Appendix B

Generalized spherical harmonics

The generalized scalar spherical harmonics (Phinney and Burridge, 1973) are defined by:

\[ Y_{\ell}^{N \cdot m}(\theta, \phi) = P_{\ell}^{N \cdot m}(\cos \theta) e^{im\phi} \]  

(B.1)

with the associated Legendre function

\[ P_{\ell}^{N \cdot m}(x) = \frac{(-1)^{l-N}}{2^l(l-N)!} \sqrt{(l-N)!(l+m)!} \]
\[ \times (1-x)^{(m-N)/2}(1+x)^{(m+N)/2} \]
\[ \times \frac{d^{l-m}}{dx^{l-m}} [(1-x)^{-N}(1+x)^{l+N}] \]  

(B.2)

They are normalised:

\[ \int \int Y_{\ell}^{N \cdot m}(\theta, \phi) Y_{\ell'}^{N' \cdot m'}(\theta, \phi) d\Omega = \frac{4\pi}{2\ell+1} \delta_{\ell\ell'} \delta_{mm'} \]  

(B.3)

In the particular case \( N = 0 \) we have:

\[ Y_{\ell}^{0 \cdot m}(\theta, \phi) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell}^{m}(\theta, \phi) \]  

(B.4)

with

\[ Y_{\ell}^{m}(\theta, \phi) = P_{\ell}^{m}(\cos \theta) e^{im\phi} \]  

(B.5)

The components of any tensor can be decomposed on a generalized scalar spherical harmonic basis:

\[ \Lambda^{\alpha,\beta,\gamma \ldots}(r, \theta, \phi) = \sum_{s=-|N|}^{N} \sum_{t=-s}^{s} \Lambda_{st}^{\alpha,\beta,\gamma \ldots}(r) Y_{s}^{N \cdot t}(\theta, \phi) \]  

(B.6)

with \( N = \alpha + \beta + \gamma + \ldots \)