2. Energy Extraction

This chapter describes the fundamentals of energy transfer by a wind turbine. In section 1 the maximum power that can be extracted from a fluid flow is discussed. The classic result for an actuator disk is that the power extracted equals the kinetic power transferred. This is a consequence of disregarding the flow around it. When we include this flow we get the balance below, having the practical consequence that an actuator disk representing a wind turbine in optimum operation transfers 50% more kinetic energy than it extracts and that this amount is dissipated into heat.

\[
- U \cdot D = -(U+U_i) \cdot D + U_i \cdot D
\]

| kinetic power transferred | power extracted | rate of heat production |

Lanchester [46] proved that the velocity at an actuator disk should be the average of that far upwind and that far downwind, but adds to this that in practice the tips of a rotor emit vortices that also represent kinetic energy. If these flows of energy are included, the energy per second increases so that the speed at the force should be higher than average.

We see no reason to doubt this plausible explanation and to introduce another, based on the concept of edge-forces on the tips of the rotor blades [45]. We question the concept wherein the edge-forces transfer momentum but no energy. First of all, from the above energy balance it follows that any axial force appears in the energy balance, and second, the axial force at the tips will accelerate the flow in the direction of the force and inevitably have induced drag or will transfer energy. The experiment with a rotor [45] in hover, to confirm the edge-force concept, was not reliable. The heat production referred to above was neglected, re-circulation may have been significant and the velocity changes used for the momentum transfer estimate were not measured in the far wake, so that the momentum exchange was not completed.

Section 2 deals with induction by presenting models of the phenomenon and by showing that correcting for the aspect ratio, for induced drag and application of Blade Element Momentum
Theory all have the same significance for a wind turbine. This is not generally known, and may lead to double corrections as proposed in [26] or to the idea that the aspect ratio correction includes the tip correction [45].

Section 3 deals with tip corrections. Prandtl’s tip correction addresses the azimuthal non-uniformity of disk loading, but does not correct for the flow around the tips or for the flow around the edges of an actuator disk. Lanchester [46] stated ‘At the disk edge, it is manifestly impossible to maintain any finite pressure difference between the front and the rear faces.’ So in fact a concept for a second tip correction is proposed that affects even an actuator disk.

Section 4 briefly reviews airfoil aerodynamics. They are basic for the detailed treatment of the aerodynamics on rotating blades given in section 5.

Here we estimated the effects of rotation on flow separation by arguing that the separation layer is thick, therefore the velocity gradients are small and viscosity can be neglected. With the argument that the chord-wise speed and its derivative normal to the wall is 0 at the separation line, the terms with the chord-wise speed or accelerations disappear and we must conclude that the chord-wise pressure gradient balances the Coriolis force. By doing so we get a simple set of equations that can be solved analytically. We oppose the classic model of Snel [52,53]. He uses boundary layer theory, which is invalid in separated flow [51,64]. As a consequence he neglects precisely those terms which we estimate to be dominant.
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>[-]</td>
<td>axial induction factor</td>
</tr>
<tr>
<td>$a'$</td>
<td>[-]</td>
<td>tangential induction factor</td>
</tr>
<tr>
<td>$A$</td>
<td>[m$^2$]</td>
<td>surface of the actuator disk</td>
</tr>
<tr>
<td>$b$</td>
<td>[m]</td>
<td>half of the span of the airfoil</td>
</tr>
<tr>
<td>$c$</td>
<td>[m]</td>
<td>chord</td>
</tr>
<tr>
<td>$C_D$</td>
<td>[-]</td>
<td>axial force coefficient</td>
</tr>
<tr>
<td>$C_H$</td>
<td>[-]</td>
<td>total pressure head coefficient</td>
</tr>
<tr>
<td>$C_{heat}$</td>
<td>[-]</td>
<td>dissipated heat coefficient</td>
</tr>
<tr>
<td>$c_{sep}$</td>
<td>[m]</td>
<td>separated length of the chord</td>
</tr>
<tr>
<td>$c_{di}$</td>
<td>[-]</td>
<td>induced drag coefficient</td>
</tr>
<tr>
<td>$c_l$</td>
<td>[-]</td>
<td>lift coefficient $L/(\frac{1}{2}\rho v^2)$</td>
</tr>
<tr>
<td>$C_P$</td>
<td>[-]</td>
<td>power coefficient</td>
</tr>
<tr>
<td>$c_p$</td>
<td>[-]</td>
<td>pressure coefficient $p/(\frac{1}{2}\rho v^2)$</td>
</tr>
<tr>
<td>$D$</td>
<td>[N/m]</td>
<td>drag force per unit span</td>
</tr>
<tr>
<td>$D_{ax}$</td>
<td>[N]</td>
<td>axial force exerted by the actuator disk</td>
</tr>
<tr>
<td>$D_i$</td>
<td>[N/m]</td>
<td>induced drag force per unit span</td>
</tr>
<tr>
<td>$D_N$</td>
<td>[N]</td>
<td>normalisation for axial force $\frac{1}{2}\rho AU^2$</td>
</tr>
<tr>
<td>$f$</td>
<td>[-]</td>
<td>stalled fraction of the chord $c_{sep}/c$</td>
</tr>
<tr>
<td>$d\tau$</td>
<td>[m$^3$]</td>
<td>infinitely small element of volume</td>
</tr>
<tr>
<td>$F$</td>
<td>[N/kg]</td>
<td>external force per unit of mass</td>
</tr>
<tr>
<td>$F_r$</td>
<td>[N/kg]</td>
<td>external force per unit of mass in the $r$-direction</td>
</tr>
<tr>
<td>$F_\theta$</td>
<td>[N/kg]</td>
<td>external force per unit of mass in the $\theta$-direction</td>
</tr>
<tr>
<td>$F_z$</td>
<td>[N/kg]</td>
<td>external force per unit of mass in the $z$-direction</td>
</tr>
<tr>
<td>$i$</td>
<td>[rad]</td>
<td>induced angle of attack</td>
</tr>
<tr>
<td>$L$</td>
<td>[N/m]</td>
<td>lift force per unit span</td>
</tr>
<tr>
<td>$\dot{m}$</td>
<td>[kg/s]</td>
<td>mass flow of the wind, in section 2.2.2 it is the mass flow per unit span in kg/ms to which the momentum transfer per unit span is confined.</td>
</tr>
<tr>
<td>$P$</td>
<td>[W]</td>
<td>power</td>
</tr>
<tr>
<td>$P_{flow}$</td>
<td>[W/m]</td>
<td>kinetic power extracted from the flow per unit airfoil span</td>
</tr>
<tr>
<td>$P_N$</td>
<td>[W]</td>
<td>normalisation power $\frac{1}{2}\rho AU^3$</td>
</tr>
<tr>
<td>$p$</td>
<td>[N/m$^2$]</td>
<td>pressure</td>
</tr>
<tr>
<td>$p_0$</td>
<td>[N/m$^2$]</td>
<td>atmospheric pressure</td>
</tr>
<tr>
<td>$p^+$</td>
<td>[N/m$^2$]</td>
<td>pressure on upwind site of the actuator disk</td>
</tr>
<tr>
<td>$p^-$</td>
<td>[N/m$^2$]</td>
<td>pressure on downwind side of the actuator disk</td>
</tr>
<tr>
<td>$p_d$</td>
<td>[N/m$^2$]</td>
<td>dynamic pressure $\frac{1}{2}\rho U^2$</td>
</tr>
<tr>
<td>$r$</td>
<td>[m]</td>
<td>radial position</td>
</tr>
<tr>
<td>$R$</td>
<td>[m]</td>
<td>radius of the turbine rotor</td>
</tr>
<tr>
<td>$s$</td>
<td>[-]</td>
<td>location of the separation point</td>
</tr>
<tr>
<td>$t$</td>
<td>[s]</td>
<td>time</td>
</tr>
<tr>
<td>$U$</td>
<td>[m/s]</td>
<td>wind speed</td>
</tr>
<tr>
<td>$U_D$</td>
<td>[m/s]</td>
<td>wind speed at the disk</td>
</tr>
<tr>
<td>$U_i$</td>
<td>[m/s]</td>
<td>induced velocity</td>
</tr>
</tbody>
</table>
Flow Separation on Wind Turbine Blades

$U_W$ [m/s] wind speed in the far wake
$V$ [m/s] wind speed in the very far wake
$v$ [m/s] velocity of the airfoil
$v_r$ [m/s] flow velocity in the $r$-direction in the rotating frame of reference
$v_\theta$ [m/s] flow velocity in the $\theta$-direction in the rotating frame of reference
$v_z$ [m/s] flow velocity in the $z$-direction in the rotating frame of reference
$W$ [m/s] resultant inflow velocity
$x$ [m] position in the direction of the chord
$y$ [m] position in the direction of the span
$z$ [m] position normal to the blade surface

$\alpha$ [rad] angle of attack
$\alpha_0$ [rad] zero lift angle of attack
$\beta$ [rad] local pitch angle including twist
$\Gamma$ [m$^2$/s] circulation
$\delta$ [m] boundary layer thickness
$\Delta U$ [m/s] velocity change in very far wake due to actuator disk.
$\Delta P_s$ [W] kinetic power extracted from the flow through the stream tube.
$\Delta P$ [W] kinetic power extracted from the total flow.
$\varepsilon$ [-] fraction of the total mass flow $\dot{m}$ through the actuator disk
$\nabla$ [m$^{-1}$] nabla-operator ($\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$)
$\rho$ [kg/m$^3$] air density $\approx 1.25$ kg/m$^3$
$\mu$ [Ns/m$^2$] dynamic viscosity of air $\approx 17.1 \cdot 10^{-6}$ Ns/m$^2$
$\tau$ [N/m$^2$] shear stress
$\eta$ [-] efficiency of kinetic energy transfer
$\phi$ [rad] geometric angle of attack
$\theta$ [rad] position in chord-wise direction
$\lambda$ [-] tip speed ratio $\Omega R/U$
$\lambda_r$ [-] aspect ratio $(2b^2)/bc$
$\lambda_r$ [-] local speed ratio $\Omega r/U$
$\Sigma$ [m] cross section of inflow per unit span to which momentum change is confined
$\omega$ [s$^{-1}$] vorticity
$\Omega$ [rad/s] rotor angular frequency
2.1 Maximum Energy Transfer

The theory predicting the maximum useful power that can be extracted from a fluid flow was first published by F.W. Lanchester [46] in 1915. In most cases however, this theory is attributed to A. Betz, who published the same argument in 1920 [7]. To do justice to the first author, we will speak of the 'Lanchester-Betz' limit. The first subsection briefly describes the model, in which Lanchester analyses the actuator disk introduced by Froude in 1889 [32]. The second subsection adds a new aspect to the classic model: the inherent viscous losses of an actuator disk. It will be shown that an actuator disk operating in wind turbine mode extracts more energy from the fluid than can be transferred into useful energy. At the Lanchester-Betz limit the decrease of the kinetic energy in the wind is converted by $2/3$ into useful power and by $1/3$ into heat. The heat is produced by the viscous force of the outer flow on the stream tube that just encloses the flow through the actuator disk. The analysis shows that there is no necessity to add edge-forces to the actuator disk model [45].

2.1.1 The Lanchester-Betz Limit

This section summarises a text written by Glauert [34], to which physical arguments are added. First the actual wind turbine will be replaced by a so-called actuator disk which was introduced by Froude (see figure 2.1). This actuator disk is an abstract theoretical analogue of a wind turbine being used in momentum theory. The disk has a surface $A$, equal to the swept area of the wind turbine, and it is oriented perpendicular to the wind. The disk does not consist of several rotor blades but has a homogeneous structure. The undisturbed wind speed is $U$, at the actuator disk it is $U_D=(1-a)U$ and in the far wake it is $(1-2a)U$. The parameter $a$ is called the induction factor which takes into account the decrease of the wind speed when it passes through the permeable actuator disk. The mass flow through this disk is $\rho A(1-a)U$ and it is driven by the difference in pressure $p^+$ on the upwind side of the disk and $p^-$ on the downwind side. So the pressure at the disk is discontinuous and the disk is subject to a net axial force $D_{ax} = A(p^+-p^-)$. This force is also exerted on the fluid and thus it should be equal to the change of the flow of momentum. From conservation of mass it follows that the stream tube just enclosing the flow through the actuator disk has a constant

![Figure 2.1](image-url)  
*Figure 2.1* Froude’s actuator model. The stream tube consists of a slipstream behind the disk, but has no velocity discontinuity in front of the disk.
mass flow \( \rho A (1-a)U \) at all cross sections from far upstream to far downstream. The figure shows this stream tube and its expansion. Behind the actuator we have a clear slipstream, but in front of it such a boundary does not exist, therefore we dashed the slip stream contour here. As this mass flow is constant, the change of momentum should be attributed to a velocity difference between the flow in the far wake and the undisturbed wind speed far upstream:

\[
p^+ - p^- = \rho (1-a)U (U - U_w).
\]

Upwind and downwind of the actuator disk, the kinetic energy in the flow is transferred into 'pressure' energy. So the actuator disk does not directly extract kinetic energy. The disk slows down the flow which causes a pressure difference over the disk. The extracted energy comes from the product of the pressure difference and the volume flow through the disk. Application of Bernoulli's relation that \( p + \frac{1}{2} \rho U^2 = \) constant along a streamline (when no power is extracted), yields for the flow upwind and downwind respectively:

\[
\frac{1}{2} \rho U^2 + p_o = \frac{1}{2} \rho (1-a)^2 U^2 + p^+,
\]

\[
\frac{1}{2} \rho U_w^2 + p_o = \frac{1}{2} \rho (1-a)^2 U^2 + p^-,
\]

where \( p_o \) is the undisturbed atmospheric pressure. By subtracting equations 2.2 and 2.3 it follows that:

\[
p^+ - p^- = \frac{1}{2} \rho (U^2 - U_w^2).
\]

The combination of equations 2.1 and 2.4 demonstrates that the velocity decrease in front of the disk equals that behind the disk:

\[
U_w = (1 - 2a)U, \quad U_D = (1-a)U.
\]

The remarkable fact that half the acceleration must take place in front of the disk and half behind it will be discussed in sections 2.1.2 and 2.2. The absolute values for \( p^+ \) and \( p^- \) are found to be:

\[
p^+ = p_o + \frac{1}{2} \rho U^2 (2a - a^2) = p_o + p_{dyn} (2a - a^2),
\]

\[
p^- = p_o - \frac{1}{2} \rho U^2 (2a - 3a^2) = p_o - p_{dyn} (2a - 3a^2),
\]

where the free stream dynamic pressure \( p_{dyn} = \frac{1}{2} \rho U^2 \) is used. It should be noted that the increase of the pressure on the upwind side is larger than the decrease of the pressure on the downwind side. This suggests that the pressure field far from the turbine can be modelled as the sum of a dipole and a monopole or source.

The extracted power is equal to the difference of the kinetic energy in the flow far upstream, minus the kinetic energy in the flow far downstream, multiplied by the mass flow \( \rho A (1-a)U \).

Far upstream the velocity is \( U \) and far downstream it is \( (1-2a)U \). Thus we find for the power:

\[
P = 4a(1-a)^2 \frac{1}{2} \rho AU^3 = 4a(1-a)^2 P_{ad}.
\]
in which \( P_N (\frac{1}{2} \rho AU^3) \) is the flow of kinetic energy through a cross section of size \( A \) perpendicular to the undisturbed wind. It follows that only the axial induction factor \( a \) determines the fraction of the power extracted by the wind turbine. From \( \frac{dP}{da} = 0 \) we find that the maximum fraction extracted is \( \frac{16}{27} \), which corresponds to \( a = \frac{1}{3} \). This maximum was derived by Lanchester in 1915. If both the maximum power and the corresponding axial force are normalised with \( P_N \) and \( D_N = \frac{1}{2} \rho AU^2 \) respectively, then it follows that:

\[
C_P = 4a(1-a)^2 = \frac{16}{27}, \\
C_D = 4a(1-a) = \frac{8}{9},
\]

for the power and the axial force coefficients respectively.

Equation 2.9 only gives the fraction of \( P_N \) that can be converted into useful power. It should not be confused with the efficiency of the turbine. When we read the literature of almost a century ago we find the following text on efficiency written by Betz, 1920 [7]: ‘Eine Fläche welche dem Winde einen gewissen Widerstand entgegensetzt, dadurch seine Geschwindigkeit, also seine kinetische Energie, vermindert und diese ihm entzogene kinetische Energie verlustlos in nutzbare Form überführt.' But in the same paper Betz states that a turbine on an airplane translating with velocity \( U \) and axial force \( D_{ax} \) has efficiency \( P/(U \cdot D_{ax}) \), which is 1-\( a \). However Betz says about this: ‘Diese Definition befriedigt nun zwar das theoretische Bedürfnis, da die Axialkraft eine Größe ist, die für die wirtschaftliche Beurteilung eines Windmotors nur untergeordnete Bedeutung hat'. Glauert 1934 [34] confirms this by stating that it is necessary to distinguish between a windmill driven by the speed of an airplane and a windmill on the ground driven by the wind. In the first case the efficiency is meaningful, but for the latter only the extracted energy is relevant. So, in classic theory the efficiency of a wind turbine (1-\( a \)) is considered unimportant, which probably was one reason for not paying attention to the physical effect which caused the loss.

In recent literature we find that the decrease of the flow of kinetic energy equals the useful power produced by the actuator disk. Spera 1994 [57], Hunt 1981 [42] and Wilson and Lissaman 1974 [65] normalise the power produced by \( (1-a)P_N \) instead of \( P_N \) since the mass flow through the actuator is \( (1-a)UA \) and not \( UA \). So they hold that the power in the flow is converted with an efficiency (defined as power output/power input) = \( C_P/(1-a) = 4a(1-a) \), which is \( \frac{8}{9} \) at the Lanchester-Betz limit. This means that they limit themselves to the wind that flows through the actuator disk. They find that \( \frac{1}{9} \) of the kinetic energy remained in the flow and thus \( \frac{8}{9} \) was converted into useful power, where the conversion is assumed to have an efficiency of 100%.

In the next section the power transfer by an actuator disk will be calculated for the case in which the outer flow is included.

### 2.1.2 Heat Generation

In the actuator disk model, the power extracted by the axial force is \( -(1-a)U \cdot D_{ax} \). However, if the same actuator disk, exerting a force \( D_{ax} \), is fixed on an airplane moving with speed \( U \), the power required to move the disk would be \( -U \cdot D_{ax} \). So it takes more power to drive the disk
than the maximum power that can be generated by the disk. This difference is understood when the flow around the actuator is also included in the analysis. It then follows that the energy conversion by an actuator disk has an inherent dissipation of kinetic energy into heat.

Kinetic Power Transfer by an Axial Force

Let $\dot{m}$ be the indefinite but large mass flow in the wind, in which an actuator disk is placed perpendicular to the flow direction (see figure 2.2). Only a fraction $\varepsilon$ of $\dot{m}$ flows through the stream tube that just encloses the actuator disk, which exerts a finite axial force $D_{ax}$ on the flow against the flow direction. In the far wake, the momentum and the energy relations will be:

$$D_{ax} = -\dot{m}2aU,$$  \hfill (2.11)

$$\Delta P_s = \frac{1}{2} \varepsilon \dot{m} (U^2 - (1-2a)^2 U^2) = -(1-a)U \cdot D_{ax}.$$  \hfill (2.12)

Where $\Delta P_s$ refers to the change of the kinetic power in the flow in the stream tube when it crosses the actuator disk. We now provide the actuator disk model with a very far wake, defined as the location beyond the far wake, where the velocity distribution has become uniform again. The definition of the far wake remains classic, namely the location where the axial force has stopped transferring momentum to the flow, or in other words, where the stream tube is no longer expanding. The velocity is $(1-2a)U$ in the far wake and $U$ outside the wake. The smoothing of the velocity profile behind the far wake is due to turbulent mixing and viscous shear, which will eventually make all velocities equal to a common speed $V$ in the very far wake. During this process no external force acts on the flow, so momentum is conserved and the flow does not expand further.

Comparing the flow far upwind $\dot{m}U$ with that in the very far wake $\dot{m}V$, the difference in the flow of momentum should be equal to the axial force.

$$D_{ax} = \dot{m}(U - V).$$  \hfill (2.13)

**figure 2.2** Introduction of the very far wake and viscous dissipation. When the outer flow and that inside the stream tube mix, heat is generated and the slipstream vanishes while it contracts.
We can express $V$ in $U$, $a$ and $\varepsilon$ by using the momentum balance between the far wake and the very far wake. The momentum in the outer flow of the far wake is $(1-\varepsilon)\dot{m}U$, and in the stream tube it is $\varepsilon\dot{m}(1-2a)U$, which together should be equal to $\dot{m}V$ to conserve momentum, or

$$V = (1-\varepsilon)U + \varepsilon(1-2a)U = (1-2a\varepsilon)U.$$  

(2.14)

The velocity change obtained from the momentum relation 2.13 is connected to the change of the kinetic power in the wind, by

$$\Delta P = \dot{m}(U^2 - V^2) = -(1-a\varepsilon)U \cdot D_{ax},$$

(2.15)

To clarify: this is the change of the kinetic power in the flow due to the axial force when the outer flow is included, whereas equation 2.12 expresses that change when the outer flow is excluded. In practice the mass flow $\dot{m}$ is large but finite, so that the fraction of $\dot{m}$ going through the disk, $\varepsilon$, is much smaller than 1 and $\Delta P$ is close to $-D_{ax}U$. So, the decrease of flow of kinetic energy by a force $D_{ax}$ approaches the scalar product of the undisturbed wind speed $-U$ and $D_{ax}$ and not the often used product of the local velocity $-(1-a)U$ and the force $D_{ax}$. The latter corresponds to the power extracted from the flow.

**Dissipation into Heat**

In the process of mixing between the far wake and the very far wake, the kinetic power in the flow will not be conserved, but it will be partially converted into heat. This heat is generated by the viscous force that accelerates the flow in the stream tube to the velocity $V$ in the very far wake. In this process the flow inside the stream tube gains less kinetic energy than the outer flow loses. In the far wake the kinetic power inside the stream tube is $\frac{1}{2}\varepsilon\dot{m}(1-2a)^2U^2$ and in the outer flow it is $\frac{1}{2}(1-\varepsilon)\dot{m}U^2$. In the very far wake the kinetic power is $\frac{1}{2}\dot{m}V^2$. The difference has to be the heat generated;

$$P_{heat} = \frac{1}{2}\dot{m}\left\{[\varepsilon(1-2a)^2U^2 + (1-\varepsilon)U^2] - V^2\right\} = -(1-\varepsilon)aU \cdot D_{ax}.$$  

(2.16)

Of course, this is also equal to $\Delta P - \Delta P_s$.

If we want to normalise to $P_N = \frac{1}{2}\rho AU^3$, as in the previous section, the mass flow through the actuator disk $(1-a)\rho AU$ has to be replaced by $\varepsilon\dot{m}$. So we use $P_N = \frac{1}{2}\varepsilon\dot{m}U^2/(1-a) = -D_{ax}U/(4a(1-a))$. Since the mass flow through the actuator disk is much smaller than the flow outside the wake, we take the limit $\varepsilon \rightarrow 0$, and find the following power coefficients,

$$C_H = \frac{\Delta P}{P_N} = 4a(1-a),$$  

(2.17)

$$C_p = \frac{\Delta P}{P_N} = 4a(1-a)^2,$$  

(2.18)
\[ C_{\text{heat}} = \frac{P_{\text{heat}}}{P_N} \approx 4a^2(1 - a). \] (2.19)

Here \( C_H \) refers to the transferred kinetic power, \( C_P \) to the kinetic power actually extracted and \( C_{\text{heat}} \) to the power in the viscous heating. \( C_P \) is the commonly used (classic) power coefficient.

It follows that the maximum efficiency for the process of transfer of kinetic energy into useful power by an actuator disk \( \eta \) is:

\[ \eta = \frac{C_P}{C_H} \approx 1 - a, \] (2.20)

which is in agreement with Betz’s result [7]. Our calculation makes clear that an actuator disk does not convert all transferred kinetic energy into useful energy. The energy balance reads:

\[ C_H = C_P + C_{\text{heat}}. \] (2.21)

As mentioned before, the maximum extractable useful power from the flow is obtained for \( a = \frac{1}{3} \). In that case a fraction \( C_H = \frac{24}{27} \) of the flow of kinetic energy \( P_N \) is transferred. From this, \( \frac{2}{3} \) is extracted as useful power, and \( \frac{1}{3} \) is dissipated as heat. Figure 2.3 shows schematically the power transfer by an actuator disk representing a wind turbine. We introduced \( U_i = -aU \) for the induction velocity in order to make the model more general, so that the situation for an actuator disk representing a propeller is also included.

\[ \text{transferred kinetic flow} \quad a = \frac{1}{3} \rightarrow \frac{24}{27}P_N \]

\[ - D_{ax} \cdot U = 4a(1 - a)P_N \]

\[ \text{generated heat flow} \quad a = \frac{1}{3} \rightarrow \frac{8}{27}P_N \]

\[ U_i \cdot D_{ax} = 4a^2(1 - a)P_N \]

\[ \text{useful / engine power} \quad a = \frac{1}{3} \rightarrow \frac{16}{27}P_N \]

\[ -(U + U_i) \cdot D_{ax} = 4a(1 - a)^2 P_N \]

**Figure 2.3** Schematic view of the kinetic energy transfer by an actuator disk.
Energy Extraction

For an actuator disk representing a rotor in hover \((U=0)\) it follows from equations 2.17 to 2.19 and from figure 2.3 that the power required to yield \(D_{ax}\) is \(U_i\cdot D_{ax}\) and that this power is entirely converted into heat. It first turns up as kinetic energy, which is eventually dissipated via turbulent mixing and viscous shear as heat. For an actuator in propeller state \((U\geq0\text{ and } U_i\geq0)\), the engine power is \((U+U_i)\cdot D_{ax}\) and the heat produced is \(U_i\cdot D_{ax}\) and the kinetic energy of the flow is increased by \(U\cdot D_{ax}\). (The co-ordinate system is still attached to the actuator).

We conclude that the inherent limitation to the efficiency of energy extraction by an actuator disk is determined by dissipation as heat. This dissipation is \(a/(1-a)\) times the extracted useful energy. The heat capacity of the mass flow through a wind turbine is so large that the heat generated will hardly affect the temperature. To give an example: a wind turbine operating at 10 m/s at the Lanchester-Betz limit will transfer 44.4J of kinetic energy per unit of air mass into 29.6J of useful work and 14.8J of heat. This heat raises the temperature by only 0.015°C. In practice it will be even less since the heat generated is not limited to the flow inside the stream tube.

Edge-Forces
Adding forces to the edges of the actuator disk has been proposed by Van Kuik [45]. These forces would transfer momentum without having an effect on the energy relations. In this way he explains a 10-15% increase of the velocity through the disk and at the same time a 1 degree increase of the angle of attack along the span of actual wind turbine blades. This proposal is therefore relevant to the present thesis.

Our heat-analysis implies that any measurement of the extracted, or fed, power based on the decrease of the total pressure (represented by the transferred kinetic power in the scheme) in the wake of a wind turbine depends on the position of measurement. If we measure the velocities induced by a rotor in hover, the sensors should be close to the rotor, otherwise the velocity pattern will be affected by dissipation or turbulent mixing. But, the closer to the rotor, the more the total pressure depends on the dynamics of the blade passages. This sets high demands on the sensors. On the other hand, if we want to know the total change of momentum from velocity measurements, the sensors should be far behind the rotor, in the far wake, since only there has the momentum exchange taken place fully. This difficulty can be illustrated by Van Kuik’s interesting measurement on a rotor in hover [45]. Here the velocity sensors (hot-wires) were at 0.5\(R\) behind the rotor, where the velocity discontinuity at the boundary of the slip stream is already vanishing or, in Van Kuik’s words: ‘Figure 4.10 (in his thesis) shows that the vortex cores are not visible any more as the vortex structure has desintegrated.’ We propose that the disappearance is due to turbulent mixing and viscous dissipation. If we estimate how much kinetic power was lost (by calculating the kinetic power by assuming that the velocity does not decrease up to the stream tube boundary and using Van Kuik’s figure 4.8), we find this to be approximately 16%. So this is approximately the loss of total pressure flow at this position and it is as much as the effect to be validated. Besides to this we have uncertainty in the estimated momentum change Van Kuik tries to validate regarding the position of measurement and possible re-circulation.

We have shown however that any axial force, doing useful work or not, does transfer energy (for \(U\neq0\)) if the outer flow is included. This is not in contradiction with Van Kuik’s proposed edge-forces for an actuator disk, which is in fact an extension of the proven theory on cylinder symmetric concentrators. However in practice, for a wind turbine without a cylinder
Flow Separation on Wind Turbine Blades

symmetric concentrator or tip-vanes [41], when we have a select number of blades, possible axial edge-forces at the blade tips acting on a certain span-wise flow would bend this flow in the direction of the forces according to state of the art induction theory (next section). So the flow aligns with the forces to some extent and subsequently the forces transfer energy, which is in contradiction with Van Kuik’s concept. For this reason we used classic momentum theory (without edge-forces) in our simulations of rotor behaviour.

The $\frac{1}{2}$ Factor
The velocity at the actuator disk is assumed to be half the sum of that far upwind and far downwind (eq. 2.5). Only then were both the momentum and energy balance met. But in this energy balance only the kinetic energy was considered, while in fact all types of energy should be included. In the classic stream tube theory we assumed a uniform disk, which may not be true in practice. Near the centre of rotation, and near the tips, the velocity distribution immediately downwind of the rotor will not be uniform. Velocity differences will surely lead to viscous dissipation, in this case also between the rotor and the far wake. This heat should be included in the energy balance, otherwise the velocity at the location of the force, when calculated from the relation - force times local speed equals change of kinetic energy - will be too low. Lanchester [46] analysed the situation of a real rotor, where the tips are emitting vortices that contain kinetic energy, which will not remain in the fluid far downstream. But this energy had to be produced, so the transfer of kinetic energy is larger than eq. 2.12 for an actuator disk without tips. When the speed at the disk is calculated so that it includes the energy emitted by the vortices it should be higher than $\frac{1}{2}$ of the sum of the velocity far upwind and far downwind. We did not include this argument in our further analysis, because it was not yet available in a quantitative form.

Practice
The above analysis does not put the Lanchester-Betz limit in a different light, since the maximum extractable useful energy of a wind turbine remains unchanged. But for a wind turbine park as a whole (present park optimisation studies are based on momentum balances and thus deal correctly with the dissipated heat), our model clarifies what determines the loss. And we conclude that the maximum extractable useful energy shall not occur when all turbines operate individually at maximum output. By choosing the induction factor 10% below the optimum, the power coefficient decreases less than 1%, while the efficiency rises more than 3%. In the turbulent wake state in particular, when $a$ is approximately 0.4-0.5, the efficiency $(1-a)$ becomes rather low, thus other wind turbines in the wake get a lower power input. This could be reason to operate turbines at the upwind side of a park below the optimum for $a$, and certainly not in the turbulent wake state, so that the production of the park as a whole increases.
2.2 Induction

In this section the concept 'induction' will be discussed. Induction takes the differences between a three-dimensional steady or unsteady situation of practice and the two-dimensional test situation in a wind tunnel into account. The concept is also used to derive the potential theoretical contribution to the drag force, which is the induced drag. It is useful therefore to start with definitions concerning induction in aerodynamics. Section 1 then deals with the induced velocities which were proposed by Prandtl for a finite airfoil. Section 2 discusses induction related to a wind turbine. Section 3 involves the classic blade element momentum theory.

Induced velocities and vorticity
The velocity field around an aerodynamic object, which experiences forces perpendicular to the flow direction (lift forces), can be described mathematically by a vorticity distribution. But, vorticity is only a way to describe a velocity field, it is not the cause of the velocity field. Vortices do not induce velocities; they are equivalent to certain velocity patterns.

Pressure distribution and velocity field
In inviscid flow, the flow field is determined by the Euler equation which describes the interaction between pressure distribution, external forces and velocity field and assumes that no internal friction (viscosity) exists. When an aerodynamic object is placed in a fluid in motion a pressure distribution over the surface of the object comes into being. This pressure distribution is in agreement with the velocity field around the object. The words ‘in agreement with’ were used to emphasise the mutual interaction between pressure distribution on the object and velocity field around the object instead of a causal connection. In summary: an object in combination with a flow causes a combination of a velocity field and a pressure distribution. The resulting velocity field can be described as the sum of the undisturbed fluid motion and the motion described by a vorticity distribution.

Induced velocities in 'Prandtl-terms'
Vortices describe induced velocities, but only a specific portion of them are ‘induction velocities’ in Prandtl-terms. This portion accounts for the difference between the three-dimensional steady or unsteady practical situations and the 'two-dimensional steady' wind tunnel situation. The difference consists in general of three types of vortices, shown in figure 2.4. The first is the trailing vorticity of the tips of a finite airfoil (what BT induces at BB); the second the vorticity shed from the airfoil when the bound circulation changes over time (what PP induces at BB); and the third is the absence of the (shed) vorticity outside the span of the airfoil (what

![figure 2.4 Prandtl's induction velocities.](image)
Flow Separation on Wind Turbine Blades

SP, which is the shed vorticity of AB, induces at BB. This contribution does not exist in the 3d-situation, but is present in the 2d-situation. So the difference has to be corrected. For a precise definition of the third contribution we refer to Van Holten [41]. It is especially important for helicopter rotors with their strong variations in circulation with azimuth. In the case of a wind turbine it is normally sufficient to account for the tip vortices only. It should be noted that the velocity pattern described by the bound vorticity that was present in the wind tunnel is excluded from the induction velocities in 'Prandtl terms'.

2.2.1 Prandtl Finite Airfoil Induction

This section gives a summary of Prandtl’s reasoning for obtaining a general expression for the induced drag of finite airfoils. The fact that lift is necessarily accompanied by induced drag was first pointed out by Lanchester; later Prandtl developed a rigorous system of mathematical equations which will be explained below. The text is based on the contribution of von Karmán and Burgers in [44].

When a finite airfoil of span $2b$ exerts a lift force per unit span $L$ on the flow, this force is balanced by an equal momentum change. If this change of momentum is confined to a certain area $\Sigma$ per unit span perpendicular to the flow direction then it results in a downward velocity $u_0$ which equals $L/\rho v \Sigma$ (see figure 2.5). The downward motion is associated with a flow of kinetic energy per unit span.

\[ P_{\text{flow}} = \frac{1}{2} \rho v \Sigma u_0^2 = \frac{L^2}{2 \rho v \Sigma} = D_i v. \]  

(2.22)

This power per unit span is produced by the so-called induced resistance of the airfoil $D_i$ per unit span. The power loss due to the motion of the airfoil times $D_i$ equals the flow of kinetic energy of the downward flow per unit span, as was shown in equation 2.22. (We know that we should in fact account for the total power loss, thus also static pressure changes, kinetic energy changes in any direction and possibly heat produced.) It can be derived (see [44]) that $\Sigma$ has the maximum value $\pi b$, when $2b$ is the span of the airfoil. It follows that:

\[ u_0 = \frac{L}{\rho v \pi b}. \]  

(2.23)

This maximum corresponds to a minimum induced drag. The minimum drag and minimum drag coefficient read respectively:

\[ D_i = \frac{L^2}{2 \rho v^2 \pi b}, \quad c_{\text{ii}} = \frac{c_i^2}{\pi \lambda}, \]  

(2.24)
in which \( c_l = L/(\frac{1}{2}\rho v^2 c) \) is the lift coefficient, \( c \) is the chord of the airfoil and \( \lambda = (2h)^2/bc \) is the aspect ratio. It is assumed that half of the downward velocity is imparted to the air before it reaches the airfoil and half is imparted after it has passed the airfoil. The same relation was found for the entire rotor (see equation 2.5). Lanchester explains this using the following argument for a fluid which is initially at rest:

'Let \( m = \text{the mass of fluid per second, and } V \text{ its ultimate velocity; then } mV^2/2 \text{ is the energy or work done per second. And the momentum per second of the stream } = mV, \text{ which is also the force by which the flow is impelled. And this force must (to comply with the energy condition) move through a distance per second, in other words act with a velocity } U \text{ such that:}'

\[
UmV = \frac{mV^2}{2}, \quad \text{or} \quad U = \frac{V}{2},
\]

(2.25)

Lanchester also proves the validity for any nonzero initial velocity; the change of the velocity where the force acts is given by half of the total change. It follows that the downward velocity at the airfoil \( u = \frac{1}{2}u_0 \). If we combine this velocity with the undisturbed velocity \( v \) we obtain the resultant velocity \( \sqrt{(v^2+u^2)} \) which is inclined under an angle \( \tan(i) = v/u \). In practice \( u \) is much smaller than \( v \), therefore the approximation \( i = v/u \) is acceptable. The conclusion is that the effective angle of incidence \( \alpha \) differs from the geometric angle of attack \( \phi \) by the angle \( i \):

\[
\alpha = \phi - i.
\]

(2.26)

In summary: the inflow direction is inclined by an angle \( i \) compared to the geometrical inflow direction and thus the lift force has a component in the backward direction. This component equals the induced drag. The induced drag times the velocity of the airfoil equals the flow of kinetic energy in the downstream.

### 2.2.2 Induction for a Wind Turbine

This section deals with the induction of a wind turbine rotor and relates it to the above for a finite airfoil. It will be shown that the induced drag of wind turbine blades is implicitly taken into account in momentum theory.

**Induced Drag**

We will follow Prandtl's analysis for a finite airfoil to derive the induced drag for a wind turbine blade section. The situation is slightly more complicated due to the speed of flow itself starting with \( U \) instead of 0. Figure 2.6 gives an overview. Assume that the blade section has its own speed \( v \) and that the wind speed is \( U \). The blade exerts a lift force per unit span \( L \) on the flow. The lift force is tilted forward under an angle \( \varphi = \arctan(U/v) \approx U/v \), if \( U < v \).
Thus the power extracted from the flow in this initial situation is \( L \cdot U \), which has to be compared with 0 for Prandtl's finite airfoil. The lift force will be balanced by the change of momentum of the mass flow per unit span. This mass flow \( \dot{m} \) refers to the mass flow through the cross section \( \Sigma \) (and not to the indefinite flow referred to in 2.1.2). The resulting velocity change will be \( \Delta U = L/\dot{m} \). The kinetic power of the inflow was \( \frac{1}{2} \dot{m} U^2 \) and decreased to \( \frac{1}{2} \dot{m} (U-\Delta U)^2 \) when passing the airfoil. Thus the kinetic power extracted from the flow per unit span \( P_{\text{flow}} \) is:

\[
P_{\text{flow}} = \dot{m} U \Delta U - \frac{\dot{m}(\Delta U)^2}{2} = U L - \frac{L^2}{2 \dot{m}}. \tag{2.27}
\]

The expression on the right hand side follows after a substitution of \( \Delta U \) by \( L/\dot{m} \). So, the power extracted by the lift force \( U \cdot L \) exceeds the power extracted from the flow by \( L^2/(2 \dot{m}) \). This error will be corrected by the introduction of the induced angle of attack \( i \). The induced angle should tilt the lift force backwards until the power generated by the lift is decreased with the power surplus. Thus if we assume that \( i \) is small, then \( L \cdot i \cdot v \) should equal \( L^2/(2 \dot{m}) \), or:

\[
i = \frac{L}{2mv} = \frac{\Delta U}{2v}. \tag{2.28}
\]

It follows that the induced angle of attack decreases the geometric angle of attack by means of half the induced velocity in the far wake, in agreement with equation 2.5 of section 2.1.1 and with the argument of Lanchester. So the introduction of Prandtl's induced drag via the induced angle of attack \( i \) is equivalent to the effect of the induction factors in blade element momentum theory, which is described in the next section. This is not generally known. Reference is often made to Viterna and Corrigan [62] who propose a correction for the induced drag in addition to the effect of induction velocities calculated using blade element momentum theory. This means that they correct for induction twice.

**Aspect Ratio**

The performance of a finite airfoil diminishes by a decreasing aspect ratio. The smaller the aspect ratio the larger the ratio of the lift force and the mass flow on which the force is exerted. So the velocity in the down flow increases and thus the induced drag. We should emphasise that the aspect ratio correction is equivalent to a correction for induction velocities. In fact the aspect ratio is just the geometric factor that determines the induced drag via \( c_{dI} = c_l^2/\pi \lambda \). Thus it is already part of blade element momentum theory.

2.2.3 Blade Element Momentum Theory

This theory, sometimes referred to as strip theory, is W. Froude's [33]. It differs from momentum theory in that the forces on the flow are produced by the blades of a propeller, or wind turbine rotor, instead of an actuator disk. The theory, found in much of the literature [34, 63, 65], is based on the assumption that no interference exists between successive blade elements. In short, the theory offers a calculation scheme that iteratively brings the forces on the airfoil sections at a certain radial position into agreement with the momentum changes of
the flow through the annulus at that radial position. It yields both the forces and the axial and tangential induction factors $a$ and $a'$. The axial force causes the flow to slow down by $aU$ at the rotor disk and $2aU$ in the far wake. The torque exerted by the flow on the rotor will cause the flow to rotate in the opposite direction with rotation speed $a'\Omega$ at the rotor and $2a'\Omega$ in the far wake.

One assumes a rotor with $N$ blades and airfoil sections at radial position $r$ with chord $c$. When the rotor speed is $\Omega$ and the undisturbed wind speed $U$, the velocity component at the blade sections are:

$$U_{ax} = (1-a)U, \quad U_{tan} = (1+a')\Omega r.$$  \hspace{1cm} (2.29)

The axial and tangential induction factors $a$ and $a'$ first get an initial value, for example 0. From these velocities the inflow conditions are obtained, namely the resultant velocity $W$ and the angle of attack $\alpha$ with

$$W = \sqrt{U_{ax}^2 + U_{tan}^2}, \quad \alpha = \arctan \frac{U_{ax}}{U_{tan}} - \beta,$$  \hspace{1cm} (2.30)

where $\beta$ is the blade pitch angle. Using tables for $c_l(\alpha)$ and $c_d(\alpha)$, the lift $L$ and the drag force $D$ are found,

$$L = \frac{1}{2} \rho W^2 c_l N c \Delta r, \quad D = \frac{1}{2} \rho W^2 c_d N c \Delta r,$$  \hspace{1cm} (2.31)

which can be expressed as an axial and tangential force

$$F_{ax} = L \cos \alpha + D \sin \alpha, \quad F_{tan} = L \sin \alpha - D \cos \alpha.$$  \hspace{1cm} (2.32)

These forces should balance the axial and change of tangential momentum of the mass flow through an annulus of cross section $2\pi r \Delta r$:

$$2\pi r \Delta r U (1-a) 2a U = F_{ax}, \quad 2\pi r \Delta r U (1-a) 2a' \Omega r = F_{tan}. \hspace{1cm} (2.33)$$

In this way one can find a new estimate for $a$ and $a'$, but these values are still based on the condition of undisturbed inflow. One has to go through this procedure a number of times to find more correct values for the forces and induction factors. By doing so for many radial positions and many wind speeds, the rotor performance can be calculated. The geometry can subsequently be changed until optimum performance is obtained. Relevant changes include the local pitch angle $\beta$, the chord $c$, the airfoil that determines the tables for $c_l$ and $c_d$ and the rotor speed.

Strip theory cannot deal with yawed conditions and wind shear, which often do occur in practice. It is therefore common practice to extend the calculation scheme by dividing the swept area not only in radial, but also in $k$ azimuthal sections. The mass flow will decrease by $1/k$ and the number of blades in an azimuthal section becomes $N/k$. Now the wind speed input can vary with altitude, to represent shear, and the relative direction of motion of the blades and the wind can be accounted for, to represent yaw.
2.3 Tip Correction

The flow through an actuator disk does not depend on azimuth. This disk is a theoretical concept, whereas in practice one has 2 or 3 blades on which the force is exerted. That force will therefore vary with time at any fixed azimuthal position. The smaller the ratio of the tip velocity and the wind velocity, \( \frac{\Omega R}{U} \), and the fewer the blades, the greater becomes the pitch of the tip vortices and thus the variation of the induced velocities with azimuth. A correction for the non-uniform disk loading was proposed by Prandtl in 1919. It will be explained in the first section. The second section deals qualitatively with another tip correction that is required even in the case of the actuator disk.

2.3.1 Prandtl Tip Correction

This correction addresses the azimuthal non-uniformity of the disk loading. A small number of blades covering the entire swept area would not need the correction and an infinite number of blades in only one quarter of the swept area would need it.

Prandtl’s model replaces the helices of trailing tip vortices with a series of parallel disks at a uniform spacing equal to the normal distance between successive tip vortices at the slipstream boundary (see figure 2.7). For the precise formulation reference is made to Glauert [34]. Glauert explains Prandtl’s model as follows: 'In the interior of the slipstream the velocity imparted to the air by the successive sheets of this membrane will have important axial and rotational components but the radial component will be negligibly small. Near the boundary of the slipstream however, the air will tend to flow around the edges of the vortex sheets and will acquire an important radial velocity also.' The method of estimating the effect of this radial flow has been the following. A reduction factor \( f \) must be applied to the momentum equation for the flow at radius \( r \), since it represents the fact that only a fraction \( f \) of the air between the successive vortex disks of the slipstream receives the full effect of the motion of these disks. If the induction factor \( a \) is defined as the value which applies when the blade passes, then the average induction factor will be \( af \). At the locus of the blade the induction is \( aU \), but on average the induction is \( afU \). The momentum balance including the Prandtl tip correction yields the axial force:

\[
D_{ax,r} = \rho A_r (1 - a_r f_r) U \cdot 2a_r f_r U \tag{2.34}
\]
Here the index \( r \) is added to the variables \( D_{ax}, A, a \) and \( f \), in order to denote that they refer to an annulus and not to the entire rotor. The reduction factor \( f \) is found to be:

\[
f_r = \frac{2}{\pi} \arccos \left( e^{-\frac{\pi(R-r)}{d}} \right),
\]

(2.35)

with

\[
\frac{\pi(R-r)}{d} = \frac{(R-r)NW}{2R(1-a)U},
\]

(2.36)

in which \( R \) is the blade radius and \( r \) is the radial position, \( d \) is the spacing between the solid disks, \( N \) is the number of blades and \( W \) is the resultant velocity. It can be seen that \( f_r \), the tip correction, vanishes when \( N \), the number of blades, becomes very large and we approach the theoretical concept of the actuator disk.

### 2.3.2 Tip Correction for an Actuator Disk

In theoretical treatments several definitions of the actuator disk are used. For example, Johnson [43] discusses both uniformly loaded and non-uniformly loaded actuator disks. In the case of the usually applied uniform load distribution on the disk, a pressure singularity exists at the edge of the disk. For an extensive study of this singularity we refer to Van Kuik [45]. Lanchester [46] already opposed this concept: 'At the edge it is manifestly impossible to maintain any finite pressure difference between the front and the rear faces'. One would expect that the gradient from the high pressure side to the low pressure side, would drive the flow around the edges of the disk. This flow around the tip or edge has to exist even for an actuator disk representing a wind turbine with an infinite number of blades. It will equalise the pressure discontinuity so that the loading per unit of surface on the disk decreases to zero when the edge is approached. The decrease of disk loading directly corresponds to a decrease of the extracted power. Therefore, the flow around the edges (see figure 2.8) is parasitic. It can be compared to the loss of lift of an airplane due to the span-wise flow around the tips. This loss of lift was originally called the tip correction for finite airfoils. It is discussed for example in Hoerner [38] where the parasitic flow is accounted for via a reduction of the geometric blade span to an effective blade span. If we compare this situation to the case of a wind turbine, then it is expected that the wake will contract first behind the disk and than will rapidly expand again (see figure 2.7). Such a contraction has also been confirmed by experiments (see [60]).
It should be mentioned that the tip correction for rotors is, in the existing literature, entirely attributed to the effect of the finite number of blades. For example, Johnson [43], Spera [57] and Glauert [34] attribute the tip correction wholly to the effect of a finite number of blades. In their theory the actuator disk has no loss of lift at the edges. A loss of lift at the tips of rotor blades is mentioned by Freris, but not worked out in his formulas for the tip correction [31].

So, two corrections for the tip of wind turbine blades can be distinguished. First that by Prandtl for the azimuthal variation of the induced velocity. Second a correction for the loss of lift and thus a loss of transferred power due to the span-wise/axial flow around the blade tips. The latter correction is also required for an actuator disk. It has been stated that the correction for the aspect ratio includes the tip correction [45], but this is not correct. The aspect ratio corrects for the induced drag, while the flow around the tips means that the geometric aspect ratio should itself be corrected to obtain a smaller effective aspect ratio. For a wind turbine this means that the physical diameter should be corrected to a somewhat smaller effective diameter.
Presently it is still impossible to calculate the lift and drag characteristics of an airfoil accurately. Especially beyond the stall angle, the calculations can be off by tens of percents. For that reason airfoil characteristics still have to be determined in wind tunnels, under the assumption that in practice the airfoil will show the same behaviour as in a wind tunnel. But as shown in the preceding section, induction effects should be taken into account to make the field situation comparable to the wind tunnel situation. Section 2.5 will explain that also rotational effects need to be accounted for. This section deals with two-dimensional characteristics of airfoil sections, which involve angle of attack, lift, drag and stall. Figure 2.9 introduces the chord, the thickness and the camber of a profile; the camber line is the line with equal distance to the lower and the upper sides of the airfoil.

2.4.1 The Angle of Attack

The two-dimensional steady angle of attack is defined as the geometric angle between the undisturbed stream lines and the chord line of the profile (figure 2.10). The lift force is by definition directed perpendicular to the undisturbed inflow direction. Undisturbed flow is defined as the flow without the influence of the profile. In two-dimensional steady flow the changes to the flow field are only induced by bound vorticity. It should be noted that the velocities induced by the bound vorticity are not part of induction velocities in 'Prandtl terms' (see section 2.2). The definition of the two-dimensional steady angle of attack is convenient in practice. In a wind tunnel the angle between the tunnel walls and the chord line almost equals the two-dimensional steady angle of attack (almost, since small corrections are required for the pressure distribution over the tunnel walls).

2.4 Blade Aerodynamics
2.4.2 Lift and Drag

In two-dimensional steady flow the force exerted on an object consists of a component, perpendicular to the undisturbed flow, which is by definition the lift, and a component parallel to that flow which is by definition the drag. In the unsteady three-dimensional situations that occur in practice, these definitions refer to the direction of the sum of the undisturbed flow velocity and the induction velocities in Prandtl's terms. Lift is described in theory as the force exerted by a fluid flow on a bound vortex, given that the fluid flows perpendicular to the vorticity vector. The vorticity is defined as $\omega = \nabla \times \mathbf{v}$. The total vorticity or circulation $\Gamma$ in a surface $S$ is the integral of the local vorticity over $S$:

$$\Gamma = \oint \omega \cdot dS = \int \mathbf{v} \cdot dC$$  \hspace{1cm} (2.37)

The equivalence of the integral over the surface $S$ and the integral along the closed curve $C$ follows from Stokes' theorem. The difference between vorticity and circulation is, that vorticity is a property of an infinitesimal element of fluid, while circulation is an integral property. The physical meaning of circulation becomes clear when a line (hence a 2d-situation) of constant vorticity $\omega$ is considered. If $C$ is a circle of radius $r$ perpendicular to the line of vorticity in the centre, it follows that $v = \Gamma/(2\pi r)$. The lift force per unit length is related to the circulation and the inflow velocity via Joukowski's theorem:

$$L = \rho v \times \Gamma, \hspace{1cm} (2.38)$$

According to Joukowski's hypothesis, the effect of viscosity in the boundary layer is to cause precisely that circulation so that the stagnation point at the rear of the airfoil corresponds to the sharp trailing edge of the airfoil. For a description of this process reference is made to Batchelor [6]. Joukowski's hypothesis implies that the circulation around an airfoil under small inflow angles is almost proportional to this inflow angle. Airfoils have camber since it yields a slightly better performance regarding the lift over drag ratio. The camber also causes the lift curve of the airfoil to shift over a certain angle $\alpha_0$, which is the angle of attack at zero lift (see figure 2.11). Both the lift and drag per unit of span are conventionally given as dimensionless quantities after normalisation by the product of dynamic pressure and the chord $c$ of the airfoil. The lift and drag coefficient are respectively defined by:

$$c_l = \frac{L}{\frac{1}{2} \rho v^2 c} \approx 2\pi(\alpha - \alpha_0), \quad c_d = \frac{D}{\frac{1}{2} \rho v^2 c}, \hspace{1cm} (2.39)$$
where $D$ is the drag force per unit span. Equation 2.39 presents the general expressions for the lift and drag coefficients and the theoretical value for the first of the situations of thin airfoils at small angle of attack. In practice the slope $d\eta/d\alpha$ is approximately 5.7 instead of the theoretical $2\pi$. Typical relations for the lift and drag coefficients as a function of the angle of attack are presented in figure 2.11. In this figure it can be seen that in practice the lift curve deviates a great deal from the theoretical curve beyond approximately $\alpha=10^\circ$. The reason is that the flow on the suction side of the airfoil does not reach the trailing edge any more. At a certain distance from the trailing edge it comes to a standstill, causing reversal of the flow and separation. These effects, which will be explained in the next section, cause a loss of lift and a sudden increase of drag.

### 2.4.3 Stall

One sometimes holds [38] that 'an airfoil is said to stall when the lift decreases with increasing angle of attack'. But at angles beyond the stall angle, the lift first decreases and then increases again and develops a secondary maximum at an angle of attack of approximately $\pi/4$. Moreover under conditions of rotation the airfoil behaviour can change considerably and the $L(\alpha)$ could even become a monotonous rising function up to an angle of attack of approximately $\pi/4$. And for example in [64] stall is again defined to be equal to boundary layer separation. This demonstrates that the term 'stall' is not clearly defined. The phenomena in the flow that cause the loss of lift have a clearer meaning. These phenomena are reversed flow and separation.

#### Separation and Reversed Flow

Separation refers to detachment of the boundary layer from the airfoil. The explanation for a two-dimensional situation is as follows. With increasing angle of attack the circulation increases and the suction peak near the leading edge becomes deeper. This means that the velocity just outside the boundary layer near the suction peak becomes very high. The suction peak is located near the stagnation point where the boundary layer is still very thin. Thus the velocity gradient in the boundary layer, and thereby the viscous shear stress, becomes very high. This viscous shear converts kinetic energy from the boundary layer flow into heat. When the flow has passed the suction peak, four quantities/effects will determine whether it will

![Qualitative representation of separation types.](figure 2.12)
reach the trailing edge. First, the remained speed (kinetic energy); second the trajectory of the viscous shear over the surface; third the trajectory of the adverse pressure gradient which decelerates the flow; and fourth the momentum that will be transferred from the main stream via viscous and turbulent stress in the boundary layer. The integral of the shear stress from the stagnation point to a certain position further downstream determines the kinetic energy losses. With increasing angle of attack the suction peak becomes deeper since the curvature of the flow around the leading edge increases. The deeper this suction the more kinetic energy is lost by shear, and at a certain angle the flow does not reach the trailing edge any more. At a certain position it comes to a standstill (not only on the surface but also above it; mathematically formulated this means that the velocity gradient normal to the wall is zero (see eq. 2.40)) and this position is called the separation line. The occurrence of two separation lines is possible in practice (see figure 2.12). Downstream from the separation line the pressure gradient accelerates the air towards the suction peak and this causes reversed flow.

Three types of separation can be distinguished. Two of them concern 'two-dimensional' flow and the third concerns the rotating case, to be discussed in section 2.5. The 'two-dimensional' types are leading edge separation and trailing edge separation.

**Leading Edge Separation**

This type of separation has two appearances: the long bubble and the short bubble. The long bubble type of leading edge separation and turbulent reattachment downstream, is a laminar separation type and gives a gradual decrease of the lift curve slope. The bubble grows with increasing angle of attack towards the trailing edge. It occurs on thin airfoil sections in combination with low Reynolds numbers $< 5 \cdot 10^5$. At higher Reynolds number this separation has less effect although the physical mechanism remains the same: laminar separation and immediate turbulent reattachment within the first 1% of the chord. Due to the condition of the low Reynolds number this separation type is not expected to be significant on rotors above 5m diameter.

A short bubble type can also be formed on the leading edge, which is quickly reattached. Above a certain angle of attack such bubbles suddenly burst and cause a sudden lift drop and drag increase. This occurs on thin airfoil sections with a round nose and low camber, for example the NACA 63-012 (see figure 2.13). Wind turbine blades in general have camber and are thicker, thus they probably will not suffer from this abrupt type of separation.

**Trailing Edge Separation**

The trailing edge type of separation is a gradual type of separation, which starts at the trailing edge and moves forward with increasing angle of attack. This occurs on thick or cambered airfoils with a round nose, which have a less deep suction peak, so the trailing edge becomes the preferential location for the onset of separation. This stall type is usually observed on
airfoil sections for wind turbine rotors in the wind tunnel. Airfoil sections for wind turbine blades range from 15% thickness to approximately 35% and the Reynolds number is approximately $5 \cdot 10^6$ for a 1MW rotor of 60m diameter.

**Effective Airfoil Shape**

The effective shape of an airfoil is the geometrical shape to which the displacement thickness has been added. This displacement thickness accounts for viscosity effects when the flow around the airfoil is described by Euler’s potential theory. Important deviations between the lift characteristics of an airfoil and those predicted by potential theory occur when the displacement thickness becomes large, which corresponds to the occurrence of separation. As reversed flow is always the consequence of separation, the occurrence of reversed flow can be used to measure the onset of significant deviation of the lift curve slope. Thus the initial occurrence of reversed flow denotes the onset of large deviations from the theoretical lift $2\pi(\alpha-\alpha_0)$. With the stall flag technique we can observe such a beginning of trailing edge separation.

**Turbulent Separation**

The boundary layer near the location of separation is often thought to behave as indicated in figure 2.14 [51]. Here, one streamline intersects the wall at the point of separation $s$. The location of $s$ is determined by the condition that the velocity gradient normal to the wall vanishes there:

$$\left(\frac{\partial v}{\partial z}\right)_{wall} = 0,$$

in which $z$ is the distance to the wall. This way of seeing things differs from the description given by Betz 1935 in [8]: ‘Very often another phenomenon can be observed in the period of transition between normal and disturbed (separated) flow, the two states of affairs continually interchanging.’ Recently Simpson 1996 [55] came up with a more realistic model of turbulent separation. He argued that the criterion of the vanishing velocity gradient is too narrow for separation and that separation begins intermittently at a given location. The flow reversal occurs only a fraction of the time. At progressively downstream locations, the fraction of the time that the flow moves downstream is progressively less. Quantitative definitions were proposed on the basis of the fraction of the time that the flow moves downstream. Incipient detachment (ID) is defined as 1% of the time reversed flow, intermittent transitory detachment (ITD) as 20% of the time reversed flow and transitory detachment (TD) corresponds to 50% of the time reversed flow. ID corresponds to the practical situation in which flow markers such as tufts move occasionally in the reversed direction. The Simpson model agrees with stall flag observations described in this thesis (section 3.2.5) and is also confirmed by recent PIV observations in the wind tunnel [40].

![Figure 2.14](image-url)
Flow Separation on Wind Turbine Blades

2.5 Rotational Effects

This section deals with the effects of blade rotation on the aerodynamics.

2.5.1 Fundamental Equations in a Rotating Frame of Reference

When it is assumed that the flow about wind turbine blades is incompressible and that the viscous stress is linearly proportional to the velocity gradients, which are both generally accepted assumptions, the fundamental continuity equation and the Navier-Stokes equation for the velocity \( v \) read:

\[
\nabla \cdot v = 0, \tag{2.41}
\]

\[
\frac{D v}{D t} = F - \frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla^2 v. \tag{2.42}
\]

Here, \( F \) is the external force per unit mass, and \( \mu \) is the dynamic viscosity, whereas \( p \) and \( \rho \) are as usual the pressure and the mass density of air.

To apply these equations to the situation of a rotating wind turbine blade, we will write them in cylinder co-ordinates. For the continuity equation this yields:

\[
\frac{\partial v_\theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( rv_\theta \right) + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0, \tag{2.43}
\]

and for the equations of motion in the direction of azimuth \( \theta \), radius \( r \) and axis \( z \) (figure 2.15):

\[
\frac{\partial v_\theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( rv_\theta \right) + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_z}{\partial z} = F_\theta - \frac{\partial p}{\rho \partial \theta} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial^2 v_\theta}{\partial r \partial \theta} + \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} \right), \tag{2.44}
\]

\[
\frac{\partial v_r}{\partial t} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( rv_\theta \right) - \frac{v_\theta^2}{r} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = F_r - \frac{\partial p}{\rho \partial r} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial r \partial \theta} + \frac{\partial^2 v_r}{\partial r^2 \partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} \right), \tag{2.45}
\]

\[
\frac{\partial v_z}{\partial t} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( rv_\theta \right) + \frac{v_\theta}{r} \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial z} = F_z - \frac{\partial p}{\rho \partial z} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{\partial^2 v_z}{\partial r \partial \theta} + \frac{\partial^2 v_z}{\partial r^2 \partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right). \tag{2.46}
\]
Forces in the Rotating Frame of Reference

For an incompressible fluid with only one phase, the external forces in the inertial (non-rotating) reference system are usually zero. In practice the only external force is gravitational, but that force is balanced by the hydrostatic pressure gradient, so that both are left out of the equations. In a rotating frame the centrifugal and Coriolis forces appear. An observer on the blade notices radial and azimuthal accelerations on passing air elements \(d\tau\). Therefore the centrifugal and Coriolis forces are real forces in the rotating frame of reference. If the angular velocity of the frame of reference is \(\Omega\) then the centrifugal force equals \(\rho d\tau \Omega^2 r\). When the particle is moving in the rotating reference system with velocity vector \(v\), then the Coriolis force equals \(2\rho d\tau \Omega v\). The vector \(\Omega\) only has a \(z\)-component, and thus the Coriolis accelerations are: \(2v_\theta \Omega_\theta - 2v_\rho \Omega r\), in which \(\theta\) and \(r\) are the unit vectors in the \(\theta\) and \(r\)-direction respectively. They act on the mass element in addition to other inertial forces, which, however can be left out, as explained above. So, the Coriolis force acts in the \(\theta\)-direction and \(r\)-direction, and thus the first term on the right-hand side of equation 2.44 can be replaced by \(2\Omega v\). As the centrifugal force works in the \(r\)-direction, the first term on the right-hand side of equation 2.45 can be replaced by a centrifugal contribution \(r\Omega^2\) and a Coriolis contribution \(-2v_\theta \Omega\). In the above, it is assumed that the wing rotates in the \(r,\theta\)-plane given by \(z=0\). But in practice the rotor blades have a small cone angle and therefore the tip rotates at a slightly negative value of \(z\). The centrifugal and Coriolis force are thus assumed to work in the plane of the boundary layer. In short, the relevant external forces per unit of mass are:

\[
F_\rho = 2\Omega v_\rho , \quad F_r = r\Omega^2 - 2\Omega v_\rho \quad \text{and} \quad F_z = 0 .
\]  

(2.47)

2.5.2 Boundary Layer Assumptions

In the flow about rotating wind turbine blades the rate of downstream convection (in the \(\theta\)-direction) is much larger than the rate of transverse viscous diffusion, which means that viscosity only plays a significant role in a thin so-called boundary layer around the object. This insight will be used to estimate the order of magnitude of terms in equations 2.43-2.46. Terms of small order will then be neglected. The thickness of the boundary layer can be estimated as follows. At the wall the velocity is 0 and at a certain distance, say \(\delta\), perpendicular to the wall the flow velocity will be \(v_\theta\). The velocity gradient perpendicular to the wall is therefore approximately \(v_\theta/\delta\) and the shear stress \(\tau = -\mu v_\theta/\delta\). The derivative of this stress \(\partial\tau/\partial y\) equals the convective deceleration of the flow \(\rho v_\theta r(\partial v_\theta/\partial \theta)\), where \(\partial v_\theta/\partial \theta = v_\theta(c/r)\) and \(c\) is the chord of the airfoil. Thus \(\partial\tau/\partial y = -\mu v_\theta^2 \delta^2 \approx \rho v_\theta^2/c\), or \(\delta \approx \sqrt{\mu c/\rho v_\theta}\), which is very small since \(\mu_{\text{air}} \approx 17.1 \cdot 10^{-6} \text{ Pa s}\).

It follows that the shear layer of thickness \(\delta\) is small compared to the chord \(c=r\theta\). The \(z\)-direction is perpendicular to the boundary layer where most velocity changes take place. The velocity derivatives in the \(z\)-direction are therefore relatively large: \(\partial v_\theta/\partial z\) is of the order \(v_\theta/\delta\). Outside the boundary layer the second derivative of \(v_\theta\) in the \(z\)-direction is zero. Thus inside the boundary layer the second derivative equals the change of the first derivative, which was of the order \(v_\theta/\delta\). Therefore the second derivative \(\partial^2 v_\theta/\partial z^2\) is of the order \(v_\theta/\delta^2\). These results will be used to find the significant terms which yield the boundary layer equations.
2.5.3 Attached Flow on a Rotating Blade

For a wind turbine blade with attached flow, a typical value for the ratio of the tip speed $\Omega R$ and the axial wind speed $V$, $\lambda = \Omega R/V$, is approximately 7. That means that the inflow speed is close to the speed of the blade element itself being given by the radial position times the angular speed. This is true for radial positions $r > R/\lambda$, thus for approximately $0.3R$ and larger. In this range the pressure distribution on the blade is roughly proportional to $\frac{1}{2} \rho v_c^2$, which is approximately $\frac{1}{2} \rho \Omega^2 r^2$. The radial pressure gradient will therefore be approximately $\rho \Omega^2 r$ and due to this pressure gradient an element of air in the boundary will be accelerated in the radial direction with an acceleration of approximately $\Omega^2 r$. The given element will remain approximately $c/v_\theta = c/(\Omega r)$ in the boundary layer and thus will develop a radial speed $v_r$ of approximately $\Omega^2 r c/(\Omega r) = \Omega c$. Thus the order of magnitude of $v_r$ is $\Omega c$ and, in a similar way, $\partial v_r/\partial z$ and $\partial^2 v_r/\partial z^2$ are found to be of the order $\Omega c/\delta$ and $\Omega c/\delta^2$ respectively.

By substitution of $v_\theta$ and $v_r$ in the continuity equation and assuming $r >> c$ it follows that $v_z$ is approximately $\Omega r \delta c$, because it should balance the largest term which is $\partial v_\theta/(r \partial \theta)$. The table below lists all estimates:

<table>
<thead>
<tr>
<th>parameter estimate</th>
<th>parameter estimate</th>
<th>parameter estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$ $\sqrt{\mu c / \rho v_\theta}$</td>
<td>$p$ $\frac{1}{2} \rho \Omega^2 r^2$</td>
<td>$\partial p/\partial r$ $\rho \Omega^2 r^2$</td>
</tr>
<tr>
<td>$v_\theta$ $\Omega r$</td>
<td>$\partial v_\theta/\partial z$ $\Omega r/\delta$</td>
<td>$\partial^2 v_\theta/\partial z^2$ $\Omega r/\delta^2$</td>
</tr>
<tr>
<td>$v_r$ $\Omega c$</td>
<td>$\partial v_r/\partial z$ $\Omega c/\delta$</td>
<td>$\partial^2 v_r/\partial z^2$ $\Omega c/\delta^2$</td>
</tr>
<tr>
<td>$v_z$ $\Omega r \delta c$</td>
<td>$\partial v_z/\partial z$ $\Omega r/c$</td>
<td></td>
</tr>
<tr>
<td>$\Delta \theta$ $c/r$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**table 1** Parameters and estimated orders of magnitude.

Now the Navier-Stokes equations can be written in terms of estimates instead of derivatives and unspecified forces. We will do so by giving the order of magnitude under each term. The order of magnitude of the pressure terms follows from the equations and is therefore set by the other terms. For the equation of continuity and those of $\theta$, $r$ and $z$-motion respectively, it follows that:

$$\frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{r} + \frac{v_r}{r} = 0$$

$$\begin{pmatrix} \frac{\Omega r}{c} \\ \frac{\Omega r}{c} \end{pmatrix} \begin{pmatrix} \frac{\Omega c}{r} \\ \frac{\Omega c}{r} \end{pmatrix}.$$  \hspace{1cm} (2.48)

$$\frac{v_r \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{r \partial \theta} + \frac{v_r \partial v_\theta}{r} + \frac{v_z \partial v_r}{\partial z} = 2v_r \Omega - \frac{\partial p}{\rho r \partial \theta} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial v_\theta}{r \partial r} + \frac{\partial^2 v_\theta}{r^2 \partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} \right)$$

$$\begin{pmatrix} \frac{\Omega^2 r^2}{c} \\ \frac{\Omega^2 r^2}{c} \end{pmatrix} \begin{pmatrix} \frac{\Omega^2 c}{r} \\ \frac{\Omega^2 c}{r} \end{pmatrix}.$$  \hspace{1cm} (2.49)

The last term in the last equation is much larger than the three preceding ones, since $\delta << c, r$, so that it is the only one of the viscous terms that needs to be retained. Further,
\[
\frac{v_r \partial v_r}{\partial r} + \frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_r \frac{\partial v_r}{\partial z} = r\Omega^2 - 2v_\theta \omega - \frac{\partial p}{\partial r} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{r^2 \partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} \right)
\]

(2.50)

\[
\left( \frac{\Omega^2}{r^2} \right) \left( \Omega^2 r \right) \left( \Omega^2 r \right) \left( \Omega^2 r \right) \left( \text{set} \right) \frac{\mu}{\rho} \left( \frac{\Omega c}{r^2} \right) \left( \frac{\Omega c}{r^2} \right) \left( \frac{\Omega c}{r^2} \right)
\]

Again the first three viscous terms are much smaller than the fourth term. Finally,

\[
\frac{v_r \partial v_r}{\partial r} + \frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{\partial r} = -\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{r^2 \partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} \right)
\]

(2.51)

\[
\left( \frac{\delta \Omega^2 r}{c^2} \right) \left( \delta \Omega^2 r \right) \left( \frac{\delta \Omega^2 r}{c^2} \right) \left( \text{set} \right) \frac{\mu}{\rho} \left( \frac{\Omega c}{rc^2} \right) \left( \frac{\Omega c}{rc^2} \right) \left( \frac{\Omega c}{rc^2} \right)
\]

Again the first three of the four viscous terms in the \( z \)-momentum equation can be neglected. In comparison with equations 2.49 and 2.50 all terms on the left hand side and the remaining viscous term are smaller by a factor of \( \delta c \). So, the entire \( z \)-momentum equation can be neglected with respect to the other equations and we obtain the steady boundary layer equations:

\[
\frac{\partial v_\theta}{\partial r} + \frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{\partial r} = 0
\]

(2.52)

\[
\left( \frac{\Omega r}{c} \right) \left( \frac{\Omega r}{c} \right) \left( \frac{\Omega c}{r} \right) \left( \frac{\Omega c}{r} \right)
\]

\[
\frac{v_r \partial v_r}{\partial r} + \frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{\partial r} = 2v_\theta \omega - \frac{\partial p}{\partial \theta} + \frac{\mu}{\rho} \frac{\partial^2 v_\theta}{\partial z^2}
\]

(2.53)

\[
\left( \frac{\Omega^2}{c^2} \right) \left( \frac{\Omega^2}{c^2} \right) \left( \frac{\Omega^2}{c^2} \right) \left( \text{set} \right) \frac{\mu}{\rho} \left( \frac{\Omega c}{\delta^2} \right)
\]

\[
\frac{v_r \partial v_r}{\partial r} + \frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_r}{\partial r} = r\Omega^2 - 2v_\theta \omega - \frac{\partial p}{\partial r} + \frac{\mu}{\rho} \frac{\partial^2 v_r}{\partial z^2}
\]

(2.54)

\[
\left( \frac{\Omega^2}{r^2} \right) \left( \Omega^2 r \right) \left( \Omega^2 r \right) \left( \Omega^2 r \right) \left( \text{set} \right) \frac{\mu}{\rho} \left( \frac{\Omega c}{\delta^2} \right)
\]

The Analysis of Fogarty

Fogarty [30] further reduced this set of equations for the case of a rotating boundary layer. He argued that several terms are approximately \((r/c)^2\) larger than other terms. At the root of wind turbine blades where \( r = c \), all terms have the same order of magnitude, however at larger radial position, where \( r > c \), even more terms can be neglected. It should be noted that the omission of these terms reduces the problem to a two-dimensional situation described by the continuity
Flow Separation on Wind Turbine Blades

and the $\theta$-momentum equations. The problem described by the two equations and the boundary conditions can be solved by any two-dimensional laminar boundary layer algorithm. The conclusions are that rotation does not influence attached flow and that the location of separation is not affected either. For that situation Fogarty described attached flow on rotating blades with only:

$$\frac{\partial v_{\theta}}{r \partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$  \hspace{1cm} (2.55)$$

however, Fogarty noted that the small effects of rotation, predicted by the simple equations, were contrary to experience. He speculated that the engineer's observations concerned separated flow, that the effects might be larger on profiles with strong pressure gradients, that blade rotation might have more influence close to the tip and that rotational effects on a turbulent boundary layer might be more profound.

The Analysis of Banks and Gadd

In 1963 Banks and Gadd [5] found that rotation has a delaying effect on laminar separation. They assumed that the chord-wise velocity decreased linearly (by a factor $k$) from the leading edge according to $v_{\theta} = \Omega r (1-k\theta)$. If the decrease of this velocity was very large ($k \rightarrow \infty$), the rotation did not have an appreciable effect on separation. However, when the decrease was small ($k < 0.7$) separation was postponed more than 10%, with the result that the pressure rise between the leading edge and the separation line was increased. Below a certain critical value for $k$ ($k \approx 0.55$) separation would never occur.

We do not think that this delay is an important phenomenon. To estimate the delay of separation we assume that velocity decreases linearly over the chord length. It then follows that $k = r/c$, which for wind turbine blades has a minimum value between approximately 2 and 4 at the maximum chord position. Since this is much larger than the above 0.7 or 0.55, the delay will be negligible in practice.

2.5.4 Rotational Effects on Flow Separation; Snel's Analysis

In discussing the case of separated flow, Snel is implicitly using the following line of argument to find his model for separated flow on rotating blades [52, 53, personal communication]. His arguments refer to the boundary layer equations 2.52, 2.53 and 2.54.

A1 The fluid in the boundary layer is moving with the blade, thus $v_{\theta} \ll \Omega r$.

This means for the equation of motion in the $\theta$-direction that:

A2 The Coriolis term $2v_r \Omega$ is larger than the co-ordinate curvature term $v_r v_{\theta}/r$.  

A3 The viscous stress and the pressure gradient are small compared to the case of attached flow. The Coriolis term is dominant and should be balanced by the convective terms on the left-hand side.

And for the $r$-direction:

A4 The centrifugal force $\Omega^2 r$ is dominant along with the radial pressure gradient, since the pressure is of the order $\Omega^2 r^2$.

A5 The convective term with $v_r$ again is smaller than the other two convective terms; from the continuity equation it follows that the remaining terms with $v_z$ and $v_\theta$ are of the same order.

A6 The convective terms should balance the terms on the right-hand side, thus $(v_\theta/r)(\partial v_r/\partial \theta) \approx \Omega^2 r$ and it follows that $v_r \approx \Omega^2 r c/v_\theta$.

Returning to the $\theta$-direction Snel argues that:

A7 The convective term $v_r \partial v_\theta/\partial r$ should be much smaller than the Coriolis term.

A8 The remaining convective terms are of the same order. This is implied by the continuity equation. Since, if the terms with $v_r$ are smaller than the other terms, these other terms yield $v_\theta/c = v_\theta/\delta$; and by substituting $v_r = v_\theta \delta/c$ in the convective term with $v_z$ of equation 2.53 we see that the statement holds.

A9 It follows that $v_\theta \partial v_\theta/\partial r \approx \Omega^2 r c/v_\theta$ and $v_r^2$ is of the order $\Omega^2 c v_r$.

And for the $r$-direction Snel finds finally:

A10 By substituting the relation found in A9 in the $r$-momentum equation, it follows that $v_\theta \approx \Omega c^{2/3} r^{1/3}$ and $v_r \approx \Omega c^{1/3} r^{2/3}$.

A11 $v_r/\rho v_\theta \approx (r/c)^{1/3}$, which agrees with A7.

A12 If $c = r$, then it follows that $v_\theta = v_\theta$; when $v_r \approx \Omega c^{1/3} r^{2/3} \approx \Omega r$ thus $v_\theta = \Omega r$ which is in contradiction with the primary assumption; so the approximations are only valid for $r/c >> 1$.

A13 $v_z \approx \delta \Omega c^{1/3} r^{-1/3}$; this follows from the substitution of $v_r$ for $v_\theta$ in the continuity equation, which yields $\Omega c^{2/3} r^{-2/3} + \Omega r^{-1/3} c^{1/3} + v_\theta/\delta = 0$.

<table>
<thead>
<tr>
<th>parameter</th>
<th>estimate</th>
<th>parameter</th>
<th>estimate</th>
<th>parameter</th>
<th>estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_\theta$</td>
<td>$\Omega c^{2/3} r^{1/3}$</td>
<td>$p$</td>
<td>$\frac{1}{2} \rho \Omega^2 r^2$</td>
<td>$\partial p/\partial r$</td>
<td>$\rho \Omega^2 r$</td>
</tr>
<tr>
<td>$v_r$</td>
<td>$\Omega c^{1/3} r^{2/3}$</td>
<td>$\partial v_\theta/\partial z$</td>
<td>$v_\theta/\delta$</td>
<td>$\partial^2 v_\theta/\partial z^2$</td>
<td>$v_\theta \delta^2$</td>
</tr>
<tr>
<td>$v_z$</td>
<td>$\delta \Omega c^{1/3} r^{1/3}$</td>
<td>$\partial v_r/\partial z$</td>
<td>$v_r/\delta$</td>
<td>$\partial^2 v_r/\partial z^2$</td>
<td>$v_r \delta^2$</td>
</tr>
<tr>
<td>$v_\theta$</td>
<td>$c/r$</td>
<td>$\partial v_\theta/\partial z$</td>
<td>$v_\theta/\delta$</td>
<td>$\partial^2 v_\theta/\partial z^2$</td>
<td>$v_\theta \delta^2$</td>
</tr>
</tbody>
</table>

**table 2** Parameters and orders of magnitude for separated flow according to Snel.

The estimated parameters are listed in table 2. They are used in equations 2.57, 2.58 and 2.59 for the boundary layer to find the order of magnitude for each term.

$$
\frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_z}{\partial z} + \frac{\partial v_r}{r \partial r} + \frac{v_z}{r} = 0
$$

(2.57)
Flow Separation on Wind Turbine Blades

\[
\frac{v_r \partial v_\theta}{\partial r} + \frac{v_\theta \partial v_\theta}{r \partial \theta} + \frac{v_r \partial v_\theta}{r} + \frac{v_r \partial v_\theta}{\partial z} = 2v_r \Omega - \frac{\partial p}{\rho r \partial \theta} + \frac{\mu}{\rho} \frac{\partial^2 v_\theta}{\partial z^2} \quad (2.58)
\]

\[
(\Omega^2 c) (\Omega^2 c^{1/3} r^{2/3}) (\Omega^2 c) (\Omega^2 c^{1/3} r^{2/3}) (\Omega^2 c^{1/3} r^{2/3}) (\Omega^2 c^{1/3} r^{2/3}) \quad \text{(set)} \quad \frac{\mu}{\rho} (\Omega c^{2/3} r^{1/3})
\]

\[
\frac{v_r \partial v_r}{\partial r} + \frac{v_\theta \partial v_r}{r \partial \theta} - \frac{v_r^2}{r} + \frac{v_r \partial v_r}{\partial z} = r \Omega^2 - \frac{\partial p}{\rho \partial r} + \frac{\mu}{\rho} \frac{\partial^2 v_r}{\partial z^2} \quad (2.59)
\]

\[
(\Omega^2 c^{2/3} r^{1/3}) (\Omega^2 r) (\Omega^2 c^{4/3} r^{-1/3}) (\Omega^2 r) (\Omega^2 r) (\Omega^2 r) \left( \frac{\mu \Omega^2 c^{2/3} r^{4/3}}{\rho \delta^2} \right)
\]

All terms of order \((c/r)^{2/3}\) compared to the other terms become small when \(r \gg c\). When they are neglected we get:

\[
\frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_z}{\partial z} = 0, \quad (2.60)
\]

\[
\frac{v_\theta \partial v_\theta}{\partial r} + \frac{v_r \partial v_\theta}{r \partial \theta} + \frac{v_r \partial v_\theta}{\partial z} = 2v_r \Omega - \frac{\partial p}{\rho r \partial \theta} + \frac{\mu}{\rho} \frac{\partial^2 v_\theta}{\partial z^2}, \quad (2.61)
\]

\[
\frac{v_\theta \partial v_r}{r \partial \theta} + \frac{v_r \partial v_r}{\partial z} = r \Omega^2 - \frac{\partial p}{\rho \partial r} + \frac{\mu}{\rho} \frac{\partial^2 v_r}{\partial z^2}. \quad (2.62)
\]

The difference between this set of equations for separated flow and the set obtained by Fogarty for attached flow resides in the Coriolis term in equation (2.61). This term acts as a pressure gradient directing the flow towards the trailing edge. It should be noted that the neglect of terms in the case of detached flow is less justified than in the case of attached flow, since \((c/r)^{2/3}\) decreases more slowly than \((c/r)^2\). Snel concludes that the parameters describing the difference between rotating and translating airfoils are \(\lambda r/R\) and \(c/r\) and that in the case of attached flow changes are expected for \(r \approx c\), and that in the case of stalled flow at larger radii also.

The equations derived by Snel have been implemented in a program that solves the two-dimensional boundary layer equations. The calculated results have been compared with experimental data and both sets have been approximated by a single engineering result for the relation between the three-dimensional lift coefficient \(c_{l,3d}\) and the two-dimensional lift coefficient \(c_{l,2d}\):

\[
c_{l,3d} = c_{l,2d} + a \left( \frac{c}{r} \right)^b (2\pi (\alpha - \alpha_0) - c_{l,2d}), \quad \text{with} \quad a = 3.1 \quad \text{and} \quad b = 2, \quad (2.63)
\]

Here \(a\) and \(b\) are fitted parameters, \(\alpha\) is the angle of attack and \(\alpha_0\) is the zero lift angle.
Questions Concerning Snel’s Model

In the next section we will present an alternative model and therefore we give here our motivations for our doubts about Snel’s model. We have in fact 4 questions:

1. Can boundary layer theory be used in separated flow?
2. Is the order of partial velocity differentials such as $\frac{\partial v_r}{\partial \theta}$ equal to $O(v_r/c)$?
3. What is the argument for neglecting the radial convective acceleration in the $r$-equation?
4. Is the model consistent?

**ad. 1** To answer the first questions we will follow the example in Schlichting [51], page 24-26. Here the flat plate in parallel flow at zero incidence is analysed. The length of the plate is $L$, the undisturbed speed is $U$, (in the $x$-direction) the boundary layer thickness $\delta$. In the boundary layer, *which has not separated*, the main physical argument is that the frictional forces are comparable to the inertia forces. The velocity gradient in the flow direction $\frac{\partial u}{\partial x}$ is proportional to $U/L$, hence the inertia force $\rho u \frac{\partial u}{\partial x}$ is of the order $\rho U^2/L$. The velocity gradient perpendicular to the wall is of the order $U/\delta$, so that the frictional force $\mu \frac{\partial^2 u}{\partial x^2}$ is proportional to $\mu U/\delta^2$. Since friction and inertia forces are comparable we get: $\mu U/\delta^2 \approx \rho U^2/L$, the separated area.

Now we ask what happens when the flow over the plate separates due to a positive pressure gradient. We return to our co-ordinates system where $r\theta$ compares to the $x$-direction and $v_\theta$ can be compared with $U$. Then the velocity has decreased due to friction and due to the pressure gradient until it comes to a standstill at the separation line. Beyond this point the pressure gradient drives the air backwards. Therefore at the separation line the flow must move away normal to the wall and separates. By definition the speed in the $\theta$-direction has become 0 here and the velocity gradient perpendicular to the wall is 0 too. Therefore, when approaching the stagnation line and beyond it in the separated area, the boundary layer assumptions no longer apply. Id est: the inertia force $\rho v_\theta \frac{\partial v_\theta}{\partial (r\theta)} (\rho u \frac{\partial u}{\partial x}$ for the flat plate) can no longer be estimated using $\rho v_\theta^2/c$ ($\rho U^2/L$ for the flat plate) and, since the velocity gradient is small the frictional forces become negligible. So we showed that boundary layer theory is invalid, both in separated flow and when approaching separation. This is also mentioned in literature on separated flow [41, 64].

**ad. 2** In figure 2.16 we plotted the chord-wise and radial velocity over a separated airfoil. In boundary layer theory one may estimate the order of magnitude of the chord-wise velocity gradient from $\frac{\partial v_\theta}{\partial \theta} = O(v_\theta/c)$, but

**figure 2.16** In separated flow, estimation of the order of magnitude such as used in boundary layer theory is invalid.
only in attached flow. In separated flow this may give errors. For example the order of the partial velocity gradient $\partial v_r/(r \partial \theta)$ cannot be estimated by $O(v_r/c)$, with the argument that the radial speed is approximately $v_r$ inside the separated area and approximately 0 outside the separated area. Snel’s analysis uses such estimates for the second and fourth term of equation 2.59. In fact, when the partial derivative is estimated inside the dead-water region, we find $\partial v_r/(r \partial \theta) \approx 0$.

One might come up with the statement that the order of the partial derivatives in the dead-water region is still equal to that in the attached region, although the magnitudes can be neglected. The order of a term is not only decisive for its magnitude, the coefficient is also important. Snel’s analysis is based on orders only.

**ad. 3** In Snel’s model the convective term with $v_r$ is smaller than the other terms (A5), but the argument for this is lacking. In our model this term appears to be dominant.

**ad. 4** In Snel’s model $\theta$-equation leads to $v_\theta^2 = O(\Omega c v_r)$ (A9). Then from the $r$-equation he has no reason (see ad. 3) to assume that any acceleration term is off smaller order than $O(\Omega^2 r)$ so that $v_\theta \partial v_r/\partial r = O(\Omega^2 r)$ and thus $v_r = O(\Omega r)$. If we substitute this in the term $v_\theta \partial v_r/(r \partial \theta)$ of the $\theta$-equation, it follows that $v_\theta = O(\Omega c)$. However, the group of estimates obtained, $v_r = O(\Omega r)$, $v_\theta = O(\Omega c)$ and $v_\theta^2 = O(\Omega c v_r)$, is inconsistent.

### 2.5.5 Rotational Effects on Flow Separation: Our Analysis

Snel’s model gave the first estimate of three-dimensional effects in stall, which have been valuable understanding rotor behaviour. The reason for an alternative model was to include the often observed and intuitively expected radial flow, which is not dominant Snel’s model. Our model is valid in the separated flow and shows that the separated air flows in a radial stream with $v_r$ as the dominant velocity. Furthermore, the new model is not based on the boundary layer theory: we use the full set of equations and do not use the property of boundary layers in which partial velocity gradients can be estimated with the ratio of differences, such as $\partial v_r/(r \partial \theta) = (v_r/c)$, which is invalid in separated flow. The model describes the separated flow on rotating blades without any effect of viscosity, which seems to be a paradox. However by studying the physics of flow separation this becomes clear. Separation occurs because the air is coming to a standstill in the main flow direction due to friction and the positive pressure gradient. Then in 2d-flow, beyond the point of separation, a dead-water region is formed. Here the frictional forces are negligible and the accelerations and the pressure gradients are small, which is illustrated by figure 3.20. Some back-flow will cause the flow to move normal to the wall at the separation line. In 3d-flow however, we have still the situation that the flow comes to a standstill in the chord-wise direction and separates due to the back-flow at a slightly larger chord-wise position. As in 2d-flow, near and beyond the stagnation line, the gradient of the chord-wise velocity normal to the wall is very small or even zero, so viscous effects and chord-wise accelerations are negligible, otherwise separation would not have occurred. This means that the pressure gradient and the Coriolis force must balance in the 3d-separated area. The difference with the 2d-situation is that a radial pressure gradient and a radial external force are also present and accelerate the separated flow in the radial direction.
Heuristics of Flow Separation about Rotating Blades

B1 Due to viscous drag and the positive pressure gradient some air in the boundary layer will be decelerated in the chord-wise direction so that it becomes detached.

B2 In the separated area the ‘boundary layer is thick’, so that the velocity gradients are small. In this case the role of viscosity becomes less important and the equations become of the Euler type.

B3 When the flow has separated, it has come to a standstill in the chord-wise direction, and the chord-wise velocity and velocity gradient normal to the wall are small. So the chord-wise acceleration and the frictional forces are small too. Since other large chord-wise forces exist, namely the chord-wise pressure gradient and the Coriolis forces, they must be balancing.

B4 In the absence of a thin layer with large velocity gradients, partial derivatives in the $z$-direction will not be much different from those in the other directions.

B5 The centrifugal acceleration and the radial pressure gradient drive the separated air in the radial direction. The first acceleration is $\Omega^2 r$ and the second depends on the span-wise variation of the angle of attack and of $\lambda$, but will be of the same order.

B6 Separated air moving over the blade in the radial direction can enter attached flow at a larger radial position and thereby advances stall to a certain extent, but eventually it will leave the blade in the $\theta$-direction (figure 2.17).

B7 The above radial flow experiences three chord-wise forces: the Coriolis force acting towards the trailing edge, the chord-wise pressure gradient acting towards the leading edge and a turbulent mixing stress as a result of the interaction of the chord-wise flow above the boundary layer with the radial flow. The latter effect forces the flow towards the trailing edge.

B8 The turbulent mixing shear on the upper side of the separated flow is comparable to that of the two-dimensional case. And in that case it is negligible since the pressure distribution is flat in the separated area (see for example figure 3.20).

B9 The remaining counteracting chord-wise forces (Coriolis and pressure gradient) are stabilising pure radial outflow, otherwise the flow could not have separated (B3).

B10 The chord-wise Coriolis acceleration is constant over the chord, which means that the chord-wise pressure gradient should be constant. This predicts a triangular shape for the pressure distribution in the separated area, as observed in experiments [9].

B11 The radial flow of separated air is fed at the blade root but also from both the leading edge and trailing edge. These sources are hard to quantify but their effect will be large. For this reason we cannot neglect any term in the continuity equation.

B12 In this model the $z$-direction is only relevant for the control of the chord-wise pressure gradient via the displacement thickness and thus $v_z$ remains small.

Mathematical Description of the Model

The stream of separated air is of the order of the chord in the $\theta$- and $z$-direction and as long as the blade in the radial direction. Its flux can therefore not be described by the boundary layer equations. We have to start with the complete set of fundamental equations 2.43 to 2.46. B2 suggests that terms with viscosity can be neglected. B3 implies that the chord-wise velocity can be neglected in the separated area, which is reason to neglect all terms with $v_\theta$ except that in the continuity equation. B4 is reason to neglect the remaining $z$-derivatives in equations 2.45 and 2.46. We further assume that the flow is steady and that the external mass forces given by equation 2.47 are relevant. Using these approximations it follows that in equation 2.46, the term $v_z \partial v_z / \partial r$ is the only convective term left and the entire equations is of smaller
order than the continuity equation and equations 2.44 and 2.45. The latter two remain in a much reduced form:

\[
\frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0, 
\]

(2.64)

\[
\frac{\partial p}{\rho \partial \theta} = 2v_r \Omega. 
\]

(2.65)

It follows from equation 2.65 that the chord-wise pressure gradient is a constant in the chord-wise direction if the radial velocity is constant in this area. This explains the often observed triangularly shaped pressure distributions.

\[
v_r \frac{\partial v_r}{\partial r} = r \Omega^2 - \frac{\partial p}{\rho \partial r} = r \Omega^2 (1 - c_p - r \frac{\partial c_p}{2 \partial r})
\]

(2.66)

It should be noted that equation 2.66 for the motion in the \( r \)-direction retains precisely the term \( v_r \frac{\partial v_r}{\partial r} \) that was neglected in Snel’s model (see equation 2.62). Moreover Snel's continuity equation does not contain the terms with \( v_r \), which are thought to be relevant in the present model.

This extremely simple model is useful for identifying the leading terms. To obtain a first estimate, \( p \) was substituted by \( \frac{1}{2} \rho \Omega^2 r^2 \) in equation 2.66. Coefficient \( c_p \) will vary from almost 0 at the trailing edge to a value if \( c_{p,sep} \) at the separation line. (\( c_{p,sep} = -3 \) estimated from pressure distributions in reference [3]). We assume that \( \frac{\partial c_p}{\partial r} = 0 \) and that separation is initiated at the trailing edge and find that:

\[
v_r = r \Omega \sqrt{1 - c_p} \Rightarrow \Omega r < v_r < 2 \Omega r,
\]

(2.67)

so \( v_r \) is approximately \( \Omega r \) at the trailing edge to approximately \( 2 \Omega r \) at the separation line. This can be substituted in the equation for the \( \theta \)-direction in which \( p \) is also substituted by \( \frac{1}{2} \rho \Omega^2 r^2 \). If the air is separated over a fraction \( f \) of the chord, the chord-wise pressure gradient and the increase of the pressure coefficient at the stagnation line are restricted by:

\[
\frac{4}{r} < \frac{\partial c_p}{r \partial \theta} < \frac{8}{r} \quad \text{or} \quad -\frac{4 f c}{r} > \Delta c_p > -\frac{8 f c}{r}. 
\]

(2.68)

This equation describes the decrease of the pressure from the trailing edge towards the separation line. If we assume that the pressure coefficient is 0 at the trailing edge, it equals approximately \( \Delta c_p \) at the separation line. In case of two-dimensional stall the pressure coefficient remains almost constant between the trailing edge and the stagnation line. Thus due to rotation the pressure coefficient in the separated area is on the average \( \Delta c_p / 2 \) higher and the lift coefficient of the section increases by:

\[
\frac{4}{r} < \frac{\partial c_p}{r \partial \theta} < \frac{8}{r} \quad \text{or} \quad -\frac{4 f c}{r} > \Delta c_p > -\frac{8 f c}{r}. 
\]
\[ \Delta c_l = \frac{\Delta c_{pl} f}{2} \quad \text{or} \quad \frac{2 f^2 c}{r} < \Delta c_l < \frac{4 f^2 c}{r}. \]  

(2.69)

We predict a linear decrease of the pressure from the trailing edge to the separation line, whereas in the case of two-dimensional flow that is constant in the separated area. In our view the adverse pressure gradient which causes separation is therefore lower. As a result of this, the separation line will be closer to the trailing edge compared to 2d-flow; this reduces the wake of the blade and less drag will be experienced.

**Orders of Magnitude**

In our model we found that \( v_r \approx \Omega r \) and when we substitute this in the continuity equation we find that \( v_\theta \approx v_z \approx \Omega c \). This can physically be interpreted as follows: mass conservation demands that the radial stream which accelerates in the radial direction, should contract in the other directions. If we substitute the estimates of the velocities in equations 2.48 to 2.51 (were we neglected the viscous terms), it follows that the system is consistent. All terms on the left-hand side of 2.49 are approximately \( \Omega^2 c \), while the ones on the right hand side are approximately \( \Omega^2 r \) (the pressure term should equal the Coriolis term since that is the only one left!). In equation 2.50 the terms \( v_r \partial v_r / \partial r \), \( r \Omega^2 \) and \( -\partial p / (\rho \partial r) \) are approximately \( \Omega^2 r \), the term \( 2 v_\theta \Omega \) is approximately \( \Omega^2 c \), the term \( v_\theta^2 / r \) is approximately \( \Omega^2 c^2 / r \) and the terms \( v_\theta \partial v_r / (r \partial \theta) \) and \( v_r \partial v_r / \partial z \) are approximately 0. To understand the last estimate we use our argument that \( \partial v_r / (r \partial \theta) \) cannot be approximated with \( v_r / c \) (see section 2.5.2, ad.2). Our model is valid inside the separated area where \( v_r \) is almost constant and the gradient is approximately 0.

### 2.5.6 Extension of the Heuristics with \( \theta-z \) Rotation

The separated air above the blade will, according to the no-slip condition at the wall, form a boundary layer with a much lower radial velocity. Since this radial velocity gives rise to the Coriolis force directed towards the trailing edge this force is much reduced while the pressure gradient accelerating the flow towards the leading edge will be impressed on the boundary layer. So close to the wall the flow will accelerate towards the leading edge. At the upper side of the separated area the flow will move towards the trailing edge due to the turbulent mixing stress with the main flow. These effects will cause the flow in the separated area to spin around an axis parallel to the blade axis (next to the radial acceleration described in the section above). We call this spinning motion \( \theta-z \) rotation and deal with it as if it were independent of the above set of equations 2.64, 2.65 and 2.66.

**B13** Turbulent mixing on the upper side of the separated stream adds chord-wise momentum. The chord-wise pressure gradient acts on the entire separated stream and balances the Coriolis force and the chord-wise stress due to the turbulent mixing. This means that air in the separated stream near the blade surface is driven towards the leading edge, and the air at the upper side of the separated stream will accelerate towards the trailing edge. As a result, the air will be spinning (see figure 2.17).

**B14** The spinning motion takes places in the \( \theta, z \)-plane so that the boundary layer equations cannot describe it: it requires the equation of motion in the \( z \)-direction.

**B15:** The spinning motion in the separated stream implies that the radial velocity in the radial stream becomes more or less uniform.
Flow Separation on Wind Turbine Blades

B16: A negative pressure gradient towards the centre of the spinning motion is also required to produce the required centripetal forces.

Mathematical Model for $\theta$-z Rotation
The spinning motion can be described with the pressure gradients responsible for the centripetal force and the balancing convective acceleration terms. If we restrict ourselves to the terms required for spinning motion, we get for continuity and the $\theta$ and the $z$-equations of motion respectively:

$$\frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$

(2.70)

$$v_z \frac{\partial v_\theta}{\partial z} = -\frac{\partial p}{\rho r \partial \theta},$$

(2.71)

$$v_\theta \frac{\partial v_z}{r \partial \theta} = -\frac{\partial p}{\rho \partial z}.$$  

(2.72)

The $\theta$-z rotation is a mechanism that is assumed to be insignificant compared to the radial flow effects described by equations 2.64 and 2.66. However, in this thesis the existence of the $\theta$-z rotation is important because this phenomenon is responsible for the signal of stall flags with hinges parallel to the blade axis.

![Heuristic model on the stream of separated air and the $\theta$-z rotation.](image-url)

**Figure 2.17** Heuristic model on the stream of separated air and the $\theta$-z rotation.
Discussion of the Models on Separation

If we compare the above analysis with Snel's, the main differences are the role of radial flow and the inviscid approach. We think the radial flow is dominant because the equilibrium between Coriolis acceleration and chord-wise pressure gradient (the condition for separation) cannot give chord-wise motion. Snel’s model is based on boundary layer theory, but this is not valid as we showed and as is mentioned in standard literature. It gives errors, since most partial differentials of acceleration terms were linearised to find the orders of magnitude, which is not justified. As a consequence, in Snel’s model the radial convective acceleration is neglected and the other convective terms are estimated to be large. In our model, using the condition of separation, the other two convective terms are neglected and the radial convective term is the largest. The dominant role of the radial motion of separated air has been confirmed with laser Doppler measurements \[10\].

We neglect the viscous terms by arguing that the separated layer is thick, so we reach outside the range of validity of the boundary layer concept. The extension of our model with the \( \theta-z \) rotation required terms from the Navier-Stokes equations, which are not part of the boundary layer equations.

The terms we select yield a simpler set of equations, which can even be solved analytically. We predict the increase of the lift coefficient to be proportional to \( c/r \), which agrees with Sorensen’s computational results [56]. The model also explains the triangular shape of pressure distributions on rotating blades in stall analytically [9].