Chapter 3

Two-Dimensional Turbulence and Accretion Disks

Abstract. It has been suggested that two-dimensional turbulence is operating in thin astrophysical accretion disks and would give rise to long-lived vortices. This hypothesis is investigated by studying the development of 2D-turbulence under some of the typical circumstances that occur in thin non-selfgravitating accretion disks. This is done with what is in essence a shallow-water code. It is found that the influence of Coriolis forces is limited but that the shearing flow alters 2D-turbulence significantly. Only prograde vortices are expected in an accretion disk and, moreover, they should be strong enough to withstand the shear. Also, the maximum radial extend of the vortices is probably limited to the thickness of the disk. This implies that the flow geometry is not flat and the Rossby number not small so that a 2D-treatment of the fluid dynamics is formally only correct if the flow is in hydrostatic equilibrium in the direction perpendicular to the disk. Therefore it is questionable whether 2D-vortices really abound in accretion disks, and our results suggest why many numerical simulations of accretion disks do not show long-lived 2D-vortices. However, the existence of such vortices is not ruled out.

M.D. Nauta
submitted to Geophysical and Astrophysical Fluid Dynamics

45
3.1 Introduction

From a fluid dynamical point of view, one of the most interesting astrophysical objects is an accretion disk. It comes in different flavors, but the standard is a plasma disk around a compact star such as a white dwarf, neutron star, or black hole. Another manifestation is a gaseous and dusty disk around (proto-) stars that have just formed. Our solar system is thought to be a remnant of such a protostellar disk. These disks derive their name from transport of material through the disk onto the central object: they accrete material. Accretion rates have been determined observationally (e.g. Rutten et al., 1992) and were found to be much larger than can be explained by simple transport processes such as molecular viscosity. That is why it was suggested that accretion disks are turbulent (e.g. Shakura and Sunyaev, 1973). Lately, most research concentrates on magnetohydrodynamically (MHD) driven turbulence. For a recent review of accretion disks see Papaloizou and Lin (1995) and Lin and Papaloizou (1996).

A special version of turbulence, namely two-dimensional hydrodynamic turbulence, was proposed to be active in thin accretion disks because such disks are thin and rapidly rotating (Dubrulle and Valdettaro, 1992). Two-dimensional turbulence is characterized by an inverse energy cascade (Kraichnan and Montgomery, 1980) and the formation of coherent structures or vortices (McWilliams, 1984) (Sect. 3.3), and it is invoked to explain certain phenomena in planetary atmospheres and oceans, e.g. the robustness and persistence of Jovian vortices (Dowling, 1995). Similarly, the existence of large-scale vortices has also been suggested in thin accretion disks (Abramowicz et al., 1992; Kuijpers, 1995; Adams and Watkins, 1995). Unfortunately, lack of spatial resolution prohibits direct observational verification of this hypothesis.

The above mentioned characteristics of 2D-turbulence apply to an isotropic incompressible flow. For many 2D-flows in nature these assumptions do not hold. Violation of these assumptions can lead to different phenomena. For example from Geophysical Fluid Dynamics (GFD) it is known that variation of the Coriolis force with latitude (the $\beta$-effect) introduces an anisotropy and leads to the growth of longitudinally elongated structures (the Rhines effect; Rhines, 1975). Also in an accretion disk the assumption of an isotropic incompressible 2D-flow is clearly not justified as the flow pattern is highly directive and (some) motions are highly supersonic. Even if gross simplifications are made - such as assuming 2D-flow (because of stratification or the Taylor-Proudman theorem) and using a polytropic relation instead of an energy equation - the character of 2D-turbulence may still differ from the standard picture by, for example, the shear of the Keplerian motion. It is the aim of this paper to investigate the influence on 2D-turbulence of some of the typical circumstances in an accretion disk. The main question I would like to answer is why only so few studies (only Bracco et al., 1998 to the author’s knowledge) do find vortices in their (2D-) simulations of accretion disks, whereas it is claimed that vortices are a robust feature of 2D-flow. For this purpose, the 2D-polytropic gas equations that are often used in theoretical studies of disk dynamics, (e.g. Goldreich et al., 1986), are solved with what is in essence a shallow-water code (Arakawa and Lamb, 1981). The equations
and numerical procedure are briefly outlined in Sect. 3.2. A thorough discussion and testing of the code was presented in a previous paper (Nauta and Tóth, 1998). Ordinary 2D-turbulence is discussed in Sect. 3.3, which is mainly written for those unfamiliar with the subject. It serves as the standard to which the other simulations are compared. The influence of Coriolis forces is investigated in Sect. 3.4. Next, a background shear flow is introduced in Sect. 3.5. The paper ends with a discussion on the implications of the observed phenomena for an accretion disk.

3.2 Equations

Two-dimensional turbulence is usually (e.g. McWilliams, 1984; Santangelo et al., 1989; McWilliams, 1990) studied by numerically solving the vorticity equation (the curl of the momentum equation):

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \vec{v}) = \text{Dif},$$

(3.1)

on a square periodic grid for a given initial condition. All variables have their usual meaning: $\omega = \nabla \times \vec{v}$ is the vorticity and Dif stands for some diffusion process. In case of molecular diffusion Eq. (3.1) is the 2D incompressible Navier-Stokes equation. However, if Dif represents the influence on the resolved scales of sub-grid scale motions, then it is not clear that Dif is described by a Laplacian operator, and often a higher order differential operator is chosen to simulate flow with an effectively higher Reynolds number. This is called hyperviscosity.

In principle Eq. (3.1) could be used to study the flow in an accretion disk. However, to arrive at this equation so many approximations have to be made that possibly significant physical effects (e.g. compressibility) could have been neglected. At the opposite end are the 3D MHD equations with self-gravity and radiative transfer. They are highly relevant for an accretion disk, but they are also extremely complicated to solve and it is hard to unravel the different physical processes going on at the same time from their solutions. Here an “intermediate” approach is chosen, the 2D-polytropic gas equations:

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \vec{v}) = \text{Dif},$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + 2\Omega \hat{z} \times \vec{v} = -\frac{1}{\Sigma} \nabla P + \nabla \Phi_G + \text{Dif},$$

(3.2)

Here $\Sigma$ is the surface density, $P$ the height integrated pressure and $\Phi_G = GM/r + \Omega^2 r^2/2$ describes the effective gravity of the central compact object. This simplified set of equations describes an accretion disk in a frame rotating with fixed angular rotation speed $\Omega \hat{z}$ (and they have been used before by for example Goldreich et al., 1986). The simplifications implicit to the 2D-polytropic gas equations consist of neglecting magnetic fields, radiative transfer and self-gravity of the gas and integrat-
ing over the vertical thickness of the disk. To eliminate baroclinic effects, a poly-
tropic equation of state between surface density $\Sigma$ and height integrated pressure $P$
is used. It short-circuits the energy equation so that only the continuity and momen-
tum equations need to be solved. If the adiabatic index, $\gamma = 2$, is chosen, then the
2D-polytropic gas equations are identical to the shallow-water equations which have
been used before in 2D-turbulence simulations (e.g. Cho and Polvani, 1996). Now
the surface density is to be interpreted as the height of the free surface. The shallow-
water equations are often considered to form the most simple dynamical model of a
planetary atmosphere or ocean. Similarly, the 2D-polytropic non-selfgravitating gas
equations can be considered as the most simple model of an accretion disk.

Equations (3.2) are written in a rotating coordinate system. This introduces the
Coriolis acceleration and centrifugal force. A solution in which the fluid motion is
circular and determined by a balance of centrifugal force with gravity of the central
compact object, causing the velocity to be Keplerian, is:

$$v_k = \left(\frac{GM}{r}\right)^{1/2} - \Omega r,$$

where $\Omega = (GM/R^3)^{1/2}$ and $R$ is a fixed reference value. A velocity field superposed
on this flow is required to explain the large radial transport of mass and angular
momentum observed in astrophysical accretion disks.

For a planetary atmosphere or ocean, the fluid layer on which the turbulence is
superimposed is a thin layer in solid body rotation on the surface of a sphere. The
influence of these conditions on 2D-turbulence was investigated by Cho and Polvani
(1996a). For an accretion disk the turbulence is superimposed on a thin layer in
Keplerian rotation. It follows that at least two conditions may change the character
of 2D-turbulence from its standard behavior: rotation of the coordinate frame (also
studied by Polvani et al. (1994) and others) and shear of the Keplerian motion. The
influence of these is studied in an elementary way: I take a 2D-turbulence simulation
and add only Coriolis forces (this is the influence of a rotating coordinate frame) in
Sect. 3.4, and in another simulation I add a uniform shear flow in Sect. 3.5. The
application to accretion disks is the subject of Sect. 3.6.

### 3.2.1 Technicalities

The equations are made dimensionless by introducing a typical length ($L$), velocity
($V$), time ($T$) and density (the unperturbed value is set to unity so that the total den-
sity is $1 + \Sigma$) scales. The resulting non-dimensional numbers are discussed at the
initial conditions (Froude or Mach number ($M_a$) and the scaling of the time) and
at the sections where they are first introduced (Burger number ($B$) in Sect. 3.4, non-
dimensional shear in Sect. 3.5). Simulations were performed on a square Cartesian
grid of $[-1,1] \times [-1,1]$ with $256^2$ grid points. The algorithm used is in essence a
shallow-water code (Arakawa and Lamb, 1981) and, because of the close similarity
between the shallow-water equations and the 2D-polytropic gas equations, can
also be used for the latter. The reason for not choosing a shock capturing scheme is
that the present algorithm allows for higher Reynolds number simulations (Nauta and Tóth, 1998). It also conserves potential enstrophy (average of vorticity squared divided by free surface height) for ideal flow. Two periodic and two free-slip reflective boundaries were used. As hyperviscosity a $\nu \nabla^8$ operator is used on the right hand side of all three differential equations (3.2). The diffusion is such that a wave of wavelength four grid cells is suppressed in approximately one initial eddy turnover time.

In the analysis of turbulence, spectral information plays an important role; especially the analysis of energy spectra has a long tradition. Two technical points regarding the calculation of spectra merit closed inspection.

The kinetic plus internal energy in the polytropic 2D-gas equations is:

$$E = \iint \left( \frac{1}{2} (1 + \Sigma) |\vec{v}|^2 + \frac{1}{M_a^2} \frac{1}{\gamma (\gamma - 1)} (1 + \Sigma)^\gamma \right) d^2 \vec{x}.$$  

Since the energy is not a quadratic function of the flow variables it is far from clear how energy spectra should be constructed for this case. Even for shallow-water ($\gamma = 2$) the kinetic part of the energy is not quadratic. Previously this was solved by using the kinetic energy per unit mass (Polvani et al., 1994) or a linearization procedure (Farge and Sadourny, 1989). For the spectra presented here the difference between the two methods is small. The spectra shown are calculated following the approach of Farge and Sadourny. They assumed small perturbations (small Froude number ($M_a$) and small deviations in free surface height) and linearized the shallow-water equations (so $\gamma = 2$). For this new set of equations the energy is a quadratic quantity:

$$E' = \frac{1}{2} \iint \left( |\vec{v}|^2 + \frac{1}{M_a^2} \Sigma^2 \right) d^2 \vec{x},$$

from which a spectrum can be calculated. Because of two quadratic conserved quantities, the energy and the potential enstrophy, the flow field can be split into three distinct eigenmodes which are called: one potentio-vortical and two inertio-gravitational modes. The potentio-vortical mode contains all the potential vorticity (the absolute vorticity divided by the surface density) and is incompressible. The two inertio-gravitational modes are compressible and do not contain any potential vorticity. As these eigenmodes facilitate the analysis and the results did not depend very strongly on $\gamma$ all simulations presented in this paper are performed with the shallow-water equations unless stated otherwise. As a result, the nomenclature is adjusted: when the shallow-water equations are used terms like surface density and Mach number are replaced with their equivalent terms: free surface height and Froude number respectively.

There is another problem in the construction of the energy spectra. Flow variables are not periodic in the direction perpendicular to the closed boundaries. Most notably the stream function can be different on both closed boundaries (which implies that there is a net flow through the periodic boundaries). When Fourier transformed this jump results in significant power in the high wavenumber bins. To remove this artifact the stream function ($\psi$) and velocity potential ($\chi$) ($\vec{v} = \hat{z} \times \nabla \psi + \nabla \chi$) are
multiplied with a function that is flat in the middle and decreases smoothly to zero (as \((1 - \cos(x))/2, 0 \leq x \leq \pi\)) in the 20 grid cells next to the closed boundaries. The modified stream function and velocity potential are used to calculate spectra, from Farge and Sadourny (1989):

\[
E'_0(k_n) = \frac{1}{2} \sum_{k_n \leq |\vec{k}| < k_{n+1}} \frac{[k^2 \psi(\vec{k}) + \phi(\vec{k})/(M_B B^{1/2})]^2}{1/B + k^2},
\]

\[
E'_g(k_n) = \frac{1}{2} \sum_{k_n \leq |\vec{k}| < k_{n+1}} k^2 \left( |\chi|^2 + \frac{[\psi(\vec{k})/B^{1/2} - \phi(\vec{k})/M_B]^2}{1/B + k^2} \right),
\]

where \(\vec{k}\) is the wave vector, \(\phi\) the free surface height.

The initial condition is constructed with the help of a stream function which is defined in Fourier space by the relation:

\[
E(k) = E'_0(k) = 2\pi k \frac{1}{2} \left( \frac{1}{B} + k^2 \right) \left| \psi(\vec{k}) \right|^2 \propto \frac{k}{1 + (k/k_0)^4},
\]

which has the same spectral energy distribution as in McWilliams (1984). The phases of the Fourier components of the stream function are chosen randomly. By choosing \(k_0\) relatively large (\(k_0 = 10\pi\) resulting in 10 waves on the simulation domain) a clear inverse energy cascade can be seen. The amplitude of the stream function determines the amplitude of the velocity field or the Froude number. Here a maximum Froude number of approximately 0.1 is used which results in an rms value of 0.036 for the given initial spectrum. The initial velocity potential is chosen zero because the usual 2D-turbulence simulations are incompressible and here we want to add extra elements one by one. The initial distribution of free surface height is chosen such that in the subsequent evolution the amplitude of the excited inertia-gravitational waves is small. The free surface height distribution that satisfies this condition is called a balanced equilibrium, and to calculate it the simple balancing procedure of Farge and Sadourny (1989) is used, though more elaborate balancing procedures are known (Polvani et al., 1994). The initial enstrophy \(S_0\) (spatial average of vorticity squared) determines the "initial eddy turnover time" \(t_{eddy} = T = 2\pi/(S_0)^{1/2}\), the time scale that is used throughout this article.

### 3.3 Standard two-dimensional turbulence

If fluid motion is restricted to two dimensions then the vorticity vector is directed perpendicular to the flow. All vortex tubes are parallel which implies that they do not stretch each other. In this way the process responsible for transfer of energy to small length scales in 3D-turbulence is absent from 2D-flow. It is a well known result (e.g. Pedlosky, 1987) that in 2D-incompressible turbulence the energy goes to large length scales or small wave numbers. This process is called the inverse
energy cascade. Numerical simulations (e.g. McWilliams, 1984) show that during the inverse cascade coherent structures or vortices form. This is nowadays considered the most characteristic process of 2D-incompressible turbulent flows.

It was shown (Farge and Sadourny, 1989) that 2D-turbulence in shallow-water shares many of its characteristics with 2D-incompressible turbulence. Shallow-water differs from 2D-incompressible flow in that there is a free surface. Local changes in the thickness of the fluid layer introduce 2D-compressibility ($\partial v_x/\partial x + \partial v_y/\partial y = -\partial v_z/\partial z \neq 0$). In this section a low Froude number shallow-water simulation is used to demonstrate the characteristics of 2D-turbulence. Eq. (3.2) is solved without Coriolis acceleration ($B = \infty$) and effective gravitational potential. The results serve as "the standard" reference material for the simulations in the following sections.

The formation of coherent structures is clearly seen in potential vorticity or vortensity ($\nabla \times \vec{\omega}/\Sigma$), Fig. 3.1. The initial distribution looks very random. Initially the potential vorticity is stretched, creating thin vorticity structures. This corresponds to the enstrophy cascade that transfers enstrophy towards the dissipation regime at

![Figure 3.1: Potential vorticity distribution at four different times (left to right, top to bottom). The initial potential vorticity is distributed randomly. After 0.75 eddy turnover times some of the vorticity is elongated into thin structures transferring enstrophy towards high wave numbers. At 2.8 turnover times the coherent structures are clearly visible. A small number of them is left after approximately 120 eddy turnover times.](image-url)
high wave numbers. A bit later part of the vorticity rolls up to form vortices. These vortices interact with each other in the form of (partial) mergers and (partial) strainings. Merging only occurs among vortices of the same sign and is often interpreted as manifestation of the inverse energy cascade in configuration space, creating large scale structures. After many mergers only a limited number of vortices remain. This is already the case after a couple of eddy turnover times. Eventually these vortices interact with each other to result in two vortices of opposite sign (Matthaeus et al., 1991), but this takes longer than the time over which the present simulation was evolved.

Figure 3.2: The total energy (upper curve) and potential enstrophy (lower curve) as a function of time. The data are normalized with their initial values. The time $t_{eddy}$ is defined in Sect. 3.2.1.

Characteristic for 2D-turbulence is also the evolution of energy and potential enstrophy, Fig. 3.2. Because of the inverse energy cascade little energy flows into the dissipation regime so that the total energy is almost conserved. The energy plotted is the total energy which includes the energy in the inertio-gravitational components which can cascade in the “ordinary” direction and be dissipated by diffusion. However, because of the initially balanced state, the contribution of this energy to the total is negligible and the total energy is almost fully comprised of potentio-vortical energy. On the other hand, potential enstrophy does cascade in the normal direction and is dissipated. A considerable decrease in potential enstrophy is observed. The enstrophy decrease slows down as soon as vortices form, because non-linear enstrophy transfer is prohibited inside vortices (Babiano et al., 1987; Benzi et al., 1986).

The inverse energy cascade can be observed in the spectrum, Fig. 3.3. Plotted is the energy in the potentio-vortical component. The initial energy spectrum peaks at a wavenumber of $3^{-1/3}k_0 \approx 7.6 \pi$. At the end of the simulation the energy peaks at the lowest possible wavenumber $\pi$. Also, from the shape of the spectrum it is clear that the energy at lower wavenumbers grows at the cost of the energy at higher wave numbers. The slope of the spectrum is approximately -4 which deviates from what a Kolmogoroff style of analysis would give (-3). This has been attributed to the coherent vortices (e.g. Benzi et al., 1987) and to intermittency (localization of phenomena in space and/or time).
The intermittency of the turbulence can also be measured in real space with the kurtosis of the relative vorticity ($\nabla \times \vec{v}$). The probability density function of the relative vorticity becomes a strongly peaked function around zero, while there are tails to positive and negative relative vorticities due to the vorticity in the coherent structures that hardly decays. So the distribution deviates strongly from Gaussian and it is this quality that is measured by the kurtosis (Press et al., 1992). It increases with time, as shown in Fig. 3.4.

Since vortices appear to form the elementary building blocks of 2D-turbulence, their statistical properties were determined as a function of time (McWilliams, 1990). Here this is done as well but with a different vortex recognition algorithm, see the Appendix. Fig. 3.5 shows the number of vortices as a function of time. Due to mergers and hyperviscosity the number of vortices drops as a power law with exponent -0.70, which is in agreement with McWilliams (1990). Power laws for other observables were predicted and claimed to be found by Carnevale et al. (1991). The average vortex radius, circulation, the enstrophy and vorticity kurtosis should exhibit
power law behavior in time. This follows from combining the observed power law in the number of vortices as a function of time, a dimensional analysis and conservation of energy and peak vorticity (of the vortices). The theory was heavily criticized (Dritschel, 1993) and in the present simulations the predicted power law exponents could not be verified.

Summarizing, the low Froude number shallow-water simulation is well equipped to demonstrate the formation of coherent structures, the intermittency and the cascading in energy and enstrophy such as are typical for 2D-turbulence.

3.4 Two-dimensional turbulence and Coriolis forces

Two-dimensional turbulence as it was studied in the previous section forms an idealized paradigm for large-scale geophysical turbulence. It is easy to make it a bit more realistic by adding solid body rotation, so that Coriolis forces should be taken into account (which also play a role in an accretion disk). Eq. (3.2) is solved with Coriolis acceleration but without effective gravitational potential. The f-plane approximation is made implying that variation of the Coriolis parameter or planetary vorticity \( f = 2 \Omega \) with a latitude coordinate is not considered. Influence of rotation on 2D-turbulence is expected to become visible as the Coriolis parameter is increased because the force balance within vortices changes from cyclostrophic (pressure gradients balancing centrifugal forces) to geostrophic (pressure gradients balancing Coriolis forces).

More specifically, rotation of the coordinate frame introduces a new time scale \( 1/f \) and, when combined with the gravity wave speed \( c_s \), a new intrinsic length scale:

\[
L_D = \frac{c_s}{f},
\]

called the Rossby radius of deformation. Traditionally this length scale is interpreted as the scale at which the Coriolis force becomes of the same order of magnitude as the pressure gradient. Perhaps a better interpretation follows from an expansion of
the (dimensional) potential vorticity in small parameters:

\[ \frac{f + \nabla \times \vec{v}}{\Sigma} \approx \frac{f}{\Sigma} + \frac{\nabla \times \vec{v}}{\Sigma} - \frac{f\Sigma_1}{\Sigma_0}. \]

The third term on the rhs (which describes the contribution of vortex stretching to the potential vorticity) is of the same order of magnitude as the second (which describes the contribution of relative vorticity to the potential vorticity) if the length scale of the phenomenon is of the order of the Rossby deformation radius (Pedlosky, 1987, p.92). (In this scale analysis geostrophic equilibrium is used.) The square ratio of the Rossby deformation radius to the typical length scale of the flow (\(L\)) is called the Burger number:

\[ B \equiv \left( \frac{L_D}{L} \right)^2. \]

To investigate the influence of the Coriolis force (or, equivalently, a finite Rossby deformation radius or finite Burger number), several simulations were done with different values for the Coriolis parameter \(f\). The initial energy spectra were the same for all simulations. Similar experiments have been done before by Farge and Sadoumy (1989) and especially the work of Polvani et al. (1994) is valuable. This topic has also been investigated with a reduced set of equations (Cushman-Roisin and Tang, 1990; Larichev and McWilliams, 1991). The main findings are illustrated on the basis of my simulation with Rossby deformation radius 0.031 or Burger number 0.16, where the energy centroid of the initial energy spectrum was used to get a typical length scale. This small Burger number simulation is compared with the standard which has a Burger number of infinity. Other experiments were done at Burger numbers of 0.64 and 10.2.

Comparison of the potential vorticity for the small Burger number simulation after approximately 120 initial eddy turnover times, Fig. 3.6, with that of the standard, Fig. 3.1, shows that there are many more vortices in the small Burger number.

Figure 3.6: Potential vorticity distribution of a simulation with small Burger number (0.16) after approximately 120 initial eddy turnover times.
Figure 3.7: Cross section through two vortices. On the left, a vortex larger than its Rossby deformation radius, on the right, from the standard simulation with infinite deformation radius. Relative vorticity is indicated by the solid line, velocity by the dashed line, the arrows are discussed further down the text. The difference in profile is very clear: the vortex on the left is shielded as it is surrounded by a ring of opposite vorticity so that the velocity drops off rapidly, while the vortex on the right is surrounded by potential flow.

The potential vorticity distribution in Fig. 3.6 suggests that the inverse energy cascade is slowed down at small Burger numbers. This can be quantified by measuring the number of vortices as a function of time. In the standard simulation this number drops as a power law with index -0.7, Fig. 3.5, while here the index is -0.41. The reduced cascade is corroborated by a slower decay of potential vorticity and a slower growth of the vorticity kurtosis (not shown). The slow down of the inverse energy cascade is also very clear from the spectral evolution, Fig. 3.8. The peak in the spectral distribution only shifts from 7.6π to 4π during the time interval studied, while in standard decaying 2D-turbulence it is at the lowest wavenumber where
most of the energy is contained.

The slowing down of the inverse energy cascade raises a question: Is there an upper limit to the size of the vortices that can form? The vortex size is often related to the peak in the spectral energy distribution (Polvani et al., 1994) and the peak position almost comes to a standstill. The existing literature is rather unclear on this point. That is why the vortices were identified by an algorithm (see the Appendix) and their radii measured, Fig. 3.9. The measurements show that the largest vortices grow to several times the Rossby deformation radius and even the average radius gets larger than this value. So I don’t find that the Rossby deformation radius sets an upper limit to the size of the vortices. The simulations at other Burger number indicate

Figure 3.8: The energy spectrum of the potentio-vortical component after approximately 120 eddy turnover times. The dotted line indicates the standard.

Figure 3.9: On the left, the average vortex radius (solid) and the radius of the largest vortex (dashed), both expressed in the number of deformation radii, as a function of time. On the right, the vortex radius of a particular vortex. It undergoes several mergers, at t=25 it merges with the vortex indicated with the dotted line and at t=90 with that indicated with the dashed line. During merging the vortex radius can substantially increase, but without mergers the radius hardly changes.
that the physical growth rate of the vortex radius depends little on this parameter. That the radial growth is more than just viscous widening, follows from tracking the behavior of one particular vortex which undergoes several merger events, Fig. 3.9.

It is observed that the vortex radius mainly increases due to merging with other vortices. If there is no merger (25 \(<\ t\ <\ 90\)), then the radius increases (the solid line) or decreases (the dashed line) only a little bit, indicating that during these phases viscosity is not very important in determining the vortex radius. It should also be noted that the radii before and after merging imply that areas of the vortices add up (for this particular merger). This is contrary to what is supposed to occur in 2D incompressible turbulence where \( R^4_{\text{before}} + R^4_{\text{before}} = R^4_{\text{after}} \) (Carnevale \textit{et al.}, 1991).

Also the fact that the average vortex radius growth rate is independent of Burger number indicates that at lower Burger number, when there are fewer mergers, the growth of the radius per merger is larger. This was also noted by Waugh (1992) in contour dynamics simulations of the merging of two quasi-geostrophic vortices.

These are not the only indications that the merging process is different at small Burger number. The vorticity profile of a typical vortex has already been partly discussed in Fig. 3.7, where the emphasis was on shielding. But the vortex core itself is also different: at small Burger number, there is a kink halfway the vorticity core, which is indicated by arrows in Fig. 3.7. A lot of vortices have a similar kink. One of the most extreme examples is shown in Fig. 3.10. The large negative vortices clearly exhibit strange kinks while the small positive vortex is fairly regular. It turns out that the kinks are a very persistent phenomena and are not the result of a recent merger. It looks as if it results from merging of a small intense and a large weak vortex many turnover times ago. Apparently, after the merger, there is little redistribution of vorticity if the vortex is larger than the Rossby deformation radius. As a result, large vortices look less robust and they have shapes that deviate considerably from a circular monopolar vortex which was also noted by others (Polvani \textit{et al.}, 1994). Unfortunately I could not turn this lack of robustness into a criterion that sets

![Figure 3.10](image_url)

Figure 3.10: A cut through the relative vorticity field. The large anti-cyclonic vortices show a strange kink (indicated with arrows) in their vorticity distribution, while the small cyclonic vortex has a regular profile.
limits on the properties of the vortices that are expected.

In the literature there is only one such a report as far as I know. With a reduced set of equations Cushman-Roisin and Tang (1990) found that cyclonic vortices with huge perturbations in free surface height become unstable and break up if they get larger than the Rossby radius of deformation. In experiments with the full shallow-water equations with a single Gaussian vortex I could confirm this result, but the instability turns out to be very sensitive to the depth of the depression. The instability is very clear if the free surface height drops to some 10 percent of its average value, but at 25 percent the instability could not be noticed. Such large depressions are very hard to reach with decaying shallow-water turbulence experiments, and that is why Polvani et al. (1994) could not verify it.

To summarize, the inverse energy cascade is slowed down when vortices grow larger than the Rossby deformation radius. The reason is that a typical vortex becomes shielded. However this does not imply that the vortex growth stops, because the merging leads to more efficient growth at small Burger number. So the Rossby deformation radius does not impose an upper limit to the vortex size. Vortices larger than the Rossby deformation radius do look less robust, but this could not be translated to a maximum size of the vortex that can develop.

3.5 Two-dimensional turbulence and a shearing background flow

The simulations presented so far address turbulence in the absence of a background shear. In an accretion disk, however, the turbulence is superposed on Keplerian shear flow. How 2D-turbulence develops when it is subjected to a shear flow, is a topic which received little attention.

Here I restrict myself to uniform shear flows (with a linear velocity profile) because these are unlikely to support 2D-turbulence so that, just as in the previous sections, decaying turbulence and not forced turbulence is studied. For 2D-incompressible flow it is easy to show (Toh et al., 1991) that any turbulence superposed on a uniform shear flow must be decaying. Because the turbulence here is 2D-compressible similar reasoning can only make it plausible that turbulence decays. This follows from the enstrophy equation. Multiply the ideal vorticity equation by the vorticity:

\[ \frac{\partial \omega^2}{\partial t} = -2\omega \nabla \cdot (\omega \vec{v}) = -\nabla \cdot \omega^2 \vec{v} - \omega^2 \nabla \cdot \vec{v}. \]

Next split the flow in contributions from the uniform shear flow (indicated with index 0) and the turbulence (indicated with index 1):

\[ \frac{\partial \omega^2}{\partial t} = 2\omega_0 \nabla \cdot (\omega_0 + \omega_1) \vec{v} - \nabla \cdot (\omega_0 + \omega_1)^2 \vec{v} - \omega_0^2 \nabla \cdot \vec{v} - 2\omega_1 \omega_1 \nabla \cdot \vec{v} - \omega_1^2 \nabla \cdot \vec{v}, \]

where the first term on the right hand side comes from the time derivative of the vorticity which has been expressed as a spatial derivative with the help of the vorticity.
equation. Then integrate over the area and use the boundary conditions (periodic or free slip closed boundaries are used):

$$\frac{\partial}{\partial t} \int \int \omega_1^2 dA = -2\omega_0 \int \int \omega_1 \nabla \cdot \vec{v}_1 dA - \int \int \omega_1^2 \nabla \cdot \vec{v}_1 dA.$$

The result shows that turbulence can only grow at the expense of the shear flow if there is a negative correlation between vorticity and divergence of the turbulence. Since simulations are started from a balanced state so that initially \(\nabla \cdot \vec{v}\) is small, it is expected that the right hand side is approximately zero. Also at later times, when non-linear interactions or the overreflection instability have created a non-vanishing \(\nabla \cdot \vec{v}\), the vorticity and divergence are found to be approximately uncorrelated. If a diffusion process is incorporated then an extra negative term is added on the right hand side of the equations above, indicating that the enstrophy of the turbulence is a decaying function of time. This behavior is also observed on average when the enstrophy is monitored during the simulations. It exhibits small amplitude, short term fluctuations around a decaying curve. So, just like in the previous sections, the turbulence decays.

Simulations were done with a modified version of the numerical algorithm because the (large) shear leads to serious dispersion errors. It was found that the partly fourth order modification (Takano and Wurtele, 1982) of the Arakawa-Lamb scheme gave considerably better results for these kind of simulations than the second order scheme. Therefore, the partly forth order, algorithm is used throughout this section.

The equations solved are just the continuity and momentum equations, Eqs. (3.2), without contributions from Coriolis acceleration and effective gravity but including shear flow; \(\vec{v}\) consists of two parts: the linear shear flow \(\vec{v}_0 = sy\hat{x}\) in the x-direction and the turbulence \(\vec{v}_1\). The new non-dimensional parameter in this problem is the ratio of shear speed (over the typical length scale \(L\)) over typical turbulent speed: \(V_0/V\), where \(V_0 = s \times L\).

Simulations over 50 initial eddy turnover times were performed for the same initial turbulent field but with different values of the shear. The shear \(V_0/V\) was varied between the values 0 and 3.6, where \(V/L\) was determined from the rms initial relative vorticity. The potential vorticity distribution of four simulations is shown in Fig. 3.11. For small values of the shear \((V_0/V = 0.06)\) the potential vorticity looks very similar to that of ordinary 2D-turbulence without shear. When the shear is increased \((V_0/V = 0.11)\), retrograde vortices are removed. These are the white vortices with vorticity of the opposite sign to that of the shear of the background flow. This behavior is well-known (Kida, 1981) and has also been observed in previous studies (Marcus, 1988; Toh et al., 1991). If the shear gets larger than 0.135 times the peak vorticity of a vortex, then that eddy is torn apart (Legras and Dritschel, 1993). The reason is that the vorticity is distributed beyond the separatrix of the stream function, Fig. 3.12. This allows (potential) vorticity to be removed from the vortex (a process called vortex stripping) and be carried away to infinity, thus degrading the vortex. For vortices of the opposite sign (the black ones in Fig. 3.11), the position of the separatrix of the stream function is well outside the vorticity distribution. Therefore, these vortices survive at these values of the shear.
Figure 3.11: Potential vorticity distribution for four different values (left to right, top to bottom) of the shear ($V_0/V = 0.06, 0.11$ and $1.8$) after 50 and ($V_0/V = 3.6$) after only 12 eddy turnover times.

Figure 3.12: Zoom in of the lower left corner of the potential vorticity distribution of the simulation with shear=$0.11$ in Fig. 3.11. Superimposed are contours of the stream function: on the left in a frame co-moving with the retrograde vortex, on the right co-moving with the prograde one. It is clear that the separatrix of the stream function is at the edge of the retrograde vortex while the potential vorticity of the prograde vortex is well inside its separatrix. The slight misalignment between streamlines and potential vorticity can be attributed to uncertainty in the advection speed of the vortices.
A further increase in shear \((V_0/V = 1.8)\) leads to the destruction of all retrograde vortices. The prograde vortices are now strongly deformed in the shape of an ellipse. An animation of the potential vorticity shows that the mobility of the vortices in the direction perpendicular to the shear decreases. The prograde vortices stay at approximately the same y-coordinate throughout their lifetimes. There could be several explanations for this phenomenon. Firstly, the result of the elliptical deformation of the prograde vortices is that the velocity induced by the vortex perpendicular to the elongation decreases compared to that of a circular vortex of the same circulation. So vortices at different y-coordinates feel less of each other. Secondly, vortices of opposite sign have the tendency to propel each other through the fluid. This is most clearly exhibited by dipolar vortices (which consist of one positive and one negative vorticity core). Vortices of the same sign only rotate around each other (and merge), but remain localized. With the removal of all retrograde vortices by the shear, it is expected that the mobility of the prograde vortices drops. Thirdly, it is found that the vortices at higher shear are on average weaker, that is they have a lower circulation. This is, at least partly, the result of viscosity. The vortices develop thinner structures which diffuse more easily. For another part it may be intrinsic, namely that vortex merging leads to a larger waste in the presence of a background shear flow. As weaker vortices have weaker interactions, this can also explain the decreased mobility. The drop in mobility in the x-direction is of course compensated by the advection of the shear flow.

The decreased mobility implies that there are fewer encounters with other vortices of the same sign that can be swallowed. So we expect to see fewer mergers and an increase in the number of vortices with increasing shear. In Fig. 3.13 the number of prograde vortices after 50 initial eddy turnover times is plotted as a function of shear. We see a marked increase in the number of vortices when the shear increases from 0.11 to 0.66. This seems to be related to the decreased mobility of the vortices in the direction perpendicular to the shear. For very small values of the shear, from

![Figure 3.13: The number of prograde vortices as a function of the background shear after 50 initial eddy turnover times. The vortices were identified by eye. The data points are connected by straight line segments.](image-url)
0 to 0.11, the number of vortices drops. This may be due to an increased inverse energy cascade when a small amount of shear is added forcing eddies onto each others neighbors, a phenomenon known from forced 2D-turbulence (Toh et al., 1991). For larger values of the shear, larger than 0.66, the number of vortices drops slowly.

The final increase of shear \((V_0/V = 3.6)\) gets rid of all vortices, Fig. 3.11. There could be several reasons for this.

![Figure 3.14](image-url)

Figure 3.14: On the left, vortex elongation as a function of peak vorticity from one of the simulations. The drawn curve is from a model based on continuity. On the right a sketch of a vortex superimposed on a shear flow. Indicated are the ingredients of the continuity model for vortex elongation.

One phenomenon already observed at lower values of the shear is that vortices adjust their shape and become elongated. Fig. 3.14 shows the measured elongation, semi major axis divided by semi minor axis, of the vortices. The drawn curve is from a model based on continuity (of the potential vorticity transport). Suppose the vortex adds a velocity of magnitude \(v_1\) to the shear flow \(v_0\), Fig. 3.14. Then the net transport of potential vorticity through the semi-major axis is \(v_1 \cdot a\). This must equal transport through the semi-minor axis \((v_1 + v_0) \cdot b\). So the elongation of the vortex is \(a/b = (v_1 + v_0)/v_1 \approx (\omega + s)/\omega\) where \(\omega\) is the peak vorticity of the vortex. Considering the simplicity of the argument, the continuity model (drawn line) is a decent fit to the data. What is apparent from Fig. 3.14 is that when the peak vorticity of the vortex drops significantly below the shear of the background flow, the vortex’s elongation rises dramatically. It is questionable if a vortex with an aspect ratio of 5 or more can be discriminated from a filament. Such highly elongated structures are not characterized properly by the predicate vortex.

Even if prograde vortices are somehow able to withstand the elongation process, then at high enough values of the shear they will be prone to vortex stripping, the same process that removes retrograde vortices at smaller shear, Fig. 3.12. The exact value of the shear at which vortex stripping starts is hard to calculate because the position of the separatrix also depends on the position and strength of other vortices.

Finally, a third reason for the lack of vortices at large shear is that viscosity is more effective when vortices develop thin structures.

Closely related to the elongation effect is a timing argument against large scale
vortices in strong shear flows. If the time to shear structures apart is much longer than the vortex rotation period:

\[
\left( \frac{dv_0}{dy} \right)^{-1} \gg 4\pi \left( \frac{1}{r} \frac{dv_1}{dr} \right)^{-1},
\]

then vortices can form. Here the velocity \( v_0 \) is to be interpreted as due to the shear flow and \( v_1 \) due to the vortex. In the opposite case, eddies will be torn apart by the shear.

These considerations indicate that there is a minimum threshold on the peak vorticity for vortices to exist in a uniform shear flow. The hope that this threshold can be reached by vortex merging is in vain because one of the fundamental observations from decaying 2D-turbulence experiments is that the peak vorticity of a vortex stays almost constant. (The vortex scaling theory of Carnevale et al. (1991) is based on this fact.) This, in combination with the apparent lower limit on peak vorticity of vortices in shear flow, leads to a significant difference between 2D-turbulence with and without shear flow. In the absence of shear, no matter how small the initial turbulent fluctuations, vortices form. The strength of the fluctuations determines the time scale of formation. In the presence of a shear flow, the perturbations have to be large enough, namely of the order of the shear in the background flow, otherwise no vortices form at all.

Finally, there is a limit to which the turbulence decay experiments did not extend but which could be important. The gravity wave speed can be combined with the shear to form an intrinsic length scale, \( L_s = c_s/s \), rather similar to the Rossby deformation radius. If the typical length scale of the perturbations in the y-direction is larger than \( L_s \) then the difference in the background shear over the typical distance is larger than the gravity wave speed. All vortices observed are considerably smaller than \( L_s \) (in their cross-stream extent). In experiments on single vortices in shear flows, those that were close to this limit broke up into two smaller vortices. The author has not been able to produce a vortex larger than \( L_s \). This intrinsic length scale, \( L_s \), seems to put an upper limit to the cross-stream size of vortices that are to be expected in shear flows.

3.6 Discussion

In this section the implications of the phenomena described in the previous sections for 2D-turbulence in an accretion disk are discussed.

3.6.1 Rotation of the disk

In Sect. 3.4 the influence of the Coriolis force on shallow-water turbulence was studied. It was found that the inclusion of such a simple new element (and 2D-compressibility) could already change the character of standard 2D-turbulence considerably. 2D-turbulence changes its character if the typical length scale gets larger than the
Rossby radius of deformation:

\[ L_D = \frac{c_s}{f}. \]

For an accretion disk the equivalent of the planetary vorticity \( f \) is the vorticity of the Keplerian disk \( f = 0.5(GM/r^3)^{1/2} \) and this can be used to determine the Rossby deformation radius. It can also be argued that \( f \) is twice the rotation frequency. Then the Rossby deformation radius is four times larger. To settle this controversy an experiment was done in which an azimuthally symmetric density perturbation was allowed to develop freely in an accretion disk (1.5D model). This excites compressional waves that move to the inner and outer edge of the disk. What is left is a bump in the density (in geostrophic equilibrium) with a typical width given by the Rossby deformation radius. The Rossby deformation radius determined in this way was found to be in between both estimates given above: \( L_D = \frac{rc_s}{v_{\text{kep}}} \). This is to be compared with the typical half-thickness \( H \) of a thin accretion disk which follows from hydrostatic equilibrium in the direction perpendicular to the disk (z-axis):

\[ GMz/(r^2 + z^2)^{3/2} = -(c_{\text{sio}}^2/\rho)(\partial \rho/\partial z) \]

which approximates to \( GMH/r^3 = c_{\text{sio}}^2/H \). It is observed that the Rossby deformation radius is of the order of the half thickness of the disk. So if the vortices get larger than the half thickness of the disk then shallow-water turbulence becomes different from standard 2D-turbulence.

The most important conclusion from Sect. 3.4 is that vortices can grow larger than the Rossby deformation radius. So the Coriolis force poses no objection to large-scale vortices in accretion disks. Most changes in the character of 2D-turbulence at small Burger number result from a change in the vorticity profile of a typical vortex. Vortices tend to be shielded. It is unclear if something similar can occur in an accretion disk as a shear flow was left out of the simulations in Sect. 3.4 while in Sect. 3.5 it was shown that a shear flow is inhospitable for the wrong kind of vorticity. It is, therefore, quite well possible that the shielding due to a small Burger number is removed by the shear in an accretion disk. This makes it hard to apply the other results from Sect. 3.4 to an accretion disk.

### 3.6.2 Keplerian shear

In Sect. 3.5, shallow-water turbulence superimposed on a uniform shear flow was studied. Several of the conclusions drawn there also apply to 2D-turbulence in an accretion disk, as will be discussed in this section.

The most obvious observation from the shear flow experiments is that there is an asymmetry between prograde and retrograde vortices, Fig. 3.11. Retrograde vortices have the tendency to be stripped and torn apart. This behavior is known from a single vortex in a shear flow where only those vortices with peak vorticity larger then 7.4 times the shear (Legras and Dritschel, 1993) survive. In the experiments done here where many vortices interact, the requirement on peak vorticity is even higher. For example, in many experiments where the initial vorticity fluctuations were up to 17 times larger than the shear, no retrograde vortices were found after 50 initial eddy turnover times. So an accretion disk is much more likely to have prograde vortices than retrograde. This implies that the vortices will be dominantly
of anti-cyclonic nature.

Prograde vortices were found to withstand shear much better. But at high values of the shear, larger than the peak vorticity in the vortices, they are also lost. It was argued that vortices get elongated to such an extent that it is hard if not impossible to identify them as vortices. If the prograde vortices are not stretched then they are stripped, just as retrograde vortices at lower shear. It has not been possible to give a firm bound on the ratio peak vorticity to shear below which vortex stripping starts, because the position of the separatrix with respect to the vorticity distribution is determined not only by the relative strength of the shear but also by the relative strength and position of other vortices. So, the simulations suggest that there is a kind of lower limit to the peak vorticity of prograde vortices as well if they are supposed to withstand a given shear. The peak vorticity should be of the order of the shear or larger. This fact can be used to calculate a Rossby number for vortices in an accretion disk because the Rossby number is the ratio of relative to planetary vorticity. The relative vorticity of the vortices is of the order of the shear or larger, which is $3v_{\text{kep}}/2r$ in an accretion disk. In the previous Sect. 3.6.1 it is argued that the “planetary vorticity” of a Keplerian flow is $v_{\text{kep}}/r$ so that we find a Rossby number of 1.5 or larger. So these vortices are not in geostrophic equilibrium, an assumption made in previous work (Adams and Watkins, 1995). Strong (anti-cyclonic) vortices with large Rossby numbers are probably low pressure areas where the centrifugal force is balanced by the pressure force. For Rossby numbers of order unity the Coriolis force alone might be large enough to balance the centrifugal force, so that the perturbation in pressure is small.

With the simple continuity model for vortex elongation (semi-major over semi-minor axis $a/b \approx (\omega + s)/\omega$), the Rossby number of a vortex can be connected to its shape. If the background shear is the result of a Keplerian flow then we can relate the shear again to the “planetary vorticity”. This gives an estimate for the Rossby number of the vortex:

$$R_o = \frac{\omega}{f} \approx \frac{1.5}{a/b - 1}.$$  

This relation implies that geostrophic vortices in an accretion disk hardly look like Jovian or Earth like vortices, but are very elongated structures that are easily taken as filaments. They should also be far apart otherwise they are prone to vortex stripping. This implies that they will have few interactions with each other. From the point of view of mass and angular momentum transport in the radial direction through an accretion disk, this is probably not a very interesting situation. Efficient transport probably requires intense interactions among vortices and it is questionable if geostrophic vortices could live through this.

Another problem in accretion disks is how to create the required intense vortices. Rossby numbers of order unity and larger imply that we are dealing with huge disturbances of the Keplerian profile. At the inner edge of the accretion disk it may be in contact with a central compact object and large perturbations in vorticity can be expected (although not necessarily only in the component perpendicular to the disk). This is equally true for the outer edge where a jet of overflowing material impinges on the disk. But in the middle of the disk, there seems to be little cause
for strong vorticity perturbations. Perhaps they can be created by magnetic fields or the interaction of shock waves. The combination of an incident and a reflected shock and a Mach stem (which together form the letter Y) is known to produce strong shear layers (e.g., Srivastava, 1994) that might break up into vortices, but this is mere speculation. Besides, high resolution 2D-simulations of shocks in accretion disks (Godon, 1997) gave no indication of long-lived vortices.

Shear does not only put conditions on the strength of the surviving vortices but also on their size. Vortices in cross-stream direction larger than the intrinsic length scale $L_s = c_s/s$ have not been observed in the experiments. If the Keplerian shear is substituted, one obtains:

$$L_s = \frac{2c_s}{3\nu_{\text{kep}}}r,$$

which again is of the order of the thickness of the accretion disk. So it is not obvious how to produce long lived 2D-vortices that are larger in cross-stream direction than the thickness of the accretion disk.

For vortices in an accretion disk that have elongations of order 2 or 3 (which is common for vortices on Earth and Jupiter), the Rossby number is not small and they are also not larger than the thickness of the disk. This points in the direction of an internal inconsistency. From the start it was assumed that the fluid motion is essentially two-dimensional. Fluids are known to behave as if they are two-dimensional under a number of different conditions, the most important of which is that the Rossby number is small. Another is that the flow geometry is thin. If these conditions are violated, as seems to be the case for vortices in accretion disks, then it must be the stable stratification in the direction perpendicular to the disk that keeps the dynamics two-dimensional. If this condition is not satisfied, then there is no reason to expect long-lived two-dimensional vortices to be abundant. This does not exclude that they exist but it now comes as no surprise that most numerical studies of accretion disks do not show vortices. Firstly, often the perturbations in vorticity have been chosen too small to develop into vortices. Secondly, the resolution is often too low to resolve vortices smaller than the thickness of the disk (and I don’t expect larger vortices). Thirdly, in 3D-simulations that don’t show hydrostatic equilibrium in the vertical direction, long lived 2D-vortices are not expected.

The conclusions reached here are rather different from those of Bracco et al. (1998). They also noted the threshold on initial vorticity perturbation for vortices to develop in Keplerian shear. However, neither did they relate this to the peak vorticity of the vortices, nor did they come to the conclusion that the Rossby number is of order unity or larger in vortices (although this is visible in their Fig. 2). Also because they mainly did 2D-incompressible calculations, they didn’t find that $L_s$ acts as an upper limit to the vortex size. These are the reasons why they are much more optimistic about long-lived 2D-vortices in real accretion disks than the author of this article.

Acknowledgements. I want to thank prof. J. Kuijpers and prof. J.T.F Zimmerman for their help with this paper.
Appendix: A vortex recognition algorithm

An automatic vortex recognition algorithm was described by McWilliams (1990). Its main disadvantage is that it uses only one vorticity contour to determine if a flow structure is a vortex. Here it was preferred to use all points that belong to the candidate vortex for that decision. Therefore, a new algorithm to recognize vortices was constructed. Many of the steps are extensively described by McWilliams (1990). The main difference is that the Weiss field is used instead of the vorticity, and that the accuracy of a Gaussian fit replaces the vortex shape analysis.

- Determine the Weiss field (Brachet et al., 1988) out of the stream function. The Weiss field is strain squared minus the vorticity squared: \( S_1^2 + S_2^2 - \omega^2 \), where \( S_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, S_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \).
- Determine the local minima.
- If the minima are within 3 grid cells from each other then keep the strongest.
- Determine for each minimum the grid points that fall within the 0.1× minimum contour of the Weiss field. This contour sets the boundary of the vortex.
- Remove the candidate vortices that are too large or too small (6 grid cells).
- From those candidates that overlap only keep the one with the strongest minimum.
- Fit a Gaussian profile to the grid points of the candidate.
- Calculate the chi-squared value of the fit; if it is too large then remove the candidate vortex.

Candidates that pass these tests are accepted as a vortex.