit of it, thereby explicitly breaking covariance of the boundary theory. An interesting question is whether it is possible to do an eikonal approximation in a covariant way, or whether it is possible to restore covariance afterwards, as discussed in the previous chapter. See also [113, 116, 33]. In particular, as we saw before, in the simplified 2+1 – dimensional setup, restoring Lorentz covariance is tantamount to going beyond the extreme eikonal regime.

It would be extremely interesting if we could find an analog of (3.52) in the context of the AdS/CFT correspondence. This would amount to identifying the operators X^a in the CFT and to interpreting them from the bulk point of view. Based on previous considerations by 't Hooft and a computation of the trajectories of massless particles outlined in Appendix A.1, they are expected to correspond to the positions of colliding particles, however a careful analysis is required. This could most easily be done using the techniques in [35] where boundary sources and operators are related to the coefficients of the perturbative expansion of bulk fields.

3.7.1 Boundary description of scalar fields

So far we have discussed single particles in an AdS background and interactions between quantum mechanical particles by means of shock-waves. One would however be ultimately interested in considering second quantised fields that interact gravitationally. In flat space, computing an S-matrix and extracting from it the amplitude for scattering between massless particles is a relatively straightforward task even if the interactions are gravitational [114, 113]. In AdS, however, things are much more complicated due to the presence of the timelike boundary and the impossibility to separate wavepackets. These problems can be sidestepped by imposing appropriate boundary conditions on the fields and ensuring that the S-matrix is unitary [9]. However, this is not possible for all the modes, and in the context of the AdS/CFT correspondence we are interested in considering both normalisable and non-normalisable modes. For other discussions of the AdS S-matrix, see [97, 106, 53]. In this chapter we will not consider this issue, but rather concentrate on the CFT duals of scalar fields with generic boundary conditions.

As a first step towards considering the full quantum mechanics of scalar fields interacting gravitationally in AdS, we consider scalar fields on an AdS-shock-wave background. We concentrate on conformally coupled scalar fields. These have the nice property that their equation of motion is invariant under Weyl rescalings, up to a certain weight. The Klein-Gordon equation for these fields is

$$\left(\Box_G - \frac{d-2}{4(d-1)}\, R[G]\right)\phi(y) = 0, \tag{3.63}$$

and so in the AdS-shock-wave background they have mass $m^2 = -\frac{d^2-1}{4\ell^2}$. In this section and in the following chapters, d will denote the dimension of the boundary of the $(d{+}1)$-dimensional bulk AdS.

We perform a conformal transformation by which we remove the double pole of the metric:

$$\mathrm{d}s^2 = G_{\mu\nu}\, \mathrm{d}y^\mu \mathrm{d}y^\nu = \frac{1}{\Omega(y)^2}\, \bar{G}_{\mu\nu} \mathrm{d}y^\mu \mathrm{d}y^\nu, \tag{3.64}$$

G being the AdS-shockwave metric (3.56). The Klein-Gordon equation transforms into

$$\bar{\Box}\, \bar{\phi}(y) = 0, \tag{3.65}$$

calculated in the metric $\bar{G}_{\mu\nu}$, and

$$\bar{\phi}(y) = \Omega^{\frac{1-d}{2}}\ \phi(y). \tag{3.66}$$

There is no curvature term in (3.65) because $R[\bar{G}] = 0$. For the metric $\bar{g}_{\mu\nu}$, the Laplacian factorises into a flat piece plus a shock-wave part,

$$\bar{\Box}\, \bar{\phi}(y) = \eta^{\mu\nu} \partial_\mu \partial_\nu \bar{\phi}(y) - p\, \delta(u)\, F(\rho)\, \partial_v^2 \bar{\phi}(y) = 0. \tag{3.67}$$

Equation (3.67) is difficult to solve in general due to the transverse derivatives[1] but it can be readily solved in the eikonal approximation. A simple plane-wave solution is given by

$$\phi(y) \quad = \quad \Omega(y)^{\frac{d-1}{2}}\ \exp\left[ikv + ikp\, \theta(u)\, F(\rho)\right], \tag{3.68}$$

[1]For exact solutions, see [49].

as one would expect from a computation of trajectories: the only effect of the shock-wave is a shift of the wave function over a distance given by the shift. The full solution gives, in the eikonal approximation,

$$\phi(y) = \Omega(y)^{\frac{d-1}{2}} \int \mathrm{d}^d k\, a(k)\, e^{ipk\,\theta(u)F(\rho)+ik_\mu y^\mu} + \text{c.c.}, \tag{3.69}$$

where $k_\mu^2 = 0$. Note that this sense of eikonal approximation is the same as in previous sections – all transverse derivatives are set to zero.

To interpret this classical field from the CFT point of view, it is easiest to go to Poincare co-ordinates where the boundary is at $r = 0$. The above field then has the following expansion [35]

$$\phi(r, x) = r^{\frac{d-1}{2}} \phi_0(x) + \cdots \tag{3.70}$$

as it approaches the boundary. This is the expected behaviour for a field of mass $m^2 = -\frac{d^2-1}{4\ell^2}$. As explained in the introduction, for a field of such mass the value of the field at infinity can have an interpretation either as the expectation value of the operator dual to the field or as a source. The coefficient at order $r^{\frac{d+1}{2}}$ is then interpreted as the source or as the dual operator respectively.

Let us consider the Δ_+-theory, where ϕ_0 corresponds to an operator of dimension $\Delta = \frac{d-1}{2}$,

$$\langle O(x) \rangle = -\phi_0(x). \tag{3.71}$$

The expression for ϕ_0 can be obtained from (3.69). Notice that as $r \to 0$ the step function approaches $\theta(u) \to \theta(\frac{t^2-\vec{x}^2}{t})$. This means that the operator $O(x)$ has different expectation values on either side of the light-cone, $|t| > |\vec{x}|$ and $|t| < |\vec{x}|$, and furthermore there is a reflection as $t \to -t$. The operator acquires a certain "dressing" inside the light-cone. In the Δ_--theory, where ϕ_0 is interpreted as a source for $O(x)$, we see that the effect of the shock-wave is to introduce an explicit time-dependence in the source.

As pointed out in [72] shock-waves in AdS correspond to states with a stress-energy tensor concentrated on the light cone. We have found that when we also turn on a source for an operator of dimension $\Delta = \frac{d-1}{2}$ in the back-ground of these light-cone states, the operator aquires different values on either side of the light-cone. Elaborating this a little bit further along the lines of [13] let us add that there is a map between the creation and the annihilation operators of the field ϕ and the composite operators in terms of which $O(x)$ is expanded. This however assumes a well-defined field theory for the scalar field ϕ in AdS, which we certainly have not constructed here (see however [9, 106, 97, 12]). One has to find a complete set of operators that generate the Hilbert space of the boundary theory and that have a well-defined inner product. This imposes additional conditions on the solutions (3.69) for them to be normalisable, like the quantisation of the frequencies. It would be most interesting to work out all these details, and to have an explicit field theory realisation of these phenomena.

The next step would be to consider gravitationally interacting fields in this AdS background. In reference [80] it was shown that fields interacting by means of shock-waves on a black hole horizon satisfy an exchange algebra, of the form:

$$\phi_{\rm out}(y)\phi_{\rm in}(x) = \exp\left[if^{ab}(x-y)\frac{\partial}{\partial x^a}\frac{\partial}{\partial y^b}\right]\phi_{\rm in}(x)\phi_{\rm out}(y), \tag{3.72}$$

where $f^{ab}(x-y) = \epsilon^{ab}f(x-y)$ depends on the transverse distance between the points x and y. Here the non-commutativity of the fields was ascribed to the fluctuations of the horizon due to in-coming and out-going shock-waves. In chapter 2, an alternative derivation of this exchange algebra has been given for Minkowski space. The derivation does not use the presence of a horizon, but only the fact that creation operators create particles that carry shock-waves with them and thus produce shifts on the back-ground space-time. This is closely related to the form (3.69) of the solutions of the Klein-Gordon equation, which up to a conformal factor is the same in AdS and in Minkowski space. Therefore it seems reasonable to expect that a similar kind of non-commutative behaviour is to be found in AdS. It would be interesting to interpret this in terms of operators in the CFT. Note however that when performing such a derivation one can no longer ignore the problem of correct quantisation of fields in AdS.

It seems likely that yet another way to derive the algebra (3.72) is by coupling our boundary action not to point particles but to scalar fields whose energy-momentum tensor is concentrated mainly in the longitudinal space.

When considering point particle fields in AdS and comparing them to quantities in the CFT, the UV/IR duality plays a crucial role [13, 109, 94]. Bulk translations correspond to boundary rescalings. This will be particularly important if one develops an S-matrix theory, like we do in the following section for asymptotically flat spaces. In the usual Poincare co-ordinates of AdS, it is easy to see that a constant rescaling of the bulk co-ordinate $r \to e^{\lambda}r$ together with a Weyl rescaling of the boundary metric is a symmetry of the metric. In the commonly used AdS co-ordinates where the double pole is only in the warp factor,

$$\mathrm{d}s^2 = \mathrm{d}y^2 + e^{2y}g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.73}$$

a translation $y \to y+\lambda$ induces a Weyl rescaling on the boundary, $g_{ij} \to e^{2\lambda}g_{ij}$. This symmetry is of course not a symmetry of the full theory due to infrared divergences, as we will study in more detail in chapter 4.

3.8 Comments and Conclusions

Our analysis is a semi-classical analysis in the sense that we have setup a path-integral involving $S_\parallel$ that in addition involves only the fluctuations with insertion of fields all taking place on the boundary. Thus we have actually constructed a general proof of a particular form of holography – that corresponding to interactions of massless particles via gravitational shock-wave dynamics encoded in a theory of fluctuations on the boundary.

We would like to point out that our derivation requires no specific gauge choice. This agrees with [122]. However we do impose the requirement on our metric that it is of an approximately $2+(d-2)$ block-diagonal form with small off-diagonal components $h_{i\alpha}$.

We have seen that in the course of our construction it was indeed very important to retain the small off-diagonal $h_{i\alpha}$ as it was precisely due to this that the constraint $R_{i\alpha}$ was derived and which was of importance to remove all bulk terms in the theory.

The fact that in this eikonal limit the theory becomes holographic in the sense described above is due not *only* to the fact that we treat essentially as a classical background the transverse metric but also to the crucial fact that one has an additional constraint to impose. As already remarked this constraint arises from the linear fluctuations in $h_{i\alpha}$ at order ϵ^{-1} in the rescaled action and is therefore associated to small off-diagonal pieces of the metric. In the end there is therefore no complete decoupling of transverse and longitudinal components of the metric as they are tied together by the non-trivial constraint $R_{i\alpha} = 0$ (3.44).

The off-diagonal constraint essentially restricts the variations of our solutions in the transverse directions. This is where the dependence on the transverse direction is really taken into account. If follows that to effectively obtain a boundary theory one simply imposes this constraint on the transverse dynamics. We could rephrase the state of affairs by saying that Einstein gravity in the eikonal is a topological theory on a two-dimensional manifold embedded in d dimensions, provided some constraints are imposed on the "lapse function" $V_{i\alpha}$ which allows one to move from one plane to another by means of transverse deformations.

We also recall that as stressed by 't Hooft the gravitational interactions close to the horizon of a black hole or more generally at high energies are precisely described by shock wave configurations associated to boosted particles. They have non-trivial back-reaction effects, bringing about a shift in the geodesics of the outgoing particles which induces a form of non-commutativity at the quantum level. This has been observed in the analysis of [122] and similarly occurs here in the particular subset of cases considered for which the boundary action is quadratic.

A particularly interesting result is the boundary action that we found for the asymptotically de Sitter case. This action is defined on a space-like surface which is the past or the future boundary of de Sitter space. This suggests that observables in a holographic description of de Sitter space can somehow be defined as correlation functions of a theory living on a space-like surface [22]. On the other hand, in [67] it is argued that such a prescription may have little physical meaning as there is no physical observer that can collect together the data of measurements on such a surface. We regard the solution of this conundrum as an important challenge for future research.

Throughout this chapter we have worked with the Einstein-Hilbert action

without including higher curvature corrections. However our method is perfectly applicable for these higher order terms also, and in fact considering them is important when the energy is increased above $1/\ell_{\mathrm{Pl}}$. When embedding our theory in a specific string theory, one also has to include additional matter fields. Notice, however, that for the case $d = 5$ it should be straightforward to embed our results in string theory by condidering backgrounds with a constant dilaton and a covariantly constant self-dual 5-form compactified for example on an S^5. This is left for future research.

Chapter 4

Holographic Reconstruction of Space-time in the AdS/CFT Correspondence

The contents of this chapter are based on [35]. For a review and explicit examples see also [101], and for a short account of the AdS/CFT correspondence see section 1.3 of the introduction and also reference [1].

We develop a systematic method for renormalising the AdS/CFT prescription for computing correlation functions. This involves regularising the bulk on-shell supergravity action in a covariant way, computing all divergences, adding counter-terms to cancel them and then removing the regulator. We explicitly work out the case of pure gravity up to six dimensions and of gravity coupled to scalars, but the techniques can be easily applied for other matter fields. The method can also be viewed as providing a holographic reconstruction of the bulk space-time metric and of bulk fields on this space-time, out of conformal field theory data. Knowing which sources are turned on is sufficient in order to obtain an asymptotic expansion of the bulk metric and of bulk fields near the boundary to high enough order so that all infrared divergences of the on-shell action are obtained. To continue the holographic reconstruction of the bulk fields one needs new CFT data: the expectation value of the dual operator. In particular, in order to obtain the bulk metric one needs to know the expectation value of stress-energy tensor of the boundary theory. We provide completely explicit formulae for the holographic stress-energy tensors up to six dimensions. We show that both the gravitational and matter conformal anomalies of the boundary theory are correctly reproduced. We also obtain the conformal transformation properties of the boundary stress-energy tensors.

4.1 Introduction and summary of the results

Holography states that a $(d+1)$-dimensional gravitational theory[1] (referred to as the bulk theory) should have a description in terms of a d-dimensional field theory (referred to as the boundary theory) with one degree of freedom per Planck area [112, 105]. The arguments leading to the holographic principle use rather generic properties of gravitational physics, indicating that holography should be a feature of any quantum theory of gravity. Nevertheless it has been proved a difficult task to find examples where holography is realised, let alone to develop a precise dictionary between bulk and boundary physics. The AdS/CFT correspondence [87] provides such a realisation [125, 109] with a rather precise computational framework [64, 125]. It is, therefore, desirable to sharpen the existing dictionary between bulk/boundary physics as much as possible. In particular, one of the issues one would like to understand is how space-time is built holographically out of field theory data.

The prescription of [64, 125] gives a concrete proposal for a holographic computation of physical observables. In particular, the partition function of string theory compactified on AdS spaces with prescribed boundary conditions for the bulk fields is equal to the generating functional of conformal field theory correlation functions, the boundary value of fields being now interpreted as sources for operators of the dual conformal field theory (CFT). String theory on anti-de Sitter (AdS) spaces is still incompletely understood. At low energies, however, the theory becomes a gauged supergravity with an AdS ground state coupled to Kaluza-Klein (KK) modes. On the field theory side, this corresponds to the large N and strong 't Hooft coupling regime of the CFT. So in the AdS/CFT context the question is how one can reconstruct the bulk space-time out of CFT data. One can also pose the converse question: given a bulk space-time, what properties of the dual CFT can one read off?

The prescription of [64, 125] equates the on-shell value of the supergravity action with the generating functional of connected graphs of composite operators, see (1.49)-(1.50). Both sides of this correspondence, however, suffer from infinities —infrared divergences on the supergravity side and ultraviolet divergences on the CFT side. Thus, the prescription of [64, 125] should more properly be viewed as an equality between bare quantities. One needs to renormalise the theory to obtain a correspondence between finite quantities. It is one of the aims of this chapter to present a systematic way of performing such a renormalisation.

The CFT data[2] that we will use are: which operators are turned on, and what is their vacuum expectation value. Since the boundary metric (or, more properly, the boundary conformal structure) couples to the boundary stress-energy tensor, the reconstruction of the bulk metric to leading order involves a detailed knowledge of the way the energy-momentum tensor is encoded holographically. There is by now an extended literature on the study of the stress-energy tensor in the context of the AdS/CFT correspondence starting from [11, 91]. We will build on these and other related works [43, 88, 83].

[1] In this and the next chapter, we use the convention the boundary is d-dimensional whereas the bulk is $(d+1)$-dimensional

[2] We assume that the CFT we are discussing has an AdS dual. Our results only depend on the space-time dimension and apply to all cases where the AdS/CFT duality is applicable, so we shall not specify any particular CFT model.

Our starting point will be the calculation of the infrared divergences of the on-shell gravitational action [70]. Minimally subtracting the divergences by adding counter-terms [70] leads straightforwardly to the results in [11, 43, 83]. After the subtractions have been made one can remove the (infrared) regulator and obtain a completely explicit formula for the expectation value of the dual stress-energy tensor in terms of the gravitational solution.

We will mostly concentrate on the gravitational sector, i.e. on the reconstruction of the bulk metric, but we will also discuss the coupling to scalars. Our approach will be to build perturbatively an Einstein manifold of constant negative curvature (which we will sometimes refer to as an asymptotically AdS space) as well as a solution to the scalar field equations on this manifold out of CFT data. The CFT data we start from is what sources are turned on. We will include a source for the dual stress-energy tensor as well as sources for scalar composite operators. This means that in the bulk we need to solve the gravitational equations coupled to scalars given a conformal structure at infinity and appropriate Dirichlet boundary conditions for the scalars. It is well-known that if one considers the standard Euclidean AdS (i.e., with isometry $SO(1, d+1)$), the scalar field equation with Dirichlet boundary conditions has a unique solution. In the Lorentzian case, because of the existence of normalisable modes, the solution ceases to be unique. Likewise, the Dirichlet boundary condition problem for (Euclidean) gravity has a unique smooth solution (up to diffeomorphisms) in the case the bulk manifold is topologically a ball and the boundary conformal structure is sufficiently close to the standard one [58]. However, given a boundary topology there may be more than one Einstein manifold with this boundary. For example, if the boundary has the topology of $S^1 \times S^{d-1}$, there are two possible bulk manifolds [69, 125]: one which is obtained from standard AdS by global identifications and is topologically $S^1 \times R^d$, and another, the Schwarzschild-AdS black hole, which is topologically $R^2 \times S^{d-1}$.

We will make no assumption on the global structure of the space nor its signature. The CFT should provide additional data in order to retrieve this information. Indeed, we will see that only the information about the sources leaves undetermined the part of the solution which is sensitive to global issues and/or the signature of space-time. To determine that part one needs new CFT data. To leading order these are the expectation values of the CFT operators.

In particular, in the case of pure gravity, we find that generically a boundary conformal structure is not sufficient in order to obtain the bulk metric. One needs more CFT data. To leading order one needs to specify the expectation value of the boundary stress-energy tensor. Since the gravitational field equation is a second order differential equation, one may expect that these data are sufficient in order to specify the full solution. However, higher point functions of the stress-energy tensor may be necessary if higher derivatives corrections such as R^2-terms are included in the action. We emphasise that we make no assumption about the regularity of the solution. Under additional assumptions the metric may be determined by fewer data. For example, as we mentioned above, under certain assumptions on the topology and the boundary conformal structure one obtains a unique smooth solution [58]. Another example is the case when one restricts oneself to conformally flat bulk metrics. Then a conformally flat boundary metric does yield a unique bulk metric, up to diffeomorphisms and global identifications [102].

Turning things around, given a specific solution, we present formulae for the expectation values of the dual CFT operators. In particular, in the case the operator is the stress-energy tensor, our formulae have a "dual" meaning [11]: both as the expectation value of the stress-energy tensor of the dual CFT and as the quasi-local stress-energy tensor of Brown and York [25]. We provide very explicit formulae for the stress-energy tensor associated with any solution of Einstein's equations with negative constant curvature.

Let us summarise these results for space-time dimension up to six[3]. The first step is to rewrite the solution in the Graham-Fefferman co-ordinate system [44]

$$\mathrm{d}s^2 = G_{\mu\nu}\mathrm{d}x^\mu dx^\nu = \frac{l^2}{r^2}\left(\mathrm{d}r^2 + g_{ij}(x,r)\mathrm{d}x^i\mathrm{d}x^j\right), \tag{4.1}$$

where

$$g(x,r) = g_{(0)} + r^2 g_{(2)} + \cdots + r^d g_{(d)} + h_{(d)} r^d \log r^2 + \mathcal{O}(r^{d+1}). \tag{4.2}$$

The logarithmic term appears only in even dimensions. l is a parameter of dimension of length related to the cosmological constant as $\Lambda = -\frac{d(d-1)}{2l^2}$. Any asymptotically AdS metric can be brought in the form (4.1) near the boundary ([58], see also [59, 57]). Once this co-ordinate system has been reached, the expectation value of the boundary stress-energy tensor reads

$$\langle T_{ij}\rangle = \frac{dl^{d-1}}{16\pi G_{\mathrm{N}}}\, g_{(d)ij} + X_{ij}[g_{(n)}], \tag{4.3}$$

where $X_{ij}[g_{(n)}]$ is a function of $g_{(n)}$ with $n < d$. Its exact form depends on the space-time dimension and it reflects the conformal anomalies of the boundary conformal field theory. In odd (boundary) dimensions, where there are no gravitational conformal anomalies, X_{ij} is equal to zero. The expression for $X_{ij}[g_{(n)}]$ for $d = 2, 4, 6$ can be read off from the formulae that will be given in (4.33), (4.38) and (4.39), respectively. The universal part of (4.3) (i.e. with X_{ij} omitted) was obtained previously in [91]. Actually, to obtain the dual stress-energy tensor it is sufficient to only know $g_{(0)}$ and $g_{(d)}$ as $g_{(n)}$ with $n < d$ are uniquely determined from $g_{(0)}$, as we will see. The coefficient $h_{(d)}$ of the logarithmic term in the case of even d is also directly related to the conformal anomaly: it is proportional to the metric variation of the conformal anomaly.

It was pointed out in [11] that this prescription for calculating the boundary stress-energy tensor provides also a novel way, free of divergences[4], of computing the gravitational quasi-local stress-energy tensor of Brown and York [25]. Conformal anomalies reflect infrared divergences in the gravitational sector [70]. Because of these divergences one cannot maintain the full group of isometries even asymptotically. In particular, the isometries of AdS that rescale the radial co-ordinate (these correspond to dilations in the CFT) are broken by infrared divergences. Because of this fact, bulk solutions that

[3]In this chapter the dimension we refer to is the dimension of the boundary. So, $d = 6$ corresponds to asymptotically AdS_7.

[4]We emphasise, however, that one has to subtract the logarithmic divergences in even dimensions in order for the stress-energy tensor to be finite.

are related by diffeomorphisms that yield a conformal transformation in the boundary do not necessarily have the same mass. Assigning zero mass to the space-time with boundary R^d, one obtains that, due to the conformal anomaly, the solution with boundary $R \times S^{d-1}$ has non-zero mass. This parallels exactly the discussion in field theory. In that case, starting from the CFT on R^d with vanishing expectation value of the stress-energy tensor, one obtains the Casimir energy of the CFT on $R \times S^{d-1}$ by a conformal transformation [27]. The agreement between the gravitational ground-state energy and the Casimir energy of the CFT is a direct consequence of the fact that the conformal anomaly computed by weakly coupled gauge theory and by supergravity agree [70]. It should be noted that, as emphasised in [11], agreement between gravity/field theory for the ground state energy is achieved only after all ambiguities are fixed in the same manner on both sides.

A conformal transformation in the boundary theory is realised in the bulk as a special diffeomorphism that preserves the form of the co-ordinate system (4.1) [75]. Using these diffeomorphisms one can easily study how the (quantum, i.e., with the effects of the conformal anomaly taken into account) stress-energy tensor transforms under conformal transformations. Our results, when restricted to the cases studied in the literature [27], are in agreement with them. We note that the present determination is considerably easier than the one in [27].

Let us briefly discuss in more detail how conformal invariance is broken. As is well-known [87], the bulk metric does not quite induce a metric on the boundary, but only a conformal class of metrics. Since the metric has a double pole on the boundary [44], one can define a metric by extracting this pole. That is, pick a positive function r with a single zero at the boundary. The induced boundary metric is then given by $g_{(0)} = r^2 G|_{\partial M}$ where ∂M is the boundary of the manifold M. However, there is an obvious arbitrariness in this definition in that any other function $r' = e^w r$ with a single zero gives an equally valid boundary metric. Therefore, the metric on the boundary is defined up to a conformal transformation. This already indicates that the holographic dual should be a conformal theory, and is very similar to how in the eikonal regime of quantum gravity bulk time translations give rise to Lorentz boosts of the boundary theory.

On the other hand, infrared divergences break the symmetries of the bulk. To renormalise the theory we introduce a cut-off on the radial variable at $r = \epsilon$. One can then renormalise the action by adding covariant counter-terms which are evaluated at the cut-off $r = \epsilon$. When sending the cut-off to zero, the action should be finite. However, the presence of a logarithmic divergence gives rise to an anomaly when we perform a conformal transformation on $g_{(0)}$, $g'_{(0)} = e^{2\sigma} g_{(0)}$. This is a special kind of bulk diffeomorphism and so one would naively expect it to be a symmetry of the action. But it transforms as [70, 101]:

$$S_{\rm ren}[e^{2\sigma} g_{(0)}] = S_{\rm ren}[g_{(0)}] + \mathcal{A}[g_{(0)}, \sigma], \tag{4.4}$$

where the anomaly $\mathcal{A}$ is a conformally invariant functional of the metric [70] and it precisely corresponds to the conformal anomalies found on the gauge theory side.

The fact that infrared divergences break bulk diffeormorphisms means that only diffeomorphisms that do not induce a Weyl rescaling on the boundary are true symmetries of the theory. This implies that bulk solutions which are related by a diffeomorphism may have different dual stress-tensors when the diffeomorphism induces a conformal transformation on the boundary.

The discussion is qualitatively the same when one adds matter to the system. We discuss scalar fields but the discussion generalises straightforwardly to other kinds of matter. We study both the case when the gravitational background is fixed and the case when gravity is dynamical.

Let us summarise the results for the case of scalar fields in a fixed gravitational background (given by a metric of the form (4.1)). We look for solutions of massive scalar fields with mass $m^2 = (\Delta - d)\Delta$ that near the boundary have the form (in the co-ordinate system (4.1))

$$\begin{aligned} \Phi(x,r) \quad &= \quad r^{d-\Delta}\left(\phi_{(0)} + r^2\phi_{(2)} + \cdots + r^{2\Delta-d}\phi_{(2\Delta-d)} + \right. \\ &\qquad \left. + r^{2\Delta-d}\log r^2 \psi_{(2\Delta-d)}\right) + \mathcal{O}(r^{\Delta+1}). \end{aligned} \tag{4.5}$$

The logarithmic terms appears only when $2\Delta - d$ is an integer and we only consider this case in this chapter. We find that $\phi_{(n)}$, with $n < 2\Delta - d$, and $\psi_{(2\Delta-d)}$ are uniquely determined from the scalar field equation. This information is sufficient for a complete determination of the infrared divergences of the on-shell bulk action. In particular, the logarithmic term $\psi_{(2\Delta-d)}$ in (4.5) is directly related to matter conformal anomalies. These conformal anomalies were shown not to renormalise in [96]. We indeed find exact agreement with the computation in [96]. Adding counter-terms to cancel the infrared divergences we obtain the renormalised on-shell action. We stress that even in the case of a free massive scalar field in a fixed AdS background one needs counter-terms in order for the on-shell action to be finite (see (4.75)). The coefficient $\phi_{(2\Delta-d)}$ is left undetermined by the field equations. It is determined, however, by the expectation value of the dual operator $O(x)$. Differentiating the renormalised on-shell action one finds (up to terms contributing contact terms in the 2-point function)

$$\langle O(x) \rangle = (2\Delta - d)\phi_{(2\Delta-d)}(x). \tag{4.6}$$

This relation, with the precise proportionality coefficient, has first been derived in [81]. The value of the proportionality coefficient is crucial in order to obtain the correct normalisation of the 2-point function in standard AdS background [46].

In the case when the bulk geometry is dynamical we find that, for scalars that correspond to irrelevant operators, our perturbative treatment is consistent only if one considers single insertions of the irrelevant operator, i.e. the source is treated as an infinitesimal parameter, in agreement with the discussion in [125]. For scalars that correspond to marginal and relevant operators one can compute perturbatively the back-reaction of the scalars to the gravitational background. One can then regularise and renormalise as in the discussion of pure gravity or scalars in a fixed background. For illustrative purposes we analyse a simple example.

This chapter is organised as follows. In the next section we discuss the Dirichlet problem for AdS gravity and we obtain an asymptotic solution for a given boundary metric (up to six dimensions). In section 4.3 we use these solutions to obtain the infrared divergences of the on-shell gravitational action. After renormalising the on-shell action by adding counter-terms, we compute the holographic stress-energy tensor. Section 4.4 is devoted to the study of the conformal transformation properties of the boundary stress-energy tensor. In section 4.5 we extend the analysis of sections 4.2 and 4.3 to include matter. In appendices C.1 and C.4 we give the explicit form of the solutions discussed in section 4.2 and section 4.5. Appendix C.2 contains the explicit form of the counter-terms discussed in section 4.3. In appendix C.3 we present a proof that the coefficient of the logarithmic term in the metric (present in even boundary dimensions) is proportional to the metric variation of the conformal anomaly.

4.2 Dirichlet boundary problem for AdS gravity

The Einstein-Hilbert action for a theory on a manifold M with boundary ∂M is given by[5]

$$S_{\rm gr}[G] = \frac{1}{16\pi G_{\rm N}}[\int_M {\rm d}^{d+1}x\,\sqrt{G}\,(R[G]+2\Lambda) - \int_{\partial M} {\rm d}^d x\,\sqrt{\gamma}\,2K], \tag{4.7}$$

where K is the trace of the second fundamental form (see (B.5)) and γ is the induced metric on the boundary. The boundary term is necessary in order to get an action which only depends on first derivatives of the metric [51], and it guarantees that the variational problem with Dirichlet boundary conditions is well-defined.

According to the prescription of [64, 125], the conformal field theory effective action is given by evaluating the on-shell action functional. The field specifying the boundary conditions for the metric is regarded as a source for the boundary operator. We therefore need to obtain solutions to Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = \Lambda G_{\mu\nu}, \tag{4.8}$$

subject to appropriate Dirichlet boundary conditions.

As explained above, metrics $G_{\mu\nu}$ that satisfy (4.8) have a second order pole at infinity. Therefore, they do not induce a metric at infinity but only a conformal class. This is achieved by introducing a defining function r, i.e. a positive function in the interior of the manifold M that has a single zero and non-vanishing derivative at the boundary. Then one obtains a metric at the boundary by $g_{(0)} = r^2 G|_{\partial M}$ [6].

We are interested in solving (4.8) given a conformal structure at infinity. This can be achieved by working in the co-ordinate system (4.1) introduced by Feffermam and Graham [44]. The metric in (4.1) contains only even powers of r up to the order we are

[5]Our curvature conventions are as follows: $R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l + \Gamma_{ip}{}^l\Gamma_{jk}{}^p - (i \leftrightarrow j)$ and $R_{ij} = R_{ikj}{}^k$. We these conventions the curvature of AdS comes out positive, but we will still use the terminology "space of constant negative curvature". Notice also that we take $\int {\rm d}^{d+1}x = \int {\rm d}^d x \int_0^\infty {\rm d}r$ and the boundary is at $r = 0$ (in the co-ordinate system (4.1)). The minus sign in front of the trace of the second fundamental form is correlated with the choice of having $r = 0$ in the lower end of the radial integration.

[6]Throughout this chapter the metric $g_{(0)}$ is assumed to be non-degenerate. For studies of the AdS/CFT correspondence in cases where $g_{(0)}$ is degenerate we refer to [23, 110].

interested in [44] (see also [59, 57]). For this reason, it is convenient to use the variable $\rho = r^2$ [70], [7]

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = l^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j\right),$$
$$g(x,\rho) = g_{(0)} + \cdots + \rho^{d/2}g_{(d)} + h_{(d)}\rho^{d/2}\log\rho + ..., \quad (4.9)$$

where the logarithmic piece appears only for even d. The sub-index in the metric expansion (and in all other expansions that appear in this chapter) indicates the number of derivatives involved in that term, i.e. $g_{(2)}$ contains two derivatives, $g_{(4)}$ four derivatives, etc., as one can see from the explicit expressions given in appendix C.1. It follows that the perturbative expansion in ρ is also a low energy expansion. We set $l = 1$ from now on. One can easily reinstate the factors of l by dimensional analysis.

One can check that the curvature of G satisfies

$$R_{\kappa\lambda\mu\nu}[G] = (G_{\kappa\mu}G_{\lambda\nu} - G_{\kappa\nu}G_{\lambda\mu}) + \mathcal{O}(\rho). \quad (4.10)$$

In this sense the metric is asymptotically anti-de Sitter. The Dirichlet problem for Einstein metrics satisfying (4.10) exactly (i.e. not only to leading order in ρ) was solved in [102].

In the co-ordinate system (4.9), Einstein's equations read [70]

$$\begin{aligned}\rho\left[2g'' - 2g'g^{-1}g' + \mathrm{Tr}\,(g^{-1}g')\,g'\right] + \mathrm{Ric}(g) - (d-2)\,g' - \mathrm{Tr}\,(g^{-1}g')\,g &= 0,\\ \nabla_i\,\mathrm{Tr}\,(g^{-1}g') - \nabla^j g'_{ij} &= 0,\\ \mathrm{Tr}\,(g^{-1}g'') - \frac{1}{2}\mathrm{Tr}\,(g^{-1}g'g^{-1}g') &= 0 \quad (4.11)\end{aligned}$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g, and $\mathrm{Ric}(g)$ is the Ricci tensor of g.

These equations are solved order by order in ρ. This is achieved by differentiating the equations with respect to ρ and then setting $\rho = 0$. For even d, this process would have broken down at order $d/2$ if the logarithm was not introduced in (4.9). $h_{(d)}$ is traceless, $\mathrm{Tr}\ g_{(0)}^{-1}h_{(d)} = 0$, and covariantly conserved, $\nabla^i h_{(d)ij} = 0$. We show in appendix C.3 that $h_{(d)}$ is proportional to the metric variation of the corresponding conformal anomaly, i.e. it is proportional to the stress-energy tensor of the theory with action the conformal anomaly. In any dimension, only the trace of $g_{(d)}$ and its covariant divergence are determined. Here is where extra data from the CFT are needed: as we shall see, the undetermined part is specified by the expectation value of the dual stress-energy tensor.

We collect in appendix C.1 the results for $g_{(n)}$, $h_{(d)}$ as well as the results for the trace and divergence $g_{(d)}$. In dimension d the latter are the only constraints that equations (4.11) yield for $g_{(d)}$. From this information we can parametrise the indeterminacy by finding the most general $g_{(d)}$ that has the determined trace and divergence.

In $d = 2$ and $d = 4$ the equation for the coefficient $g_{(d)}$ has the form of a conservation law

$$\nabla^i g_{(d)ij} = \nabla^i A_{(d)ij}\ \ , \qquad d = 2, 4, \quad (4.12)$$

[7] Greek indices, $\mu, \nu, ..$ are used for $d+1$-dimensional indices, Latin ones, $i, j, ..$ for d-dimensional ones. To distinguish the curvatures of the various metrics introduced in (4.9) we will often use the notation $R_{ij}[g]$ to indicate that this is the Ricci tensor of the metric g, etc.

where $A_{(d)ij}$ is a symmetric tensor explicitly constructed from the coefficients $g_{(n)}$, $n < d$. The precise form of the tensor $A_{(d)ij}$ is given in appendix C.1 (eq.(C.4)). The integration of this equation obviously involves an "integration constant" $t_{ij}(x)$, a symmetric covariantly conserved tensor the precise form of which cannot be determined from Einstein's equations.

In two dimensions, we get [102] (see also [15])

$$g_{(2)ij} = \frac{1}{2}(R\, g_{(0)ij} + t_{ij}), \tag{4.13}$$

where the symmetric tensor t_{ij} should satisfy

$$\nabla^i t_{ij} = 0, \qquad \mathrm{Tr}\ t = -R. \tag{4.14}$$

In four dimensions we obtain[8]

$$g_{(4)ij} = \frac{1}{8} g_{(0)ij}\, [(\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g_{(2)}^2] + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4} g_{(2)ij}\ \mathrm{Tr}\ g_{(2)} + t_{ij}. \tag{4.15}$$

The tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0, \qquad \mathrm{Tr}\ t = -\frac{1}{4}[(\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g_{(2)}^2]. \tag{4.16}$$

In six dimensions the equation determining the coefficient $g_{(6)}$ is more subtle than the one in (4.12). It is given by

$$\nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6}\mathrm{Tr}\,(g_{(4)} \nabla_j g_{(2)}), \tag{4.17}$$

where the tensor $A_{(6)ij}$ is given in (C.4). It contains a part which is antisymmetric in the indices i and j. Since $g_{(6)ij}$ is by definition a symmetric tensor the integration of equation (4.17) is not straightforward. Moreover, it is not obvious that the last term in (4.17) takes a form of divergence of some local tensor. Nevertheless, this is indeed the case as we now show. Let us define the tensor S_{ij},

$$\begin{aligned} S_{ij} = {} & \nabla^2 C_{ij} - 2R^{k\ \,l}_{\ i\ \,j} C_{kl} + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} + \frac{1}{10}(\nabla_i \nabla_j B - g_{(0)ij}\nabla^2 B) \\ & + \frac{2}{5} g_{(2)ij} B + g_{(0)ij}(-\frac{2}{3}\mathrm{Tr}\ g_{(2)}^3 - \frac{4}{15}(\mathrm{Tr}\ g_{(2)})^3 + \frac{3}{5}\mathrm{Tr}\ g_{(2)} \mathrm{Tr}\ g_{(2)}^2) \quad , \end{aligned} \tag{4.18}$$

where

$$C_{ij} = (g_{(4)} - \frac{1}{2} g_{(2)}^2 + \frac{1}{4} g_{(2)} \mathrm{Tr}\ g_{(2)})_{ij} + \frac{1}{8} g_{(0)ij} B \quad , \quad B = \mathrm{Tr}\ g_2^2 - (\mathrm{Tr}\ g_2)^2 \quad .$$

The tensor S_{ij} is a local function of the Riemann tensor. Its divergence and trace read

$$\nabla^i S_{ij} = -4\mathrm{Tr}\,(g_{(4)} \nabla_j g_{(2)}) \quad , \quad \mathrm{Tr}\, S = -8\mathrm{Tr}\,(g_{(2)} g_{(4)}) \quad . \tag{4.19}$$

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[8]From now on we will suppress factors of $g_{(0)}$. For instance, Tr $g_{(2)} g_{(4)}$ = Tr $[g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(4)}]$. Unless we explicitly mention the contrary, indices will be raised and lowered with the metric $g_{(0)}$, and all contractions will be made with this metric.

With the help of the tensor S_{ij} the equation (4.17) can be integrated in a way similar to the $d = 2, 4$ cases. One obtains

$$g_{(6)ij} = A_{(6)ij} - \frac{1}{24} S_{ij} + t_{ij} \quad . \tag{4.20}$$

Notice that tensor S_{ij} contains an antisymmetric part which cancels the antisymmetric part of the tensor $A_{(6)ij}$ so that $g_{(6)ij}$ and t_{ij} are symmetric tensors, as they should. The symmetric tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0 \quad , \quad \mathrm{Tr}\, t = -\frac{1}{3}[\frac{1}{8}(\mathrm{Tr}\, g_{(2)})^3 - \frac{3}{8}\mathrm{Tr}\, g_{(2)} \mathrm{Tr}\, g_{(2)}^2 + \frac{1}{2}\mathrm{Tr}\, g_{(2)}^3 - \mathrm{Tr}\, g_{(2)} g_{(4)}] \quad . \tag{4.21}$$

Notice that in all three cases, $d = 2, 4, 6$, the trace of t_{ij} is proportional to the holographic conformal anomaly. As we will see in the next section, the symmetric tensors t_{ij} are directly related to the expectation value of the boundary stress-energy tensor.

When d is odd the only constraint on the coefficient $g_{(d)ij}(x)$ is that it is conserved and traceless

$$\nabla^i g_{(d)ij} = 0 \quad , \qquad \mathrm{Tr}\ g_{(d)} = 0 \quad . \tag{4.22}$$

So that we may identify

$$g_{(d)ij} = t_{ij} \quad . \tag{4.23}$$

4.3 The holographic stress-energy tensor

We have seen in the previous section that given a conformal structure at infinity we can determine an asymptotic expansion of the metric up to order $\rho^{d/2}$. We will now show that this term is determined by the expectation value of the dual stress-energy tensor.

According to the AdS/CFT prescription, the expectation value of the boundary stress-energy tensor is determined by functionally differentiating the on-shell gravitational action with respect to the boundary metric. The on-shell gravitational action, however, diverges. To regulate the theory we restrict the bulk integral to the region $\rho \geq \epsilon$ and we evaluate the boundary term at $\rho = \epsilon$. The regulated action is given by

$$\begin{aligned} S_{\mathrm{gr,reg}} &= \frac{1}{16\pi G_{\mathrm{N}}} \left[\int_{\rho \geq \epsilon} \mathrm{d}^{d+1}x \sqrt{G}\,(R[G] + 2\Lambda) - \int_{\rho=\epsilon} \mathrm{d}^d x \sqrt{\gamma}\, 2K \right] = \\ &= \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^d x \left[\int_\epsilon \mathrm{d}\rho\, \frac{d}{\rho^{d/2+1}} \sqrt{\det g(x,\rho)} \right. \\ &+ \left. \frac{1}{\rho^{d/2}} (-2d\sqrt{\det g(x,\rho)} + 4\rho\partial_\rho \sqrt{\det g(x,\rho)})|_{\rho=\epsilon} \right], \end{aligned} \tag{4.24}$$

Evaluating (4.24) for the solution we obtained in the previous section we find that the divergences appears as $1/\epsilon^k$ poles plus a logarithmic divergence [70],

$$\begin{aligned} S_{\mathrm{gr,reg}} &= \frac{l}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^d x \sqrt{\det g_{(0)}} \Big(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-2)} \\ &- \log\epsilon\, a_{(d)} \Big) + \mathcal{O}(\epsilon^0), \end{aligned} \tag{4.25}$$

where the coefficients $a_{(n)}$ are local covariant expressions of the metric $g_{(0)}$ and its curvature tensor. We give the explicit expressions, up to the order we are interested in, in appendix C.

We now obtain the renormalised action by subtracting the divergent terms, and then removing the regulator,

$$\begin{aligned} S_{\rm gr,ren}[g_{(0)}] &= \lim_{\epsilon\to 0} \frac{1}{16\pi G_{\rm N}}[S_{\rm gr,reg} - \int {\rm d}^d x \sqrt{\det g_{(0)}} \Big(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots \\ &+ \epsilon^{-1} a_{(d-2)} - \log\epsilon\, a_{(d)}\Big)]. \end{aligned} \tag{4.26}$$

The expectation value of the stress-energy tensor of the dual theory is given by

$$\langle T_{ij}\rangle = \frac{2}{\sqrt{\det\gamma_{(0)}}}\frac{\partial S_{\rm gr,ren}}{\partial g^{ij}_{(0)}} = \lim_{\epsilon\to 0}\frac{2}{\sqrt{\det g(x,\epsilon)}}\frac{\partial S_{\rm gr,ren}}{\partial g^{ij}(x,\epsilon)} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-1}}\,T_{ij}[\gamma]\right), \tag{4.27}$$

where $T_{ij}[\gamma]$ is the stress-energy tensor of the theory at $\rho=\epsilon$ described by the action in (4.26) but before the limit $\epsilon\to 0$ is taken ($\gamma_{ij} = 1/\epsilon\, g_{ij}(x,\epsilon)$ is the induced metric at $\rho=\epsilon$). Notice that the asymptotic expansion of the metric only allows for the determination of the divergences of the on-shell action. We can still obtain, however, a formula for $\langle T_{ij}\rangle$ in terms of $g_{(n)}$ since, as (4.27) shows, we only need to know the first $\epsilon^{d/2-1}$ orders in the expansion of $T_{ij}[\gamma]$.

The stress-energy tensor $T_{ij}[\gamma]$ contains two contributions,

$$T_{ij}[\gamma] = T^{\rm reg}_{ij} + T^{\rm ct}_{ij}, \tag{4.28}$$

$T^{\rm reg}_{ij}$ comes from the regulated action in (4.24) and $T^{\rm ct}_{ij}$ is due to the counter-terms. The first contribution is equal to

$$\begin{aligned} T^{\rm reg}_{ij}[\gamma] &= -\frac{1}{8\pi G_{\rm N}}(K_{ij} - K\gamma_{ij}) \\ &= -\frac{1}{8\pi G_{\rm N}}\,(-\partial_\epsilon g_{ij}(x,\epsilon) + g_{ij}(x,\epsilon)\,{\rm Tr}\,[g^{-1}(x,\epsilon)\partial_\epsilon g(x,\epsilon)] \\ &+ \frac{1-d}{\epsilon} g_{ij}(x,\epsilon)). \end{aligned} \tag{4.29}$$

The contribution due to counter-terms can be obtained from the results in appendix C.2. It is given by

$$\begin{aligned} T^{\rm ct}_{ij} &= -\frac{1}{8\pi G_{\rm N}}\Bigg((d-1)\gamma_{ij} + \frac{1}{(d-2)}(R_{ij} - \frac{1}{2}R\gamma_{ij}) + \\ &-\frac{1}{(d-4)(d-2)^2}[-\nabla^2 R_{ij} + 2R_{ikjl}R^{kl} + \frac{d-2}{2(d-1)}\nabla_i\nabla_j R - \frac{d}{2(d-1)}RR_{ij} \\ &-\frac{1}{2}\gamma_{ij}(R_{kl}R^{kl} - \frac{d}{4(d-1)}R^2 - \frac{1}{d-1}\nabla^2 R)] - T^a_{ij}\log\epsilon\Bigg), \end{aligned} \tag{4.30}$$

where T^a_{ij} is the stress-energy tensor of the action $\int {\rm d}^d x\,\sqrt{\det\gamma}\, a_{(d)}$. As is shown in Appendix C.3, T^a_{ij} is proportional to the tensor $h_{(d)ij}$ appearing in the expansion (4.9).

The stress tensor $T_{ij}[g_{(0)}]$ is covariantly conserved with respect to the metric $g_{(0)ij}$. To see this, notice that each of $T_{ij}^{\rm reg}$ and $T_{ij}^{\rm ct}$ is separately covariantly conserved with respect to the induced metric γ_{ij} at $\rho = \epsilon$: for $T_{ij}^{\rm reg}$ one can check this by using the second equation in (4.11), for $T_{ij}^{\rm ct}$ this follows from the fact that it was obtained by varying a local covariant counter-term. Since all divergences cancel in (4.27), we obtain that the finite part in (4.27) is conserved with respect to the metric $g_{(0)ij}$.

We are now ready to calculate T_{ij}. By construction (and we will verify this below) the divergent pieces cancel between $T^{\rm reg}$ and $T^{\rm ct}$.

4.3.1 $d = 2$

In two dimensions we obtain

$$\langle T_{ij} \rangle = \frac{l}{16\pi G_{\rm N}}\, t_{ij}, \tag{4.31}$$

where we have used (4.13) and (4.14) and the fact that $T_{ij}^a = 0$ since $\int R$ is a topological invariant (and reinstated the factor of l). As promised, t_{ij} is directly related to the boundary stress-energy tensor. Taking the trace we obtain

$$\langle T_i^i \rangle = -\frac{c}{24\pi}\, R, \tag{4.32}$$

where $c = 3l/2G_{\rm N}$, which is the correct conformal anomaly [24].

Using our results, one can immediately obtain the stress-energy tensor of the boundary theory associated with a given solution G of the three dimensional Einstein equations: one needs to write the metric in the co-ordinate system (4.9) and then use the formula

$$\langle T_{ij} \rangle = \frac{2l}{16\pi G_{\rm N}}\, (g_{(2)ij} - g_{(0)ij}\, {\rm Tr}\ g_{(2)}). \tag{4.33}$$

From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G.

4.3.2 $d = 4$

To obtain T_{ij} we first need to rewrite the expressions in $T^{\rm ct}$ in terms of $\gamma_{(0)}$. This can be done with the help of the relation

$$\begin{aligned} R_{ij}[\gamma] \;&=\; R_{ij}[\gamma_{(0)}] + \frac{1}{4}\,\epsilon \left(2R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 R g_{(0)ij} \right) \\ &+\; \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.34}$$

After some algebra one obtains,

$$\begin{aligned} \langle T_{ij}[g_{(0)}] \rangle \;=\; & -\frac{1}{8\pi G_{\rm N}} \lim_{\epsilon\to 0} \Bigg[\frac{1}{\epsilon}(-g_{(2)ij} + g_{(0)ij}{\rm Tr}\ g_{(2)} + \frac{1}{2}R_{ij} - \frac{1}{4}g_{(0)ij}R) \\ & + \log\epsilon\, (-2h_{(4)ij} - T^a_{ij}) - 2g_{(4)ij} - h_{(4)ij} - g_{(2)ij}{\rm Tr}\ g_{(2)} - \frac{1}{2}g_{(0)ij}{\rm Tr}\ g_{(2)}^2 \end{aligned}$$

$$+\frac{1}{8}(R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 R g_{(0)ij})$$
$$-\frac{1}{4}g_{(2)ij}R + \frac{1}{8}g_{(0)ij}(R_{kl}R^{kl} - \frac{1}{6}R^2)\Big]. \qquad (4.35)$$

Using the explicit expression for $g_{(2)}$ and $h_{(4)}$ given in (C.1) and (C.6) one finds that both the $1/\epsilon$ pole and the logarithmic divergence cancel. Notice that had we not subtracted the logarithmic divergence from the action, the resulting stress-energy tensor would have been singular in the limit $\epsilon \to 0$.

Using (4.15) and (4.16) and after some algebra we obtain

$$\langle T_{ij}\rangle = -\frac{1}{8\pi G_{\rm N}}[-2t_{ij} - 3h_{(4)}]. \qquad (4.36)$$

Taking the trace we get

$$\langle T^i_i\rangle = \frac{1}{16\pi G_{\rm N}}(-2a_{(4)}), \qquad (4.37)$$

which is the correct conformal anomaly [70].

Notice that since $h_{(4)ij} = -\frac{1}{2}T^a_{ij}$ the contribution in the boundary stress-energy tensor proportional to $h_{(4)ij}$ is scheme-dependent. Adding a local finite counter-term proportional to the trace anomaly will change the coefficient of this term. One may remove this contribution from the boundary stress energy tensor by a choice of scheme.

Finally, one can obtain the energy-momentum tensor of the boundary theory for a given solution G of the five dimensional Einstein equations with negative cosmological constant. It is given by

$$\langle T_{ij}\rangle = \frac{4}{16\pi G_{\rm N}}[g_{(4)ij} - \frac{1}{8}g_{(0)ij}[(\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g^2_{(2)}] - \frac{1}{2}(g^2_{(2)})_{ij} + \frac{1}{4}g_{(2)ij}\mathrm{Tr}\ g_{(2)}], \qquad (4.38)$$

where we have omitted the scheme-dependent $h_{(4)}$-terms. From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G.

4.3.3 $d = 6$

The calculation of the boundary stress tensor in the $d = 6$ case goes along the same lines as in $d = 2$ and $d = 4$ cases although it is technically involved. Up to a local traceless covariantly conserved term (proportional to $h_{(6)}$) the result is

$$\langle T_{ij}\rangle = \frac{3}{8\pi G_{\rm N}}\,(g_{(6)ij} - A_{(6)ij} + \frac{1}{24}S_{ij}) \ . \qquad (4.39)$$

where $A_{(6)ij}$ is given in (C.4) and S_{ij} in (4.18). It is covariantly conserved and has the correct trace

$$\langle T^i_i\rangle = \frac{1}{8\pi G_{\rm N}}(-a_{(6)}) \ , \qquad (4.40)$$

reproducing correctly the conformal anomaly in six dimensions [70].

Given an asymptotically AdS solution in six dimensions equation (4.39) yields the quasi-local stress-energy tensor associated with it.

4.3.4 $d = 2k + 1$

In this case one can check that the counter-terms only cancel infinities. Evaluating the finite part we get

$$\langle T_{ij} \rangle = \frac{d}{16\pi G_{\rm N}}\, g_{(d)ij}, \tag{4.41}$$

where $g_{(d)ij}$ can be identified with a traceless covariantly conserved tensor t_{ij}. In odd boundary dimensions there are no gravitational conformal anomalies, and indeed (4.41) is traceless. As in all previous cases, one can also read (4.41) as giving the quasi-local stress-energy tensor associated with a given solution of Einstein's equations.

4.3.5 Conformally flat bulk metrics

In this subsection we discuss a special case where the bulk metric can be determined to all orders given only a boundary metric. It was shown in [102] that, given a conformally flat boundary metric, equations (4.11) can be integrated to all orders if the bulk Weyl tensor vanishes[9]. We show that the extra condition in the bulk metric singles out a specific vacuum of the CFT.

The solution obtained in [102] is given by

$$g(x,\rho) = g_{(0)}(x) + g_{(2)}(x)\rho + g_{(4)}(x)\rho^2 \quad , \quad g_{(4)} = \frac{1}{4}(g_{(2)})^2, \tag{4.42}$$

where $g_{(2)}$ is given in (C.1) (we consider $d > 2$), and all other coefficients $g_{(n)}$, $n > 4$ vanish. Since $g_{(4)}$ and $g_{(6)}$ are now known, one can obtain a local formula for the dual stress-energy tensor in terms of the curvature by using (4.15) and (4.20).

In $d = 4$, using (4.15) and $g_{(4)} = \frac{1}{4}(g_{(2)})^2$, one obtains

$$t_{ij} = t^{\rm cf}_{ij} \equiv -\frac{1}{4}(g_{(2)})^2_{ij} + \frac{1}{4} g_{(2)ij} {\rm Tr}\ g_{(2)} - \frac{1}{8} g_{(0)ij}[({\rm Tr}\ g_{(2)})^2 - {\rm Tr}\ g^2_{(2)}] \quad . \tag{4.43}$$

It is easy to check that trace of $t^{\rm cf}_{ij}$ reproduces (4.16). Furthermore, by virtue of the Bianchi identities, one can show that $t^{\rm cf}_{ij}$ is covariantly conserved. It is well-known that the stress-energy tensor of a quantum field theory on a conformally flat space-time is a local function of the curvature tensor (see for example the book by Birrell and Davies, [20]). Our equation (4.43) reproduces the corresponding expression given in [20].

In $d = 6$, using (4.20) and $g_{(6)} = 0$ we find

$$\begin{aligned} t_{ij} = t^{\rm cf}_{ij} \equiv [&\frac{1}{4} g^3_{(2)} - \frac{1}{4} g^2_{(2)} {\rm Tr}\ g_{(2)} + \frac{1}{8} g_{(2)} ({\rm Tr}\, g_{(2)})^2 - \frac{1}{8} g_{(2)} {\rm Tr}\ g_{(2)} \\ &+ g_{(0)}(\frac{1}{8} {\rm Tr}\ g_{(2)} {\rm Tr}\ g^2_{(2)} - \frac{1}{12} {\rm Tr}\ g^3_{(2)} - \frac{1}{24}({\rm Tr}\ g_{(2)})^3)]_{ij} \quad . \end{aligned} \tag{4.44}$$

[9]In [102] it was proven that if the bulk metric satisfies Einstein's equations and it has a vanishing Weyl tensor, then the corresponding boundary metric has to be conformally flat. The converse is not necessarily true: one can have Einstein metrics with non-vanishing Weyl tensor which induce a conformally flat metric in the boundary.

One can verify that the trace of $t^{\rm cf}_{ij}$ reproduces (4.21) (taking into account that $g_{(4)} = \frac{1}{4}g^2_{(2)}$ and that $t^{\rm cf}_{ij}$ is covariantly conserved (by virtue of the Bianchi identities)).

Following the analysis in the previous subsections we obtain

$$\langle T_{ij}\rangle = \frac{d}{16\pi G_{\rm N}}\, t^{\rm cf}_{ij}. \tag{4.45}$$

So, we explicitly see that the global condition we imposed on the bulk metric implies that we have picked a particular vacuum in the conformal field theory.

Note that the tensors $t^{\rm cf}_{ij}$ in (4.43), (4.44) are local polynomial functions of the Ricci scalar and the Ricci tensor (but not of the Riemann tensor) of the metric $g_{(0)ij}$. It is perhaps an expected but still a surprising result that in conformally flat backgrounds the anomalous stress tensor is a local function of the curvature.

4.4 Conformal transformation properties of the stress-energy tensor

In this section we discuss the conformal transformation properties of the stress-energy tensor. These can be obtained by noting [75] that conformal transformations in the boundary originate from specific diffeomorphisms that preserve the form of the metric (4.9). Under these diffeomorphisms $g_{ij}(x,\rho)$ transforms infinitesimally as [75]

$$\delta g_{ij}(x,\rho) = 2\sigma(1-\rho\partial_\rho)\, g_{ij}(x,\rho) + \nabla_i a_j(x,\rho) + \nabla_j a_i(x,\rho), \tag{4.46}$$

where $a_j(x,\rho)$ is obtained from the equation

$$a^i(x,\rho) = \frac{1}{2}\int_0^\rho {\rm d}\rho' g^{ij}(x,\rho')\partial_j\sigma(x). \tag{4.47}$$

This can be integrated perturbatively in ρ,

$$a^i(x,\rho) = \sum_{k=1} a^i_{(k)}\rho^k. \tag{4.48}$$

We will need the first two terms in this expansion,

$$a^i_{(1)} = \frac{1}{2}\partial^i\sigma, \qquad a^i_{(2)} = -\frac{1}{4}g^{ij}_{(2)}\partial_j\sigma. \tag{4.49}$$

We can now obtain the way the $g_{(n)}$'s transform under conformal transformations [75]

$$\begin{aligned}
\delta g_{(0)ij} &= 2\sigma g_{(0)ij}, \\
\delta g_{(2)ij} &= \nabla_i a_{(1)j} + \nabla_j a_{(1)i}, \\
\delta g_{(3)ij} &= -\sigma g_{(3)ij}, \\
\delta g_{(4)ij} &= -2\sigma(g_{(4)} + h_{(4)}) + a^k_{(1)}\nabla_k g_{(2)ij} + \nabla_i a_{(2)j} + \nabla_j a_{(2)i} \\
&\quad + g_{(2)ik}\nabla_j a^k_{(1)} + g_{(2)jk}\nabla_i a^k_{(1)}, \\
\delta g_{(5)ij} &= -3\sigma g_{(3)ij},
\end{aligned} \tag{4.50}$$

where the term $h_{(4)}$ in $g_{(4)}$ is only present when $d = 4$. One can check from the explicit expressions for $g_{(2)}$ and $g_{(4)}$ in (C.1) that they indeed transform as (4.50). An alternative way to derive the transformation rules above is to start from (C.1) and perform a conformal variation. In [75] the variations (4.50) were integrated leading to (C.1) up to conformally invariant terms.

Equipped with these results and the explicit form of the energy-momentum tensors, we can now easily calculate how the quantum stress-energy tensor transforms under conformal transformations. We use the term "quantum stress-energy tensor" because it incorporates the conformal anomaly. In the literature such transformation rules were obtained [27] by first integrating the conformal anomaly to an effective action. This effective action is a functional of the initial metric g and of the conformal factor σ. It can be shown that the difference between the stress-energy tensor of the theory on the manifold with metric $ge^{2\sigma}$ and the one on the manifold with metric g is given by the stress-energy tensor derived by varying the effective action with respect to g.

In any dimension the stress-energy tensor transforms *classically* under conformal transformations as

$$\delta\langle T_{\mu\nu}\rangle = -(d-2)\,\sigma\,\langle T_{\mu\nu}\rangle. \tag{4.51}$$

This transformation law is modified by the quantum conformal anomaly. In odd dimensions, where there is no conformal anomaly, the classical transformation rule (4.51) holds also at the quantum level. Indeed, for odd d, and by using (4.41) and (4.50), one easily verifies that the holographic stress-energy tensor transforms correctly.

In even dimensions, the transformation (4.51) is modified. In $d = 2$, it is well-known that one gets an extra contribution proportional to the central charge. Indeed, using (4.33) and the formulae above we obtain

$$\delta\langle T_{ij}\rangle = \frac{l}{8\pi G_{\rm N}}\,(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) = \frac{c}{12}\,(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma), \tag{4.52}$$

which is the correct transformation rule.

In $d = 4$ we obtain,

$$\begin{aligned}\delta\langle T_{ij}\rangle &= -2\sigma\langle T_{ij}\rangle \\ &+ \frac{1}{4\pi G_{\rm N}}\Big(-2\sigma h_{(4)} + \frac{1}{4}\nabla^k\sigma[\nabla_k R_{ij} - \frac{1}{2}(\nabla_i R_{jk} + \nabla_j R_{ik}) - \frac{1}{6}\nabla_k R g_{(0)ij}] \\ &\quad + \frac{1}{48}(\nabla_i\sigma\nabla_j R + \nabla_i\sigma\nabla_j R) + \frac{1}{12}R(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) \\ &\quad + \frac{1}{8}[R_{ij}\nabla^2\sigma - (R_{ik}\nabla^k\nabla_j\sigma + R_{jk}\nabla^k\nabla_i\sigma) + g_{(0)ij}R_{kl}\nabla^k\nabla^l\sigma]\Big).\end{aligned} \tag{4.53}$$

The only other result known to us is the result in [27], where they computed the finite conformal transformation of the stress-energy tensor but for a conformally flat metric $g_{(0)}$. For conformally flat backgrounds, $h_{(4)}$ vanishes because it is the metric variation of a topological invariant. The terms proportional to a single derivative of σ vanish by virtue of Bianchi identities and the fact that the Weyl tensor vanishes for conformally flat metrics. The remaining terms, which only contain second derivatives of σ, can be shown to coincide with the infinitesimal version of (4.23) in [27].

One can obtain the conformal transformation of the stress-energy tensor in $d = 6$ in a similar fashion but we shall not present this result here.

4.5 Matter

In the previous sections we examined how space-time is reconstructed (to leading order) holographically out of CFT data. In this section we wish to examine how field theory describing matter on this space-time is encoded in the CFT. We will discuss scalar fields but the techniques are readily applicable to other kinds of matter.

The method we will use is the same as in the case of pure gravity, i.e. we will start by specifying the sources that are turned on, find how far we can go with only this information and then input more CFT data. We will find the same pattern: knowledge of the sources allows only for determination of the divergent part of the action. The leading finite part (which depends on global issues and/or the signature of space-time) is determined by the expectation value of the dual operator. We would like to stress that in the approach we follow, i.e. regularise, subtract all infinities by adding counter-terms and finally remove the regulator to obtain the renormalised action, all normalisations of the physical correlation functions are fixed and are consistent with Ward identities.

Other papers that discuss similar issues include [8, 93, 92, 111].

In order to couple gravity to matter, one has to solve the coupled system of Einstein's equations and the matter field equations. This is non-trivial, as in general it is hard to solve them exactly. In particular, it is not enough to have a solution $G_{\mu\nu}$ of Einstein's equations given some matter fields, denoted collectively by $\Phi(x)$, which enter Einstein's equations through the stress-energy tensor $T_{\mu\nu}$. One also has to ensure that the fields $\Phi(x)$ remain a solution of the matter field equations for the metric $G_{\mu\nu}$ with back-reaction. A simple example where this is the case is the shock-wave solution discussed in chapter 2. This solution is an exact solution of Einstein's equations with stress-energy tensor $T_{vv} = -p\,\delta(v)\,\delta^{(d-2)}(x)$. Now a straightforward analysis of the geodesics in the shock-wave metric, (2.19), shows that the null geodesic $v = 0$, $x^i = 0$ remains a null geodesic in the shock-wave metric. The reason is that the shock-wave metric still has an isometry along the u-direction. So, the stress-energy tensor does not change and the shock-wave solution solves both the Einstein and the matter field equations exactly: there is no gravitational self-interaction.

In general, however, it is hard to find exact solutions and one takes a perturbative approach, assuming that the matter content perturbs the space-time only slightly. So, as long as the geometry is not too violently modified, one can set up a perturbative expansion where the expansion parameter is the Planck length divided by the typical length scale set by matter. So, one usually neglects the second-order back-reaction which is produced by the changes in the matter field equations induced by the first order back-reaction. This is the approach we will pursue here.

In addition, since we look for perturbative solutions of Einstein's equations near the boundary, also the matter system should have perturbative solutions near the boundary. In other words, we need a perturbative expansion of the stress-energy tensor in r. The existence of perturbative solutions of Einstein's equations sets constraints on the allowed behaviour of the stress-energy tensor near the boundary. For the scalar fields of mass m that we will study in the next section, this implies $m^2 \leq 0$. With these constraints, it is easy to check that the leading behaviour of the stress-energy tensor does not change when we take into account the back-reaction on the metric. This is because, from the CFT point of view, turning on a source $\phi_{(0)}$ or giving a non-vanishing expectation value to the operator $O(x)$ of dimension Δ to which the source couples only changes the expectation value of the stress-energy tensor, but not the metric $g_{(0)}$. In other words, we still have a genuine Dirichlet problem in the bulk and only normalisable solutions change. It is possible to find the general expansion of the stress-energy tensor in r up to the desired order and including an arbitrary number of back-reaction steps, but in most cases the second-order back-reaction effects do not affect the bulk metric to the order we are interested in.

4.5.1 Coupling gravity to matter

In this section we make some preliminary remarks concerning the existence of solutions of Einstein's equations coupled to matter. We do this very generally, without assuming any specific matter model.

The local analysis in the previous sections revealed that undeterminacies in the bulk metric in asymptotically AdS spaces are directly related to information about expectation values of operators in the CFT. For future reference, let us write the three components of Einstein's equations as follows:

$$\begin{aligned} E_{ij} &= 0 \\ E_{ri} &= 0 \\ E_{rr} &= 0 \end{aligned} \tag{4.54}$$

where

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, G_{\mu\nu}\, R - \Lambda\, G_{\mu\nu} + 8\pi G_{\text{N}} T_{\mu\nu}. \tag{4.55}$$

The three components of (4.54) are of course the three components of (4.11) coupled to arbitrary matter.

Now an essential fact in our analysis of the previous sections was that the (ij)-component of Einstein's equations (4.54) left undetermined the metric coefficient $g_{(d)}$. Its trace was determined by the third of (4.54), and the second of (4.54) gave additional information about the traceless part of $g_{(d)}$. This seems to be at odds with the fact that Einstein's equations have some degeneracy related to co-ordinate invariance, and the first and third of (4.54) together with the Bianchi identities are generally sufficient to solve the second one. We will see that this general expectation is only true up to "integration constants". It is interesting to study this in some detail, as the information missing from the second of (4.54) was exactly the information about the dual stress-energy tensor. Indeed, as we shall now see, one can prove that under certain constraints the first and third of (4.54) are enough to satisfy the second of (4.54) *up to a certain coefficient.* This coefficient is exactly the one that specifies the dual stress-energy tensor.

The same is true for the third of (4.54): a certain integration constant has to be set to zero, and this in turn gives the right value for the conformal anomaly. Our only restrictions are that (4.54) has perturbative solutions in r, and that we work with the lowest-order supergravity action without α'-corrections.

In this section we work in the r-co-ordinate system (4.1). It is convenient to first work out the Ricci tensor in (4.54):

$$\begin{aligned} R_{ij} &= R_{ij}(g) + \frac{d}{r^2}\, g_{ij} - \frac{d-1}{2r}\, g'_{ij} + \frac{1}{2}\, g''_{ij} - \frac{1}{2}(g' g^{-1} g')_{ij} \\ &+ \frac{1}{4}\, g'_{ij} \,\mathrm{Tr}\,(g^{-1}g') - \frac{1}{2r}\, g_{ij}\, \mathrm{Tr}\,(g^{-1}g') \\ R_{ir} &= \frac{1}{2}(g^{-1})^{jk}\left(\nabla_i g'_{jk} - \nabla_k g'_{ij}\right) \\ R_{rr} &= \frac{d}{r^2} - \frac{1}{2r}\,\mathrm{Tr}\,(g^{-1}g') + \frac{1}{2}\,\mathrm{Tr}\,(g^{-1}g'') - \frac{1}{4}\,\mathrm{Tr}\,(g^{-1}g')^2. \end{aligned} \tag{4.56}$$

We see from (4.56) and (4.54) that, for the existence of perturbative solutions, the stress-energy tensor is not allowed to diverge worse than $1/r$. Thus, we consider the perturbative expansion: $T_{\mu\nu} = \frac{1}{r}\, T_{(-1)\mu\nu} + T_{(0)\mu\nu} + \ldots$ For T_{ir} we have the stronger requirement $T_{(-1)ir} = 0$. In later sections we will make some comments on stress-energy tensors that have a more violent decay near the boundary. The stress-energy tensor can also contain logarithmic contributions, but usually these appear at higher order and we will not consider them here.

In the co-ordinate system (4.1), the Bianchi identities take the following form:

$$[(d-1) - \frac{r}{2}\,\mathrm{Tr}\,(g^{-1}g')]E_{ir} - rE'_{ir} = r\nabla^k E_{ik} \tag{4.57}$$

$$\begin{aligned} [(d-2) - \frac{r}{2}\,\mathrm{Tr}\,(g^{-1}g')]rE_{rr} - r^2 E'_{rr} &= r\mathrm{Tr}\,(g^{-1}E) - \frac{r^2}{2}\,\mathrm{Tr}\,(g^{-1}g'g^{-1}E) + \\ &\quad + r^2\nabla^k E_{rk}, \end{aligned} \tag{4.58}$$

Substituting our ansatz for the metric, (4.9), for the first Bianchi identity at lowest order we get:

$$(d-1)E_{ir}|_{r=0} = E_{(-1)ij}. \tag{4.59}$$

Now if the first Einstein equation is satisfied at lowest order, $E_{(-1)ij} = 0$, then so is the second, $E_{(0)ir} = 0$.

Now we can use induction to see whether, if $E_{ij} = 0$ to all orders, $E_{ir} = 0$ is true to all orders as well. We take successive derivatives of (4.58), which at order n gives the expression:

$$\sum_{k=0}^{n+1} a_k^{n+1}(r) E_{ir}^{(k)}|_{r=0} = 0, \tag{4.60}$$

a being some coefficient with the property $a_{n+1}^{n+1}(r=0) = 0$. The vanishing of (4.60) would be enough to ensure $E_{ir} = 0$ at each order. However, if some a_n^{n+1} vanishes, the equation cannot be solved and so at that order we may need to introduce logarithmic terms. This happens exactly for $n+1=d$. So, the perturbative analysis reveals that $E_{ij}=0$ ensures $E_{ir}=0$ only up to order $d-1$. Let us analyse this in some more detail.

Assuming $E_{ij}=0$ to all orders, (4.58) reduces to

$$[(d-1)-\frac{r}{2}\,\mathrm{Tr}\,(g^{-1}g')]E_{ir}-rE'_{ir} \;=\; 0. \tag{4.61}$$

This we can integrate exactly, getting

$$E_{ir} \;=\; c_i\, r^{d-1}e^{-H(r)}, \tag{4.62}$$

where $H(r)=\frac{1}{2}\int \mathrm{d}r\mathrm{Tr}\,(g^{-1}g')$ and therefore it has the same regular power expansion as g. We thus see that, in general, we need to impose the additional constraint $c_i=0$ for $E_{ir}=0$ to be true. This is equivalent to setting

$$c_i \;=\; E_{ir}^{(d-1)}|_{r=0}=0. \tag{4.63}$$

The fact that (4.63) is met at order $d-1$ is not accidental. This is exactly the same behaviour we encountered in the vacuum case. So, it is true that the first of Einstein's equations together with the Bianchi equation imply the second Einstein equation, only if (4.63) is satisfied. The latter condition in turn implies that the d-th derivative of g is not specified by the first Einstein equations and has to be imposed additionally. Thus, the second of Einstein's equations gives us information about the traceless part of $g_{(d)}$.

The same analysis can be done for the second Bianchi identity (4.58). We get

$$\begin{aligned}[(d-2)-\frac{r}{2}\,\mathrm{Tr}\,(g^{-1}g')]rE_{rr}-r^2E'_{rr} \;&=\; r\mathrm{Tr}\,(g^{-1}E)-\frac{r^2}{2}\,\mathrm{Tr}\,(g^{-1}g'g^{-1}E)+\\ &+\; r^2\nabla^k E_{rk}.\end{aligned} \tag{4.64}$$

Now, assuming $E_{ij}=0$ and $c_i=0$ implies $E_{ir}=0$ by the previous argument, and this gives an equation for E_{rr} with the following exact solution

$$E_{rr} \;=\; D\, r^{d-2}e^{-H(r)}. \tag{4.65}$$

Thus, we also have to impose $D=0$, which gives the condition

$$D \;=\; E_{rr}^{(d-2)}|_{r=0}=0. \tag{4.66}$$

This ensures $E_{rr}=0$ to all orders and imposes a further constraint on the d-th derivative of g, which now has to satisfy (4.63) and (4.66): in the vacuum case, the latter condition determines the trace of $g_{(d)}$.

To summarise, we have found that $E_{ij}=0$ and the Bianchi identities are not enough to have a solution of Einstein's equations. One needs to set to zero two additional integration constants, and these determine (part of) the coefficient $g_{(d)}$ in the expansion of the metric. Notice, however, that setting these integration constants to zero only ensures the existence of a solution of Einstein's equations, but does not necessarily specify all the coefficients of the metric uniquely. In fact, as we saw in the previous sections, the traceless part of $g_{(d)}$ is still undetermined.

It of course remains to be shown that the first of (4.54) indeed has solutions to all orders given an arbitrary boundary condition $g_{(0)}$. For maximally symmetric spaces this was done in [102].

4.5.2 Dirichlet boundary problem for scalar fields in a fixed background

In this section we consider scalars on a fixed gravitational background. This is taken to be of the generic form (4.9). In most of the literature the fixed metric was taken to be that of standard AdS, but with not much more effort one can consider the general case.

The action for a massive scalar is given by

$$S_{\rm M} = \frac{1}{2}\int \mathrm{d}^{d+1}x\,\sqrt{G}\,\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right), \tag{4.67}$$

where $G_{\mu\nu}$ has an expansion of the form (4.9).

We take the scalar field Φ to have an expansion of the form

$$\Phi(x,\rho) = \rho^{(d-\Delta)/2}\,\phi(x,\rho), \qquad \phi(x,\rho) = \phi_{(0)} + \phi_{(2)}\rho + \ldots\,, \tag{4.68}$$

where Δ is the conformal dimension of the dual operator. We take the dimension Δ to be quantised as $\Delta = \frac{d}{2} + k, k = 0, 1, \ldots$ This is often the case for operators of protected dimension. For the case of scalars that correspond to operators of dimensions $\frac{d}{2} - 1 \le \Delta < \frac{d}{2}$ we refer to [81]. Inserting (4.68) in the field equation,

$$(-\Box_G + m^2)\Phi = 0, \tag{4.69}$$

where $\Box_G\Phi = \frac{1}{\sqrt{G}}\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu\Phi)$, we obtain that the mass m^2 and the conformal dimension Δ are related as $m^2 = (\Delta - d)\Delta$, as explained in the introduction, see (1.52). ϕ satisfies

$$[-(d-\Delta)\partial_\rho \log g\,\phi + 2(2\Delta - d - 2)\partial_\rho\phi - \Box_g\phi] + \rho[-2\partial_\rho\log g\,\partial_\rho\phi - 4\partial_\rho^2\phi] = 0. \tag{4.70}$$

Given $\phi_{(0)}$ one can determine recursively $\phi_{(n)}, n > 0$. This is achieved by differentiating (4.70) and setting ρ equal to zero. We give the result for the first couple of orders in appendix C.4. This process breaks down at order $\Delta - d/2$ (provided this is an integer, which we assume throughout this section) because the coefficient of $\phi_{(2\Delta-d)}$ (the field to be determined) becomes zero. This is exactly analogous to the situation encountered for even d in the gravitational sector. Exactly the same way as there, we introduce at this order a logarithmic term, i.e. the expansion of Φ now reads,

$$\Phi = \rho^{(d-\Delta)/2}\,(\phi_{(0)} + \rho\phi_{(2)} + \ldots) + \rho^{\Delta/2}\,(\phi_{(2\Delta-d)} + \log\rho\,\psi_{(2\Delta-d)} + \ldots). \tag{4.71}$$

The equation (4.70) now determines all terms up to $\phi_{(2\Delta-d-2)}$, the coefficient of the logarithmic term $\psi_{(2\Delta-d)}$, but leaves undetermined $\phi_{(2\Delta-d)}$. This is analogous to the situation discussed in section 4.2, where the term $g_{(d)}$ was undetermined. It is well known [12, 13, 81] that precisely at order $\rho^{\Delta/2}$ one finds the expectation value of the dual operator. We will review this argument below, and also derive the exact proportionality coefficient. Our result is in agreement with [81].

We proceed to regularise and then renormalise the theory. We regulate by integrating in the bulk from $\rho \geq \epsilon$,[10]

$$
\begin{aligned}
S_{\mathrm{M,reg}} &= \frac{1}{2}\int_{\rho\geq\epsilon} \mathrm{d}^{d+1}x\,\sqrt{G}\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right) \\
&= -\int_{\rho=\epsilon} \mathrm{d}^d x\,\sqrt{g(x,\epsilon)}\epsilon^{-\Delta+d/2}\,[\frac{1}{2}\,(d-\Delta)\phi^2(x,\epsilon) + \epsilon\,\phi(x,\epsilon)\partial_\epsilon\phi(x,\epsilon)] \qquad (4.72)\\
&= \int \mathrm{d}^d x\,\sqrt{g_{(0)}}\,[\epsilon^{-\Delta+d/2}a^{\mathrm{M}}_{(0)} + \epsilon^{-\Delta+d/2+1}a^{\mathrm{M}}_{(2)} + \ldots + \epsilon\,a^{\mathrm{M}}_{(2\Delta-d+2)} \\
&- \log\epsilon\,a_{(2\Delta-d)}] + \mathcal{O}(\epsilon^0).
\end{aligned}
$$

Clearly, with $\Delta - d/2$ a positive integer there is a finite number of divergent terms. The logarithmic divergence appears exactly when $\Delta = d/2 + k, k = 0, 1, .., $ in agreement with the analysis in [96], and is directly related to the logarithmic term in (4.71). The first few of the power law divergences read

$$
a^{\mathrm{M}}_{(0)} = -\frac{1}{2}(d-\Delta)\phi^2_{(0)}, \qquad a^{\mathrm{M}}_{(2)} = -\frac{1}{4}\mathrm{Tr}\;g_{(2)}\,\phi^2_{(0)} + (d-\Delta+1)\,\phi_{(0)}\phi_{(2)}. \qquad (4.73)
$$

Given a field of specific dimension it is straightforward to compute all divergent terms.

We now proceed to obtain the renormalised action by adding counter-terms to cancel the infinities,

$$
\begin{aligned}
S_{\mathrm{M,ren}} &= \lim_{\epsilon\to 0}[S_{\mathrm{M,reg}} - \int \mathrm{d}^d x\,\sqrt{g_{(0)}}\,[\epsilon^{-\Delta+d/2}a^{\mathrm{M}}_{(0)} + \epsilon^{-\Delta+d/2+1}a^{\mathrm{M}}_{(2)} + \ldots + \epsilon\,a^{\mathrm{M}}_{(2\Delta-d+2)} \\
&- \log\epsilon\,a_{(2\Delta-d)}]. \qquad (4.74)
\end{aligned}
$$

Exactly as in the case of pure gravity, and since the regulated theory lives at $\rho = \epsilon$, one needs to rewrite the counter-terms in terms of the field living at $\rho = \epsilon$, i.e. the induced metric $\gamma_{ij}(x,\epsilon)$ and the field $\Phi(x,\epsilon)$, or equivalently $g_{ij}(x,\epsilon)$ and $\phi(x,\epsilon)$. This is straightforward but somewhat tedious: one needs to invert the relation between ϕ and $\phi_{(0)}$ and between g_{ij} and $g_{(0)ij}$ to sufficiently high order. This then allows to express all $\phi_{(n)}$, and therefore all $a^{\mathrm{M}}_{(n)}$, in terms of $\phi(x,\epsilon)$ and $g_{ij}(x,\epsilon)$ (the $\phi_{(n)}$'s are determined in terms of $\phi_{(0)}$ and $g_{(0)}$ by solving (4.70) iteratively). Explicitly, the first two orders read

$$
\begin{aligned}
S_{\mathrm{M,ren}} &= \lim_{\epsilon\to 0}\Bigg[\frac{1}{2}\int_{\rho\geq\epsilon} \mathrm{d}^{d+1}x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right) \qquad (4.75)\\
&+ \int_{\rho=\epsilon}\sqrt{\gamma}\,[\frac{(d-\Delta)}{2}\Phi^2(x,\epsilon) + \frac{1}{2(2\Delta-d-2)}\,(\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon) \\
&+ \frac{d-\Delta}{2(d-1)}R[\gamma]\Phi^2(x,\epsilon)) + \ldots]\Bigg].
\end{aligned}
$$

[10] This regularisation for scalar fields in a fixed AdS background was considered in [90, 46]. In these papers the divergences were computed in momentum space, but no counter-terms were added to cancel them. Addition of boundary counter-terms to cancel infinities for scalar fields was considered in [28], and more recently in [81].

The addition of the first counter-term was discussed in [81]. The action (4.75) with only the counter-terms written explicitly is finite for fields of $\Delta < d/2+2$. As remarked above, it is straightforward to obtain all counter-terms needed in order to make the action finite for any field of any mass. These counter-terms contain also logarithmic subtractions that lead to the conformal anomalies discussed in [96]. For instance, if $\Delta = \frac{1}{2}d+1$, the coefficient $[2(2\Delta-d-2)]^{-1}$ in (4.75) is replaced by $-\frac{1}{4}\log\epsilon$. An alternative way to derive the counter-terms is to demand that the expectation value $\langle O\rangle$ is finite. This holds in the case of pure gravity too, i.e. the counter-terms can also be derived by requiring finiteness of $\langle T_{\mu\nu}\rangle$ [11].

The expectation value of the dual operator is given by

$$\langle O(x)\rangle = -\frac{1}{\sqrt{\det g_{(0)}}}\frac{\delta S_{\text{M,ren}}}{\delta\phi_{(0)}} = -\lim_{\epsilon\to 0}\frac{1}{\sqrt{\det g(x,\epsilon)}}\frac{\delta S_{\text{M,ren}}}{\delta\phi(x,\epsilon)}. \tag{4.76}$$

Exactly as in the case of pure gravity, the expectation value receives a contribution both from the regulated part and from the counter-terms. We obtain,

$$\langle O(x)\rangle = (2\Delta-d)\,\phi_{(2\Delta-d)} + F(\phi_{(n)},\psi_{(2\Delta-d)},g_{(m)}), \qquad n<2\Delta-d, \tag{4.77}$$

where we used that $\phi_{(2\Delta-d)}$ is linear in $\phi_{(0)}$ (notice that the action (4.67) does not include interactions). $F(\phi_{(n)},\psi_{(2\Delta-d)},g_{(m)})$ is a local function of $\phi_{(n)}$ with $n<2\Delta-d$, $\psi_{(2\Delta-d)}$ and $g_{(m)}$. These terms are related to contact terms in correlation functions of O with itself and with the stress-energy tensor. Its exact form is straightforward but somewhat tedious to obtain (just use (4.75) and (4.76)).

As we have promised, we have shown that the coefficient $\phi_{(2\Delta-d)}$ is related with the expectation value of the dual CFT operator. In the case that the background geometry is the standard Euclidean AdS one can readily obtain $\phi_{(2\Delta-d)}$ from the unique solution of the scalar field equation with given Dirichlet boundary conditions. One finds that $\phi_{(2\Delta-d)}$ is proportional to (an integral involving) $\phi_{(0)}$. Therefore, $\phi_{(2\Delta-d)}$ carries information about the 2-point function. The factor $(\Delta-d/2)$ is crucial in order for the 2-point function to be normalised correctly [46]. We refer to [81] for a detailed discussion of this point.

We finish this section by calculating the conformal anomaly associated with the scalar fields and in the case the background is (locally) standard AdS (i.e. $g_{(n)}=0$, for $0<n<d$). Equation (4.70) simplifies and can be easily solved. One gets

$$\begin{aligned}
\phi_{(2n)} &= \frac{1}{2n(2\Delta-d-2n)}\,\Box_0\phi_{(2n-2)}, \\
\psi_{(2\Delta-d)} &= -\frac{1}{2(2\Delta-d)}\,\Box_0\phi_{(2\Delta-d-2)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}\,(\Box_0)^k\phi_{(0)},
\end{aligned} \tag{4.78}$$

where $k=\Delta-\frac{d}{2}$ and $\Box_0$ is the Laplacian of $g_{(0)}$. The regularised action written in terms of the fields at $\rho=\epsilon$ contains the following explicit logarithmic divergence:

$$S_{\text{M,reg}} = -\int_{\rho=\epsilon} \mathrm{d}^d x\,\sqrt{\gamma}\,[\log\epsilon\,(\Delta-\frac{d}{2})\,\phi(x,\epsilon)\,\psi_{(2\Delta-d)}(x,\epsilon)+\cdots], \tag{4.79}$$

where the dots indicate power law divergent and finite terms, $\psi_{(2\Delta-d)}(x,\epsilon)$ is given by (4.78) with $g_{(0)}$ replaced by γ and $\phi_{(0)}$ by $\phi(x,\epsilon)$. Using the same argument as in [70] we obtain the matter conformal anomaly,

$$\mathcal{A}_{\rm M} = \frac{1}{2}\left(\frac{1}{2^{2k-2}(\Gamma(k))^2}\right)\phi_{(0)}(\Box_0)^k\phi_{(0)}. \tag{4.80}$$

This agrees exactly with the anomaly calculated in [96] (compare with formulae (10), (37) in [96]).

4.5.3 Scalars coupled to gravity

In the previous section we ignored the back-reaction of the scalars to the bulk geometry. The purpose of this section is to discuss this issue. The action is now the sum of (4.7) and (4.67),

$$S = S_{\rm gr} + S_{\rm M}. \tag{4.81}$$

The gravitational field equation in the presence of matter reads

$$R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda)G_{\mu\nu} = -8\pi G_{\rm N} T_{\mu\nu}. \tag{4.82}$$

In the co-ordinate system (4.9) and with the scalar field having the expansion in (4.71), these equations read

$$\begin{aligned}
\rho\,[2g''_{ij} - 2(g'g^{-1}g')_{ij} + {\rm Tr}\,(g^{-1}g')\,g'_{ij}\,] \quad &+ \quad R_{ij}(g) - (d-2)\,g'_{ij} - {\rm Tr}\,(g^{-1}g')\,g_{ij} = \qquad (4.83)\\
&= \quad -8\pi G_{\rm N}\,\rho^{d-\Delta-1}\left[\frac{(\Delta-d)\Delta}{d-1}\,\phi^2\,g_{ij} + \rho\,\partial_i\phi\partial_j\phi\right],\\
\nabla_i{\rm Tr}\,(g^{-1}g') - \nabla^j g'_{ij} \quad &= \quad -16\pi G_{\rm N}\,\rho^{d-\Delta-1}\left[\frac{d-\Delta}{2}\,\phi\partial_i\phi + \rho\,\partial_\rho\phi\partial_i\phi\right],\\
{\rm Tr}\,(g^{-1}g'') - \frac{1}{2}{\rm Tr}\,(g^{-1}g'g^{-1}g') \quad &= \quad -16\pi G_{\rm N}\,\rho^{d-\Delta-2}\left[\frac{d(\Delta-d)(\Delta-d+1)}{4(d-1)}\,\phi^2\right.\\
&+ \quad \left.(d-\Delta)\,\rho\,\phi\partial_\rho\phi + \rho^2\,(\partial_\rho\phi)^2\,\right],
\end{aligned}$$

If $\Delta > d$, the right-hand side diverges near the boundary whereas the left-hand side is finite. Operators with dimension $\Delta > d$ are irrelevant operators. Correlation functions of these operators have a very complicated singularity structure at coincident points. As remarked in [125], one can avoid such problems by considering the sources to be infinitesimal and to have disjoint support, so that these operators are never at coincident points. Requiring that the equations in (4.83) are satisfied to leading order in ρ yields

$$\phi^2_{(0)} = 0, \tag{4.84}$$

which is indeed the prescription advocated in [125].

If $\Delta \leq d$, which means that we deal with marginal or relevant operators, one can perturbatively calculate the back-reaction of the scalars to the bulk metric. At which order the leading back-reaction appears depends on the mass of the field. For fields that correspond to operators of dimension $\Delta = d - k$ the leading back-reaction appears at order ρ^k, except when $k = 0$ (marginal operators), where the leading back-reaction is at order ρ.

Let us see how conformal anomalies arise in this context. The logarithmic divergences are coming from the regulated on-shell value of the bulk integral in (4.81). The latter reads

$$\begin{aligned} S_{\rm reg}({\rm bulk}) &= \int_{\rho \geq \epsilon} {\rm d}\rho\, {\rm d}^d x \sqrt{G}\, [\frac{d}{8\pi G_{\rm N}} - \frac{m^2}{d-1}\, \Phi^2] \\ &= \int_{\rho \geq \epsilon} {\rm d}\rho\, {\rm d}^d x\, \frac{1}{\rho} \sqrt{g(x,\rho)}\, [\frac{d\, \rho^{-d/2}}{16\pi G_{\rm N}} - \frac{m^2\, \rho^{-k}}{2(d-1)}\, \phi^2(x,\rho)\,], \end{aligned} \qquad (4.85)$$

where $k = \Delta - d/2$. We see that gravitational conformal anomalies are expected when d is even and matter conformal anomalies when k is a positive integer, as it should.

In the presence of sources the expectation value of the boundary stress-energy tensor is not conserved but rather it satisfies a Ward identity that relates its covariant divergence to the expectation value of the operators that couple to the sources. To see this consider the generating functional

$$Z_{\rm CFT}[g_{(0)}, \phi_{(0)}] = \big\langle \exp \int {\rm d}^d x \sqrt{g_{(0)}}\, [\frac{1}{2}\, g^{ij}_{(0)} T_{ij} - \phi_{(0)} O] \big\rangle. \qquad (4.86)$$

Invariance under infinitesimal diffeomorphisms,

$$\delta g_{(0)ij} = \nabla_i \xi_j + \nabla_j \xi_i, \qquad (4.87)$$

yields the Ward identity,

$$\nabla^j \langle T_{ij} \rangle = \langle O \rangle\, \partial_i \phi_{(0)}. \qquad (4.88)$$

As we have remarked before, $\langle T_{ij} \rangle$ has a dual meaning [11], both as the expectation value of the dual stress-energy tensor and as the quasi-local stress-energy tensor of Brown and York. The Ward identity (4.88) has a natural explanation from the latter point in view as well. According to [25] the quasi-local stress-energy tensor is not conserved in the presence of matter but it satisfies

$$\nabla^j \langle T_{ij} \rangle = -\tau_{i\rho}, \qquad (4.89)$$

where $\tau_{i\rho}$ expresses the flow of matter energy-momentum through the boundary. Evidently, (4.88) is of the form (4.89).

Solving the coupled system of equations (4.83) and (4.70) is straightforward but somewhat tedious. The details differ from case to case. For illustrative purposes we present a sample calculation: we consider the case of two-dimensional massless scalar field ($d = \Delta = 2, k = 1$).

The equations to be solved are (4.70) and (4.83) with $d = \Delta = 2$ and the expansion of the metric and the scalar field are given by (4.9) and (4.71) (again with $d = \Delta = 2$), respectively. Equation (4.70) determines $\psi_{(2)}$,

$$\psi_{(2)} = -\frac{1}{4}\Box_0\phi_{(0)}. \tag{4.90}$$

Equations (4.83) determine $h_{(2)}$, the trace of the $g_{(2)}$ and provide a relation between the divergence of $g_{(2)}$ and $\phi_{(2)}$,

$$\begin{aligned} h_{(2)} &= -4\pi G_{\rm N}\left(\partial_i\phi_{(0)}\partial_j\phi_{(0)} - \frac{1}{2}\,g_{(0)ij}\,(\partial\phi_{(0)})^2\right), \\ \text{Tr } g_{(2)} &= \frac{1}{2}\,R + 4\pi G_{\rm N}\,(\partial\phi_{(0)})^2, \\ \nabla^i g_{(2)ij} &= \partial_i \text{Tr } g_{(2)} + 16\pi G_{\rm N}\,\phi_{(2)}\partial_i\phi_{(0)}. \end{aligned} \tag{4.91}$$

Notice that $g_{(2)}$ and $\phi_{(2)}$ are still undetermined and are related to the expectation values of the dual operators (4.27) and (4.77), respectively. Notice that $h_{(2)}$ is equal to the stress-energy tensor of a massless two-dimensional scalar.

Going back to (4.85), we see that the second term drops out (since $m^2 = 0$) and one can use the result already obtained in the gravitational sector,

$$\begin{aligned} \mathcal{A} &= \frac{1}{16\pi G_{\rm N}}(-2a_{(2)}) = \frac{1}{16\pi G_{\rm N}}(-2\text{Tr } g_{(2)}) \\ &= -\frac{1}{16\pi G_{\rm N}}R + \frac{1}{2}\phi_{(0)}\Box_0\phi_{(0)} - \frac{1}{2}\,\nabla_i(\phi_{(0)}\nabla^i\phi_{(0)}), \end{aligned} \tag{4.92}$$

which is the correct conformal anomaly [70, 96] (the last term can be removed by adding a covariant counter-term).

The renormalised boundary stress tensor reads

$$\langle T_{ij}(x)\rangle = \frac{1}{8\pi G_{\rm N}}\left(g_{(2)ij} + h_{(2)ij} - g_{(0)ij}\text{Tr } g_{(2)}\right)(x). \tag{4.93}$$

Its trace gives correctly the conformal anomaly (4.92). On the other hand, taking the covariant derivative of (4.93) we get

$$\begin{aligned} \nabla^j\langle T_{ij}\rangle &= \langle O(x)\rangle\,\partial_i\phi_0(x) \quad , \\ \langle O(x)\rangle &= 2(\phi_2(x) + \psi_2(x)). \end{aligned} \tag{4.94}$$

in agreement with equations (4.88) and (4.77).

4.5.4 Pointlike particles

The method developed in the previous subsections is quite generic and can be applied to other matter fields. Although we have not worked out all the details, in this section we give a further example for illustrative purposes: we consider pointlike particles. This is in our opinion a very important example for our understanding of holography in the

AdS/CFT correspondence, and we hope to report the full details elsewhere. Indeed, one can do interesting gedanken experiments with point particles and black holes in AdS [98, 108, 85] to test the causality and locality properties of the boundary theory.

So we couple the Einstein-Hilbert action to the action for a pointlike particle. One then needs to solve Einstein's equations coupled to the geodesic equation and the constraint

$$G_{\mu\nu}(z)\dot{z}^\mu \dot{z}^\nu = -\varepsilon \tag{4.95}$$

($\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for massless particles). In the massive case, we get the following stress-energy tensor:

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{|G(x)|}} \int \mathrm{d}t\, \delta^{(d+1)}(x - z(t))\, \dot{z}^\mu \dot{z}^\nu. \tag{4.96}$$

In the massless case, the stress-energy tensor is given by (2.5). We have also analysed the tachyonic case, but we will not present the results here.

We are interested in computing the back-reaction effects of the particle on the metric near the boundary. This will allow us to compute the expectation value of the stress-energy tensor of the dual theory [35], which will depend in a crucial manner on the boundary conditions on the position and the speed of the particle. Therefore we are interested in the asymptotic behaviour of the stress-energy tensor as $r \to 0$. This is given by the part of the trajectory satisfying $r(t) \to 0$. Hence, the problem of finding the asymptotics of the stress-energy tensor translates itself into finding the region of the trajectory γ near the boundary. For the massless particle and the tachyon it is known that they can travel from the boundary to the bulk and viceversa, so we expect that there are values of t corresponding to $r = 0$. However, the particle with positive mass squared never reaches the boundary, and so we expect it not to contribute to the stress-energy tensor at $r = 0$. As we will see, this turns out to be true also for Einstein spaces with arbitrary boundary metric.

The strategy will be the following. To identify the region of t for which $r(t) \to 0$, we solve the geodesic equation perturbatively in r and find the solutions $r(t)$ and $z^i(t)$ perturbatively in some function of t. If there are such solutions, the perturbative expansion makes sense; if there are not, the geodesic equation cannot be solved perturbatively near the boundary.

The massless particle

To lowest order, the geodesic equations for massless particles are solved by

$$\begin{aligned} r(t) &= \frac{1}{c(t-d)} \\ z^i(t) &= z_0^i + r(t) v^i, \end{aligned} \tag{4.97}$$

where v^i is now a timelike vector, $g_{ij} v^i v^j = -1$, defined in general by $v^i(r) \equiv \frac{\mathrm{d}z^i}{\mathrm{d}r}$. In this case, the stress-energy tensor can be cast in the form

$$T_{\mu\nu} = \frac{pc\ell^2}{\sqrt{g}} \left(\frac{r}{\ell}\right)^{d+3} v_\mu v_\nu\, \delta^{(d)}(x - z(r)), \tag{4.98}$$

where $v^\mu(r)$ is defined by $v^\mu(r) = (1, v^i(r))$. It is null in the space-time metric and satisfies

$$\begin{aligned} \partial_\mu v^\mu &= 0 \\ v^\mu \partial_\mu \delta^{(d)}(x - z(r)) &= 0. \end{aligned} \tag{4.99}$$

All components of the above stress-energy tensor are proportional to r^{d-1} in leading order in r as $r \to 0$. Therefore, it will contribute to $g_{(d)}$ but not to $h_{(d)}$, just as in the tachyonic case. It is now also straightforward to compute the dual stress-energy tensor. This will have an interesting behaviour [72]: to start with, unless one chooses very special boundary conditions, the effective Hamiltonian will be time-dependent due to the covariance of our formulae in the boundary co-ordinates. Notice that to get agreement with the results in [72], where the stress-energy is centred on the light-cone, one may need to first perform a co-ordinate transformation. As mentioned in the previous sections, such a co-ordinate transformation changes the value of the stress-energy tensor if it induces a boundary Weyl rescaling.

The massive particle

The bulk trajectory of a particle with positive mass squared is given by

$$r(t) = \frac{r_0}{|\cos(t/\ell + c)|}, \tag{4.100}$$

and, like in the tachyonic case, r_0 and c are to be determined by the boundary conditions only. In this case, however, we see that $r(t)$ can never be zero unless $r_0 = 0$, in which case the particle stays forever at the boundary and never reaches the bulk. Therefore, a perturbative solution of the geodesic equation in powers of r does not make sense in this case, as the world-line of the particle actually never reaches the boundary. Therefore, one can only hope to solve the geodesic equation for simple exact solutions of the vacuum Einstein equations. For example, it is an elementary exercise to solve for the case of a flat boundary, the trajectory being given by (4.100). In that case, one finds an expression for the stress-energy tensor analogous to that for the tachyon, but now involving a step function $\theta(r - r_0)$, hence with support only on the region $r > r_0$.

In this case, the particle contributes only a finite piece to the action.

4.6 Conclusions

Most of the discussions in the literature on the AdS/CFT correspondence are concerned with obtaining conformal field theory correlation functions using supergravity. Here we started investigating the converse question: how can one obtain information about the bulk theory from CFT correlation functions? How does one decode the hologram?

Answering these questions in all generality, but within the context of the AdS/CFT duality, entails developing a precise dictionary between bulk and boundary physics. A prescription for relating bulk/boundary observables is already available [64, 125], and one would expect that it would allow us to reconstruct the bulk space-time from the boundary CFT. The prescription of [64, 125], however, relates infinite quantities. One of the main results presented here is the systematic development of a renormalised version of this prescription. Equipped with it, and with no other assumption (except that the CFT has an AdS dual), we then proceeded to reconstruct the bulk space-time metric and bulk scalar fields to the first non-trivial order.

Our approach to the problem is to start from the boundary and try to build iteratively bulk solutions. Within this approach, the pattern we find is the following:

• Sources in the CFT determine an asymptotic expansion of the corresponding bulk field near the boundary to high enough order so that *all infrared divergences* of the bulk on-shell action can be computed. This then allows to obtain a renormalised on-shell action by adding boundary counter-terms to cancel the infrared divergences.

• Bulk solutions can be extended one order further by using the 1-point function of the corresponding dual CFT operator.

In the case the bulk field is the metric, our results show that a conformal structure at infinity is not in general sufficient in order to obtain a bulk metric. The first additional information one needs is the expectation value of the boundary stress-energy tensor.

As a by-product, we have obtained ready-to-use formulae for the Brown-York quasi-local stress-energy tensor for arbitrary solution of Einstein's equations with negative cosmological constant up to six dimensions. The six-dimensional result is particularly interesting because, via AdS/CFT, it provides new information about the still mysterious $(2,0)$ theory. Furthermore, we have obtained the conformal transformation properties of the stress-energy tensors. These transformation rules incorporate the trace anomaly and provide a generalisation to $d > 2$ of the well-known Schwartzian derivative contribution in the conformal transformation rule of the stress-energy tensor in $d = 2$.

Our discussion extends straightforwardly to the case of different matter. We expect that in all cases obstructions in extending the solution to the deep interior region will be resolved by additional CFT data. An interesting case to study in this framework is point particles. Reconstructing the trajectory of the bulk point particle out of CFT data will present a model of how holography works with time dependent processes. Furthermore, following [72], one could study the interplay between causality and holography. Another extension is to study renormalisation group flows using the present formalism. This amounts to extending the discussion in section 4.2 by adding a potential for the scalars. Another application of our results is in the context of Randall-Sundrum (RS) scenarios [99]. Incorporating such a scenario in string theory, in the case the bulk space is AdS, may yield a connection with the AdS/CFT duality [123, 126]. As advocated in [126], one may view the RS scenario as $4d$ gravity coupled to a cut-off CFT. The regulated theory in our discussion provides a dual description of a cut-off CFT. In this context, the counter-terms are re-interpreted as providing the action for the bulk modes localised on the brane [102, 62, 55]. We see, for instance, that the counter-terms in (4.75) can be re-interpreted as an action for a bulk scalar mode localised on the brane (see, e.g., [31]). This is the subject of study in the next chapter.

Chapter 5

Warped Compactifications and the Holographic Stress Tensor

The contents of this chapter are based on [36]. We study gravitational aspects of brane-world scenarios. We show that the bulk Einstein equations together with the junction condition imply that the induced metric on the brane satisfies the full non-linear Einstein equations with a specific effective stress-energy tensor. This result holds for any value of the bulk cosmological constant. The analysis is done by either placing the brane close to infinity or by considering the local geometry near the brane. In the case that the bulk space-time is asymptotically AdS, we show that the effective stress-energy tensor is equal to the sum of the stress-energy tensor of matter localised on the brane and of the holographic stress-energy tensor appearing in the AdS/CFT duality. In addition, there are specific higher-curvature corrections to Einstein's equations. We analyse in detail the case of asymptotically flat space-time. We obtain asymptotic solutions of Einstein's equations and show that the effective Newton's constant on the brane depends on the position of the brane.

5.1 Warped Compactifications and AdS/CFT holography

The previous chapters dealt mainly with holography from the point of view of a bulk observer. We used the existence of a holographic dual to find how information about the boundary is encoded in the bulk. Now we change perspective and ask ourselves where the boundary observer finds the information about the bulk geometry and fields. We do this in the context of warped compactifications, where the boundary observer lives on a brane[1]. We find that the information about the bulk, and in particular global information that is not captured by the local analysis, is encoded in the stress tensor on

[1]The sense in which "holography" is used here differs from the original sense. Here the boundary theory is a gravitational theory, and there is not a duality between bulk and boundary, but rather an embedding of the boundary in the bulk.

the brane.

In the AdS/CFT correspondence, the supergravity partition function is related to the generating functional of conformal field theory (CFT) correlation functions as

$$Z[\Phi] = \int_{\phi} D\Phi \, \exp(iS[\Phi]) = W_{\text{CFT}}[\phi], \tag{5.1}$$

where Φ denotes collectively all fields and ϕ is a field parametrising the boundary condition of Φ at infinity. In the conformal field theory the boundary fields ϕ are interpreted as sources for CFT operators. In particular, the metric at infinity, $g_{(0)}$, is considered as the source for the stress-energy tensor of the dual CFT. The relation (5.1) suffers from divergences and has to be regularised and renormalised.

On the CFT side, there are UV divergences when operators come to coincident points. These correspond to IR divergences on the gravitational side. To regulate the gravitational theory one may cut-off the asymptotically AdS space-time at some radius $\rho = \epsilon$ near the boundary. One can then compute all infrared divergences. The renormalised theory is obtained by adding counter-terms to cancel the infinities and then removing the cut-off.

One may, however, wish to consider situations where the infrared cut-off is kept finite instead of being sent to zero. This is the case in warped compactifications, where the AdS space-time is cut-off by the presence of a brane. In this case, (5.1) does not have any infrared divergences and so one does not need to add counter-terms.

In the cut-off space-time, the induced metric at the boundary γ corresponds to a normalisable mode and so one should integrate over it:

$$\int D\gamma_\epsilon \int_{\gamma_\epsilon} DG \exp(iS[G]) = \int D\gamma_\epsilon \, W_{\text{CFT}}[\gamma, \epsilon], \tag{5.2}$$

Under these circumstances, gravity becomes dynamical on the brane, and the brane theory is a CFT coupled to dynamical gravity.

Consider a space-time M with a boundary ∂M. The action in (5.2) is given by[2]

$$\begin{aligned} S[\Phi, G] \quad = \quad & \frac{1}{16\pi G_{d+1}} [\int_M \mathrm{d}^{d+1}x \, \sqrt{G} \, (R[G] + 2\Lambda) - \int_{\partial M} \mathrm{d}^d x \, \sqrt{\gamma} \, 2K] \\ & + \int_M \mathrm{d}^{d+1}x \, \sqrt{G} \, \mathcal{L}^{\text{bulk}} + \int_{\partial M} \mathrm{d}^d x \, \sqrt{\gamma} \, \mathcal{L}^{\text{bdry}} \end{aligned} \tag{5.3}$$

where $\mathcal{L}^{\text{bulk}}$ denotes the Lagrangian for bulk matter and $\mathcal{L}^{\text{bdry}}$ the Lagrangian for matter living on the boundary. Einstein's equations read[3]:

$$R_{\mu\nu}[G] - \frac{1}{2} (R[G] + 2\Lambda) \, G_{\mu\nu} \quad = \quad -8\pi G_{d+1} \, T^{\text{bulk}}_{\mu\nu}[G] \tag{5.4}$$

$$K_{ij}[\gamma] - \gamma_{ij} \, K[\gamma] \quad = \quad 8\pi G_{d+1} \, T^{\text{bdry}}_{ij}[\gamma]. \tag{5.5}$$

[2]Our curvature conventions are as follows: $R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l + \Gamma_{ip}{}^l \Gamma_{jk}{}^p - (i \leftrightarrow j)$ and $R_{ij} = R_{ikj}{}^k$. With these conventions the curvature of AdS comes out positive, but we will still use the terminology "space of constant negative curvature". Notice also that we take $\int \mathrm{d}^{d+1}x = \int \mathrm{d}^d x \int_0^\infty \mathrm{d}\rho$ and the boundary is at $\rho = 0$. The minus sign in front of the trace of the second fundamental form is correlated with the choice of having $\rho = 0$ in the lower end of the radial integration.

[3]The different signs in the right hand side of these two equations is related to our conventions discussed in the previous footnote.

These equations describe the case the bulk space-time ends on the brane. This is in fact half of the Randall-Sundrum (RS) space-time [99]. In the RS scenario one glues on the other side of the brane an identical space-time. Then the substitution

$$K_{ij} \to \lim_{\delta \to 0} [K_{ij}(\rho = \epsilon + \delta) - K_{ij}(\rho = \epsilon - \delta)],$$

in (5.5) yields the junction condition (see, for example, [21] for a derivation). $\rho = \epsilon$ is the position of the brane. In the RS context, $K_{ij}(\rho = \epsilon + \delta) = -K_{ij}(\rho = \epsilon - \delta)$ due to the Z_2-symmetry, so the net effect is to get back (5.5) but with an extra factor of two. In the remainder we will work with equations (5.4) and (5.5) and we will refer to (5.5) as the junction condition.

The usual way [99] of establishing localisation of gravity on the brane is to study small fluctuations around a given configuration (such as a flat brane in AdS space) which solves equations (5.4). The equations for small gravitational fluctuations around the solution take the form of a quantum mechanical problem. In terms of the effective quantum mechanical problem the existence of a localised graviton translates into the existence of a normalisable zero-mode solution (this solution is the wave function associated to the graviton localised on the brane). In addition to the zero mode there are additional massive modes. One still has to show that these modes do not drastically change the physics, i.e. that they yield sub-leading corrections relative to the zero mode. Note that the question of normalisability of the zero mode depends on global properties of the gravitational solution. If the bulk space is asymptotically flat there is still a zero-mode but it is not normalisable. There may still be a quasi-localisation due to a collection of low-energy Kaluza-Klein modes [60, 29, 42].

The analysis just described is at the linearised level. It is technically involved in this approach to go beyond the linear approximation and demonstrate the full non-linear structure of the gravity localised on the brane. In this chapter we use the AdS/CFT duality in order to achieve this goal. Previous works that use the AdS/CFT duality in the RS context include [123, 62, 55, 68, 41, 6, 54, 31].

It has been shown in [44, 70] that given a metric $g_{(0)}$ on the boundary of AdS one can obtain an asymptotic expansion of the bulk metric near the boundary up to certain order in the radial co-ordinate (which is regarded as the small parameter in the expansion). The next order coefficient is left undetermined by the bulk field equations [44]. This coefficient is determined once a symmetric covariantly conserved tensor $T_{ij}^{\text{CFT}}(x)$ with trace equal to the holographic Weyl anomaly is supplied. The tensor $T_{ij}^{\text{CFT}}(x)$ is the holographic stress tensor of the dual conformal field theory [35] (see also [16]). Notice that the CFT stress energy tensor encodes global information too. In particular, regularity of the bulk solution sometimes uniquely fixes $T_{ij}^{\text{CFT}}(x)$.

Let us consider a brane placed close to the AdS boundary. Then one can solve (5.4) by simply considering the asymptotic solution described in the previous paragraph. The junction condition (5.5) then becomes Einstein's equation for the induced metric on the brane. The right-hand side in Einstein's equations is equal to the stress-energy tensor due to matter localised on the brane plus the CFT stress-energy tensor. In fact, irrespectively of the value of the bulk cosmological constant, Einstein's equations in the bulk plus the junction condition effectively impose Einstein's equations on the brane. This result first appeared in [100]. In particular in all cases the gravitational equations on the brane involve a "holographic stress-energy tensor". This can be taken to holographically represent the bulk space-time.

This chapter is organised as follows. In the next section we adopt the results from the AdS/CFT duality to brane-world scenarios. In particular, we put a brane near the boundary of AdS and obtain the equation that the induced metric on the brane satisfies. In section 5.3 we place a brane at some (arbitrarily chosen) position in the bulk and analyse the equations near the brane, i.e. we consider the radial distance from the brane as a small parameter. These considerations are valid for any bulk cosmological constant. In section 5.4 we consider the case of a brane placed near infinity of an asymptotically flat bulk space-time. Finally, in section 5.5 we study bulk metrics that are conformally flat.

In this chapter we only perform a local analysis. Global issues are important and need to be addressed in order to establish localisation of the graviton on the brane. This important issue is left for future study.

5.2 Brane gravity from the asymptotic analysis of AdS space

The asymptotic solutions of the bulk Einstein equation (5.4) in vacuum were worked out in [70] to sufficiently high order. These solutions are best found by writing the bulk metric in the Fefferman-Graham form [44] used throughout the previous chapter (see (4.9)):

$$\mathrm{d}s^2 = \frac{l^2}{4\rho^2}\,\mathrm{d}\rho^2 + \frac{l^2}{\rho}\,g_{ij}(\rho,x)\mathrm{d}x^i\mathrm{d}x^j, \tag{5.6}$$

where the metric g_{ij} has the expansion

$$g(\rho,x) = g_{(0)} + \rho g_{(2)} + \cdots + \rho^{d/2}g_{(d)} + h_{(d)}\rho^{d/2}\log\rho + \mathcal{O}(\rho^{(d+1)/2}). \tag{5.7}$$

l^2 is related to the cosmological constant as $\Lambda = -d(d-1)/2l^2$. Given $g_{(0)}$ all coefficients up to $g_{(d)}$ can be found as local functions of $g_{(0)}$. The coefficient $g_{(d)}$ is undetermined from the gravity equations, and it is related to the stress-energy tensor of the dual CFT:

$$\langle T_{ij}\rangle_{\rm CFT} = \frac{dl^{d-1}}{16\pi G_{d+1}}\,g_{(d)ij} + X^{(d)}_{ij}[g_{(j)}], \tag{5.8}$$

where $X^{(d)}_{ij}[g_{(j)}]$ is a known function of the lower-order coefficients $g_{(j)}, j<d$ [35] (see [101] for a review). The gravitational equations imply that $\langle T_{ij}\rangle_{\rm CFT}$ is covariantly conserved and its trace reproduces the conformal anomaly of the boundary CFT.

Let us place a brane close to infinity at constant $\rho=\epsilon$, where ϵ is small enough for the expansion (5.7) to be a good approximation for the metric in the bulk. Using the results of chapter 4 [35], it is now a simple matter (using (4.27)-(4.30)) to see that the junction condition gives Einstein's equation on the brane. For a 3-brane we get:

$$R_{ij}[\gamma] \quad - \quad \frac{1}{2}\gamma_{ij}\,(R[\gamma]-\frac{12}{l^2}) + \frac{1}{4}l^2\log\epsilon\left(\frac{1}{12}\nabla_i\nabla_i R[\gamma] - \frac{1}{4}\,\nabla^2 R_{ij}[\gamma] + \frac{1}{24}\,\gamma_{ij}\nabla^2 R[\gamma]\right.$$

$$+ \quad \frac{1}{2}\, R^{kl}[\gamma] R_{ikjl}[\gamma] - \frac{1}{6}\, R[\gamma] R_{ij}[\gamma] + \frac{1}{24}\, \gamma_{ij}\, R^2[\gamma] - \frac{1}{8}\, \gamma_{ij}\, R^{kl}[\gamma] R_{kl}[\gamma] \Big)$$
$$= \quad -16\pi G_5 \frac{1}{l}\,(\langle T_{ij}[\gamma]\rangle_{\rm CFT} + T^{\rm bdry}_{ij}[\gamma]), \tag{5.9}$$

where we kept only terms $\mathcal{O}(R^2)$, and there is an explicit dependence on the cut-off through the logarithmic term.

There are several comments in order here:

- In deriving (5.9) it was essential that we added no counter-terms to the action. Had we added counter-terms, then all the curvature terms in the above formula would have been cancelled. Indeed, these precisely come from the infrared divergent part of the action.
- In the effective Einstein equations the bulk space-time is represented by the holographic stress-energy tensor. In other words, the Brane-World has a purely d-dimensional description where the bulk space-time has been replaced by the cut-off CFT. The CFT couples to matter on the brane only through gravitational interactions.
- The effective Newton's constant is given by

$$G_4 = \frac{2G_5}{l} \tag{5.10}$$

  In the context of the two-sided RS scenario one should divide this result by two (see the discussion after (5.5)).
- The AdS/CFT duality predicts specific R^2-terms. The terms in (5.9) are derivable from the local action: $\int \mathrm{d}^d x\, a_{(4)}$, where $a_{(4)}$ is the holographic trace anomaly in four dimensions.
- The original expansion in the cut-off becomes an expansion in the brane curvature.

It is straightforward to extend these results to higher dimensions using the results in chapter 4.

In $(2+1)$ dimensions the series in (5.7) terminates at the ρ^2-term, and one has the exact expression [102]

$$g(x,\rho) = (g_{(0)} + \frac{1}{2} g_{(2)}\rho)^2 \;, \quad g_{(2)} = \frac{1}{2}(R\, g_{(0)ij} + t_{ij}) \;\;, \tag{5.11}$$

where t_{ij} is conserved, $\nabla^j_{(0)} t_{ij} = 0$, and its trace is Tr $t = -R$. It follows that t_{ij} can be identified as the Liouville stress-energy tensor. The holographic stress-energy tensor is equal to $\langle T_{ij}\rangle = \frac{l}{16\pi G_3}\, t_{ij}$.

Placing an one-brane at $\rho = \epsilon$ and neglecting ϵ^2-terms one finds that the junction condition (5.5) implies

$$\gamma_{ij} = -8\pi G_3 (T^{\rm bdry}_{ij} + \langle T_{ij}\rangle) \;\;, \tag{5.12}$$

where $\gamma_{ij} = \frac{1}{\rho} g_{ij}(x,\rho)$ is the induced metric on the brane, and $T^{\rm bdry}_{ij}$ is the stress tensor of matter on the brane. Note that in two dimensions there is no dynamical theory for just the metric tensor. Gravity induced on the one-brane is of the scalar-tensor type.

In the presence of matter in the bulk, it was shown in the previous chapter that the bulk equations can be solved in the same way. In this case, one again reinterprets the leading in ϵ terms as giving the terms in the action that determine the dynamics on the brane. For bulk scalar fields of mass $m^2 = (\Delta - d)\Delta$, the effective brane action is:

$$\begin{aligned} S[\gamma,\Phi] &= \int \mathrm{d}^d x \sqrt{\gamma} \left[\frac{1}{2(2\Delta-d-2)} \Phi(x,\epsilon) \Box_\gamma \Phi(x,\epsilon) \right. \\ & \left. + \frac{(d-\Delta)}{2} \left(1 + \frac{1}{2(d-1)(2\Delta-d-2)} R[\gamma] \right) \Phi^2(x,\epsilon) \right], \end{aligned} \tag{5.13}$$

where again we only show the first few terms in the low energy expansion. The d-dimensional mass receives contributions both from the mass term in $(d+1)$ dimensions but also from the bending of the brane. Notice that a massless field in $d+1$ dimensions remains massless in d dimensions.

5.3 Local analysis

In the previous section we made use of the asymptotic expansion of the bulk AdS metric (5.7). A similar analysis can be done for a brane located anywhere in the bulk by considering the local geometry near the brane.

Consider the Einstein equations in the bulk

$$R_{\mu\nu} + \frac{2}{d-1} \Lambda G_{\mu\nu} = 0. \tag{5.14}$$

Near the brane one can use Gaussian normal co-ordinates. In these co-ordinates the bulk metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}(r,x) \mathrm{d}x^i \mathrm{d}x^j, \tag{5.15}$$

where r stands for the radial co-ordinate adjusted so that the brane location is at $r=0$. Then the $(ij), (rr)$ and (ri) components of Einstein equations (5.14) read

$$R_{ij}[\gamma] + \frac{2}{d-1} \Lambda \gamma_{ij} + \frac{1}{2} \partial_r^2 \gamma_{ij} - \frac{1}{2} (\partial_r \gamma \gamma^{-1} \partial_r \gamma)_{ij} + \frac{1}{4} \partial_r \gamma_{ij} \mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma) = 0 \tag{5.16}$$

$$\frac{1}{2} \partial_r (\mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma)) + \frac{1}{4} \mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma)^2 + \frac{2}{d-1} \Lambda = 0 \tag{5.17}$$

$$\nabla_j [\gamma^{-1} \partial_r \gamma - \mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma)]_i^j = 0. \tag{5.18}$$

Combining the equations (5.16) and (5.17) we find that

$$R[\gamma] + 2\Lambda + \frac{1}{4} \left([\mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma)]^2 - \mathrm{Tr}\,(\gamma^{-1} \partial_r \gamma)^2 \right) = 0. \tag{5.19}$$

Let $\gamma_{ij}(x,r)$ have the following expansion near the brane:

$$\gamma = \gamma_{(0)} + \gamma_{(1)}r + \gamma_{(2)}r^2 + ...$$

Then solving equations (5.16), (5.17) and (5.18) iteratively we find expressions relating the coefficients $\gamma_{(k)}$. From equation (5.16) we find that

$$\mathrm{Ric}[\gamma_{(0)}] + \frac{2}{d-1}\Lambda\gamma_{(0)} + \gamma_{(2)} - \frac{1}{2}\gamma_{(1)}^2 + \frac{1}{4}\gamma_{(1)}\mathrm{Tr}\,\gamma_{(1)} = 0. \tag{5.20}$$

Equation (5.17) to leading order gives

$$\mathrm{Tr}\,\gamma_{(2)} = \frac{1}{4}\mathrm{Tr}\,\gamma_{(1)}^2 - \frac{2}{d-1}\Lambda. \tag{5.21}$$

Taking the trace of (5.20) and using (5.21) one finds

$$R[\gamma_{(0)}] + 2\Lambda - \frac{1}{4}(\mathrm{Tr}\,\gamma_{(1)}^2 - (\mathrm{Tr}\,\gamma_{(1)})^2) = 0. \tag{5.22}$$

This equation can also be obtained from (5.19). Equation (5.17) to the first two orders yields

$$\nabla^j\gamma_{(1)ij} = \nabla_i\mathrm{Tr}\;\gamma_{(1)}, \tag{5.23}$$

$$\nabla^j\gamma_{(2)ij} = \frac{1}{2}\nabla_j[\gamma_{(1)}^2 - \frac{1}{2}\gamma_{(1)}\mathrm{Tr}\,\gamma_{(1)} - \frac{1}{4}\gamma_{(0)}(\mathrm{Tr}\,\gamma_{(1)}^2 - (\mathrm{Tr}\,\gamma_{(1)})^2)]_i^j. \tag{5.24}$$

Equation (5.23) can be integrated as

$$\gamma_{(1)} = t_{(1)} + \gamma_{(0)}\mathrm{Tr}\;\gamma_{(1)}, \tag{5.25}$$

where $t_{(1)ij}$ is an "integration constant" that satisfies $\nabla^i t_{(1)ij} = 0$. One can check that (5.24) is automatically satisfied when (5.20) and (5.22) are satisfied.

Forming the Einstein tensor, we obtain

$$R_{ij}[\gamma_{(0)}] - \frac{1}{2}\gamma_{(0)ij}R[\gamma_{(0)}] = \Lambda\gamma_{(0)ij} + T_{ij}, \tag{5.26}$$

where

$$T_{ij} = -\frac{2}{d-1}\Lambda\gamma_{(0)ij} - \gamma_{(2)ij} + \frac{1}{2}\gamma_{(1)ij}^2 - \frac{1}{4}\gamma_{(1)ij}\mathrm{Tr}\,\gamma_{(1)} - \frac{1}{8}\gamma_{(0)ij}[\mathrm{Tr}\,\gamma_{(1)}^2 - (\mathrm{Tr}\,\gamma_{(1)})^2]. \tag{5.27}$$

Equation (5.24) implies that T_{ij} is covariantly conserved. In addition, equation (5.21) determines the trace of T_{ij},

$$\mathrm{Tr}\;T = -2\Lambda - \frac{(d-2)}{8}\left(\mathrm{Tr}\,t_{(1)}^2 - \frac{1}{d-1}(\mathrm{Tr}\,t_{(1)})^2\right) \tag{5.28}$$

Let us now consider a physical brane with stress tensor T_{ij}^{bdry} located at $r=0$. Then in addition to equations (5.16), (5.17), (5.18) we have the junction condition (5.5). For the metric (5.15) the second fundamental form is equal to $K_{ij} = \frac{1}{2}\partial_r \gamma_{ij}$. From the junction condition (5.5) we get using the equation (5.23)

$$t_{(1)ij} = 16\pi G_{d+1} T_{ij}^{\text{bdry}}. \tag{5.29}$$

The junction condition thus identifies the undetermined covariantly conserved tensor $t_{(1)}$ in (5.25) with the stress tensor of the brane. Notice that conservation of the boundary stress-energy tensor is a necessary condition for this identification.

To summarise, we have shown that Einstein's equations in the bulk plus the junction condition lead to Einstein's equations on the brane. The effective stress-energy tensor T_{ij} represents both the bulk space-time and the matter on the brane. Its trace is determined by the matter stress-energy tensor on the brane. This is similar to the case discussed in the previous section. There the effective stress-energy tensor was a sum of the stress-energy tensor of matter localised on the brane of the $\langle T_{ij} \rangle_{\text{CFT}}$. The latter was taken to represent the bulk space-time, and its trace was fixed to be the holographic conformal anomaly.

The results in this section agree with the results obtained in [100] for $d = 4$. To see this, let

$$\gamma_{(2)ij} = -E_{ij} + \frac{1}{4}\gamma_{(1)ij}^2 - \frac{2}{d(d-1)}\Lambda \gamma_{(0)ij}, \tag{5.30}$$

and also let the boundary stress-energy tensor be equal to $T_{ij}^{\text{bdry}} = -\lambda \gamma_{ij}^{(0)} + \tau_{ij}$, where λ is the tension and τ_{ij} the matter energy momentum tensor on the brane. Equation (5.30) defines the tensor E_{ij}. A short calculation shows that it agrees with the tensor $E_{\mu\nu}$ of [100]. In particular, E_{ij} is traceless and its divergence is equal to $\nabla^j E_{ij} = K^{jk}(\nabla_i K_{jk} - \nabla_j K_{ik})$. This agrees with formula (22) of [100]. One can also verify agreement with (17)-(20) of [100].

Note that the above considerations are quite general and valid for any value of the bulk cosmological constant. Note also that when the brane matter consists of only a brane cosmological constant the brane geometry has a constant Ricci scalar.

5.4 Asymptotically flat case

In this section we perform an asymptotic analysis of Einstein's equations with zero cosmological constant similar to the one that has been done for asymptotically AdS spaces in [44, 70].

We work in Gaussian normal co-ordinates. The metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}(x,r)\,\mathrm{d}x^i \mathrm{d}x^j. \tag{5.31}$$

Einstein's equations in this co-ordinate system are given in equations (5.16), (5.17) and (5.18). We look for an asymptotic solution near infinity. Assuming that the leading part of γ near infinity is non-degenerate we find that it scales like r^2 (to prove this use (5.16)). Restricting ourselves to this case, we look for solutions of the form

$$\gamma(x,r) = r^2(g_{(0)} + g_{(2)}\frac{1}{r} + g_{(4)}\frac{1}{r^2} + \ldots)\ \ . \tag{5.32}$$

In other words, the bulk metric asymptotes to a cone with $g_{(0)}$ the metric on the base. In general, one can include logarithmic terms in (5.32). Such more general asymptotic solutions have been studied in [18, 17][4]. We restrict ourselves to (5.32).

We solve Einstein's equations

$$R_{\mu\nu} = 0, \tag{5.33}$$

perturbatively in $1/r$. The leading order equations imply [52, 103] that $g_{(0)}$ should satisfy

$$R_{(0)ij} + (d-1)g_{(0)ij} = 0 \quad . \tag{5.34}$$

This means that the space at infinity is described by an Einstein metric of constant positive scalar curvature. In particular, for Euclidean signature the standard metric on the unit sphere S^{d-1} satisfies this equation. Then the leading part of the bulk metric (5.31), (5.32) is just Euclidean R^d space. In the Lorentzian signature equation (5.34) is solved by the de Sitter space. Thus, already at leading order, we find an important difference between the cases of asymptotically flat space-time and of asymptotically AdS space-times. Whereas in the latter case one could choose the boundary metric arbitrarily, in the former case the boundary metric has to satisfy (5.34).

To next order we find

$$\nabla^j g_{(2)ij} = \nabla_i \mathrm{Tr}\, g_{(2)}, \tag{5.35}$$

$$d g_{(2)} + 2\mathrm{Ric}_{(2)} - g_{(0)} \mathrm{Tr}\, g_{(2)} = 0, \tag{5.36}$$

where

$$\mathrm{Ric}[\gamma] = \mathrm{Ric}_{(0)} + \frac{1}{r}\,\mathrm{Ric}_{(2)} + \frac{1}{r^2}\,\mathrm{Ric}_{(4)} \cdots \tag{5.37}$$

and

$$R_{(2)ij} = -\frac{1}{2}[\nabla_i\nabla_j \mathrm{Tr}\, g_{(2)} - \nabla^2 g_{(2)ij} + 2(d-1)g_{(2)ij} + 2R_{(0)ikjl}g^{kl}_{(2)}], \tag{5.38}$$

where indices raised and lowered by $g_{(0)}$. In deriving this equation, (5.34) and (5.35) were used. Then equation (5.36) becomes

$$\nabla_i\nabla_j \mathrm{Tr}\, g_{(2)} - \nabla^2 g_{(2)ij} + (d-2)g_{(2)ij} + g_{(0)ij}\mathrm{Tr}\, g_{(2)} + 2R_{(0)ikjl}\, g^{kl}_{(2)} = 0. \tag{5.39}$$

Notice that this equation leaves undetermined the trace of $g_{(2)}$. Let us define

$$t_{ij} = g_{(2)ij} - g_{(0)ij}\mathrm{Tr}\, g_{(2)}. \tag{5.40}$$

It follows from the (5.35) that $\nabla^i t_{ij} = 0$.

[4]In [18, 17] the authors look for solutions whose metric coefficients near infinity is given by an expansion in negative powers of the radial co-ordinate. Co-ordinate transformations allow one to put the metric in the form $\mathrm{d}s^2 = N^2\mathrm{d}r^2 + \gamma_{ij}(x,r)\mathrm{d}x^i\mathrm{d}x^j$, with $N = 1+\sigma(x)/r$ and $\gamma(x,r)$ as in (5.32). By a further logarithmic transformation one can reach Gaussian normal co-ordinates but at the expense of introducing logarithmic terms in $\gamma(x,r)$. Our results for $d=3$ agree with the results of [18, 17] for $\sigma(x) = 0$. We thank Kirill Krasnov for bringing these papers to our attention.

To the next order we find the equations

$$g_{(4)} = -\frac{1}{2}\mathrm{Ric}_{(4)} + \frac{1}{2}g_{(0)}\mathrm{Tr}\, g_{(4)} - \frac{1}{4}g_{(0)}\mathrm{Tr}\, g_{(2)}^2 + \frac{1}{4}g_{(2)}^2 + \frac{1}{8}g_{(2)}\mathrm{Tr}\, g_{(2)}, \quad (5.41)$$

$$\mathrm{Tr}\, g_{(4)} = \frac{1}{4}\mathrm{Tr}\, g_{(2)}^2, \quad (5.42)$$

$$\nabla^j g_{(4)ij} = \frac{1}{2}\nabla_j[g_{(2)}^2 - \frac{1}{2}g_{(2)}\mathrm{Tr}\, g_{(2)} - \frac{1}{4}g_{(0)}(\mathrm{Tr}\, g_{(2)}^2 - (\mathrm{Tr}\, g_{(2)})^2)]_i^j, \quad (5.43)$$

and

$$\begin{aligned} R_{(4)ij} = \; & \frac{1}{2}[-\frac{1}{4}\nabla_i\nabla_j \mathrm{Tr}\, g_{(2)}^2 - \nabla^k\nabla_i g_{(4)jk} - \nabla^k\nabla_j g_{(4)ik} + \nabla^2 g_{(4)ij} \\ & + g_{(2)}^{kl}[\nabla_l\nabla_i g_{(2)jk} + \nabla_l\nabla_j g_{(2)ik} - \nabla_l\nabla_k g_{(2)ij}] \\ & + \frac{1}{2}\nabla^k \mathrm{Tr}\, g_{(2)}(\nabla_i g_{(2)jk} + \nabla_j g_{(2)ik} - \nabla_k g_{(2)ij}) \\ & + \frac{1}{2}\nabla_i g_{(2)kl}\nabla_j g_{(2)}^{kl} + \nabla_k g_{(2)il}\nabla^l {g_{(2)j}}^k - \nabla_k g_{(2)il}\nabla^k {g_{(2)j}}^l]. \end{aligned} \quad (5.44)$$

It may seem that by taking the trace of (5.41) and using (5.42) and (5.41) one obtains a new equation for $g_{(2)}$. However, it turns out that the resulting equation is automatically satisfied. The same is true when taking the trace of (5.36) and using (5.38).

The equations we obtained look similar to the equations one gets in the case of asymptotically AdS space-times. There are important differences, however. In the case of asymptotically AdS space-times the equations were algebraic, and they could be solved up to order ρ^d. The coefficient $g_{(d)}$ was undetermined except for its trace and divergence. In the case at hand the equations for the coefficients are differential, and it is the trace of $g_{(2)}$ which is undetermined.

Let us comment on the logarithmic terms that can be included in our ansatz (5.32) and which were considered in [18, 17] for the case $d = 3$. We start with the metric of [18],

$$\mathrm{d}s^2 = N^2\mathrm{d}\rho^2 + \rho^2 f_{ij}(\rho, x)\,\mathrm{d}\phi^i\mathrm{d}\phi^j, \quad (5.45)$$

where N and h are given by

$$\begin{aligned} N &= 1 + \frac{\sigma(x)}{r}, \\ f_{ij} &= f_{(0)ij} + \frac{1}{r}\, f_{(2)ij} + \ldots \end{aligned} \quad (5.46)$$

The following co-ordinate transformation:

$$\begin{aligned} \rho &= r - \sigma(x)\log r + \frac{1}{r}\,\sigma^2(x) + \ldots \\ \phi^i &= x^i - \frac{1 + \log r}{r}\,\nabla^i\sigma + \ldots \end{aligned} \quad (5.47)$$

brings the metric to the Gaussian normal form:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 g_{ij}(r, x)\mathrm{d}x^i\mathrm{d}x^j. \quad (5.48)$$

However, g_{ij} now has an expansion that includes logarithmic terms:

$$g(r,x) = g_{(0)} + \frac{1}{r}\,g_{(2)} + \frac{1}{r}\,\log r\,h_{(2)} + \frac{1}{r}g_{(4)} + \dots \tag{5.49}$$

The first few coefficients $g_{(n)}$ and $h_{(2)}$ are related to the ones in (5.46) in the following way:

$$\begin{aligned} g_{(0)}(x) &= f_{(0)}(x) \\ g_{(2)}(x) &= f_{(2)}(x) - 2\nabla_i\nabla_j\sigma \\ h_{(2)}(x) &= -2(\sigma f_{(0)} + \nabla_i\nabla_j\sigma). \end{aligned} \tag{5.50}$$

Repeating the analysis above one finds that $h_{(2)}$ satisfies the same equations (5.35)-(5.36) as $g_{(2)}$ above, but for $d=3$ its trace is zero:

$$\mathrm{Tr}\,h_{(2)} = 0. \tag{5.51}$$

Filling in the expression for $h_{(2)}$ from (5.50), one finds that for $d=3$ (5.39) reduces to:

$$(\Box + 3)\sigma = 0 \tag{5.52}$$

which agrees with the result in [18]. Notice that for $d=3$ the equations for $g_{(2)}$ remain unchanged. A further co-ordinate transformation

$$\begin{aligned} \rho &= \bar\rho\,(1+\sigma) + \dots \\ \phi^i &= \bar\phi^i + \frac{1}{\rho}\,\nabla^i\sigma + \cdots \end{aligned} \tag{5.53}$$

maps our $g_{(2)}$ into the tensor k of [18], $g_{(2)} = k = f_{(1)} + 2\sigma f_{(0)}$. In the following we continue our analysis for general d and $\sigma = 0$.

Let us place the brane at a fixed large radius $r = r_0 \gg 1$. Then expanding the Einstein tensor for the induced metric γ_{ij} we find that

$$R_{ij}[\gamma] - \frac{1}{2}\,\gamma_{ij}R[\gamma] = (d-2)\left(\frac{d-1}{2r_0^2}\,\gamma_{ij} + \frac{1}{2r_0}\,t_{ij}\right) + \mathcal{O}(1/r_0^2)\ \ , \tag{5.54}$$

where t_{ij} is given in (5.40). On the other hand we have

$$K_{ij} - \gamma_{ij}K = -\frac{d-1}{r_0}\gamma_{ij} - \frac{1}{2}t_{ij} + \mathcal{O}(1/r_0^2)\ \ . \tag{5.55}$$

Notice that this is the Brown-York stress-energy tensor [25]. Thus, up to the leading divergence in $r_0 \to \infty$, t_{ij} is equal to the Brown-York stress-energy tensor. This divergence can again be cancelled by adding covariant counter-terms, along the lines of the previous chapter (see also [35]). The junction condition on the brane gives a relation between t_{ij} and the stress tensor $T^{\rm bdry}_{ij}$ of matter fields on the brane. Plugging back to (5.54) we find

$$R_{ij}[\gamma] - \frac{1}{2}\gamma_{ij}R[\gamma] = -\frac{(d-2)(d-1)}{2r_0^2}\,\gamma_{ij} - \frac{(d-2)8\pi G_{d+1}}{r_0}\,T^{\rm bdry}_{ij} + \mathcal{O}(1/r_0^2), \tag{5.56}$$

i.e. we get Einstein's equations with negative cosmological constant $\Lambda = -\frac{(d-1)(d-2)}{2r_0^2}$ and Newton's constant $G_d = \frac{(d-2)G_{d+1}}{r_0}$. The position of the brane becomes the AdS radius of gravity on the brane. Notice also that the formula for G_d is the same with formula (5.10) with l replaced by r_0.

5.5 Conformally flat metrics

In the case of asymptotically AdS spaces it was found in [102] that, imposing the vanishing of the bulk Weyl tensor, Einstein's equations could be integrated, and the perturbative expansion ended at order ρ^2 (see (4.42)). In this case, the (conformally flat) boundary condition on the metric was enough to obtain the exact solution as we found in the previous chapter. This implied that the boundary stress-energy tensor was completely determined by the background. These results can be extended for arbitrary value of the cosmological constant. Let us write the Weyl tensor in the following way:

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} - P_{\mu\nu}G_{\alpha\beta} - P_{\alpha\beta}G_{\mu\nu} + P_{\mu\beta}G_{\alpha\nu} + P_{\alpha\nu}G_{\mu\beta} \tag{5.57}$$

where

$$P_{\mu\nu} = \frac{1}{d-1}(R_{\mu\nu} - \frac{1}{2d}\, R\, G_{\mu\nu}). \tag{5.58}$$

Using Einstein's equations for generic cosmological constant,

$$R_{\mu\nu} = -\frac{2\Lambda}{d-1}\, G_{\mu\nu}, \tag{5.59}$$

one easily sees that the vanishing of the Weyl tensor implies

$$R_{\mu\nu\alpha\beta} = -\frac{2\Lambda}{d(d-1)}(G_{\mu\alpha}G_{\nu\beta} - G_{\mu\beta}G_{\nu\alpha}). \tag{5.60}$$

Hence the bulk space is a maximally symmetric space which is locally dS, AdS or flat space. The cosmological constant is $\Lambda = \frac{d(d-1)}{2l^2}\,\varepsilon$, where $\varepsilon = \pm 1$ or 0 for dS/AdS space or AF space, respectively.

The Gaussian normal co-ordinate system (5.15) is the most convenient one for the AF case. In these co-ordinates, (5.60) gives, in components:

$$\begin{aligned}
\gamma'' - \frac{1}{2}\gamma'\gamma^{-1}\gamma' &= -\frac{2\varepsilon}{l^2}\,\gamma \\
\nabla_i \gamma'_{jk} &= \nabla_k \gamma'_{ij} \\
R_{ikjl}[\gamma] &= -\frac{1}{4}(\gamma'_{ij}\gamma'_{kl} - \gamma'_{il}\gamma'_{jk}) - \frac{\varepsilon}{l^2}(\gamma_{ij}\gamma_{kl} - \gamma_{il}\gamma_{jk}).
\end{aligned} \tag{5.61}$$

Differentiating the first equation one gets, for $\varepsilon = 0$,

$$\gamma''' = 0, \tag{5.62}$$

and so the expansion stops at order r^2. In fact, one finds that the induced metric has the same following form as the boundary metric in the case of AdS [102]:

$$\gamma = (\gamma_{(0)} + \frac{r}{2}\,\gamma_{(1)})\gamma_{(0)}^{-1}(\gamma_{(0)} + \frac{r}{2}\,\gamma_{(1)}). \tag{5.63}$$

It has only two undetermined coefficients: $\gamma_{(0)}$ and $\gamma_{(1)}$. Unlike the AdS case, where the boundary metric $g_{(0)}$ gives a conformally flat bulk solution if and only if it is conformally flat, in this case there are no restrictions on $\gamma_{(0)}$. On the other hand, we do have constraints on $\gamma_{(1)}$. It satisfies the following two equations:

$$\begin{aligned} \nabla_i \gamma_{(1)jk} &= \nabla_k \gamma_{(1)ij} \\ R_{ikjl}[\gamma_{(0)}] &= -\frac{1}{4}\,(\gamma_{(1)ij}\gamma_{(1)kl} - \gamma_{(1)il}\gamma_{(1)jk}). \end{aligned} \tag{5.64}$$

Taking the trace of these equations gives back the constraints found in section 5.3 for $\Lambda = 0$. The conditions (5.64), however, are stronger. One also finds an expression for the effective stress-energy tensor purely in terms of $t_{(1)}$:

$$T_{ij} = \frac{1}{4}[t_{(1)}^2 - \frac{1}{(d-1)}\,t_{(1)}\,\mathrm{Tr}\,t_{(1)} - \frac{1}{2}\gamma_{(0)}\,\mathrm{Tr}\,t_{(1)}^2 + \frac{1}{2(d-1)}\,\gamma_{(0)}\,(\mathrm{Tr}\,t_{(1)})^2]. \tag{5.65}$$

In the brane-world scenario, this also provides a direct relation between T_{ij} and T_{ij}^{bdry}.

One can also perform the analysis in the co-ordinate system (5.32), which is more convenient to analyse the equations at infinity in the asymptotically flat case. We find the following solution:

$$g = (g_{(0)} + \frac{1}{2r}\,g_{(2)})g_{(0)}^{-1}(g_{(0)} + \frac{1}{2r}\,g_{(2)}), \tag{5.66}$$

so once again the vanishing of the Weyl tensor is enough to solve the equations. We also get additional constraints on both coefficients $g_{(0)}$ and $g_{(2)}$. The equation for $g_{(0)}$ tells us that the boundary has vanishing Weyl tensor:

$$R_{ikjl}[g_{(0)}] = -g_{(0)ij}g_{(0)kl} + g_{(0)il}g_{(0)jk}, \tag{5.67}$$

and so the space is asymptotically de Sitter [103]. We also get an additional differential equation which puts further constraints on $g_{(2)}$. Notice that both equations give back the equations in section 5.4 when contracting two indices, but again the requirements that we find here are stronger.

The co-ordinate systems (5.32) and (5.15) describe the same space and so the metric must be the same. A complicated co-ordinate transformation may however be required to go from one system to the other.

One can check that the equations can be integrated in the other cases as well. The solutions, however, become more involved in the Gaussian co-ordinate system and it is better to change the radial co-ordinate. It would be interesting to have a more detailed analysis that includes also different brane embeddings and dS space.

Appendix A

Shock-Wave Geometries

A.1 More on AdS shock-wave solutions

In this appendix we give some details of the geodesics and stress-energy tensor of massless particles in AdS, discussed in chapters 2-3 and their properties.

We write AdS space in the co-ordinate system (2.12), $y^\mu = (u, v, y^i)$, i running from 1 to $d-2$. The metric reads:

$$\mathrm{d}s^2 = \frac{4}{\Omega^2}\,\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu, \tag{A.1}$$

where the conformal factor is given by $\Omega = 1 - y^2/\ell^2$.

It is well-known that the null geodesics of two conformally related space-times are the same, up to a reparametrisation of the geodesic length. Therefore, null trajectories in the above co-ordinates will take the same form as those in Minkowski space. It is nevertheless convenient for the computation of the stress-energy tensor to see explicitly how the affine parameter changes.

The geodesic equation and the mass-shell condition give:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\eta_{\mu\nu}\dot{z}^\nu}{\Omega^2}\right) &= 2\frac{\eta_{\mu\nu}z^\nu \mathcal{L}}{\ell^2\,\Omega} \\ \mathcal{L} &= \frac{1}{\Omega^2}\,\eta_{\mu\nu}\dot{z}^\mu\dot{z}^\nu = 0. \end{aligned} \tag{A.2}$$

$\mathcal{L}$ is the Lagrange density, defined by the second of (A.2), and λ the affine parameter along the geodesic. These equations integrate to

$$\eta_{\mu\nu}\,\dot{z}^\nu = v_\mu\,\Omega^2. \tag{A.3}$$

v_μ is a constant, lightlike vector satisfying $\eta^{\mu\nu}v_\mu v_\nu = 0$ to be determined by the boundary conditions. This equation also relates the affine parameter in AdS to the affine parameter in Minkowski space.

The stress-energy tensor (2.5) now equals:

$$T_{\mu\nu} = -p\,\Omega^d\,v_\mu v_\nu \int \mathrm{d}s\,\delta^{(d)}(y - z(s)), \tag{A.4}$$

and choosing co-ordinates where momentum is purely in the v-direction, this reduces to:

$$T_{uu} \quad = \quad -p\,\Omega^d\,\delta(u-u_0)\,\delta(\rho-\rho_0), \tag{A.5}$$

where $\rho = \sum_{i=1}^{d-2} y_i^2$. Notice that in order for the metric (3.56) to be a solution of Einstein's equations with this stress-energy tensor, we need the initial condition $u_0 = 0$. It is also convenient to take $\rho_0 = 0$. Thus we get the stress-energy tensor used in chapter 3,

$$T_{uu} = -p\,\delta(u)\delta(\rho), \tag{A.6}$$

which gives rise to the delta-function in (3.57). This form for the stress-energy tensor agrees with the one computed in chapter 4, equation (4.98), in Poincare co-ordinates, and for the case $g_{(0)ij} = \eta_{ij}$. One can check this by performing the following co-ordinate transformation from y^μ to Poincare co-ordinates $x^\mu = (r, t, x^i)$:

$$\begin{aligned} u &= \frac{t^2 - r^2 - \vec{x}^2}{r+t} \\ v &= \frac{\ell^2}{r+t} \\ y^i &= \frac{\ell x^i}{r+t} \\ \Omega &= \frac{2r}{r+t} \\ r &= \frac{1}{2v}\,(\ell^2 - uv - \rho^2) \\ t &= \frac{1}{2v}\,(\ell^2 + uv + \rho^2) \\ x^i &= \frac{\ell}{v}\,y^i. \end{aligned} \tag{A.7}$$

With (A.6) at hand, one can compute the back-reaction on the AdS metric, obtaining the solution found by Horowitz and Itzhaki with the shift functions as given in (3.60). The next step is then to compute the geodesics of a test particle in the back-reaction corrected metric. The computation goes along the same lines as the one above. We do not give the details here since it is a straightforward exercise, but give only the results. We concentrate on trajectories whose initial velocities are perpendicular to the velocity of the shockwave, that is, the geodesics with $v = y^i = 0$ before the collision. This gives a head-on collision.

It turns out that the geodesic equations can again be exactly integrated, and the effect is the same as in Minkowski space: there is a shift in the v co-ordinate and a deflection in the x^i-plane which nevertheless is negligible in the eikonal approximation where the impact parameter is much larger than the Planck length. In this approximation, the shift is given by

$$\delta v = -8\pi G_{\rm N}\,p_u\,F_0\,\theta(u), \tag{A.8}$$

where F_0 is the shift function before the collision, $F_0 = F(u=0)$.

Of course the same results can be found from geodesics in Minkowski space by noting that massless geodesics are invariant under conformal transformations of the metric.

It is interesting to note that, when one considers only one particle, there is no self-interaction, and therefore the present solution to the Einstein-matter system with the given boundary conditions is exact. However, when considering two particles this is no longer true, and one has to restrict oneself to consider a "soft" test particle in the background of a "hard" particle.

A.2 The induced two-dimensional Ricci tensor

In this Appendix we outline the proof that Einstein's equations with a massless source reduce to the conditions (2.9)-(2.10). We also compute the curvature of the transverse part of the metric, equation (2.32). This computation follows [40], and for more details we refer to that paper.

The ansatz in [40] for the metric is the following:

$$\mathrm{d}\hat{s}^2 = 2A(\hat{u},\hat{v})\,\mathrm{d}\hat{v}(\mathrm{d}\hat{u} - \delta(v)\mathrm{d}\hat{v}) + g(\hat{u},\hat{v})\,h_{ij}(\hat{x}^i)\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j. \tag{A.9}$$

We also have the unperturbed metric

$$\mathrm{d}s^2 = 2A(u,v)\,\mathrm{d}u\mathrm{d}v + g(u,v)\,h_{ij}(x^i)\mathrm{d}x^i\mathrm{d}x^j, \tag{A.10}$$

which will be assumed to solve Einstein's equations. (A.9) is related to (A.10) by a shift *and* a co-ordinate transformation:

$$\begin{aligned} \hat{u} &= u + \theta f \\ \hat{v} &= v \\ \hat{x}^i &= x^i. \end{aligned} \tag{A.11}$$

The metric (A.9) should be a solution of Einstein's equations with a massless source:

$$\begin{aligned} R_{\mu\nu}[\hat{G}] &= R_{\mu\nu}[G] + \delta R_{\mu\nu}[G] = -8\pi G_{\mathrm{N}}\hat{T}_{\mu\nu} \\ R_{\mu\nu}[G] &= 0 \\ T^{\hat{u}\hat{u}} &= 4p\,\delta(\hat{v})\,\delta(\tilde{x}), \end{aligned} \tag{A.12}$$

so our massless particle travels along the null geodesic $\hat{v} = 0$, $\hat{x}^i = 0$.

Let us first work out the vacuum piece of Einstein's equations, $R_{\mu\nu}[G] = 0$. We use the formula

$$R^{\mu}{}_{i\mu j}[G] = \frac{1}{\sqrt{-G}}\,\partial_\mu\left(\sqrt{-G}\,\Gamma^{\mu}_{ij}\right) - \partial_i\partial_j\left(\log\sqrt{-G}\right) - \Gamma^{\mu}_{\nu i}\Gamma^{\nu}_{j\mu}, \tag{A.13}$$

and we have:

$$\begin{aligned} \sqrt{-G} &= Ag\sqrt{h} \\ \Gamma^{\alpha}_{ij} &= -\frac{1}{2}\,g^{\alpha\beta}h_{ij}\partial_\beta g \end{aligned}$$

$$
\begin{aligned}
\Gamma^i_{j\alpha} &= \frac{1}{2g}\,\delta^i_j \partial_\alpha g \\
\Gamma^i_{\alpha j} &= \frac{1}{2g}\,\delta^i_j \partial_\alpha g \\
\Gamma^\alpha_{\beta i} &= \Gamma_{i\alpha\beta} = 0,
\end{aligned} \tag{A.14}
$$

where the indices μ, ν run from 1 to 4, α and β take the values 1, 2, and i, j take the values $3, 4$. Plugging this in equation (A.13), we get:

$$
\begin{aligned}
R_{ij}[G] &= R_{ij}[h] - \frac{1}{2Ag}\,h_{ij}\partial_\alpha \left(Agg^{\alpha\beta}\partial_\beta g\right) - \Gamma^\alpha_{ki}\Gamma^k_{j\alpha} - \Gamma^k_{\alpha i}\Gamma^\alpha_{jk} \\
&= R_{ij}[h] - \frac{1}{A}\,h_{ij}\partial_u\partial_v g = 0.
\end{aligned} \tag{A.15}
$$

Here $R_{ij}[h]$ is the two-dimensional Ricci tensor calculated in the metric h_{ij}. This gives

$$
R_{ij}[h] = \frac{1}{A}\,\partial_u\partial_v g\, h_{ij}, \tag{A.16}
$$

which gives (2.32).

After some algebra, and using the vacuum solutions, one finds that the remaining piece of the metric only contributes the $\hat{u}\hat{u}$-component of the Ricci tensor. Einstein's equations,

$$
\delta R_{\mu\nu}[G] = -8\pi G_{\rm N}\hat{T}_{\mu\nu}, \tag{A.17}
$$

are then satisfied provided $R_{\mu\nu}[G] = 0$ and (2.9)-(2.10) hold. More details can be found in the appendices of [40].

Appendix B

Scaling and Classical Solutions of the Einstein-Hilbert Action

B.1 Scaling of curvature

In this section we give the details of the final rescaled (up to lowest order in $h_{i\alpha}$, all orders in ϵ) Ricci tensor $R_{\mu\nu}$. From here one can simply check the expansion of the Einstein-Hilbert action[1]. Higher orders in $h_{i\alpha}$ (quadratic at $1/\epsilon^2$ and at ϵ^0) are not necessary as we are not going to consider the fluctuations of the metric.

Under the rescaling (3.3), the Christoffel symbols transform as:

$$\begin{aligned}
\Gamma^{\gamma}_{\alpha\beta}(G) &= \Gamma^{\gamma}_{\alpha\beta}(\hat{G}) + \frac{1-\epsilon}{2}\, g^{i\gamma}\partial_i g_{\alpha\beta} \\
\Gamma^{\alpha}_{ij}(G) &= \frac{1}{\epsilon}\,\Gamma^{\alpha}_{ij}(\hat{G}) + \frac{\epsilon-1}{2\epsilon^2}\, g^{\alpha\beta}\partial_\beta g_{ij} \\
\Gamma^{\alpha}_{\beta i}(G) &= \Gamma^{\alpha}_{\beta i}(\hat{G}) + \frac{1-\epsilon}{2\epsilon}\,(g^{\alpha\gamma}\partial_\beta g_{\gamma i} - g^{\alpha\gamma}\partial_\gamma g_{\beta i} + g^{\alpha j}\partial_\beta g_{ij}) \\
\Gamma^{k}_{ij}(G) &= \Gamma^{k}_{ij}(\hat{G}) + \frac{\epsilon-1}{2\epsilon}\, g^{k\alpha}\partial_\alpha g_{ij} \\
\Gamma^{i}_{\alpha\beta}(G) &= \epsilon\,\Gamma^{i}_{\alpha\beta}(\hat{G}) + \frac{\epsilon(1-\epsilon)}{2}\, g^{ij}\partial_j g_{\alpha\beta} \\
\Gamma^{i}_{\alpha j}(G) &= \Gamma^{i}_{\alpha j}(\hat{G}) + \frac{\epsilon-1}{2}\,(g^{i\beta}\partial_j g_{\alpha\beta} + g^{ik}\partial_j g_{\alpha k} - g^{ik}\partial_k g_{\alpha j})
\end{aligned} \tag{B.1}$$

where $G_{\mu\nu}$ is the ϵ-dependent metric, whereas $\hat{G}_{\mu\nu}$ is the rescaled metric, which is independent of ϵ.

Working out the curvature components, we get:

$$R_{\alpha\beta}[G] \;=\; \epsilon^0(R_{\alpha\beta}[\hat{G}] - \frac{1}{2}\nabla_\beta(g^{ik}\partial_\alpha g_{ik}) - \frac{1}{4}g^{ij}\partial_\alpha g_{kj}g^{km}\partial_\beta g_{im}) +$$

[1]For notational simplicity, we denote the transverse metric by g_{ij}. In chapter 3 it is denoted by h_{ij}.

$$\begin{aligned} &+ \quad \epsilon^2(-\frac{1}{2}\nabla_i(g^{ij}\partial_j g_{\alpha\beta}) - \frac{1}{4}g^{\gamma\rho}\partial_i g_{\gamma\rho}g^{ij}\partial_j g_{\alpha\beta} + \\ &+ \quad \frac{1}{4}g^{\gamma\rho}\partial_k g_{\beta\rho}g^{ki}\partial_i g_{\alpha\gamma} + \frac{1}{4}g^{\gamma\rho}\partial_k g_{\alpha\rho}g^{ki}\partial_i g_{\beta\gamma}) \end{aligned} \tag{B.2}$$

The leading term in $R_{i\alpha}$ is at zero order in $h_{i\alpha}$ which is already sufficient for our purposes as it is always multiplied by $h_{i\alpha}$ in the action and this term arises at order ϵ^0

$$\begin{aligned} R_{i\alpha} &= \epsilon^0(\frac{1}{2}\nabla_\beta(g^{\beta\rho}\partial_i g_{\alpha\rho}) - \frac{1}{2}\nabla_\alpha(g^{\beta\rho}\partial_i g_{\beta\rho}) + \frac{1}{2}\nabla_k(g^{kj}\partial_a g_{ij}) \\ &- \frac{1}{2}\nabla_i(g^{kj}\partial_\alpha g_{kj}) + \frac{1}{4}g^{\gamma\rho}\partial_i g_{\alpha\rho}g^{kj}\partial_\gamma g_{kj} + \frac{1}{4}g^{km}\partial_\alpha g_{im}g^{\gamma\beta}\partial_k g_{\gamma\beta} \\ &- \frac{1}{2}g^{\beta\rho}\partial_\rho g_{ik}g^{kj}\partial_j g_{\alpha\beta}) \end{aligned} \tag{B.3}$$

R_{ij} is identical to $R_{\alpha\beta}$ under the interchange of Greek and Roman indices and $\epsilon \to \epsilon^{-1}$.

$$\begin{aligned} R_{ij}[G] &= \epsilon^{-2}(-\frac{1}{2}\nabla_\alpha(g^{\alpha\beta}\partial_\beta g_{ij}) - \frac{1}{4}g^{km}\partial_\alpha g_{km}g^{\alpha\beta}\partial_\beta g_{ij} \\ &+ \frac{1}{4}g^{km}\partial_\gamma g_{jm}g^{\gamma\alpha}\partial_\alpha g_{ik} + \frac{1}{4}g^{km}\partial_\gamma g_{im}g^{\gamma\alpha}\partial_\alpha g_{jk}) \\ &+ \epsilon^0(R_{ij}[\hat{G}] - \frac{1}{2}\nabla_j(g^{\alpha\gamma}\partial_i g_{\alpha\gamma}) - \frac{1}{4}g^{\alpha\beta}\partial_\alpha g_{\gamma\beta}g^{\gamma\rho}\partial_j g_{\alpha\rho}) \end{aligned} \tag{B.4}$$

B.2 Scaling of the exterior curvature

The exterior curvature part of the Einstein – Hilbert action is

$$S = \frac{1}{\ell_{\rm Pl}^{d-2}} \int \sqrt{\gamma}\, \nabla_\mu n^\mu \tag{B.5}$$

γ is the boundary metric which under rescaling is multiplied by $\ell_\parallel^2 \ell_\perp^{2(d-2)}$. The normal n will have a non-zero component only in the direction perpendicular to the boundary, parallel to the longitudinal scattering plane. Thus as the longitudinal metric scales with $\ell_\parallel^2$ the normalisation condition for n implies that it will also scale with $\ell_\parallel$. Thus,

$$\nabla_\mu n^\mu = \frac{\nabla_\alpha n^\alpha}{\ell_\parallel} + \frac{\nabla_i n^i}{\ell_\perp}. \tag{B.6}$$

The exterior curvature term of the action becomes

$$\epsilon^{d-4} S_{\partial M} = \frac{1}{\epsilon^2} \int \sqrt{\gamma}\, \nabla_\alpha n^\alpha + \frac{1}{\epsilon} \int \sqrt{\gamma}\, \nabla_i n^i. \tag{B.7}$$

As claimed in the text there is no additional contribution to the boundary action coming from the exterior curvature.

B.3 Classical solutions

In this appendix we give some more details on how to solve the equations of motion for the background, coming from the $\frac{1}{\epsilon^2}$ part of the action (3.10).

We rewrite the action (3.12) in the following form (now concentrating on the two-dimensional covariant part),

$$S = -\frac{1}{2} \int \sqrt{-g} \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \Lambda \phi^2 - \frac{1}{2} \phi^2 R[g] \right). \tag{B.8}$$

This action belongs to the class of actions considered in [14], with Lagrangian of the form

$$L = \sqrt{-g} \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \lambda \phi^{2k} - Q \phi^2 R[g] \right), \tag{B.9}$$

with the obvious values $k = 1$, $\lambda = -\frac{(d-2)}{2(d-3)} \Lambda$, $Q = \frac{(d-2)}{4(d-3)}$.

As argued in the main text, we consider static metrics of the form (3.15). The lagrangian (with $k = 1$) then reduces to the particle Lagrangian

$$L = \frac{1}{g} \left(e \phi'^2 - 4 Q e' \phi \phi' \right) - \lambda g e \phi^2. \tag{B.10}$$

The prime denotes derivatives with respect to x. It is obvious that the field g does not contribute to the dynamics - the equation of motion for g is simply an expression of reparameterization invariance in the spatial co-ordinate. In fact, all the g-depenence disappears from the equations of motion if we define a new variable $r = \int_0^x \mathrm{d}x' \, g(x')$. We then get

$$\begin{aligned} -2Q \frac{\ddot{e}}{e} + \frac{\dot{\phi} \dot{e}}{\phi e} + \frac{\ddot{\phi}}{\phi} + \lambda &= 0 \\ \frac{\ddot{\phi}}{\phi} + \left(1 + \frac{1}{4Q} \right) \left(\frac{\dot{\phi}}{\phi} \right)^2 - \frac{\lambda}{4Q} &= 0 \\ \frac{\dot{\phi}}{\phi} \left(\frac{\dot{\phi}}{\phi} - 4Q \frac{\dot{e}}{e} \right) + \lambda &= 0, \end{aligned} \tag{B.11}$$

and the dots denote derivatives with respect to r.

Substituting this solution one has for the curvature

$$R[g] = -2a^2 \left[1 + \frac{3\gamma}{4Q} + \frac{\gamma(\gamma - 4Q)}{16Q^2} \left(\frac{A - Be^{-2ar}}{A + Be^{-2ar}} \right)^2 \right] \tag{B.12}$$

where

$$a^2 = \frac{\lambda}{4Q\gamma}. \tag{B.13}$$

We see from (B.12) that among various solutions we also have the case in which the curvature is constant if either $A = 0$ or $B = 0$.

Since both cases differ only by a co-ordinate transformation, we choose $B = 0$. The longitudinal metric $g_{\alpha\beta}$ is then

$$\mathrm{d}s^2 = -(aCA^q)^2 e^{2aqr}\mathrm{d}t^2 + \mathrm{d}r^2. \tag{B.14}$$

where

$$q = 1 + \frac{\gamma}{4Q} \tag{B.15}$$

This is indeed the AdS_2 metric with the proper warp factor growing linearly in the radial co-ordinate. Our co-ordinates, however, do not cover the whole of AdS. One finds global co-ordinates by defining

$$e^{aqr} = \cos\rho, \tag{B.16}$$

where $0 \leq \rho \leq \pi/2$.

The curvature $R[g] = -\frac{2\ddot{e}}{e}$ obviously simplifies and becomes

$$R[g] = -\frac{\lambda(4Q+\gamma)}{8Q^2\gamma} \tag{B.17}$$

Furthermore, those solutions with A and B non-zero will be analogous in structure to AdS_2/Schwarzschild geometries, though the metric will have a different functional form due to the presence of the non-trivial scalar field.

In $d = 3$ there are small modifications due to the appearence of several $(d-3)$ factors in the general solutions. We can easily proceed here as follows. The one-dimensional form of the action is:

$$L = \frac{e'\phi^{2\prime}}{g} - \Lambda e g \phi^2, \tag{B.18}$$

and so the equations of motion reduce to

$$\begin{aligned} \ddot{\phi}^2 + \Lambda\phi^2 &= 0 \\ \ddot{e} + \Lambda e &= 0 \\ \frac{\dot{\phi}^2\dot{e}}{\phi^2 e} + \Lambda &= 0, \end{aligned} \tag{B.19}$$

after reabsorbing the non-dynamical field g in the definition of the parameter r, as before.

Global structure of the solutions

The metric

$$\mathrm{d}s^2 = -e(r)^2\mathrm{d}t^2 + \mathrm{d}r^2 \tag{B.20}$$

has a horizon when $e(r) = 0$. There are two possible locations of this horizon, depending on the relative sign of the initial conditions A and B.

For $B/A > 0$, $e(r)$ has a simple zero. With the following rescalings of the co-ordinates,

$$\begin{aligned} r &= \sqrt{\frac{Q\gamma}{\lambda}}\log B/A + \eta \\ t &= \frac{4Q\gamma}{C\lambda}(4AB)^{-\gamma/8Q-1/2}\tau \end{aligned} \tag{B.21}$$

the metric near the horizon is simply the Rindler space metric,

$$\mathrm{d}s^2 = -\eta^2\mathrm{d}\tau^2 + \mathrm{d}\eta^2, \tag{B.22}$$

and so locally the space is flat.

For $B/A < 0$, we rescale the co-ordinates as follows:

$$\begin{aligned} r &= \sqrt{\frac{Q\gamma}{\lambda}}\log|B/A| + \eta \\ t &= (\frac{\lambda|AB|}{Q\gamma})^{-\gamma/8Q-1/2}\tau, \end{aligned} \tag{B.23}$$

and we find the metric

$$\mathrm{d}s^2 = -\eta^{\gamma/2Q}\mathrm{d}\tau^2 + \mathrm{d}\eta^2 \tag{B.24}$$

with curvature $R = -\frac{\gamma(\gamma-4Q)}{8Q^2\eta^2}$.

Appendix C

Einstein Spaces and the Holographic Stress-Tensor

C.1 Asymptotic solution of Einstein's equations

In this appendix we collect the results for the solution of the equations (4.11) up to the order we are interested in.

From the first equation in (4.11) one determines the coefficients $g_{(n)}$, $n \neq d$, in terms of $g_{(0)}$. For our purpose we only need $g_{(2)}$ and $g_{(4)}$. There are given by

$$\begin{aligned}
g_{(2)ij} &= \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R\, g_{(0)ij}\right), \\
g_{(4)ij} &= \frac{1}{d-4}\left(-\frac{1}{8(d-1)} D_i D_j R + \frac{1}{4(d-2)} D_k D^k R_{ij}\right. \\
&\quad - \frac{1}{8(d-1)(d-2)} D_k D^k R g_{(0)ij} - \frac{1}{2(d-2)} R^{kl} R_{ikjl} \\
&\quad + \frac{d-4}{2(d-2)^2} R_i{}^k R_{kj} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\
&\quad \left. + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij}\right). \qquad \text{(C.1)}
\end{aligned}$$

All curvature expressions and covariant derivatives here are evaluated in the metric $g_{(0)}$. Thus, the above coefficients $g_{(n)}$ are functions of $g_{(0)}$ through the Riemann tensor and its derivatives. The expressions for $g_{(n)}$ are singular when $n = d$. One can obtain the trace and the divergence of $g_{(n)}$ for any n from the last two equations in (4.11). Explicitly,

$$\begin{aligned}
&\mathrm{Tr}\ g_{(4)} = \frac{1}{4}\,\mathrm{Tr}\ g_{(2)}^2, \qquad \mathrm{Tr}\ g_{(6)} = \frac{2}{3}\,\mathrm{Tr}\ g_{(2)} g_{(4)} - \frac{1}{6}\,\mathrm{Tr}\ g_{(2)}^3, \\
&\mathrm{Tr}\ g_{(3)} = 0, \qquad \mathrm{Tr}\ g_{(5)} = 0, \qquad \text{(C.2)}
\end{aligned}$$

and

$$\nabla^i g_{(2)ij} = \nabla^i A_{(2)ij}, \qquad \nabla^i g_{(3)ij} = 0, \qquad \nabla^i g_{(4)ij} = \nabla^i A_{(4)ij}$$

$$\nabla^i g_{(5)ij} = 0, \qquad \nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6}\mathrm{Tr}\,(g_{(4)}\nabla_j g_{(2)})\,, \tag{C.3}$$

where

$$\begin{aligned}
A_{(2)ij} &= g_{(0)ij}\mathrm{Tr}\; g_{(2)}, \qquad\qquad (C.4)\\
A_{(4)ij} &= -\frac{1}{8}[\mathrm{Tr}\; g_{(2)}^2 - (\mathrm{Tr}\; g_{(2)})^2]\, g_{(0)ij} + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4}\, g_{(2)ij}\,\mathrm{Tr}\; g_{(2)},\\
A_{(6)ij} &= \frac{1}{3}\Bigg(2(g_{(2)}g_{(4)})_{ij} + (g_{(4)}g_{(2)})_{ij} - (g_{(2)}^3)_{ij} + \frac{1}{8}\,[\mathrm{Tr}\; g_{(2)}^2 - (\mathrm{Tr}\; g_{(2)})^2]\, g_{(2)ij}\\
&- \mathrm{Tr}\; g_{(2)}\,[g_{(4)ij} - \frac{1}{2}(g_{(2)}^2)_{ij}]\\
&- [\frac{1}{8}\mathrm{Tr}\; g_{(2)}^2\mathrm{Tr}\; g_{(2)} - \frac{1}{24}(\mathrm{Tr}\; g_{(2)})^3 - \frac{1}{6}\mathrm{Tr}\; g_{(2)}^3 + \frac{1}{2}\mathrm{Tr}\;(g_{(2)}g_{(4)})]\, g_{(0)ij}\Bigg)\,.
\end{aligned}$$

For even $n = d$ the first equation in (4.11) determines the coefficients $h_{(d)}$. They are given by

$$\begin{aligned}
h_{(2)ij} &= 0, \qquad\qquad (C.5)\\
h_{(4)ij} &= \frac{1}{2}g_{(2)ij}^2 - \frac{1}{8}g_{(0)ij}\mathrm{Tr}\; g_{(2)}^2 + \frac{1}{8}(\nabla^k\nabla_i g_{(2)jk} + \nabla^k\nabla_j g_{(2)ik} - \nabla^2 g_{(2)ij} - \nabla_i\nabla_j\mathrm{Tr}\; g_{(2)}) \qquad (C.6)\\
&= \frac{1}{8}R_{ikjl}R^{kl} + \frac{1}{48}\nabla_i\nabla_j R - \frac{1}{16}\nabla^2 R_{ij} - \frac{1}{24}RR_{ij} + (\frac{1}{96}\nabla^2 R + \frac{1}{96}R^2 - \frac{1}{32}R_{kl}R^{kl})g_{(0)ij},\\
h_{(6)ij} &= \frac{2}{3}(g_{(4)}g_{(2)} + g_{(2)}g_{(4)})_{ij} - \frac{1}{3}g_{(2)ij}^3 - \frac{1}{6}g_{(4)ij}\mathrm{Tr}\; g_{(2)}\\
&+ \frac{1}{6}g_{(0)ij}(3\mathrm{Tr}\, g_{(6)} - 3\mathrm{Tr}\, g_{(2)}g_{(4)} + \mathrm{Tr}\, g_{(2)}^3)\\
&-\frac{1}{12}[-\frac{1}{4}\nabla_i\nabla_j\mathrm{Tr}\, g_{(2)}^2 - \nabla^k\nabla_i g_{(4)jk} - \nabla^k\nabla_j g_{(4)ik} + \nabla^2 g_{(4)ij}\\
&+g_{(2)}^{kl}[\nabla_l\nabla_i g_{(2)jk} + \nabla_l\nabla_j g_{(2)ik} - \nabla_l\nabla_k g_{(2)ij}]\\
&+\frac{1}{2}\nabla^k\mathrm{Tr}\, g_{(2)}(\nabla_i g_{(2)jk} + \nabla_j g_{(2)ik} - \nabla_k g_{(2)ij})\\
&+\frac{1}{2}\nabla_i g_{(2)kl}\nabla_j g_{(2)}^{kl} + \nabla_k g_{(2)il}\nabla^l g_{(2)j}{}^k - \nabla_k g_{(2)il}\nabla^k g_{(2)j}{}^l]. \qquad (C.7)
\end{aligned}$$

C.2 Divergences in terms of the induced metric

In this appendix we rewrite the divergent terms of the regularised action in terms of the induced metric at $\rho = \epsilon$. This is needed in order to derive the contribution of the counter-terms to the stress-energy tensor.

The coefficients $a_{(n)}$ of the divergent terms in the regulated action (4.25) are given by

$$\begin{aligned}
&a_{(0)} = 2(1-d),\\
&a_{(2)} = b_{(2)}(d)\,\mathrm{Tr}\; g_{(2)},\\
&a_{(4)} = b_{(4)}(d)\,[(\mathrm{Tr}\; g_{(2)})^2 - \mathrm{Tr}\; g_{(2)}^2],
\end{aligned}$$

$$a_{(6)} = \left(\frac{1}{8}\,\mathrm{Tr}\ g_{(2)}^3 - \frac{3}{8}\,\mathrm{Tr}\ g_{(2)}\mathrm{Tr}\ g_{(2)}^2 + \frac{1}{2}\,\mathrm{Tr}\ g_{(2)}^3 - \mathrm{Tr}\ g_{(2)}g_{(4)}\right), \qquad \text{(C.8)}$$

where $a_{(6)}$ is only valid in six dimensions and the numerical coefficients in $a_{(2)}$ and $a_{(4)}$ are given by

$$\begin{aligned} b_{(2)}(d \neq 2) &= -\frac{(d-4)(d-1)}{d-2}, \\ b_{(2)}(d = 2) &= 1, \\ b_{(4)}(d \neq 4) &= \frac{-d^2+9d-16}{4(d-4)}, \\ b_{(4)}(d = 4) &= \frac{1}{2}. \end{aligned} \qquad \text{(C.9)}$$

Notice that the coefficients $a_{(n)}$ are proportional to the expression for the conformal anomaly (in terms of $g_{(n)}$) in dimension $d = n$ [70].

The counter-terms can be rewritten in terms of the induced metric by inverting the relation between γ and $g_{(0)}$ perturbatively in ϵ. One finds

$$\begin{aligned} \sqrt{g_{(0)}} &= \epsilon^{d/2}\left(1 - \frac{1}{2}\,\epsilon\,\mathrm{Tr}\ g_{(0)}^{-1}g_{(2)} + \frac{1}{8}\,\epsilon^2\left[(\mathrm{Tr}\ g_{(0)}^{-1}g_{(2)})^2 + \mathrm{Tr}\ (g_{(0)}^{-1}g_{(2)})^2\right] + \mathcal{O}(\epsilon^3)\right)\sqrt{\gamma}, \\ \mathrm{Tr}\ g_{(2)} &= \frac{1}{2(d-1)}\frac{1}{\epsilon}\left(R[\gamma] + \frac{1}{d-2}(R_{ij}[\gamma]R^{ij}[\gamma] - \frac{1}{2(d-1)}R^2[\gamma]) + \mathcal{O}(R[\gamma]^3)\right), \\ \mathrm{Tr}\ g_{(2)}^2 &= \frac{1}{\epsilon^2}\frac{1}{(d-2)^2}\left(R_{ij}[\gamma]R^{ij}[\gamma] + \frac{-3d+4}{4(d-1)^2}R^2[\gamma] + \mathcal{O}(R[\gamma]^3)\right). \end{aligned} \qquad \text{(C.10)}$$

The terms cubic in curvatures in (C.10) give vanishing contribution in (4.27) up to six dimensions.

Putting everything together we obtain that the counter-terms, rewritten in terms of the induced metric, are given by

$$\begin{aligned} S^{\mathrm{ct}} &= -\frac{1}{16\pi G_{\mathrm{N}}}\int_{\rho=\epsilon}\sqrt{\gamma}\left[2(1-d) + \frac{1}{d-2}R \right. \\ &\quad - \left.\frac{1}{(d-4)(d-2)^2}(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2) - \log\epsilon\, a_{(d)} + \ldots\right], \end{aligned} \qquad \text{(C.11)}$$

where all quantities are now in terms of the induced metric, including the one in the logarithmic divergence. These are exactly the counter-terms in [11, 43, 83] except that these authors did not include the logarithmic divergence. Equation (C.11) should be understood as containing only divergent counter-terms in each dimension. This means that in even dimension $d = 2k$ one should include only the first k counter-terms and the logarithmic one. In odd $d = 2k+1$, only the first $k+1$ counter-terms should be included. The logarithmic counter-terms appear only for d even. The counter-terms in (C.11) render the renormalised action finite up to $d = 6$. This covers all cases relevant for the AdS/CFT correspondence. It is straightforward but tedious to compute the necessary counter-terms for $d > 6$. From (C.11) one straightforwardly obtains (4.30).

C.3 Relation between $h_{(d)}$ and the conformal anomaly $a_{(d)}$

We show in this appendix that the tensor $h_{(d)}$ appearing in the expansion of the metric in (4.9) when d is even is a multiple of the stress tensor derived from the action $\int a_{(d)}$. ($a_{(d)}$ is, up to a constant, the holographic conformal anomaly).

This can be shown by deriving the stress-energy tensor of the regulated theory at $\rho = \epsilon$ in two ways and then comparing the results. In the first derivation one starts from (4.24) and obtains the regulated stress-energy tensor as in (4.29). Expanding $T^{\rm reg}_{ij}[\gamma]$ in ϵ (keeping $g_{(0)}$ fixed) we find that there is a logarithmic divergence,

$$T^{\rm reg}_{ij}[\gamma; \log] = \frac{1}{8\pi G_{\rm N}} \log\epsilon \; (\frac{3}{2}d - 1) h_{(d)ij}. \tag{C.12}$$

On the other hand, one can derive $T^{\rm reg}_{ij}[\gamma]$ starting from (4.25). One has to first rewrite the terms in (4.25) in terms of the induced metric. This is done in the previous appendix. Once $T^{\rm reg}_{ij}[\gamma]$ has been derived, we expand in ϵ. We find the following logarithmic divergence:

$$T^{\rm reg}_{ij}[\gamma; \log] = \frac{1}{8\pi G_{\rm N}} \log\epsilon \left((1-d) h_{(d)ij} - T^a_{ij},\right), \tag{C.13}$$

where T^a_{ij} is the stress-energy tensor of the action $\int {\rm d}^d x \sqrt{\det g_{(0)}}\, a_{(d)}$. If follows that

$$h_{(d)ij} = -\frac{2}{d} T^a_{ij}. \tag{C.14}$$

We have also explicitly verified this relation by brute-force computation in $d = 4$.

C.4 Asymptotic solution of the scalar field equation

We give here the first two orders of the solution of the equation (4.70)

$$\begin{aligned}
\phi_{(2)} &= \frac{1}{2(2\Delta - d - 2)} \left(\Box_0 \phi_{(0)} + (d-\Delta)\phi_{(0)} {\rm Tr}\; g_{(2)}\right), \\
\phi_{(4)} &= \frac{1}{4(2\Delta - d - 4)} \Bigg(\Box_0 \phi_{(2)} - 2\,{\rm Tr}\; g_{(2)}\phi_{(2)} - \frac{1}{2}(d-\Delta)\left[{\rm Tr}\; g^2_{(2)}\, \phi_{(0)} - 2{\rm Tr}\; g_{(2)}\, \phi_{(2)}\right] \\
&\qquad - \frac{1}{\sqrt{g_{(0)}}}\, \partial_\mu(\sqrt{g_{(0)}}\, g^{\mu\nu}_{(2)} \partial_\nu \phi_{(0)}) + \frac{1}{2}\partial^i {\rm Tr}\; g_{(2)} \partial_j \phi_{(0)}\Bigg),
\end{aligned} \tag{C.15}$$

where in $\Box_0$ the covariant derivatives are with respect to $g_{(0)}$.

If $2\Delta - d - 2k = 0$ one needs to introduce a logarithmic term in order for the equations to have a solution, as discussed in the main text. For instance, when $\Delta = \frac{1}{2}d + 1$, $\phi_{(2)}$ is undetermined, but instead one obtains for the coefficient of the logarithmic term,

$$\psi_{(2)} = -\frac{1}{4}\left(\Box_0 \phi_{(0)} + (\frac{d}{2} - 1)\, \phi_{(0)} {\rm Tr}\; g_{(2)}\right). \tag{C.16}$$

Bibliography

[1] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N Field Theories, String Theory and Gravity”, *Phys.Rept.* **323** (2000) 183-386, hep-th/9905111

[2] P.C. Aichelburg and R.U. Sexl, “On the Gravitational Field of a Massless Particle”, *Gen. Rel. Grav.* **2** (1971) 303-312

[3] D. Amati, M. Ciafaloni and G. Veneziano, “Superstring Collisions at Planckian Energies”, *Phys. Lett.* **B197** (1987) 81;
D. Amati, M. Ciafaloni and G. Veneziano, “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions”, *Int. J. Mod. Phys.* **A3** (1989) 1615-1661

[4] D. Amati and C. Klimcik, “Strings in a Shock Wave Background and Generation of Curved Geometry from Flat-Space String Theory”, *Phys. Lett.* **B** 210 (1988) 92-96

[5] D. Amati and C. Climcik, “Nonperturbative Computation of the Weyl Anomaly for a Class of Nontrivial Backgrounds”, *Phys. Lett.* **B** 210 (1988) 92-96

[6] L. Anchordoqui, C. Nunez and K. Olsen, “Quantum Cosmology and AdS/CFT”, *JHEP* **0010** (2000) 050, hep-th/0007064

[7] G. Arcioni, S. de Haro and M. O’Loughlin, “Boundary description of Planckian scattering in curved spacetimes”, hep-th/0104039, submitted to *JHEP*

[8] I.Ya. Aref’eva and I.V. Volovich, “On the Breaking of Conformal Symmetry in the AdS/CFT Correspondence”, *Phys.Lett.* **B433** (1998) 49-55, hep-th/9804182

[9] S.J. Avis, C.J. Isham and D. Storey, “Quantum field theory in anti-de Sitter spacetime”, *Phys. Rev.* **D18** (1978) 3565-3576

[10] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri, S.F. Ross, “Supersymmetric Conical Defects: Towards a string theoretic description of black hole formation”, hep-th/0011217

[11] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” *Commun. Math. Phys.* **208** (1999) 413-428, hep-th/9902121

[12] V. Balasubramanian, P. Kraus and A. Lawrence, “Bulk vs. Boundary Dynamics in Anti-de Sitter Spacetime”, *Phys. Rev.* **D59** (1999) 046003, hep-th/9805171

[13] V. Balasubramanian, P. Kraus, A. Lawrence and S. Trivedi, "Holographic Probes of Anti-de Sitter Spacetimes", *Phys. Rev.* **D59** (1999) 104021, `hep-th/9808017`

[14] T. Banks and M. O'Loughlin, "Two-dimensional quantum gravity in Minkowski Space", *Nucl. Phys.* **B362** (1991) 649

[15] K. Bautier, "Diffeomorphisms and Weyl transformations in AdS_3 gravity", `hep-th/9910134`

[16] K. Bautier, F. Englert, M. Rooman and Ph. Spindel, "The Fefferman-Graham Ambiguity and AdS Black Holes", *Phys. Lett.* **B479** (2000) 291-298, `hep-th/0002156`

[17] R. Beig, "Integration of Einstein's equations near spatial infinity", *Proc. R. Soc. Lond.* **A 391** (1984) 295-304

[18] R. Beig and B.G. Schmidt, "Einstein's equations near Spatial Infinity", *Commun. Math. Phys.* **87** (1982) 65-80

[19] J. Bekenstein, "Black Holes and Entropy", *Phys. Rev.* **D 7** (1973) 2333-2346

[20] N.D. Birrell and P.C.W. Davies, "Quantum fields in curved space", Cambridge University Press 1982

[21] S.K. Blau, E.I Guendelman and A.H. Guth, "Dynamics of False-vacuum bubbles", *Phys. Rev.* **D35** (1987) 1747

[22] R. Bousso, "Holography in general space-times", *JHEP 9906* (1999) 28, `hep-th/9902022`.

[23] R. Britto-Pacumio, A. Strominger and A. Volovich, "Holography for Coset Spaces", *JHEP* **9911** (1999) 013, `hep-th/9905211`

[24] J.D. Brown and M. Henneaux, "Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity", *Commun. Math. Phys.* **104** (1986) 207-226

[25] J.D. Brown and J.W. York, "Quasilocal energy and conserved charges derived from the gravitational action", *Phys. Rev.* **D47** (1993), 1407-1419

[26] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, "Strings in Background Fields", *Nucl. Phys.* **B262** (1985) 593

[27] A. Cappelli and A. Coste, "On the stress tensor of conformal field theories in higher dimensions", *Nucl. Phys.* **B314** (1989) 707-740

[28] G. Chalmers and K. Schalm, "Holographic Normal Ordering and Multi-particle States in the AdS/CFT Correspondence", *Phys. Rev.* **D61** (2000) 046001, `hep-th/9901144`

[29] C. Csaki, J. Erlich and T.J. Hollowood, "Quasilocalization of gravity by resonant modes", *Phys. Rev. Lett.* **84** (2000) 5932-5935, `hep-th/0002161`

[30] J. de Boer, E. Verlinde and H. Verlinde, "On the Holographic Renormalization Group", *JHEP* **0008** (2000) 003, hep-th/9912012

[31] N.S. Deger and A. Kaya, "AdS/CFT and Randall-Sundrum Model Without a Brane", hep-th/0010141

[32] S. de Haro, "Non-Commutative Black Hole Algebra and String Theory from Gravity", *Class. and Quant. Grav.* **15** (1998) 519-535 gr-qc/9707042

[33] S. de Haro, "Planckian Scattering and Black Holes", *JHEP* **9810** (1998) 023, gr-qc/9806028

[34] S. de Haro, A. Sinkovics and K. Skenderis, in preparation

[35] S. de Haro, K. Skenderis and S.N. Solodukhin, "Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence", *Commun. Math. Phys.* **217** (2001) 595-622, hep-th/0002230

[36] S. de Haro, K. Skenderis and S.N. Solodukhin, "Gravity in Warped Compactifications and the Holographic Stress Tensor", to appear in the proceedings of the Fradkin and the Gursey memorial conferences in *Class. and Quant. Grav.*, hep-th/0011230

[37] S. de Haro and J. van Dongen, "Vermoedens", *Ned. Tijdschr. v. Natuurk.* **64/11** (1998) 284-288

[38] E. D'Hoker, D.Z. Freedman and W. Skiba, "Field Theory Tests for Correlators in the AdS/CFT Correspondence", *Phys. Rev.* **D59** (1999) 045008, hep-th/9807098

[39] Discussions with H. Verlinde

[40] T. Dray and G. 't Hooft, "The Gravitational Shock Wave of a Massless Particle", *Nucl. Phys.* **B 253** (1985) 173-188

[41] M.J. Duff and J.T. Liu, "Complementarity of the Maldacena and Randall-Sundrum Pictures", *Phys. Rev. Lett.* **85** (2000) 2052-2055, hep-th/0003237

[42] G. Dvali, G. Gabadadze and M. Porrati, "Metastable gravitons and infinite volume extra dimensions", *Phys. Lett.* **B484** (2000), 112-118, hep-th/0002190

[43] R. Emparan, C.V. Johnson and R.C. Myers, "Surface Terms as Counter-terms in the AdS/CFT Correspondence", *Phys. Rev.* **D60** (1999) 104001, hep-th/9903238

[44] C. Fefferman and C. Robin Graham, "Conformal Invariants", in *Elie Cartan et les Mathématiques d'aujourd'hui* (Astérisque, 1985) 95

[45] W. Fischler, A. Kashani-Poor, R. McNees and S. Paban, "The Acceleration of the Universe, a Challenge for String Theory", hep-th/0104181

[46] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, "Correlation functions in the CFT(d)/AdS(d+1) correpondence", *Nucl. Phys.* **B546** (1999) 96-118, hep-th/9804058

[47] M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, "Higher Order Sigma Model Counter-terms and the Effective Action for Superstrings", *Phys. Lett.* B178 (1986) 199

[48] A. Galindo and P. Pascual, "Quantum Mechanics", Springer, Berlin 1990-1991

[49] J. Garriga and E. Verdaguer, "Scattering of Quantum Particles by Gravitational Plane Waves, *Phys. Rev.* **D43** (1991) 391-401

[50] G.W. Gibbons, "Quantized Fields Propagating in Plane-Wave Spacetimes", *Commun. Math. Phys.* **45** (1975) 191-202

[51] G.W. Gibbons and S.W. Hawking, "Action integrals and partition functions in quantum gravity", *Phys. Rev.* **D15** (1977) 2752

[52] G.W. Gibbons, D.N. Page and C.N. Pope, "Einstein metrics on S^3, R^3 and R^4 bundles", *Commun. Math. Phys.* **127** (1990) 529

[53] S.B. Giddings, "The boundary S-matrix and the AdS to CFT dictionary", *Phys. Rev. Lett.* **83** (1999) 2707-2710, hep-th/9903048.

[54] S.B. Giddings and E. Katz, "Effective theories and black hole production in warped compactifications", hep-th/0009176

[55] S. B. Giddings, E. Katz, and L. Randall, "Linearized Gravity in Brane Backgrounds," *JHEP* **0003** (2000) 023, hep-th/0002091

[56] S.B. Giddings, M. Lippert, "Precursors, black holes and a locality bound", hep-th/0103231.

[57] C.R. Graham, "Volume and Area Renormalisations for Conformally Compact Einstein Metrics", to appear in *Proc. of 19th Winter School in Geometry and Physics*, Srni, Czech Republic, Jan. 1999, math.DG/9909042

[58] C. R. Graham and J.M. Lee, "Einstein Metrics with Prescribed Conformal Infinity on the Ball", *Adv. Math.* **87** (1991) 186

[59] C. R. Graham and E. Witten, "Conformal Anomaly Of Submanifold Observables In AdS/CFT Correspondence", *Nucl. Phys.* **B546** (1999) 52-64, hep-th/9901021

[60] R. Gregory, V. A. Rubakov, and S. M. Sibiryakov, "Opening Up Extra Dimensions at Ultra-Large Scales," *Phys. Rev. Lett.* **84** (2000) 5928, hep-th/0002072

[61] M.T. Grisaru and D. Zanon, "Sigma Model Superstring Corrections to the Einstein-Hilbert Action", *Phys. Lett.* **B177** (1986) 347

[62] S.S. Gubser, "AdS/CFT and gravity", hep-th/9912001

[63] D. J. Gross and E. Witten, "Superstring Modifications of Einstein's Equations", *Nucl. Phys.* B277(1986) 1

[64] S. Gubser, I. Klebanov and A. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory", *Phys. Lett.* **B428** (1998) 105-114, hep-th/9802109

[65] S.W. Hawking, "Particle Creation by Black Holes", *Commun. Math. Phys.* **43** (1975) 199-220

[66] S.W. Hawking, "Breakdown of Predictability in Gravitational Collapse", *Phys. Rev.* **D14** (1976) 2460-2473

[67] S. Hellerman, N. Kaloper and L. Susskind, "String Theory and Quintessence", hep-th/0104180

[68] S.W. Hawking, T. Hertog and H.S. Reall, "Brane New World", *Phys. Rev.* **D62** (2000) 043501, hep-th/0003052

[69] S.W. Hawking and D. Page, "Thermodynamics of black holes in anti-de Sitter space", *Commun. Math. Phys.* **87** (1983) 577

[70] M. Henningson and K. Skenderis, "The holographic Weyl anomaly", *JHEP* **9807** (1998) 023, hep-th/9806087; "Holography and the Weyl Anomaly", hep-th/9812032

[71] G. Horowitz and A.R. Steif, "Spacetime Singularities in String Theory", *Phys. Rev. Lett.* **64** (1990) 260-263

[72] G.T. Horowitz and N. Itzhaki, "Black Holes, Shock Waves, and Causality in the AdS/CFT Correspondence", *JHEP* **9902** (1999) 010, hep-th/9901012

[73] D. Kabat and M. Ortiz, "Eikonal quantum gravity and planckian scattering", *Nucl. Phys.* **B388** (1992) 570-592

[74] D. Kabat, "Validity of the Eikonal Approximation", *Comments Nucl. Part. Phys.* **20** (1992) 325-336, hep-th/9204103

[75] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, "Diffeomorphisms and Holographic Anomalies", *Class. Quant. Grav.* **17** (1990) 1129-1138, hep-th/9910267

[76] R. Jackiw, D. Kabat and M. Ortiz, "Electromagnetic Fields of a Massless Particle and the Eikonal", *Phys. Lett.* **B277** (1992) 148-152 hep-th/9112020

[77] R. Kallosh and A. Rajaraman, "Vacua of M-Theory and String Theory", *Phys. Rev.* **D58** (1998) 125003, hep-th/9805041

[78] A. Kempf and G. Mangano, "Minimal Length Uncertainty Relation and Ultraviolet Regularisation", *Phys. Rev.* **D55** (1997) 7909-7920;
A. Kempf, "On the Structure of Space-Time at the Planck Scale", Presented at International School of Subnuclear Physics: 36th Course: From the Planck Length to the Hubble Radius, Erice, Italy, 29 Aug - 7 Sep 1998. In "Erice 1998, From the Planck length to the Hubble radius", 613-622, hep-th/9810215

[79] J. Khoury and Herman Verlinde, "On Open-Closed String Duality", hep-th/0001056

[80] Y. Kiem, H. Verlinde and E. Verlinde, "Black Hole Horizons and Complementarity", *Phys. Rev.* **D52** (1995) 7053-7065, hep-th/6502074

[81] I.R. Klebanov and E. Witten, "AdS/CFT Correspondence and Symmetry Breaking", *Nucl.Phys.* **B556** (1999) 89-114, hep-th/9905104

[82] K. Krasnov, "3D Gravity, Point Particles and Liouville Theory", hep-th/0008253

[83] P. Kraus, F. Larsen and R. Siebelink, "The Gravitational Action in Asymptotically AdS and Flat Spacetimes", *Nucl. Phys.* **B563** (1999) 259-278, hep-th/9906127

[84] M. Kunzinger and R. Steinbauer, "A note on the Penrose junction conditions", *Class. Quant. Grav.* **16** (1999) 1255-1264, gr-qc/9811007

[85] D.A. Lowe and L. Thorlacius, "AdS/CFT and the Information Paradox", *Phys. Rev.* **D60** (1999) 104012, hep-th/9903237

[86] M. Maggiore, *Phys. Lett.* **B 304** (1993) 65 hep-th/9301067

[87] J.M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", *Adv. Theor. Math. Phys.* **2** (1998) 231-252, hep-th/9802150.

[88] R. Mann, "Misner string entropy", *Phys. Rev.* **D61** (2000), 084013, hep-th/9904148

[89] H.-J. Matschull, "Black Hole Creation in (2+1)-Dimensions", *Class. Quant. Grav.* **16** (1999) 1069-1095, gr-qc/9809087

[90] W. Mück and K.S. Viswanathan, "Conformal Field Theory Correlators from Classical Scalar Field Theory on AdS_{d+1}", *Phys. Rev.* **D58** (1998) 041901, hep-th/9804035

[91] R.C. Myers, "Stress Tensors and Casimir Energies in the AdS/CFT Correspondence", *Phys. Rev.* **D60** (1999) 046002, hep-th/9903203

[92] M. Nishimura and Y. Tanii, "Super Weyl Anomalies in the AdS/CFT Correspondence", *Int. J. Mod. Phys.* **A14** (1999) 3731-3744, hep-th/9904010

[93] S. Nojiri and S.D. Odintsov, "Conformal Anomaly for Dilaton Coupled Theories from AdS/CFT Correspondence", *Phys. Lett.* **B444** (1998) 92-97, hep-th/9810008

[94] A. Peet and J. Polchinski, "UV/IR Relations in AdS Dynamics", *Phys. Rev.* **D59** (1999) 065011, hep-th/9809022

[95] R. Penrose, *The geometry of impulsive gravitational waves*, in *General Relativity: Papers in honour of J. L. Synge*, L. O'Raifeartaigh, Clarendon Press, Oxford 1972

[96] A. Petkou and K. Skenderis, "A non-renormalisation theorem for conformal anomalies", *Nucl. Phys.* **B561** (1999) 100-116, hep-th/9906030

[97] J. Polchinski, "S-matrices from AdS Space-Time", hep-th/9901076

[98] Polchinski, Susskind and Toumbas, "Negative Energy, Superluminosity and Holography", *Phys. Rev.* **D60** (1999) 084006, `hep-th/9903228`;
"Space-Time Non-Commutativity and Causality", *JHEP* **0006** (2000) 044, `hep-th/0005015`

[99] L. Randall and R. Sundrum, "A Large Mass Hierarchy from a Small Extra Dimension", *Phys. Rev. Lett.* **83** (1999) 3370-3373, `hep-ph/9905221`;
"An Alternative to Compactification", *Phys. Rev. Lett.* **83** (1999) 4690-4693, `hep-th/9906064`

[100] T. Shiromizu, K. Maeda and M. Sasaki, "The Einstein Equations on the 3-Brane World", *Phys. Rev.* **D62** (2000) 024012, `gr-qc/9910076`

[101] K. Skenderis, "Asymptotically Anti-de Sitter Spacetimes and their Stress Energy Tensor", to appear in the proceedings of Strings 2000, `hep-th/0010138`

[102] K. Skenderis and S.N. Solodukhin, "Quantum effective action from the AdS/CFT correspondence", *Phys. Lett.* **B432** (2000) 316-322, `hep-th/9910023`

[103] S.N. Solodukhin, "How to make the gravitational action on non-compact space finite", *Phys. Rev.* **D62** (2000), 044016, `hep-th/9909197`

[104] R. Steinbauer, *J. Math. Phys.* **39** (1998) 2201 `gr-qc/9710119`

[105] L. Susskind, "The World as a Hologram", *J. Math. Phys.* **36** (1995) 6377, `hep-th/9409089`

[106] L. Susskind, "Holography in the Flat Space Limit", `hep-th/9901079`

[107] L. Susskind, L. Thorlacius and J. Uglum, "The Stretched horizon and Black Hole Complementarity", *Phys. Rev.* **D48** (1993) 3743-3761, `hep-th/9306069`

[108] L. Susskind and N. Toumbas, "Wilson Loops as Precursors", *Phys. Rev.* **D61** (2000) 044001, `hep-th/9909013`

[109] L. Susskind and E. Witten, "The Holographic Bound in Anti-de Sitter Space", `hep-th/9805114`

[110] M. Taylor-Robinson, "Holography for degenerate boundaries", `hep-th/0001177`

[111] M. Taylor-Robinson, "More on counter-terms in the gravitational action and anomalies", `hep-th/0002125`

[112] G. 't Hooft, "Dimensional Reduction in Quantum Gravity", in *Salamfest*, Trieste, 1993, p. 284, `gr-qc/9310026`

[113] G. 't Hooft, "The Scattering Matrix Approach for the Quantum Black Hole — an Overview", *Int. J. Mod. Phys.* **A 11** 4623 `gr-qc/9607022`

[114] G. 't Hooft, "Graviton Dominance in Ultra-High-Energy Scattering", *Phys. Lett.* **B 198** (1987) 61-63

[115] G. 't Hooft, "Horizon Operator Approach to Black Hole Quantization", to appear in the *Proceedings of The Black Hole 25 Years After, Santiago, Chile, 17-21 January 1994*, gr-qc/9402037

[116] G. 't Hooft, "TransPlanckian Particles and the Quantization of Time", *Class. Quant. Grav.* 16 (1999) 395-405, gr-qc/9805079

[117] G. 't Hooft, "Quantum Gravity as a Dissipative Deterministic System", *Class. Quant. Grav.* **16** (1999) 3263-3279, gr-qc/9903084; "Determinism and Dissipation in Quantum Gravity", Erice lecture, hep-th/0003005

[118] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions", *Nucl. Phys.* **461** (1974) 179

[119] E. Verlinde, "On RG Flow and the Cosmological Constant", *Class. Quant. Grav.* **17** (2000) 1277-1285, Contributed to Strings 99, Potsdam, Germany, 19-25 Jul 1999, hep-th/9912058

[120] E. Verlinde and H. Verlinde, "RG Flow, Gravity and the Cosmological Constant", *JHEP* **0005** (2000) 034, hep-th/9912018

[121] E. Verlinde and H. Verlinde, "Scattering at Planckian Energies", *Nucl. Phys.* **B371** (1992) 246-268, hep-th/9110017

[122] E. Verlinde and H. Verlinde, "High-Energy Scattering in Quantum Gravity", *Class. Quant. Grav.* **10** (1993) S175-S184, also in Erice 1992, Proceedings, *String quantum gravity and physics at the Planck energy scale* 262-272

[123] H. Verlinde, "Holography and Compactification", *Nucl. Phys.* **B580** (2000) 264-274, hep-th/9906182

[124] M. Welling, "Two particle Quantummechanics in 2+1 Gravity using Non Commuting Co-ordinates", *Class. Quant. Grav.* **14** (1997) 3313 gr-qc/9703058

[125] E. Witten, "Anti De Sitter Space And Holography", *Adv. Theor. Math. Phys.* **2** (1998) 253-291, hep-th/9802150

[126] E. Witten, remarks at ITP Santa Barbara conference "New dimensions in field and string theory", http://www.itp.ucsb.edu/online/susy_c99/discussion/

Samenvatting

Deze samenvatting is voor een groot deel gebaseerd op [37].

De wens de theorieën van heelal en atoom bij elkaar te brengen geeft al enige tijd vorm aan een aanzienlijk deel van de moderne theoretische fysica. Het zijn twee theorieën die mathematisch geen verband met elkaar lijken te hebben – en vooralsnog elkaar misschien zelfs uitsluiten. Toch is men op zoek naar een eenduidige beschrijving van de natuur zoals zij zich zou gedragen bij energieën (orde 10^{22} MeV) waar het onderscheid tussen zwart gat en elementair deeltje verdwijnt. Zowel quantum- als gravitatie-effecten kunnen bij zulke hoge energieën niet verwaarloosd worden en dus zal deze theorie waarschijnlijk karakteristieke elementen van de quantummechanica en de relativiteitstheorie moeten bevatten. In de juiste limiet zou ze deze theorieën moeten reproduceren.

In de jaren zeventig werd al vrij snel duidelijk dat een gequantiseerde veldentheorie van de zwaartekracht, geformuleerd op de manier waarop ook de kerninteracties tussen kerndeeltjes beschreven worden, niet de gewenste theorie kon zijn. Dit model bleek niet renormeerbaar te zijn, dat wil zeggen: de wiskundige methoden uit de veldentheorie om oneindigheden uit fysische voorspellingen te weren, zullen in deze theorie tekortschieten. In de snaartheorie, die in deze periode voor het eerst geformuleerd werd, is niet het puntdeeltje maar de snaar het fundamentele object, waardoor de divergenties vermeden kunnen worden. Snaren verschillen van puntdeeltjes in die zin dat ze uitgestrekt zijn over één dimensie.

Een tweede probleem voor een quantumveldentheorie van gravitatie is dat er volgens de klassieke theorie van Einstein objecten bestaan met een horizon: zwarte gaten. Wanneer iemand in een zwart gat valt, is de horizon de laatste plaats van waaruit hij een noodkreet kan slaken die ons zal bereiken. Voorbij de horizon is geen teugkeer mogelijk. Dit althans volgens Einstein, want in 1974 ontdekte Stephen Hawking dat quantummechanica ervoor zorgt dat deze zwarte gaten straling van een zeer lage frequentie uitzenden. De straling dankt zijn bestaan aan de horizon, die de ter plaatse zijnde deeltjes en anti-deeltjes – die in paren uit het vacuum ontstaan – van elkaar scheidt; de anti-deeltjes vallen in het gat, terwijl de vrijkomende deeltjes de Hawkingstraling vormen. Hawkingstraling is dus een quantumeffect waarbij de zwaartekracht direct betrokken is. Merkwaardig is dat het spectrum van deze straling thermisch is. Er is geen eenduidige golffunctie te bedenken die de toestand van de straling beschrijft, aldus Hawking. Wanneer een zuivere golffunctie implodeert tot een zwart gat, dat daarna thermisch gaat stralen, zal het systeem vervolgens alleen te beschrijven zijn met een dichtheidsmatrix. Deze situatie handhaaft zich als het zwarte gat volledig verdampt is, dan resteert immers niets dan thermische straling.

Dit lijkt een voorbeeld van een niet-unitaire evolutie, een meer algemene evolutie dan we kennen uit Schrödingers vergelijking, want blijkbaar maakt het zwarte gat het mogelijk dat een zuivere toestand overgaat in een gemengde toestand. Om het anders te zeggen: het zwarte gat vernietigt informatie. Bovenop de onzekerheid van Heisenberg kan nu ook niet meer voorspeld worden wat de toekomst van een zuivere toestand is (stel dat-ie tegen een zwart gat botst...). Dit bracht Hawking tot de conclusie dat alleen al de aanwezigheid van een horizon de wetten van de quantummechanica schendt: een beschrijving van een wereld met zwaartekracht (en dus met de mogelijkheid om horizons te hebben) is alleen mogelijk met toestandsmatrices.

Maar hebben we hier niet gewoon met een thermodynamische limiet te maken? Een limiet waarbij bepaalde interacties op microniveau over het hoofd gezien worden, zodat we wel bij een niet-unitaire evolutie uit moeten komen? Deze mogelijkheid is inderdaad nog steeds open, het is alleen moeilijk na te gaan waar precies in Hawkings berekening een middelingsprocedure is uitgevoerd. Hij lijkt van twee fundamentele theorieën te zijn uitgegaan, zonder dat er sprake is van het verwaarlozen van belangrijke wisselwerkingen. Toch heeft met name 't Hooft de laatste jaren getracht de vinger te leggen op datgene wat Hawking veronachtzaamd heeft.

Vrijwel alle beschrijvingen van zwarte gaten gaan ervan uit dat de entropie van het gat te identificeren is met de oppervlakte van zijn horizon. Snaartheorie heeft als enige deze entropieformule uit een microscopische beschrijving weten af te leiden.

Snaartheorie is een theorie die zwaartekracht met quantummechanica tracht te verenigen door aan te nemen dat verschillende deeltjes trillingstoestanden zijn van een fundamentele snaar. Oorspronkelijk is snaartheorie ontstaan als een poging om te verklaren waarom quarks dicht op elkaar kunnen zitten. Alleen kwam men er al gauw achter dat de snaren bij veel hogere energieën moeten leven dan we in onze versnellers kunnen bereiken. In het spectrum van de snaren zit bijvoorbeeld ook het graviton, dat alleen zichtbaar is bij de Planckenergie.

De theorie bevatte niet alleen het graviton; zij voorspelde ook een deeltje waar men van af wilde, het tachyon: een deeltje dat zich sneller dan het licht voortbeweegt. De aanname van supersymmetrie, die deeltjes van verschillende spin aan elkaar relateert, elimineert dit deeltje uit het spectrum. Ook reduceert deze symmetrie het aantal dimensies van de theorie van 26 naar 10. Deze 10 dimensies zouden dan zo opgerold zijn dat onze vierdimensionale wereld overblijft, maar met welk mechanisme dit precies gebeurt, is nog steeds een open vraag. Recent heeft men begrepen dat een zeer interessante mogelijkheid ontstaat als men aaneemt dat de extra dimensies groot zijn (dus niet opgerold), maar een bijzondere geometrie hebben. Dit soort scenario's heten "warped compactifications", kromgetrokken compactificaties of simpelweg gekromde compactificaties.

In snaartheorie bestaat een zwart gat uit p-branen. Dit zijn meer-dimensionale objecten waar snaren op kunnen eindigen (de p slaat op de dimensie: een punt deeltje is dus een 0-braan, een snaar is een 1-braan, een membraan is een 2-braan, etc.). In de limiet waarbij de massadichteid van de snaren die op de branen vastgepind zijn naar oneindig gaat, terwijl hun massa constant blijft (ze worden dus uiterst kort), heeft de geometrie in de buurt van de braan de vorm van een zogenaamde anti-de Sitter ruimte (AdS): een lege ruimte waar de kosmologische constante negatief is (een contraherend heelal dus).

De braan kan gezien worden als de rand van het anti-de Sitter heelal. Het verband van Maldacena zegt dat een veldentheorie die op de braan gedefinieerd is equivalent is met snaartheorie in de anti-de Sitter ruimte.

Voor lage energieën reduceert de snaartheorie tot de relativiteitstheorie van Einstein. Er is dus sprake van een identificatie tussen Einsteins gravitatietheorie in een contraherend heelal en een quantummechanische theorie op de vlakke rand van dit heelal. Met andere woorden, de variabelen die de zwaartekracht beschrijven kunnen op zo'n manier met elkaar gecombineerd en opgeschreven worden dat ze een quantummechanische veldentheorie beschrijven. Men spreekt dan ook over een "woordenboek" die de twee theorieën aan elkaar relateert: als je de elementaire bouwstenen van de ene theorie weet, dan kun je ook door gebruik te maken van dit woordenboek een vertaling maken naar de variabelen in de andere theorie. Het handige van Maldacena's voorstel is dat berekeningen die in de veldentheorie moeilijk zijn, nu eenvoudiger berekend kunnen worden door ze in snaartheorie in AdS uit te voeren, en vice versa. Op deze manier kan de ene theorie voorspellingen doen over de fysica van de andere theorie.

Op het eerste gezicht lijkt het verband te gek om waar te kunnen zijn. Er worden twee theorieën aan elkaar gerelateerd die in verschillende dimensies leven, zoals een Yang-Millstheorie (de theorie van quarks is ook een Yang-Millstheorie) in vier dimensies en snaartheorie in tien dimensies. Bovendien bevat snaartheorie gravitatie terwijl de andere theorie op een vlakke ruimte leeft. In 1993 stelde 't Hooft dat een van de kenmerken van een theorie van quantumgravitatie moet zijn dat het aantal dimensies gereduceerd wordt. Zo zou het oppervlak van de horizon van een zwart gat alle informatie bevatten over wat zich in het volume binnen de horizon afspeelt. Het is dus niet verbazingwekkend dat de entropie, die een maat is voor de informatie die schuilgaat in een zwart gat, evenredig is met de oppervlakte en niet met het volume van het gat. Een theorie op de horizon van een gat zou dan opgevat kunnen worden als een holografische projectie van de theorie die nodig zou zijn om de fysica achter de horizon te kunnen beschrijven. Het verband van Maldacena (ook AdS/CFT-verband genaamd, CFT staat voor de "conformal field theory" op de rand) stelt nu dat de informatie bevat door snaartheorie in de anti-de Sitter-ruimte evengoed weergegeven kan worden door een veldentheorie op de rand van zo'n ruimte. Het voorstel van Maldacena is dus ook een voorbeeld van een holografische theorie. Het begrijpen van hoe en waarom dit principe werkt is daarom een uiterst belangrijke kwestie.

Holografie is echter niet alleen in snaartheorie aanwezig. Toch zijn er behalve het AdS/CFT verband van Maldacena niet veel meer voorbeelden van holografische theorieën. Een van deze voorbeelden betreft de eigenschappen van deeltjes die in de buurt van een zwart gat wisselwerken. Voor het beschrijven van deze deeltjes kan men volstaan met de zogenaamde eikonale benadering. In deze benadering wordt aangenomen dat deeltjes frontaal en op extreem hoge energieën tegen elkaar botsen. Kenmerkend voor dit soort botsingen is dat de zwaartekracht de dominante kracht wordt, en alle andere krachten verwaarloosd kunnen worden. 't Hooft heeft aangetoond dat de theorie die men in deze benadering krijgt, een 2-dimensionale theorie is. E. en H. Verlinde hebben laten zien dat dit resultaat begrepen kan worden vanuit een vereenvoudiging van het actieprincipe dat deze wisselwerkingen beschrijft. In het eikonale regime is de typische longitudinale lengteschaal (langs de as van de bosting) klein, van de orde van de Plancklengte, terwijl de transversale fluctuaties veel groter zijn en processen in deze richting veel langzamer verlopen. In deze benadering kan men dan laten zien dat de theorie van Einstein tot een topologische theorie reduceert. E. en H. Verlinde hebben ook aangetoond dat de amplitudes die men met deze theorie krijgt, overeenkomen met de door 't Hooft eerder verkregen resultaten. Een van de interessante eigenschappen van de theorie op de rand is dat de coördinaten tussen verschillende deeltjes niet-commutatief zijn.

Dit proefschrift richt het vizier op een aantal holografische eigenschappen van zowel klassieke als quantumgravitatie en snaartheorie.

In hoofdstuk 2 van dit proefschrift bestuderen we diverse eigenschappen van het model van 't Hooft, zoals covariantie. We laten zien dat het mogelijk is de transversale effecten mee te nemen, die in de eikonale benadering verwaarloosd worden. Dit geeft een interessante niet-commutatieve algebra tussen operatoren. In 2+1 dimensies kan men bovendien laten zien dat het meenemen van transversale effecten equivalent is met het covariant formuleren van de theorie. We hebben de implicaties van de zwaartekracht voor de tweede quantisatie van deeltjes bestudeerd, en gevonden dat ook velden die veel deeltjes beschrijven niet-commutatief worden, dat wil zeggen, de volgorde waarin deze fysische grootheden gemeten worden maakt uit, op dezelfde wijze als in de gewone quantummechanica metingen van plaats en impuls elkaar beïnvloeden. Dit komt overeen met eerdere resultaten van E. en H. Verlinde in de context van zwarte gaten, maar het mechanisme waardoor de niet-commutativiteit ontstaat is verschillend.

De onderliggende motivatie voor het werk gepresenteerd in hoofdstuk 3 is dat men graag de holografisch duale theorieën van 't Hooft en van Maldacena met elkaar zou willen vergelijken. Dit lijkt belangrijk voor een goed begrip van beide theorieën. Het ligt daarom voor de hand om het eikonale regime van gravitatie te beschouwen in ruimtes met een negatieve kosmologische constante. In dat hoofdstuk wordt een veralgemenisering gegeven van de afleiding van E. en H. Verlinde dat gravitatie in de eikonale limiet topologisch wordt. We hebben gevonden dat de theorie van Einstein met willekeurige waarde van de kosmologische constante inderdaad topologisch is in de eikonale limiet. De oplossingen van de theorie op de rand zijn ook gerelateerd aan de schokgolven gevonden door Horowitz en Itzhaki. Het zou zeer interessant zijn als een expliciet verband gelegd zou kunnen worden tussen de zogenaamde "lichtkegel toestanden" die duaal zijn aan een schokgolf in AdS, en de duale theorie die wij in dit proefschrift bespreken.

In hoofdstuk 4 bestuderen we holografie in het AdS/CFT verband. We scherpen het bovengenoemde "woordenboek" tussen de twee theorieën aan. We laten zien op welke manier de informatie over de geometrie van de anti-de Sitter ruimte en de andere velden die erop leven, gecodeerd is in de CFT op de rand van AdS. We ontwikkelen ook een systematische methode om de actie te regulariseren en te renormeren.

Deze resultaten worden gebruikt in hoofdstuk 5, waar we gekromde compactificaties bestuderen. We laten zien dat de $(d+1)$-dimensionale Einstein vergelijkingen samen met een verbindingsvoorwaarde de d-dimensionale Einstein vergelijkingen op de braan opleveren met een specifieke energie-impulstensor. Dit resultaat is geldig voor willekeurige waarde van de kosmologische constante. Voor ruimtes die asymptotisch AdS zijn, is de waarde van deze energie-impulstensor gelijk aan de energie-impulstensor op de braan plus die van een CFT die op de braan leeft. Door de resultaten van hoofdstuk 4 toe te passen krijgen we ook specifieke voorspellingen voor de hogere-orde correcties op de Einsteinvergelijkingen.

De afgelopen jaren zijn er snelle ontwikkelingen gekomen op het gebied van holografie en hebben we veel meer inzicht gekregen in dit kennelijk fundamentele beginsel. Toch hebben we nog geen antwoord op vragen zoals: wat is de onderliggende reden waarom de dualiteit werkt? Hoe kan causaliteit gerespecteerd worden bij de projectie van een $(d+1)$-dimensionale naar een d-dimensionale theorie en welke rol speelt de zwaartekracht hierin? Naast deze fundamentele vragen zijn er uiteraard nog veel open vragen van meer technische aard. Het is duidelijk dat veel meer onderzoek nodig is om al deze vragen naar tevredenheid te beantwoorden.

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Curriculum Vitae

De auteur werd geboren op 23 november 1973 te Barcelona. Hij bezocht het Viaró College in Sant Cugat del Vallès (Barcelona) waar hij in 1991 het Spaanse VWO- en het International Baccalaureate-diploma behaalde.
In augustus 1991 verhuisde hij naar Nederland en begon hij de met studie natuurkunde aan de Universiteit Utrecht. In 1996 studeerde hij af bij de vakgroep Theoretische Fysica met een scriptie over zwarte gaten onder begeleiding van Prof. G. 't Hooft.
Aansluitend daarop zette hij in 1997 zijn onderzoek naar zwarte gaten en stringtheorie voort in de vorm van een promotieonderzoek, eveneens onder begeleiding van Prof. G. 't Hooft. Dit onderzoek heeft geleid tot de dissertatie die u voor u ziet. Grote delen van dit onderzoek zijn verricht in samenwerking met onderzoekers uit buitenlandse instituten. Tijdens zijn promotie heeft de auteur geassisteerd bij de werkcolleges van de vakken Quantummechanica II, Voortgezette Klassieke Mechanica en Thermische en Statistische Fysica II. Hij heeft ook deelgenomen aan internationale congressen, workshops en scholen en daarbij ook diverse voordrachten gehouden over zijn eigen onderzoek. Ook heeft hij voordrachten gegeven in onder andere Boston, Princeton, New York, Philadelphia, Chicago, Berkeley, Santa Barbara en Los Angeles. In 2000 heeft hij de stringtheorie groep van Seoul National University (Seoul, Korea) bezocht en daar ook een reeks lezingen gehouden over "High-Energy Scattering". Tijdens zijn studie- en promotieperiode is hij actief geweest in het organiseren en verzorgen van diverse culturele en wetenschappelijke activiteiten, waaronder een lezing over zwarte gaten die hij voor Studium Generale Delft heeft gehouden.
In september 2001 vertrekt hij naar Los Angeles waar hij als postdoc zijn onderzoek zal voortzetten aan de University of California at Los Angeles.