Appendix B

Generalized spherical harmonics

The generalized scalar spherical harmonics (Phinney and Burridge, 1973) are defined by :

$$Y_l^{Nm}(\theta,\phi) = P_l^{Nm}(\cos\theta)e^{im\phi} \tag{B.1}$$

with the associated Legendre function

$$\begin{aligned} P_l^{Nm}(x) &= \frac{(-1)^{l-N}}{2^l(l-N)!}\sqrt{\frac{(l-N)!(l+m)!}{(l+N)!(l-m)!}} \\ &\times (1-x)^{-(m-N)/2}(1+x)^{-(m+N)/2} \\ &\times \frac{d^{l-m}}{dx^{l-m}}\left[(1-x)^{l-N}(1+x)^{l+N}\right] \end{aligned} \tag{B.2}$$

They are normalised :

$$\int\int Y_l^{Nm}(\theta,\phi)Y_{l'}^{Nm'}(\theta,\phi)\,d\Omega = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'} \tag{B.3}$$

In the particular case $N = 0$ we have :

$$Y_l^{0m}(\theta,\phi) = \sqrt{\frac{4\pi}{2l+1}}Y_l^m(\theta,\phi) \tag{B.4}$$

with

$$Y_l^m(\theta,\phi) = P_l^m(\cos\theta)e^{im\phi} \tag{B.5}$$

The components of any tensor can be decomposed on a generalized scalar spherical harmonic basis :

$$\Lambda^{\alpha,\beta,\gamma,\cdots}(r,\theta,\phi) = \sum_{s=|N|}^{\infty}\sum_{t=-s}^{s}\Lambda_{st}^{\alpha,\beta,\gamma,\cdots}(r)Y_s^{Nt}(\theta,\phi) \tag{B.6}$$

with $N = \alpha+\beta+\gamma+...$