Chapter 5

The effect of scattering in surface wave tomography

Abstract.
We present a new technique in surface wave tomography which takes the finite-frequency of surface waves into account using first-order scattering theory in a SNREI Earth. Physically, propagating surface waves with a finite-frequency are diffracted by heterogeneity distributed on a sphere and then interfere at the receiver position. Paradoxically, surface waves have the largest sensitivity to velocity anomalies off-path the geometrical ray. The non-ray geometrical effect is increasingly important for increasing period and distance. Therefore, it is expected that the violation of ray theory in surface wave tomography is most significant for the longest periods.

We applied scattering theory to phaseshift measurements of Love waves between periods of 40 s and 150 s to obtain global phase velocity maps expanded in spherical harmonics to angular degree and order 40. These models obtained with scattering theory were compared with those constructed with ray theory. We observed that ray theory and scattering theory predict the same structure in the phase velocity maps to degree and order 25-30 for Love waves at 40 s and to degree and order 12-15 for Love waves at 150 s. A smoothness condition was included in the phaseshift inversion for phase velocity maps, so we could not access the structure with smaller length-scale of velocity anomalies in the obtained Earth models.

We carried out a synthetic experiment for phase and group velocity measurements to investigate the limits of classical ray theory in surface wave tomography. In the synthetic experiment, we computed, using the source-receiver paths in the surface wave dataset, the discrepancy between ray theoretical and scattering theoretical phase velocity measurements and group velocity measurements, respectively, for an input-model with slowness heterogeneity for increasing angular degree. We found that classical ray theory in global surface wave tomography is only applicable for structures with angular degrees smaller

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than 30 (equivalent to 1300 km) and 20 (equivalent to 2000 km) for Love waves at 40 s and 150 s, respectively. The synthetic experiment suggests that the ray theoretical great circle approximation is appropriate to use in present-day global surface wave tomography. On the other hand, in order to obtain reliable global models with a higher resolution we must take the non-ray geometrical effect of surface waves into account.

5.1 Introduction

In surface wave tomography, global as well as regional models are obtained with increasing resolution. This increase in spatial resolution allows a comparison between tomographic models and detailed tectonic features. Most techniques for surface wave tomography are based on simplified versions of ray theory; see Backus (1964), Dziewonski (1984), Woodhouse and Dziewonski (1984), Trampert and Woodhouse (1995), van der Lee and Nolet (1997) and van Heijst and Woodhouse (1999) who all apply the great circle approximation to compute Earth models from surface wave data. However, ray theory introduces an inconsistency from a methodological point of view. It is only valid if the length-scale of velocity perturbations is larger than the wavelength and the width of the Fresnel zone. This condition is often violated for high-resolution S-velocity models compiled with ray theory because the characteristic length of heterogeneity in present surface wave models is comparable with the width of Fresnel zones (Passier and Snieder, 1995).

Several examples of scattering theory to explain wave propagation in heterogeneous media are given in the literature. Yomogida and Aki (1987), Yomogida (1992), Woodward (1992), Snieder and Lomax (1996) and Spetzler and Snieder (2001) use the Rytov approximation to derive the frequency-depending timeshift. In Spetzler and Snieder (2001), it is demonstrated explicitly that the timeshift can be computed as an integration of the slowness perturbation field multiplied by a sensitivity kernel (also known as the Fréchet kernel). Furthermore, Spetzler and Snieder (2001) confirm through a numerical experiment that the regime of scattering theory is important when the length-scale of inhomogeneity is smaller than the width of the Fresnel zone. Woodhouse and Girmuis (1982) and Snieder (1993) use normal mode theory to compute the Fréchet kernel for degree $l$ and order $m$ in surface wave tomography using spherical harmonics to expand the slowness perturbation field. Marquering et al. (1998), Tong et al. (1998), Marquering et al. (1999), Dahlen et al. (2000), Hung et al. (2000) and Zhao et al. (2000) apply a cross-correlation function to introduce the frequency-dependent timeshift in body wave tomography. It is shown in several of these articles that the sensitivity kernel for 3-D wave propagation vanishes on the geometrical ray and that the maximum sensitivity to slowness perturbations is off-path the ray. However surface wave tomography is a 2-D problem and the scattering theoretical sensitivity to slowness perturbations is non-zero on the ray path.

In this study, we develop a frequency-dependent scattering theory for minor and major arc surface waves by using the first-order Rytov approximation. The theory is applicable for unconverted surface waves in a SNREI Earth model. The scattering theory can be applied to both phase and group velocity measurements. Given the same strength of inhomogeneity, diffraction of surface waves becomes increasingly important when the
dominant period in the phaseshift dataset or the source-receiver distance increases. It is shown how relative phaseshifts and timeshifts measured from surface waves are linearly related to the coefficients of the spherical harmonics for relative phase and group velocity, respectively. Relative phaseshift measurements for Love waves at 40 s and 150 s from Trampert and Woodhouse (2001) are then inverted to obtain phase velocity maps using scattering theory.

We show a synthetic experiment wherein given the source-receiver paths in the surface wave data set the relative error introduced by ray theory is computed for slowness heterogeneities with increasing angular degree. The synthetic experiment shows that the diffraction of surface waves is dominant if the structure of the Earth exceeds the angular degree 20 (corresponds to the length-scale of inhomogeneity of about 2000 km) for surface waves at 150 s and angular degree 30 (the characteristic length of heterogeneity is 1300 km) for surface waves at 40 s. This is close to the current limit of resolution using ray theory that we obtain in the phase velocity maps in this article.

In section 5.2, the width of the Fresnel zone for surface waves is derived, and it is shown how to relate surface wave measurements (i.e. relative phaseshift and timeshift) with relative phase and group velocity perturbations on a sphere using ray theory and scattering theory. In addition, the properties of the obtained Fréchet kernels are discussed. In section 5.3, the setup of the surface wave experiment using Love waves between periods of 40 s and 150 s is explained. In section 5.4, the results of the inversion of relative phaseshifts for Love and Rayleigh waves at 40 s and 150 s are given. In section 5.5, a discussion of the small-scale structures of the Earth is given, and thereby the synthetic experiment is shown. The conclusions are drawn in section 5.6.

5.2 Theory

5.2.1 The width of Fresnel zones on the sphere

Fresnel zones are defined in terms of the difference in propagation length of rays with adjacent paths. The points inside the Fresnel zone are those points giving single-scattered waves which have a detour smaller than a certain fraction of the wavelength $\lambda$ compared with the ballistic ray (e.g. Kravtsov, 1988). This fraction of the wavelength is denoted $\lambda/n$, where the number $n = 8/3$ for waves propagating in two dimensions (Spetzler and Snieder 2001). Physically, waves scattered by points inside the first Fresnel zone interfere constructively at the receiver position. In the rest of this paper, the Fresnel zone refers strictly speaking to the first Fresnel zone. It is shown in appendix A how to derive the maximum width of Fresnel zones on the sphere. The epicentral distance between a given source and receiver geometry is denoted $\Delta_{\text{off}}$. The maximum width $L_F$ of Fresnel zones in radians is then given by

$$L_F = \sqrt{\frac{3\lambda}{2} \tan \left( \frac{\Delta_{\text{off}}}{2} \right)},$$

(5.1)
where \( \Delta_{\text{off}} \in [0, \pi] \) and the wavelength is in radians. The width of Fresnel zones increases with increasing wavelength and epicentral distance. In the limit that the source-receiver distance goes towards \( \pi \), the Fresnel zone converges to the whole sphere. The formula in Eq. (5.1) is derived using second order perturbation theory. Accordingly, the tangent function goes to infinity for the source-receiver offset \( \Delta_{\text{off}} \) going to \( \pi \) (i.e., the approximation breaks down).

### 5.2.2 Phase and group velocity maps using ray theory

Trampert and Woodhouse (1995) apply the ray theoretical great circle approximation (e.g., Backus, 1964; Jordan, 1978; Dahlen, 1979) to write the relative phaseshift \( \frac{\Delta \phi}{\phi_0} \) for minor arcs (i.e., \( 0 < \Delta_{\text{off}} < \pi \)) and major arcs (i.e., \( \pi < \Delta_{\text{off}} < 2\pi \)) as the relative velocity perturbation \( \frac{\delta v}{v_0} \) averaged over the ray path between the source and receiver, hence

\[
\frac{\Delta \phi}{\phi_0} (\Delta_{\text{off}}) = -\frac{1}{\Delta_{\text{off}}} \int_{r_S}^{r_R} \frac{\delta v}{v_0} (\theta, \phi) dr,
\]

where \( dr \) is in radians. The location of the source and receiver on the unit sphere is denoted \( r_S \) and \( r_R \), respectively, and the epicentral distance between the source and receiver is \( \Delta_{\text{off}} \). In addition, Trampert and Woodhouse (1995) write the relative velocity perturbation as a summation of spherical harmonics, thus

\[
\frac{\delta v}{v_0} (\theta, \phi) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} C_{l}^m Y_l^m (\theta, \phi).
\]

The upper limit in the spherical expansion of the relative velocity perturbation is denoted \( l_{\text{max}} \), and the coefficient of spherical harmonics to angular degree \( l \) and order \( m \) for relative phase velocity is written as \( C_{l}^m \). The relative phaseshift is then expressed in spherical harmonics by inserting Eq. (5.3) in Eq. (5.2) which gives that

\[
\frac{\Delta \phi}{\phi_0} (\Delta_{\text{off}}) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} C_{l}^m K_{l,m}^{\text{ray,ph}} (\Delta_{\text{off}}),
\]

with the ray theoretical sensitivity kernel for angular degree \( l \) and order \( m \) equal to

\[
K_{l,m}^{\text{ray,ph}} (\Delta_{\text{off}}) = -\frac{1}{\Delta_{\text{off}}} \int_{r_S}^{r_R} Y_l^m (\theta, \phi) dr.
\]

Similarly, group velocity maps are retrieved from time delays obtained by bandpass-filtered surface waveforms in a frequency-band \( 2\Delta v \) around the central frequency \( v_0 \). In terms for ray theory, the timeshift \( \delta t (\Delta_{\text{off}}) = \Delta \phi (\Delta_{\text{off}}) / (2\pi v) \) at offset \( \Delta_{\text{off}} \) is given by

\[
\delta t (\Delta_{\text{off}}) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} U_{l}^m K_{l,m}^{\text{ray,gr}} (\Delta_{\text{off}}),
\]
where the coefficient of spherical harmonics to degree $l$ and order $m$ for relative group velocity is denoted $U_{l,m}$, and the sensitivity kernel for relative group velocity is

$$K_{lm}^{ray,gr} (\Delta_{off}) = -\frac{R}{v_0} \int_{r_s}^{r_k} Y_{lm}^m (\theta, \phi) dr. \quad (5.7)$$

The reference velocity at the central frequency is denoted $v_0$, and $R$ is the radius of the Earth.

Ray theory is valid when the characteristic length $a$ of heterogeneity is much larger than the wavelength $\lambda$ and the width of Fresnel zones $L_F$. Hence in non-dimensional numbers the condition for ray theory is written as:

$$\frac{\lambda}{a} \ll 1, \quad \text{and} \quad \frac{L_F}{a} \ll 1, \quad (5.8)$$

see Menke and Abbot (1990).

### 5.2.3 Phase and group velocity maps using to scattering theory

The theory of diffracted surface waves is developed for minor and major arc measurements using the first-order Rytov approximation. First the relative phaseshift for minor arcs is derived for a reference system using co-latitude coordinates with the source position at $(\pi/2, 0)$ and the receiver position at $(\pi/2, \Delta_{off})$. We derive the scattering sensitivity kernels that relate the relative phaseshift and the time residual linearly with the relative phase and group velocity, respectively, that are expanded in spherical harmonics. Then, we show how surface wave measurements for major arcs using scattering theory are derived from the developed theory for minor arcs. Finally, we show that phase and group velocity measurements for any source-receiver configuration can be computed in a fast way by rotating tabulated sensitivity kernels in the reference system.

Snieder and Nolet (1987) and Snieder and Romanowicz (1988) linearise the Lamé coefficients $\lambda$ and $\mu$ and the density $\rho$ to write the Born vector wavefield $u_1(r)$ as

$$u_1(r) = P(R, \theta_s, \phi_s)[P(R, \theta_r, \phi_r) \cdot F] \times \int_{0}^{\Delta_{off}} \int_{0}^{\pi} \frac{\exp[i(kR\Delta_2 + \frac{2\pi}{\sqrt{2kR\sin(\Delta_2)}})]}{\sqrt{2kR\sin(\Delta_2)}} V(R, \theta, \phi) \times \frac{\exp[i(kR\Delta_1 + \frac{2\pi}{\sqrt{2kR\sin(\Delta_1)}})]}{\sqrt{2kR\sin(\Delta_1)}} R^2 \sin(\theta) d\theta d\phi, \quad (5.9)$$

which is derived for wave propagation on the sphere. The adiabatic assumption (i.e. there is no mode conversion between different modes of Love and Rayleigh waves) is applied in Eq. (5.9) so there is no summation over modes and mode conversions are absent. The polarisation vector at the source $(R, \theta_s, \phi_s)$ and at the receiver $(R, \theta_r, \phi_r)$ is $P$, the wavenumber is $k$ for surface waves, the epicentral distances between the source and scatterer and between the scatterer and receiver are denoted $\Delta_1$ and $\Delta_2$, respectively, the Fourier transform of the point source function is $F$ and the scattering coefficient
is $V$. The expression in Eq. (5.9) then reads as follows; The polarised point source $\mathbf{P}(R, \theta_s, \varphi_s) \cdot \mathbf{F}$ excites the surface wave. The surface wave propagates to the scattering point $(R, \theta, \varphi)$: the phaseshift and the geometrical factor are determined by the propagating term $\exp(i (kR \Delta_1 + \frac{\pi}{4})/\sqrt{\frac{2}{k} kR \sin(\Delta_1)})$. The wavefield is scattered with an amplitude determined by the interaction term $V$. The scattered wavefield propagates to the receiver; the phaseshift and the geometrical factor are determined by the propagating term $\exp(i (kR \Delta_2 + \frac{\pi}{4})/\sqrt{\frac{2}{k} kR \sin(\Delta_2)})$. Finally the recorded signal is given by the polarisation $\mathbf{P}(R, \theta_r, \varphi_r)$. Snieder (1986) shows that for unconverted surface waves the interaction term $V$ can be written as

$$V(R, \theta, \varphi) = -\frac{k^2}{2} \frac{\delta v}{v_0} (R, \theta, \varphi), \quad (5.10)$$

where the reference phase velocity and the phase velocity perturbation is $v_0$ and $\delta v$, respectively.

Given the measurement of the $i$-component of the displacement, the phaseshift $\delta \varphi^{(i)} (\Delta_{\text{off}}, V)$ of the surface waves is obtained from

$$\delta \varphi^{(i)} (\Delta_{\text{off}}, V) = \text{Im} \{ \frac{\mu^i_1(r_r)}{\mu^i_0(r_r)} \}, \quad (5.11)$$

where the unperturbed vector wavefield $\mathbf{u}_0(r_r)$ is given by

$$\mathbf{u}_0(r_r) = \mathbf{P}(R, \theta_r, \varphi_r) \exp(i (kR \Delta_{\text{off}} + \frac{\pi}{4})/\sqrt{\frac{2}{k} kR \sin(\Delta_{\text{off}})}) \mathbf{P}(R, \theta_s, \varphi_s) \cdot \mathbf{F}, \quad (5.12)$$

for the epicentral distance $\Delta_{\text{off}}$ between the source and receiver (Snieder, 1986). The expression in Eq. (5.11) generalises the Rytov approximation (e.g. Yomogida and Aki, 1987; Snieder and Lomax, 1996; Spetzler and Snieder, 2001) to elastic waves.

The detour $\Delta_1 + \Delta_2 - \Delta_{\text{off}}$ and the geometrical factors $\sin(\Delta_1)$ and $\sin(\Delta_2)$ in Eq. (5.9) are perturbed to second and zeroth order in the path deflection $(\theta - \frac{\pi}{2})$, respectively. For a source-receiver geometry along the equator line, the detour is given by

$$\Delta_1 + \Delta_2 - \Delta_{\text{off}} \approx \frac{(\theta - \frac{\pi}{2})^2}{2} \frac{\sin(\Delta_{\text{off}})}{\sin(\varphi) \sin(\Delta_{\text{off}} - \varphi)}, \quad (5.13)$$

and the geometrical factors are

$$\sin(\Delta_1) \approx \sin(\varphi), \quad \text{and} \quad \sin(\Delta_2) \approx \sin(\Delta_{\text{off}} - \varphi), \quad (5.14)$$

(see appendix A). The relative phaseshift $\frac{\delta \varphi^{(i)}}{\varphi_0} (\Delta_{\text{off}}, V)$ for the $i^{th}$ receiver is derived by inserting Eq. (5.9), (5.10) and (5.12) in Eq. (5.11), then dividing with the phase $\varphi_0 = kR \Delta_{\text{off}}$ and finally using the Taylor approximation for the detour and for the geometrical factors in Eq. (5.13) and (5.14), respectively, thus

$$\frac{\delta \varphi^{(i)}}{\varphi_0} (\Delta_{\text{off}}, V) = \int_0^{\Delta_{\text{off}}} \int_0^{\varphi} K(R, \theta, \varphi) \frac{\delta v}{v_0} (R, \theta, \varphi) d\theta d\varphi, \quad (5.15)$$
where the wavenumber \( k = 2\pi v/v_0 \), and the sensitivity kernel \( K(R, \theta, \varphi) \) for the relative velocity perturbation field is given by

\[
K(R, \theta, \varphi) = -\frac{\sin(\theta)}{\Delta \nu} \sqrt{v R \sin(\Delta \nu)} \frac{\sin \left( \frac{2\pi v_0 (\theta - \varphi)}{v_0} \right)^2 \sin(\Delta \nu - \varphi)}{\sin(\varphi) \sin(\Delta \nu - \varphi)}
\]

The relative velocity perturbation \( \delta v(R, \theta, \phi) \) is written as an expansion of spherical harmonics as shown in Eq. (5.3). The relative phaseshift in Eq. (5.15) is then given by

\[
\frac{\delta \phi}{\varphi_0} (\Delta \nu, \nu) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} C_l^m \int_0^{\Delta \nu} \int_0^{\pi} Y_l^m(\theta, \varphi) K(R, \theta, \varphi) d\theta d\varphi
\]

The right-hand side of the relative phaseshift due to scattering in Eq. (5.17) has the same form as the ray theoretical phaseshift in Eq. (5.4), but with the scattering sensitivity kernel at frequency \( \nu \) for minor arcs given by

\[
K_{\text{scat}, ph}^{l, m} (\Delta \nu, \nu) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} U_l^m \int_0^{\Delta \nu} \int_0^{\pi} Y_l^m(\theta, \varphi) K(R, \theta, \varphi) d\theta d\varphi.
\]

In order to obtain group velocity maps using scattering theory, the frequency-averaged, timeshift \( \delta t(i) (\Delta \nu) \) in the frequency-range \( v_0 - \Delta v \) and \( v_0 + \Delta v \) is derived, hence

\[
\delta t(i) (\Delta \nu) = \frac{1}{2 \Delta v} \int_{v_0-\Delta v}^{v_0+\Delta v} \delta t(i) (\Delta \nu, \nu) d\nu
\]

where we use that \( \delta t(i) (\Delta \nu, \nu) = \delta t(i) (\Delta \nu, \nu) / (2\pi \nu) \), and the coefficients of the spherical harmonics for relative group velocity are denoted by \( U_l^m \). The minor arc, scattering theoretical sensitivity kernel for relative group velocity in Eq. (5.19) is given by

\[
K_{\text{scat}, gr}^{l, m} (\Delta \nu, \nu) = \frac{\Delta \nu R}{2 \Delta v} \int_{v_0-\Delta v}^{v_0+\Delta v} K_{\text{scat}, ph}^{l, m} (\Delta \nu, \nu) \frac{d\nu}{v_0(\nu)}.
\]

In general, the reference velocity \( v_0(\nu) \) depends upon the frequency in Eq. (5.20).

The relative phaseshift for major arcs is obtained using the scattering theory for minor arcs. This is because major arc scattering sensitivity kernels can be constructed from three scattering sensitivity kernels for minor arcs; one sensitivity kernel for the minor arc between the source \( r_s \) and the receiver anti-pod \( r_{RA} \), between the receiver anti-pod and the source anti-pod \( r_{SA} \) and between the source anti-pod and the receiver \( r_s \), respectively.
For major arcs, the scattering sensitivity kernel for relative phase velocity derived in the reference system is

\[
K_{scat, ph}^{l, m}(\Delta \text{off}, \nu) = \sum_{\nu=\pm} \left[ \frac{1}{2\pi} \left( (\Delta \text{off} - \pi) K_{scat, ph}^{l, m, S\rightarrow RA}(\Delta \text{off} - \pi, \nu) + (2\pi - \Delta \text{off}) K_{scat, ph}^{l, m, RA\rightarrow SA}(2\pi - \Delta \text{off}, \nu) + (\Delta \text{off} - \pi) K_{scat, ph}^{l, m, SA\rightarrow R}(\Delta \text{off} - \pi, \nu) \right) \right],
\]

(5.21)

where \(K_{scat, ph}^{l, m, S\rightarrow RA}(\Delta \text{off} - \pi, \nu)\), \(K_{scat, ph}^{l, m, RA\rightarrow SA}(2\pi - \Delta \text{off}, \nu)\) and \(K_{scat, ph}^{l, m, SA\rightarrow R}(\Delta \text{off} - \pi, \nu)\) are the relative phase velocity sensitivity kernels due to scattering for the minor arc between the source and receiver anti-pod, between the receiver anti-pod and the source anti-pod and between the source anti-pod and receiver, respectively. Similarly, the major arc sensitivity kernel for relative group velocity using scattering theory is given by

\[
K_{scat, gr}^{l, m}(\Delta \text{off}) = K_{scat, gr}^{l, m, S\rightarrow RA}(\Delta \text{off} - \pi) + K_{scat, gr}^{l, m, RA\rightarrow SA}(2\pi - \Delta \text{off}) + K_{scat, gr}^{l, m, SA\rightarrow R}(\Delta \text{off} - \pi).
\]

(5.22)

The expansions in Eq. (5.21) and (5.22) are derived in appendix B.

Dziewonski (1984) and Dahlen and Tromp (1998) explain how to rotate the reference system so that the source-receiver configuration, originally aligned along the equator, can be anywhere on the sphere. In appendix C, the relative phaseshift related to relative phase velocity for any minor arc, as well as, major arc with the source position \(r_S\) and the receiver position \(r_R\) is derived. The sensitivity kernel in the observed coordinate system is given by

\[
K_{scat, ph}^{l, m}(\Delta \text{off}, \nu) = \exp(im\Phi) \sum_{n=-l}^{l} \exp(in\Psi) Q_{l,m,n}(\Theta) K_{scat, ph}^{l, m, \nu}(\Delta \text{off}, \nu),
\]

(5.23)

where \(\Phi, \Psi, \Theta\) are the three Euler angles, \(Q_{l,m,n}(\Theta)\) are the elements of the rotation matrix and the sensitivity kernel \(K_{scat, ph}^{l, m, \nu}(\Delta \text{off}, \nu)\) is computed in the reference system where the source and receiver lie on the equator. This result also holds for group velocity measurements.

The regime of surface wave scattering theory is significant when the scale-length of heterogeneity is smaller than the width of the Fresnel zone (e.g. the conditions for ray theory are not satisfied). Let the characteristic length of inhomogeneity \(a = 2\pi/\lambda\) (in radians) for the angular degree \(l\). By using the condition for scattering theory, we can derive the limits of classical ray theory expressed in the angular degree of the spherical harmonics. Hence, when

\[
\frac{L_F}{a} > 1 \Rightarrow l > \sqrt{\frac{8\pi^2}{3\lambda \tan \left( \frac{2\Delta \text{off}}{2} \right)}},
\]

(5.24)

the regime of scattering theory is important.
5.2 Theory

5.2.4 The properties of the scattering sensitivity kernels

If the reference velocity is constant, the frequency integration in the sensitivity kernel due to scattering theory in Eq. (5.20) can be done analytically. The integration of the function \( \sqrt{x} \sin(ax + \pi/4) \) is

\[
\int \sqrt{x} \sin(ax + \pi/4) \, dx = \frac{\sqrt{\pi}}{2a} \left( \sin(ax) - \cos(ax) \right) + \frac{\sqrt{\pi}}{a^{3/2}} \left( C(\sqrt{2ax/\pi}) - S(\sqrt{2ax/\pi}) \right),
\]

(5.25)

where the functions \( C \) and \( S \) are the Fresnel cosine integral and sine integral, respectively. See Abramowitz and Stegun (1970) for a description of the Fresnel cosine and sine integrals. This analytical result can be used to compute the scattering sensitivity kernels in Eq. (5.20) in an efficient and accurate manner. On the other hand if the reference velocity is dispersive, the frequency integration must be carried out numerically.

It is instructive to look at the sensitivity to the relative phase and group velocity fluctuations for a minor arc surface wave and a major arc surface wave as shown in Fig. 5.1A and 5.1B. The source position is located at latitude \( (0^\circ, 0^\circ) \), and the receiver position is at \( (0^\circ, 70^\circ) \), thus the epicentral distance for the minor arc is 70\(^\circ\), while for the major arc the source-receiver distance is 290\(^\circ\). The reference velocity \( v_0 = 4779 \text{ m/s} \) which is the PREM phase velocity for Love waves at 150 s. The radius of the sphere is set to 6371 km. The sensitivity to the relative phase velocity in Fig. 5.1A is computed with Eq. (5.16). For group velocity measurements, the frequency-band is chosen proportional to the central frequency so that an optimal fit of waveforms is obtained simultaneously in the time and frequency domain. For example, surface waves at 40 s are bandpass-filtered between 20 s and 60 s while at 150 s the periodband is 150 s. Therefore, the sensitivity kernel using scattering theory for relative group velocity in Fig. 5.1B is frequency integrated between 75 s and 225 s for Love waves at the central period equal to 150 s. The sensitivity kernel for relative group velocity measurements in Fig. 5.1B is obtained by using Eq. (5.16) multiplied by \( \Delta v R/v_0(v) \) and then averaging over the frequency-band \( v_0 - \Delta v \) and \( v_0 + \Delta v \) where the frequency-dependent PREM reference velocity is taken into account. The black zones in the nearfield at the source, source anti-pod, receiver anti-pod and receiver show the singularities in the geometrical factors of the scattering sensitivity kernels for minor arcs and major arcs. The sensitivity to the relative phase and group velocity resembles those of the form of Fresnel zones for point sources. The sensitivity kernel for relative phase velocity clearly shows the first Fresnel zone, as well as, higher order Fresnel zones. Notice that the sidelobes corresponding to higher order Fresnel zones do not vanish if we take the frequency averaging, inherent to the measurement, into account. In order to obtain the phase velocity measurements from Trampert and Woodhouse (2001), the width of the bandpass-filter is 5 mHz. The form of the scattering sensitivity kernels for phase velocity measurements is also shown by Woodhouse and Girnus (1982) and Snieder (1993) who apply normal mode theory to compute the sensitivity to slowness perturbations due to scattering theory in surface wave tomography.
Figure 5.1: The scattering sensitivity kernel for relative phase and group velocity perturbations computed point by point on the sphere. The epicentral distance is 70° for the minor arc and 290° for the major arc. The source position is denoted by $S$, the receiver anti-pode position is $RA$, the source anti-pode position is $SA$ and the receiver position is $R$. The sensitivity kernel due to scattering theory for the major arc surface wave is constructed by three scattering sensitivity kernels for minor arc surface waves. A) The sensitivity kernel for relative phase velocity perturbations is calculated for Love waves with the single period at 150 s. The sensitivity kernel has sidelobes so that the first Fresnel zone and higher order Fresnel zones are visible. B) The sensitivity kernel for relative group velocity fluctuations is computed between 75 s and 225 s taking the frequency-dependence of the PREM phase velocity for Love waves into account. The sidelobes of the sensitivity kernel at different frequencies interfere destructively. The relative phase shift is therefore only sensitive to relative group velocity inside the Fresnel zone.
Notice that the sensitivity kernels in Woodhouse and Girnui (1982) and Snieder (1993) have oscillations along the great circle as a result of the interference of different surface wave orbits. The scattering sensitivity kernel for relative group velocity in Fig. 5.1B only shows the first Fresnel zone. This is because of the frequency integration that causes destructive interference of sidelobes at higher order Fresnel zones. According to ray theory, the sensitivity kernel is only non-zero on the great circle passing through the source and receiver at 0° latitude.

In Fig. 5.2, the cross sections of the scattering sensitivity kernels in Fig. 5.1A and 5.1B are plotted for different periods and epicentral offsets. The sensitivity kernels for relative phase and group velocity are calculated at the half epicentral offset where the width of the Fresnel zone is maximum. In Fig. 5.2A, the sensitivity kernels for relative phase velocity are estimated for the period at 40 s (solid line), 100 s (dotted line) and 150 s (dashed line) using the PREM model for the reference velocity, and the epicentral distance is set to 160°. In Fig. 5.2B, the sensitivity kernels for relative group velocity are computed with numerical frequency integration taking account of the PREM model for the central periods at 40 s (solid line), 100 s (dotted line) and 150 s (dashed line) with the periodbands set equal to the central period. The source-receiver distance is fixed to 160° for the curves in Fig. 5.2B. Notice that computing the scattering sensitivity kernel using the complete PREM model in the range of frequency integration yields nearly the same result as using the PREM phase velocity at central period as constant reference velocity in the frequency-range. We do not show any scattering sensitivity kernels for group velocity measurements calculated with a constant reference velocity over the frequency range of integration because they are almost indistinguishable from the scattering sensitivity kernels for group velocity measurements taking the PREM phase velocity into account. The sensitivity kernels in Fig. 5.2C and 5.2D are computed with the period fixed to 150 s, and the epicentral distance is 60° (solid line), 110° (dotted line) and 160° (dashed line). For Fig. 5.2D, the periodband of the frequency integration is equal to the central period. In brief, Fig. 5.2 shows that the width of the central lobe of the scattering sensitivity kernel increases for increasing period and source-receiver distance. In terms of ray theory, the sensitivity to relative phase and group velocity perturbations is only non-zero at 0° latitude for the given source-receiver configuration in Fig. 5.2.

Ray theory and scattering theory predict the same relative phaseshift when the length-scale of heterogeneity is larger than the width of the Fresnel zone (i.e the condition for the regime of ray theory) since it follows from expression (5.16) that

\[
\int_0^{\Delta_{\text{off}}} \int_0^\pi K(R, \theta, \phi) \frac{\delta v}{v_0} (R, \theta, \phi) d\theta d\phi = \frac{1}{\Delta_{\text{off}}} \int_0^{\Delta_{\text{off}}} \frac{\delta v}{v_0} (\theta, \phi) dr, \tag{5.26}
\]

when the characteristic length of the relative phase velocity is larger than the width of the Fresnel zone.

The maximum width \( W \) of the central lobe of the scattering sensitivity kernel is com-
Figure 5.2: Cross sections of the scattering sensitivity kernels for relative phase and group velocity perturbations computed at the half epicentral distance. A) The epicentral distance is 160° for the three curves. The cross section of the sensitivity kernel for relative phase velocity fluctuations is computed at 40 s (solid line), 100 s (dotted line) and 150 s (dashed line). B) The epicentral distance is 160° for the curves. The cross section of the sensitivity kernel for relative group velocity perturbations is computed in the period band with the central period $T_0 = 40$ s and the half period band $\Delta T = 20$ s (solid line), $T_0 = 100$ s and $\Delta T = 50$ s (dotted line) and $T_0 = 150$ s and $\Delta T = 75$ s (dashed line). C) Sensitivity kernels at 150 s for relative phase velocity fluctuations. The epicentral distance for the cross section of the scattering sensitivity kernel is 60° (solid line), 110° (dotted line) and 160° (dashed line). D) The cross section of three sensitivity kernels for relative group velocity fluctuations with the epicentral distance at 60° (solid line), 110° (dotted line) and 160° (dashed line). The sensitivity kernels for relative group velocity fluctuations are integrated between 75 s and 225 s including the frequency-dependence of the PREM phase velocity.
computed by setting the sine function in Eq. (5.16) equal to zero, hence

\[
0 = \sin \left( \frac{\pi vR}{v_0} \left( \theta - \frac{\pi}{2} \right)^2 \frac{\sin(\Delta_{\text{off}})}{\sin(\phi) \sin(\Delta_{\text{off}} - \phi)} + \frac{\pi}{4} \right) \Rightarrow \\
\left( \theta - \frac{\pi}{2} \right)^2 = \frac{3\lambda \sin(\phi) \sin(\Delta_{\text{off}} - \phi)}{4 \sin(\Delta_{\text{off}})}, \tag{5.27}
\]

where \( \lambda = v_0/(\nu R) \) is the central wavelength in radians. The maximum width \( W = 2|\theta - \pi/2| \) of the central lobe is obtained by setting \( \phi = \Delta_{\text{off}}/2 \) in Eq. (5.27) which gives that

\[
W = \sqrt{\frac{3\lambda}{2} \tan \left( \frac{\Delta_{\text{off}}}{2} \right)}. \tag{5.28}
\]

By comparing the maximum width of the central lobe in Eq. (5.28) with the width \( L_F \) of Fresnel zones on the sphere in Eq. (5.38), the number \( n \) that defines the width of the Fresnel zone is given by

\[
n = \frac{8}{3}. \tag{5.29}
\]

This result is also derived in Spetzler and Snieder (2001) in a 2-D, Cartesian coordinate system. Additionally, we identify the central lobe of the scattering sensitivity kernel as the Fresnel zone on the sphere.

According to Eq. (5.24), scattering theory is significant when the width of the Fresnel zone is larger than the length-scale of heterogeneity. We see in Fig. 5.2 that the Fresnel zone of surface waves enlarges for increasing period and epicentral offset. Therefore given the same strength of heterogeneity, scattering theory is most important for the longest period surface waves and if there are many long epicentral offsets in a given surface wave data set.

5.3 Setup of the surface wave experiment

The dataset of observed relative phaseshifts is from Trampert and Woodhouse (2001), who calculate global phase velocity maps of Love and Rayleigh waves for periods between 40 s and 150 s using the great circle approximation. The source and receiver positions corresponding to the measured phaseshifts are corrected for the ellipticity of the Earth. We use the observed relative phaseshifts to compute new phase velocity maps at 40 s and at 150 s, but using the scattering theory for fundamental-mode Love waves.

The maximum degree of the spherical expansion of the phase velocity maps is 40, thus the number of unknown model parameters to be inverted is 1681. In addition, we use the same inversion procedure as Trampert and Woodhouse (1995); an a priori Laplacian smoothness condition is implemented so that truncation problems are avoided. In this manner, using the same data set and inversion method as Trampert and Woodhouse (2001), it is possible to make a direct comparison between global phase velocity maps between periods at 40 s and 150 s inferred from ray theory and scattering theory, respectively. In Table 5.1, the a priori reference PREM velocity for Love waves at 40 s and 150 s and the number of observed relative phaseshifts are given.
### Table 5.1

<table>
<thead>
<tr>
<th>PREM velocity at 40 s (m/s)</th>
<th>4440</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREM velocity at 150 s (m/s)</td>
<td>4779</td>
</tr>
<tr>
<td>Number of obs. rel. phaseshifts</td>
<td>41016</td>
</tr>
</tbody>
</table>

*Table 5.1: The PREM reference velocity at 40 s and 150 s and the number of observed relative phasesshift measurements for Love waves.*

## 5.4 Results

In this section, we present the phase velocity maps that are obtained with ray theory and scattering theory using Love wave phase measurements between periods of 40 s and 150 s. We do not show any results retrieved from Rayleigh waves which lead to the same conclusions that we draw from the Love wave phase velocity maps. We hardly find any discrepancy between the phase velocity maps for Love waves at either 40 s and 150 s obtained from ray theory and scattering theory. The difference between the phase velocity maps compiled with scattering theory and the ones computed using ray theory are shown in Fig. 5.3A and 5.3B for the global Love wave experiment at 40 s and 150 s, respectively.

The powerspectra of the phase velocity maps in Fig. 5.4 confirm the qualitative observation that ray theory and scattering theory produce the same models. For Love waves at 40 s, the Laplacian smoothness factor $\gamma = 1 \times 10^{-4}$, while for the surface wave study at 150 s, $\gamma = 1 \times 10^{-2}$. Phase measurements for Love waves at 150 s are quite noisy which cause unrealistic small-scale structure in the phase velocity maps using too small a value for the smoothness factor. As a result of the values of the Laplacian smoothness parameter, small-scale structures for angular degrees higher than 20-25 and 10-15 (e.g. heterogeneity with a characteristic length of 1600-2000 km and 2700-4000 km, respectively) are strongly suppressed in the phase velocity maps for Love waves at 40 s and 150 s, respectively. On the other hand, the Fresnel zone for Love waves at 40 s and at 150 s with the characteristic epicentral distance equal to 100° has a maximum width of about 1400 km (angular degree $\approx 28$) and 2800 km (angular degree $\approx 14$), respectively. Hence according to the condition for the regime of ray theory in Eq. (5.8), it is approximately correct to apply ray theory for the obtained phase velocity maps (not shown in the paper).

The smoothness parameters for the scattering theoretical inversion of Love waves at 40 s and 150 s are determined in the following way; the derivation matrix $G$ (see Menke, 1989) built from the kernels $k_{i,m}$ is not necessarily the same for ray theory and scattering theory. Thus, the two theories will in general not resolve models identically for a given smoothness parameter. We require for Love waves at a given period that the trace of the resolution matrix for ray theory closely resembles to that for scattering theory. Then we can compare models built for the same number of parameters.
Figure 5.3: The difference between the phase velocity maps obtained using scattering theory and ray theory for Love wave at 40 s and 150 s. The difference in relative phase velocity are given in percent on a scale between ± 2%. Plate boundaries and hotspots are drawn with white lines and circles, respectively. The coastlines are marked with black lines on the difference between the phase velocity maps compiled using scattering theory and ray theory. A) Love wave at 40 s. The smoothness factor $\gamma = 1 \times 10^{-4}$. B) Love waves at 150 s. The smoothness factor $\gamma = 1 \times 10^{-2}$. 
Figure 5.4: The power spectra of the estimated phase velocity maps for Love waves at 40 s and 150 s using ray theory and scattering theory. The degree of the coefficients of spherical harmonics is shown on the abscissa, while the magnitude of the powerspectra is plotted on the ordinate. It is observed that the phase velocity models for Love waves at 40 s and 150 s have the same large-scale structure when using scattering theory and ray theory. However, it is not possible to obtain reliable smaller scale structures in the obtained phase velocity maps because the observed relative phaseshifts requires a rather large Laplacian smoothness factor for Love waves at 150 s. The smoothness factors applied in the inversion of phase velocity measurements for Love waves at 40 s and 150 s are the numbers in the brackets.

5.5 Discussion

In the inversion of phaseshift data for Love waves between periods at 40 s and 150 s, ray theory and scattering theory compile the same large-scale structure as shown in Fig. 5.3. Because of the large value of the smoothness parameter, it is not possible to comment on the presence of smaller scaled structures of the Earth. In order to examine the discrepancy between ray theory and scattering theory in surface wave tomography, synthetic tests should be carried out using an input-model with heterogeneity which is much smaller in size than the width of the Fresnel zone.

Spetzler and Snieder (2001) and Spetzler et al. (2001) show that scattering theory is very accurate in the prediction of timeshifts obtained from a finite-difference solution of the acoustic wave equation and from a laboratory ultrasonic wave experiment, respectively, wherein the length-scale of heterogeneity is smaller than the width of the Fresnel zone. We believe that the same holds for surface wave tomography.

In Fig. 5.5, we show with a synthetic surface wave experiment that the discrepancy between ray theory and diffraction theory in global surface wave tomography is important for heterogeneity with the angular degree larger than $l = 30$ and $l = 20$ for Love waves.
waves at 40 s and 150 s, respectively. Using the results from Spetzler and Snieder (2001) and Spetzler et al. (2001), we assume that surface wave scattering theory for phase and group velocity measurements is correct to use in models with any scale-length of velocity anomalies. We calculate in Fig. 5.5 the relative error in percent introduced by ray theory using the source and receiver positions in the dataset for Love waves, hence

$$\text{relative error} = \frac{100}{N} \sum_{i=1}^{N} \left| \frac{d_i^{\text{ray}} - d_i^{\text{scat}}}{d_i^{\text{scat}}} \right|,$$

where $N$ is the number of source-receiver geometries and $d_i^{\text{ray}}$ and $d_i^{\text{scat}}$ are the surface wave data due to ray theory and scattering theory, respectively. To avoid numerical instability, source-receiver pairs with $|d_i^{\text{scat}}| \leq 1 \times 10^{-3}$ for phase velocity measurements and $|d_i^{\text{scat}}| \leq 1$ s for group velocity measurements have not been included in Eq. (5.30). The velocity perturbation is set to 10% and the angular order $m$ is fixed to zero, while the angular degree goes from 1 to 40 corresponding to the size of velocity heterogeneity between 40000 km and 1000 km in the synthetic experiment. The ray theoretical approach based on the great circle approximation and the first order scattering theory are both linear theories, so the amplitude of the velocity perturbation does not influence the relative error in Eq. (5.30). Thus, for realistic Earth models with either a white or a red spectrum, the synthetic experiment presented in this paper indicates to which extent the ray theoretical great circle approximation differs from a more exact scattering theory. In Fig. 5.5A, the relative error due to ray theory in surface wave tomography for phase velocity measurements is calculated using the sensitivity kernel for ray theory in Eq. (5.5) and the sensitivity kernel due to diffraction theory in Eq. (5.18). In Fig. 5.5B, we show the relative error introduced by ray theory in tomographic surface wave experiments with group velocity measurements for which we have applied Eq. (5.7) for the ray theoretical sensitivity kernel and Eq. (5.20) for the scattering theoretical sensitivity kernel. The relative error due to the great circle approximation should not exceed the observational relative error in the data. The phase velocity measurements from Trampert and Woodhouse (2001) have a relative error of about 20% for Love waves at 40 s and a relative error of 40% for Love waves at 150 s. Using the results in Fig. 5.5, we see that ray theoretical surface wave tomography is limited to angular degrees smaller than $l = 30$ and $l = 20$ for Love waves at 40 s and 150 s, respectively. However, if we proceed to slightly higher angular degrees we must certainly take the non-ray geometrical effect of surface waves into account. Otherwise, we may obtain inaccurate surface wave Earth models because of the inappropriate use of ray theory.

In Fig. 5.6A to 5.6F, we present plots of the scattering theoretical phaseshift versus the ray theoretical phaseshift for Love waves at 150 s. Fig. 5.6 is similar to the plots that are found in Baig et al. (2000). The source-receiver positions in the surface wave dataset from Trampert and Woodhouse (2001) are applied. Spherical harmonics input models with the length-scale of inhomogeneity related to the angular degree $l$ are used in Fig. 5.6A ($l = 1$), 5.6B ($l = 5$), 5.6C ($l = 15$), 5.6D ($l = 20$), 5.6E ($l = 30$) and 5.6F ($l = 40$). We have chosen to plot the normalised phaseshifts calculated with scattering theory and ray theory. The solid lines indicate the error in the Love waves dataset at 150 s. We see
Figure 5.5: The synthetic experiment showing that the relative error introduced by ray theory increases for decreasing characteristic length of velocity anomalies in a global surface wave experiment with Love waves between 40 s and 150 s. The length-scale of heterogeneity is expressed in angular degree ranging between 1 and 40. The relative error between surface wave data due to ray theory and scattering is calculated using the source-receiver positions in the Love wave dataset. A) The synthetic experiment for phase velocity measurements. B) The synthetic experiment for group velocity measurements.
Figure 5.6: Plots of the scattering theoretical phaseshift versus the ray theoretical phaseshift for spherical harmonic models with the characteristic length of heterogeneity expressed written as angular degree $l$. The case of Love waves at 150 s is considered, and the source-receiver positions for the computation of the phaseshifts due to ray theory and scattering theory come from Trampert and Woodhouse (2001). The two solid lines indicate the error in the surface wave dataset for Love waves at 150 s. A) $l = 1$, B) $l = 5$, B) $l = 15$, B) $l = 20$, B) $l = 30$, B) $l = 40$. 
in Fig. 5.6A and 5.6B that there is a one-to-one correspondence between the scattering theoretical phaseshifts and the ray theoretical phaseshifts. In Fig. 5.6C ($l = 15$) and 5.7D ($l = 20$) where the angular degree of inhomogeneity is at the limit of the regime of the great circle approximation for Love waves at 150 s, it is noticed that several points of $d^{\text{scat}}$ versus $d^{\text{ray}}$ are outside the observed relative error in the dataset. It is as well seen in Fig. 5.6C and 5.6D that the points in the plot are slightly rotated anti-clockwise compared to the dashed line with slope one through origo. There is therefore a tendency for a bias towards smaller values of the scattering theoretical relative phaseshifts compared to the ray theoretical ones (see Spetzler and Snieder, 2001; Spetzler et al., 2001 for other examples of this trend). However in Fig. 5.6E and 5.6F, the picture is a bit different than in the previous plots of Fig. 5.6. The points of $d^{\text{scat}}$ versus $d^{\text{ray}}$ are spread more homogeneously around in the two plots, but there is still a tendency of an anti-clockwise rotation of the best-fitting line (not shown) through the points $(d^{\text{scat}}, d^{\text{ray}})$. In Fig. 5.6F, the best-fitting line of the points $(d^{\text{scat}}, d^{\text{ray}})$ is rather that positive scattering theoretical phaseshift correspond to negative ray theoretical phaseshift and vice versa. It means that using Love waves at 150 s to estimate small-scale structures ($l = 40$ in Fig. 5.6F) the application of the ray theoretical great circle approximation produces global maps with the wrong sign of the estimated velocity field.

In terms of wavefront healing, Nolet and Dahlen (2000) discuss scattering theory in surface wave tomography. They explain using the Gaussian beam solution to the parabolic approximation of the scalar Helmholtz equation that an inversion of phase velocity measurements is better behaved than the one using group velocity measurements. Their argument is that surface wave group velocity delays have large sidelopes compared to surface wave phase delays when the diameter of heterogeneity is of the same order of magnitude as the wavelength. The large sidelopes of the surface wave group velocity delay may therefore introduce considerable noise into the data according to Nolet and Dahlen (2000). Based on the sensitivity kernels for phase and group velocity in this paper, we rather find that the inversions of phase and group velocity measurements are both equally behaved. It is not difficult to compute the forward problem either applying the sensitivity kernel for relative phase or group velocity. The developed scattering approach for surface waves is just as easy to use as the ray theoretical great circle approximation. On a 250 MHz ultra-sparc machine, it takes 1 day, 5 days and 15 days cpu-time to compute the tabulated scattering sensitivity kernels for the analytical frequency-integration for relative group velocity, for single-frequency relative phase velocity and for the numerically frequency-integrated PREM model for relative group velocity, respectively, and to carry out the inversion of 42000 surface wave phaseshifts for a phase or group velocity map to angular degree and order 40.

### 5.6 Conclusions

We have investigated the non-ray geometrical effect in global surface wave tomography. The first-order Rytov approximation was used to derive a linear relationship between surface wave phase and group velocity measurements and relative phase and group velocity
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perturbations, respectively. The diffraction theoretical approach takes the finite-frequency
effect of surface waves into account which is not possible with conventional ray theory in
surface wave tomography. For finite-frequency surface waves, the sensitivity to relative
phase and group velocity is maximum in magnitude off-path the ray trace. In addition,
the scattering sensitivity kernel for relative phase velocity has sidelobes outside the Fres-
nel zone, while the sensitivity kernel for relative group velocity is only dominant over
the Fresnel zone. In contrast to this, ray theory predicts that the sensitivity to relative
velocity perturbations is only non-zero on the great circle path connecting the source and
receiver. Given the same strength of heterogeneity, scattering of surface waves becomes
increasingly important for increasing period and epicentral distance.

We applied phaseshift measurements for Love waves between periods at 40 s and
150 s from Trampert and Woodhouse (2001) to compile global phase velocity maps to
angular degree and order 40 using scattering theory. These models for diffraction theory
were matched with those computed with ray theory. We applied an a priori Laplacian
smoothness condition in the inversion procedure resulting that only structures to angular
degree 20-25 for Love waves at 40 s and to angular degree 10-15 for Love waves at 150 s
are present in the phase velocity maps which is close to the limit of resolution in current
global surface wave tomography. We saw that ray theory and scattering theory produce
the same tomographic models in that regime for which the conditions for ray theory are
satisfied.

We showed with a synthetic experiment where the relative error between surface wave
data using ray theory and scattering theory was computed for velocity inhomogeneity with
increasing angular degree that the scattering of surface waves is dominant at angular de-
grees larger than $l = 20$ and $l = 30$ for surface wave at 150 s and 40 s, respectively. The
regime of surface wave scattering theory corresponds to the limits of present-day resolu-
tion in surface wave tomography. Consequently, in order to obtain detailed higher degree
surface wave models using long-period surface waves or dataset with many long source-
receiver distances we must take the finite-period effect of surface waves into account.

In the USArray project, the United States will be covered with a dense array of 2000
seismographs having an uniform station spacing during the next ten years (see Levander
et al., 1999). The purpose of the USArray is to increase the resolution of tomographic
images of the North American shield. However, it is not enough to increase the data
coverage of the area of interest, but it is as well important to improve the tomographic
imaging methodology that is to be applied in inversions of data from the USArray project.

References

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5.7 Appendix A: Perturbation theory of the propagation length of scattered ray paths, the width of the Fresnel zone and the geometrical factor

According to Fig. 5.7 the epicentral distance between the source and receiver is denoted by \( \Delta_{\text{off}} \), and the epicentral distance between the source and scatterer point and the scatterer point and receiver are marked as \( \Delta_1 \) and \( \Delta_2 \), respectively. The perpendicular distance from the source-receiver geometry to the scatterer at the offset \( \phi \) is \( |\theta - \pi/2| \). Using the law of cosines on a sphere to relate \( \Delta_1 \) with \( |\theta - \pi/2| \) and \( \phi \), we obtain

\[
\cos(\Delta_1) = \cos((\theta - \pi/2)) \cos(\phi) + \sin((\theta - \pi/2)) \sin(\phi) \cos(\pi/2) \\
= \cos((\theta - \pi/2)) \cos(\phi). 
\]

Isolating \( \Delta_1 \) from Eq. (5.31) and assuming that the ray deflection \( |\theta - \pi/2| \) is small gives

\[
\Delta_1 = \arccos \left( \cos \left( \frac{|\theta - \pi/2|}{2} \cos(\phi) \right) \right) \\
\approx \arccos \left( \cos(\phi) - \frac{1}{2} (\theta - \pi/2)^2 \cos(\phi) \right) \\
\approx \phi + \frac{(\theta - \pi/2)^2}{2 \tan(\phi)}. \quad (5.32)
\]

Similarly, we have for \( \Delta_2 \) that

\[
\Delta_2 = (\Delta_{\text{off}} - \phi) + \frac{(\theta - \pi/2)^2}{2 \tan(\Delta_{\text{off}} - \phi)}. \quad (5.33)
\]
The detour (i.e. $\Delta_1 + \Delta_2 - \Delta_{\text{off}}$) is then given by

$$
\Delta_1 + \Delta_2 - \Delta_{\text{off}} = \frac{(\theta - \frac{\pi}{2})^2}{2} \left( \frac{1}{\tan(\phi)} + \frac{1}{\tan(\Delta_{\text{off}} - \phi)} \right)
$$

$$
= \frac{(\theta - \frac{\pi}{2})^2 \sin(\Delta_{\text{off}})}{2 \sin(\phi) \sin(\Delta_{\text{off}} - \phi)}. \quad (5.34)
$$

The condition for Fresnel zones on a sphere that the detour is less than the wavelength divided by a number $n$ is given by

$$
\Delta_1 + \Delta_2 - \Delta_{\text{off}} \leq \frac{\lambda}{n}, \quad (5.35)
$$

where $\lambda$ is the wavelength measured in radians. The sign of equality in Eq. (5.35) is used to calculate the Fresnel zone boundary. By inserting the detour in Eq. (5.34) in the Fresnel zone condition in Eq. (5.35), the half width $(\theta - \frac{\pi}{2})$ of Fresnel zones is derived, hence

$$
(\theta - \frac{\pi}{2}) = \sqrt{\frac{2\lambda \sin(\phi) \sin(\Delta_{\text{off}} - \phi)}{n \sin(\Delta_{\text{off}})}}, \quad (5.36)
$$

which has the largest value for $\phi = \Delta_{\text{off}}/2$. For that case, the half width of the Fresnel zone is given by

$$
(\theta - \frac{\pi}{2}) = \sqrt{\frac{\lambda}{n \tan(\Delta_{\text{off}}/2)}}. \quad (5.37)
$$
The maximum width \( L_F \) of Fresnel zones on the sphere is twice the half width \( (\theta - \pi/2) \) in Eq. (5.37), thus

\[
L_F = \sqrt{\frac{4\lambda}{n \tan \left( \frac{\Delta_{\text{off}}}{2} \right)}},
\]

(5.38)

where \( L_F \) and \( \lambda \) are measured in radians.

The geometrical factors \( \sin(\Delta_1) \) and \( \sin(\Delta_2) \) are derived to zeroth order approximation using Eq. (5.32) and Eq. (5.33), thus

\[
\sin(\Delta_1) = \sin(\varphi) \quad \text{and} \quad \sin(\Delta_2) = \sin(\Delta_{\text{off}} - \varphi),
\]

(5.39)

where it is assumed that \( (\theta - \pi)/2 = 2 \tan(\varphi) \) and \( (\theta - \pi)/2 = 2 \tan(\Delta_{\text{off}} - \varphi) \).

### 5.8 Appendix B: The scattering sensitivity kernel for major arcs

The scattering sensitivity kernel to compute phase velocity maps for major arcs \( (\pi < \Delta_{\text{off}} < 2\pi) \) can be constructed by three scattering sensitivity kernels for minor arcs. Let the scattering sensitivity kernels for the minor arcs between the source (S) and the receiver anti-pod (RA), between the receiver anti-pod and the source anti-pod (SA) and between the source anti-pod and receiver (R) be given by

\[
K_{l,m}^{\text{scat,ph,S-RA}}(\Delta_{\text{off}} - \pi, \nu) = \int_{0}^{\Delta_{\text{off}} - \pi} \int_{0}^{\pi} Y_l^m(\theta, \varphi) K_{S-RA}^{R}(R, \theta, \varphi) d\theta d\varphi,
\]

(5.40)

\[
K_{l,m}^{\text{scat,ph,RA-SA}}(2\pi - \Delta_{\text{off}}, \nu) = \int_{\Delta_{\text{off}} - \pi}^{\pi} \int_{0}^{\pi} Y_l^m(\theta, \varphi) K_{RA-SA}^{R}(R, \theta, \varphi) d\theta d\varphi,
\]

(5.41)

and

\[
K_{l,m}^{\text{scat,ph,SA-R}}(\Delta_{\text{off}} - \pi, \nu) = \int_{\Delta_{\text{off}} - \pi}^{\pi} \int_{0}^{\pi} Y_l^m(\theta, \varphi) K_{SA-R}^{R}(R, \theta, \varphi) d\theta d\varphi,
\]

(5.42)

where the sensitivity kernels \( K_{S-RA}^{R}(R, \theta, \varphi), K_{RA-SA}^{R}(R, \theta, \varphi) \) and \( K_{SA-R}^{R}(R, \theta, \varphi) \) are equivalent to the sensitivity kernel in Eq. (5.16) but having the epicentral distance substituted with \( \Delta_{\text{off}} - \pi, 2\pi - \Delta_{\text{off}} \) and \( \Delta_{\text{off}} - \pi \), respectively. In order to derive the sensitivity kernel \( K_{l,m}^{\text{scat,ph}}(\Delta_{\text{off}}, \nu) \) due to scattering theory for major arcs, the integration along the source-receiver line is split up into the three minor arc integrations. Hence,

\[
K_{l,m}^{\text{scat,ph}}(\Delta_{\text{off}}, \nu) = \frac{1}{\Delta_{\text{off}}} \left( (\Delta_{\text{off}} - \pi)K_{l,m}^{\text{scat,ph,S-RA}}(\Delta_{\text{off}} - \pi, \nu) + (2\pi - \Delta_{\text{off}})K_{l,m}^{\text{scat,ph,RA-SA}}(2\pi - \Delta_{\text{off}}, \nu) + (\Delta_{\text{off}} - \pi)K_{l,m}^{\text{scat,ph,SA-R}}(\Delta_{\text{off}} - \pi, \nu) \right),
\]

(5.43)
5.9 Appendix C: Rotation of scattering sensitivity kernels

Dziewonski (1984) and Dahlen and Tromp (1998) show that the transformation of the spherical harmonics of angular degree \(l\) and order \(m\) from a reference coordinate system to a new coordinate system is given by

\[
Y^m_l(\theta, \phi) = \exp(i m \Phi) \sum_{n=-l}^{l} \exp(i n \Psi) Q^n_m(\Theta) Y^n_l(\theta', \phi'),
\]

with the three Euler angles denoted by \(\Phi, \Psi\) and \(\Theta\), and the elements of the rotation matrix are \(Q^n_m(\Theta)\). The sensitivity kernel for minor arcs in Eq. (5.18) depends linearly on the spherical harmonics. This means that the sensitivity kernel for relative phase velocity using scattering theory can be transformed from the reference coordinate system to the observed coordinate system by using the relation for the transformation of spherical harmonics in Eq. (5.44). Let \(K^\prime(R, \theta, \phi)\) denote the sensitivity kernel in the observed coordinate system which is equivalent to the sensitivity kernel \(K(R, \theta, \phi)\) in Eq. (5.16) in the reference coordinate system. The formula in Eq. (5.44) is inserted in the scattering sensitivity kernel in Eq. (5.18). The sensitivity kernel \(K_{l,m}^{\text{scat}, \text{ph}}(\Delta_{\text{off}}, \nu)\) for the epicentral offset \(\Delta_{\text{off}}\) in the new coordinate system is then

\[
K_{l,m}^{\text{scat}, \text{ph}}(\Delta_{\text{off}}, \nu) = \int_0^{\Delta_{\text{off}}} \int_0^{\pi} Y^n_l(\theta, \phi) K^\prime(R, \theta, \phi) d\theta d\phi
\]

\[
= \exp(i m \Phi) \sum_{n=-l}^{l} \exp(i n \Psi) Q^n_m(\Theta)
\]

\[
\times \int_0^{\Delta_{\text{off}}} \int_0^{\pi} Y^n_l(\theta', \phi') K^\prime(R, \theta', \phi') d\theta' d\phi'
\]

\[
= \exp(i m \Phi) \sum_{n=-l}^{l} \exp(i n \Psi) Q^n_m(\Theta) K_{l,m}^{\text{scat}, \text{ph}}(\Delta_{\text{off}}, \nu),
\]

with the scattering sensitivity kernel for relative phase velocity given by

\[
K_{l,m}^{\text{scat}, \text{ph}}(\Delta_{\text{off}}, \nu) = \int_0^{\Delta_{\text{off}}} \int_0^{\pi} Y^n_l(\theta', \phi') K^\prime(R, \theta', \phi') d\theta' d\phi',
\]

at offset \(\Delta_{\text{off}}\) computed in the reference coordinate system.

The scattering sensitivity kernels in Eq. (5.21) and (5.22) for major arcs are composed by three scattering sensitivity kernels for minor arcs. It is therefore possible to apply the
transformation of spherical harmonics in Eq. (5.44) on each scattering sensitivity kernel for minor arcs in order to obtain the same result as in Eq. (5.45) but with the scattering kernel for major arcs computed in the reference coordinate system. In addition, the result in Eq. (5.45) is valid for major arc sensitivity kernels using scattering theory to compute group velocity maps.