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CERN 65-30
Theoretical Study
Division

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WEAK INTERACTIONS

N. Cabibbo and M. Veltman

G E N E V A

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Notes by R. Leacock

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I. INTRODUCTION

The theory of weak interactions was born around 1932 - 33 with the neutrino hypothesis of Pauli and the first successful theory of beta decay by Fermi[1]. The period of rapid development which followed the discovery of parity non-conservation, culminated in 1957 - 58 in the present scheme for weak interactions: the V-A theory of Feynman and Gell-Mann, Sudarshan and Marshak[2].

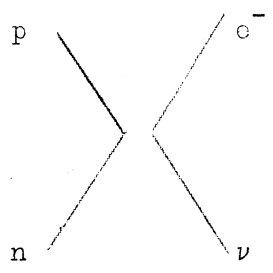
According to this theory the Hamiltonian density*) $H(x)$ is of the current-current type:

$$H(x) = \frac{G}{\sqrt{2}}\left(J_\lambda(x)\; J_\lambda^+(x)\right) \tag{1}$$

where + indicates the Hermitian conjugate. The weak interaction is assumed to be a very short-range force so that the two currents interact only at the same space-time point x. The current $J_\lambda(x)$ is the sum of $J_\lambda^\ell(x)$, the current operator for the leptons, and $J_\lambda^S(x)$, the current operator for the strongly interacting particles:

$$J_\lambda(x) = J_\lambda^\ell(x) + J_\lambda^S(x) \; . \tag{2}$$

For example, in beta decay ($n \to p + e^- + \bar{\nu}$) a lepton current $\nu \to e^-$ interacts with a nucleon current $n \to p$ according to the diagram



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*) The Hamiltonian is obtained by integrating $H(x)$ over all space:

$$H = \int d^3x H(x) \; .$$

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The term in Eq. (1) which corresponds to such a process is:

$$\frac{G}{\sqrt{2}}\left(J_\lambda^\ell J_\lambda^{s+} + J_\lambda^s J_\lambda^{\ell +}\right) . \qquad (3)$$

1. The Weak Current $J_\lambda^\ell(x)$

The explicit form of the lepton-current operator in terms of the electron, muon and neutrino fields is*):

$$J_\lambda^\ell = \left(\bar{\nu}_\mu \gamma^\lambda \ (1+\gamma^5)\,\mu\right) + \left(\bar{\nu}_e \gamma^\lambda (1+\gamma^5)\, e\right), \qquad (4)$$

The term in Eq. (1) in which J_λ^ℓ couples to itself, is the Hamiltonian for muon decay ($\mu^- \to e^- + \nu + \bar{\nu}$):

$$\frac{G}{\sqrt{2}}\left(\bar{\nu}_e \gamma^\lambda (1+\gamma^5)\, e\right)\left(\bar{\mu}\gamma^\lambda \ (1+\gamma^5)\, \nu_\mu\right) . \qquad (5)$$

Using expression (5) the muon lifetime is evaluated in terms of G and comparison with experiment gives

$$G = (1.01 \pm 0.01) \times 10^{-5} \ Mp^{-2} \qquad (6)$$

where Mp is the mass of the proton.

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*) In Eq. (4) μ, e represent the free fields of μ^- and e^- which are taken as particles. The fields are expressed in terms of creation and destruction operators. For example,

$$e(x) = \frac{1}{(2\pi)^{3/2}} \sum_i \int d^3k \left\{ a_i(k) u_i(k) e^{ikx} + b_i^*(k) v_i(k) e^{-ikx} \right\}$$

where the sum is over the two helicity states ($i = \pm 1$) which are available for each value of $\vec{k}$. $u_i(k)$ and $v_i(k)$ are Dirac spinors, and $a_i(k)$ destroys e^-; $b_i^*(k)$ creates e^+. Two introductory accounts of quantum field theory are: F. Mandl, Introduction to Quantum Field Theory, Interscience, New York (1959) and E.M. Henley and W. Thirring, Elementary Quantum Field Theory, McGraw-Hill (1962).

The coupling of J_λ^ℓ with itself gives rise to other reactions, for example,

$$\nu_e + e \rightarrow \nu_e + e ,$$
$$e^+ + e^- \rightarrow \nu_e + \bar{\nu}_e .$$

These are very difficult to observe in the laboratory.

2. The Weak Current $J_\lambda^S(x)$

It is not possible to give an expression for J_λ^S, in terms of fields*), as was done for the lepton current J_λ^ℓ, since there is no field-theoretic description of strongly interacting particles. Different approaches are used to gain information about J_λ^S; the following are the principal ones:

i) Selection rules
The study of the selection rules of J_λ^S with respect to quantities which are conserved by strong interactions. If some quantity (e.g. strangeness) is not conserved by J_λ^S it can be inquired whether the violation assumes some simple form.

ii) Dispersion relations
The main success of this approach is the discovery of the Goldberger-Treiman relations. Other applications are to the study of "weak" form factors, in close analogy to the analysis of electromagnetic form factors. The knowledge of weak form factors is of special importance in the study of elastic neutrino processes, for example, $\nu_\mu + n \rightarrow p + \mu^-$.

iii) Conserved currents
The strong interactions provide conserved quantities such as charge, strangeness, third component of isotopic spin, and baryon number. They are additive quantities: the charge of a system

*) Field theory is used here, in the restricted sense, in the building of so-called "phenomenological" expressions. A phenomenological Hamiltonian is, by definition, a Hamiltonian which gives the correct value of a certain group of transition matrix elements when used as a first-order perturbation, and the fields it contains are considered free fields (and therefore expressed in terms of creation and destruction operators).

of particles is the arithmetic sum of the individual charges. To each conserved quantity there corresponds a current:

$$Q \to j_\lambda(x) \ , \ B \to j_\lambda^B(x) \ ,$$

$$S \to j_\lambda^S(x) \ , \ I_3 \to j_\lambda^3(x) \ .$$

The currents are four-vectors; the integral over all space of the fourth component gives the conserved quantity:

$$iQ = \int d^3x \ j_4(x), \text{ etc.} \tag{7}$$

Currents which correspond to conserved quantities (briefly, conserved currents) obey a continuity equation

$$\begin{aligned} \partial_\lambda \ j_\lambda &= \frac{\partial}{\partial x_4} j_4 + \vec{\nabla} \cdot \vec{j}(x) = 0 \ , \\ \partial_\lambda \ j_\lambda^S &= \partial_\lambda j_\lambda^B = \partial_\lambda j_\lambda^3 = 0 \ . \end{aligned} \tag{8}$$

The charge symmetry of the strong interactions requires the conservation, not only of I_3, but also of the raising and lowering operators

$$I_\pm = I_1 \pm i I_2 \tag{9}$$

to which correspond the conserved currents $j_\lambda^\pm$:

$$\partial_\lambda j_\lambda^+ = \partial_\lambda j_\lambda^- = 0 \ . \tag{10}$$

The physical meaning of these currents is not as transparent as that of the currents which correspond to diagonal quantities, but it is clear that charge symmetry requires them to exist and to obey continuity equations. In conclusion, there is an elite of current operators which are singled out as being important in the description of strong interaction symmetries.

It is interesting to ask if J_λ^S, or at least some part of it, belongs to this elite. The first attempt along this line is the hypothesis that the $\Delta S = 0$ vector part of J_λ^S coincides with j_λ^+. This hypothesis [the conserved vector current theory of

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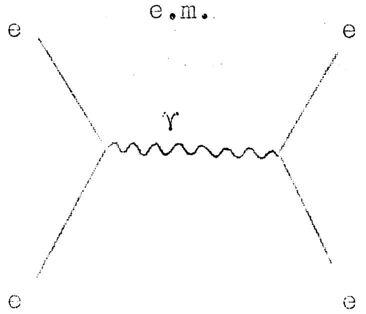
Feynman and Gell-Mann[2], Gershtejn and Zel'dovich[3]] leads to straightforward predictions which are in good agreement with experiment. More recent developments involve the use of vector currents whose existence is implied by the appropriate SU_3 invariance of the strong interactions.

In a more limited sense this approach can also be used for the axial vector part of J_λ^S and has brought a deeper understanding of the Goldberger-Treiman relations.

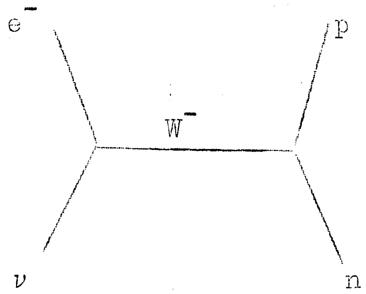
3. Vector Bosons

The exchange of a photon in electromagnetic phenomena, such as electron-electron scattering,

e.m.

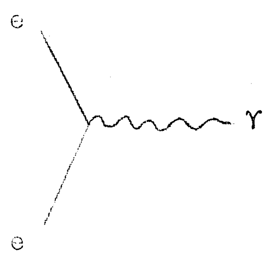


weak

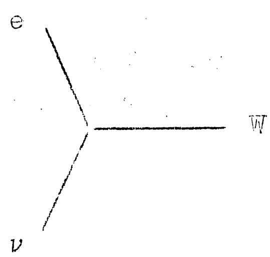


suggests the hypothesis that the weak interactions are mediated by the exchange of a charged vector meson $W^{\pm}$. The couplings for the two processes are:

e.m.



weak



$$H(x) = ej_\lambda(x)\, A_\lambda(x)\,, \qquad H(x) = gJ_\lambda(x)\, W_\lambda(x) \,. \quad (11)$$

In this view beta decay is a two-step process:

$$n \to p + W^- \quad \llcorner\!\!\to e^- + \bar{\nu}_e \ .$$

The current-current Hamiltonian density, Eq. (1), still gives an approximate phenomenological description of weak (two-step) processes if

$$\frac{G}{\sqrt{2}} = \frac{g^2}{M_W^2} \ . \qquad (12)$$

The effects which would allow an experimental distinction between Eqs. (1) and (11) are generally small and/or difficult to identify. A clear-cut proof of the existence of $W^{\pm}$ can probably come only from the direct observation of its production and decay. A promising reaction is neutrino production:

$$\nu_\mu + Z \to \mu^- + W^+ + Z \quad \llcorner\!\!\to e^+ + \nu_e\,,\ \mu^+ + \nu_\mu \ .$$

The lifetime of $W^{\pm}$ would be small ($\sim 10^{-17}$ sec) so that the over-all process looks like the production of a lepton pair $\mu^- e^+$ or $\mu^-\mu^+$. The evidence for the W meson is not compelling.

II. THE MATRIX ELEMENT

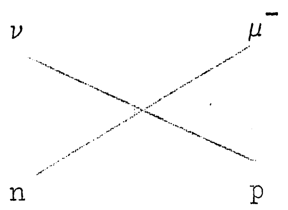
If any weak interaction process involving e, ν_e occurs, then the same process with e replaced by μ and ν_e, replaced by ν_μ is assumed to occur. This is the principle of μ-e universality. For example, the "elastic" reaction

$$\nu_e + n \to e^- + p \qquad (13)$$

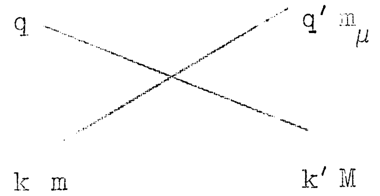
also occurs as

$$\nu_\mu + n \to \mu^- + p \ . \qquad (14)$$

The diagram for such a two-body-in - two-body-out process (a), is



(a)



(b)

where the labels of Fig. (b) are the four-momenta and masses of the particles.

The cross-section for Eq. (14) is[4]:

$$\sigma_{total} = (2\pi)^{-10} \frac{E_\nu}{|\vec{q}|} \int d_3q' \; d_3k' \delta_4(q+k-q'-k') \sum_{polarizations} |M|^2 \qquad (15)$$

where M is the matrix element for the process and where

$$|M|^2 = F(q,q',k,k',m,M) \; .$$

Calculations of cross-sections and lifetimes reduce immediately to the construction of the appropriate matrix element.

In constructing the matrix element, Lorentz invariance will be assumed. Further, all electromagnetic effects will be neglected. In this case, μ and ν will have only weak interactions which will be discussed in lowest order. The nucleons may have strong interactions among themselves. As a guiding principle, the weak interaction (with strong interactions turned off) is assumed to be pure vector-axial vector (V-A) described by the Hamiltonian density

$$H(x) = G/\sqrt{2} \left(\bar{\psi}_p \gamma^\alpha (1+\gamma^5)\psi_n\right)\left(\bar{\psi}_\mu \gamma^\alpha (1+\gamma^5)\psi_\nu\right) + \text{h.c.} \qquad (16)$$

The matrix element which results from Eq. (16) will have the form

$$M = (2\pi)^4 \; L\alpha \, \ell\alpha \qquad (17)$$

where L_α is the baryonic part of M and ℓ_α is the leptonic part. This convenient division of M is possible because the weak interaction is assumed to be a point interaction: the nucleon current and the lepton current interact only at one space-time point x. Note that because L_α and ℓ_α are Lorentz four vectors, M is a Lorentz scalar.

1. The Leptonic Factor ℓ_α

From experimental evidence the leptonic part of the weak interaction (often called the lepton current) is taken to be:

$$\left(\bar{\psi}_\mu(x)\ \gamma^\alpha(1+\gamma^5)\psi_\nu(x)\right) \tag{18}$$

where $\psi_\nu(x)$ and $\bar{\psi}_\mu(x)$ are expanded in terms of creation and destruction operators. In the language of field theory ψ_ν destroys a neutrino and $\bar{\psi}_\mu$ creates a muon. The evaluation of expression (18) between an initial neutrino state and a final muon state leads to *)

$$\ell_\alpha = \left(\bar{u}(q')\gamma^\alpha(1+\gamma^5)u(q)\right) \tag{19}$$

where the u(q) are the Dirac spinors. The variables q and q′ will not appear in the remainder of M, i.e., in L_α. At most, the momentum transfer

$$Q \equiv q' - q \tag{20}$$

will appear in L_α.

*) Since the leptons interact only weakly, the formalism of quantum field theory, at least to lowest order in the interaction, can be used to describe the muon and neutrino. The steps leading from Eq. (18) to Eq. (19) can be found in introductory texts such as that of Mandl.

2. The Baryonic Factor L_α

The discussion of the baryonic part of the matrix element is complicated by the fact that the nucleons interact strongly. The simple correspondence between Eqs. (18) and (19) for the leptonic factor is not true for the nucleonic factor. In particular, we do not expect L_α to have the form

$$\bar{u}(k')\gamma^{\alpha}(1+\gamma^5)\,u(k) \; .$$

The approach used is to write down the most general form for L_α and then to discuss each term with the available theoretical and experimental information.

The most general form for L_α is:

$$L_\alpha = \left(\bar{u}(k')\left\{a_1\gamma^{\alpha} + ia_2 k_\alpha + ia_3 k'_\alpha + b_1\gamma^{\alpha}\gamma^5 + ib_2 k_\alpha\gamma^5 + ib_3 k'_\alpha\gamma^5\right\}u(k)\right) \tag{21}$$

where the identity

$$\epsilon_{\alpha\beta\mu\nu}\,\gamma^{\nu} = \gamma^5\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu} - \delta_{\alpha\beta}\gamma^5\gamma^{\mu} - \delta_{\beta\mu}\gamma^5\gamma^{\alpha} - \delta_{\alpha\mu}\gamma^5\gamma^{\beta} \tag{22}$$

and the free particle Dirac equations*)

$$\begin{aligned} i\gamma^{\alpha}k_\alpha\, u(k) &= -\,m\,u(k) \;, \\ \bar{u}(k')\,i\gamma^{\alpha}k'_\alpha &= -\,M\,\bar{u}(k') \end{aligned} \tag{23}$$

have been used.

---

*) Asymptotically the proton and neutron are free particles represented by the spinors $\bar{u}$ and u which must satisfy the Dirac equation.

The more traditional form of L_α is*):

$$L_\alpha = \left(\bar{u}(k') \left\{ G_V \gamma^\alpha + \frac{i\mu}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q_\beta + \frac{iA}{m_\mu} Q_\alpha \right.\right.$$

$$\left.\left. + G_A \gamma^\alpha\gamma^5 + \frac{iB}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta + \frac{ib}{m_\mu} \gamma^5 Q_\alpha \right\} u(k) \right) . \qquad (24)$$

The various terms are traditionally called:

γ^α	vector,
$(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q_\beta$	weak magnetism,
Q_α	scalar,
$\gamma^\alpha\gamma^5$	axial vector,
$(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta$	axial magnetism
$\gamma^5 Q_\alpha$	induced pseudoscalar .

The coefficients G_V ... b may be functions of the Lorentz invariant quantities**)

$$k^2 = - m^2 ,$$
$$k'^2 = - M^2 ,$$
$$k_\alpha k'_\alpha = - \tfrac{1}{2}[m^2 + M^2 + Q^2] .$$

Usually, the variable Q^2 is used instead of $k_\alpha k'_\alpha$ and the dependence of G_V ... b upon m^2 and M^2 is neglected. Thus, for example,

$$G_V = G_V(Q^2) ,$$

and similarly for the other coefficients.

---

*) The eager student may try to find the relations between the coefficients in Eq. (24) and those in Eq. (21).

**) Recall that $Q^2 = (q' - q)^2 = (k - k')^2 = -m^2 - M^2 - 2k_\alpha k'_\alpha$.

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The form factor of a given term is defined as the function (normalized to 1 at $Q^2 = 0$ if not zero at that point) which gives the Q^2 dependence of that term. Thus, for example,

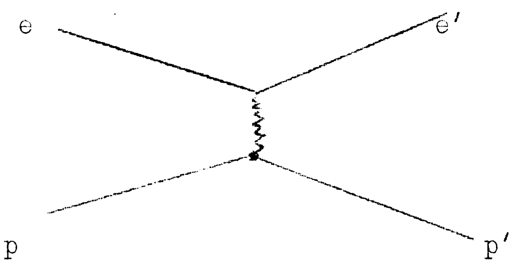
$$F_V(Q^2) = \frac{G_V(Q^2)}{G_V(0)} , \tag{25}$$

and, again, similarly for the remaining terms.

Ideally, of course, it is desired to calculate the functions $F_V \ldots F_b$ from theoretical principles. Since no complete theory of the strong interactions exists, this is not yet possible. It is possible, however, to relate the form factors of electromagnetic processes with those of weak processes. This is done in the framework of the conserved vector current theory of Feynman and Gell-Mann[2], Gershtejn and Zel'dovich[3].

III. THE CONSERVED VECTOR CURRENT THEORY

Consider electron-proton scattering described by the diagram:



As in the case of μ capture discussed above, the matrix element for this electromagnetic scattering process factors into leptonic and baryonic parts. The leptonic factor is:

$$\ell_\alpha^{e.m.} = e\left(\bar{u}_{e'}, \gamma^\alpha u_e\right) \tag{26}$$

while the baryonic factor is supposedly (before the strong interactions are turned on)

$$L_\alpha^{e.m.} = e\left(\bar{u}_p, \gamma^\alpha u_p\right). \qquad (27)$$

The most general form for the baryon factor with strong interaction is:

$$L_\alpha^{e.m.} = \left(\bar{u}_p, \left\{ e F_1^p(Q^2)\gamma^\alpha + \frac{i\mu_p}{4m} F_2^p(Q^2)\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)Q_\beta \right\} u_p\right) \qquad (28)$$

where F_1^p and F_2^p are the proton form factors. The μ_p is the proton anomalous magnetic moment. A similar formula holds in the e-n scattering process where, however, $F_1^n(0) = 0$, as the neutron has no total charge. In comparing expressions (27) and (28), one observes that in accordance with conservation of charge, the coefficient (e) of the γ^α term is <u>unchanged</u> by the presence of strong interactions (at $Q^2 = 0$). In other words, the strong interactions do not alter the value of the proton charge. In the same way, if we now, with Feynman[2]) etc., assume that the vector part of the weak interaction originates from the same expression as encountered in electromagnetism, we find that the conserved vector current hypothesis states that the coefficient (G_V) of the γ^α term of the weak baryonic factor is unchanged by the presence of strong interactions (again at $Q^2 = 0$). To make these ideas clearer it is helpful to use the formalism of isotopic spin and the current-current form of the Lagrangian.

As observed before, the weak interaction Lagrangian density may be written as a product of a lepton current with a baryon current. This is also possible for the e-p scattering process. The baryonic (or strong) currents for the two processes are:

$$J_\mu^{strong}(e.m.) = \bar{\psi}_p\gamma^\mu\psi_p + \text{pion current}, \qquad (29)$$

$$J_\mu^{strong}(weak) = \bar{\psi}_p\gamma^\mu(1+\gamma^5)\psi_n + \text{pion current}. \qquad (30)$$

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[Compare Eq. (30) with Eq. (16).] It is necessary to add the pion current to Eq. (29) because the pions are charged, and similarly, to add the pion current to Eq. (30) because the pions interact weakly. These currents may be rewritten in isotopic spin notation as *):

$$J_\mu^{\text{strong}}(\text{e.m.}) = \bar{\psi}\gamma^\mu\left(\frac{1+\tau_3}{2}\right)\psi + (\text{pion current})_3 \,, \tag{31}$$

$$J_\mu^{\text{strong}}(\text{weak}) = \bar{\psi}\gamma^\mu\tau_+\psi + (\text{pion current})_+ \,. \tag{32}$$

Note that the γ^5 term of Eq. (30) has not been included in Eq. (32). In other words, only the vector part (γ^μ) of the strong current is considered. Considering only the isovector component of Eq. (31), then

$$J_\mu^{\text{strong,isovector}}(\text{e.m.}) = \bar{\psi}\gamma^\mu\frac{\tau_3}{2}\psi + (\text{pion current})_3 \,, \tag{33}$$

$$J_\mu^{\text{strong,isovector}}(\text{weak}) = \bar{\psi}\gamma^\mu\tau_+\psi + (\text{pion current})_+ \,. \tag{34}$$

---

*) In the isospin formalism the proton and neutron four spinors ψ_p and ψ_n are treated as components of a nucleon two spinor:

$$\psi = \begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_p, \bar{\psi}_n) \,.$$

(Sometimes the convention with a minus sign for $\bar{\psi}_n$ is encountered.) The spin matrices

$$\tau_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \tau_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

can be used to form a vector in isospace

$$\bar{\psi}\tau_i\psi \,, \quad i = 1,2,3 \,.$$

The prototype isoscalar is:

$$\bar{\psi}\psi = \bar{\psi}_p\psi_p + \bar{\psi}_n\psi_n \,.$$

By simple calculation it is seen how to rewrite, in isospin notation, the electromagnetic ($\bar{\psi}_p\psi_p$) and weak ($\bar{\psi}_p\psi_n$) currents:

$$\bar{\psi}_p\psi_p = \bar{\psi}\left(\frac{1+\tau_3}{2}\right)\psi \,,$$

$$\bar{\psi}_p\psi_n = \bar{\psi}\tau_+\psi \,,$$

where $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$. (Note that the matrices γ^α have been suppressed as they do not affect the iso-character of operators.) The isovector part of the e.m. current is clearly:

$$(\bar{\psi}_p\psi_p)^{\text{isovector}} = \bar{\psi}\frac{\tau_3}{2}\psi = \frac{1}{2}(\bar{\psi}_p\psi_p - \bar{\psi}_n\psi_n) \,.$$

The conserved vector current hypothesis is that the two strong currents (e.m. and weak) are merely different components of the same isovector:

$$\vec{J}_{\mu}^{\,strong} = \bar{\psi}\gamma^{\mu}\vec{\tau}\psi + \overrightarrow{(\text{pion current})} \ . \qquad (35)$$

This may be extended to include the current which couples to anti-neutrinos; one then speaks of the strong conserved current hypothesis.

Since the electromagnetic current Eq. (33) is conserved, it follows immediately, if isospin breaking interactions, such as electromagnetic interactions, are neglected, that the general current Eq. (35) is also conserved. Thus, Eq. (34) is also conserved and from this the weak vector coupling constant is not renormalized, i.e., not altered by the strong interactions.

Since the strong matrix element factors

$$L_{\alpha}^{(e.m.)} \text{ and } L_{\alpha}^{(weak)}$$

are nucleonic matrix elements of different components of the <u>same</u> isovector operator, Eq. (35), it is possible to write*):

$$F_V(Q^2) = F_1^p(Q^2) - F_1^n(Q^2) \ , \qquad (36)$$

$$\mu = \frac{G_V}{e}\,(\mu_p - \mu_n) \ . \qquad (37)$$

These expressions relate the electromagnetic and weak form factors and provide a test of the conserved vector current hypothesis.

IV. THE PARITY, CHARGE CONJUGATION AND TIME-REVERSAL TRANSFORMATIONS

We will now consider the properties of the process $\nu + n \to \mu + p$ under P (parity), C (charge conjugation) and T (time-reversal). To this purpose we consider the process in first order in the weak interaction so that the Lagrangian and the matrix element are directly.

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*) See footnote on page 13.

connected. Thus, we neglect the effects of strong interactions. We will see later what complications arise for C and T when one considers the matrix element including strong interaction effects. As we do not know what the Lagrangian looks like (because what is observed in the matrix element is the weak interaction plus, superposed on it, all kinds of strong interaction effects) we take the most general Lagrangian within the assumptions discussed up to now. Thus, we write the interaction as a product of two currents with the leptonic current having the form $\bar{\psi}\gamma^{\alpha}(1+\gamma^5)\psi$. In the baryonic current below one finds, in order, the terms: vector, magnetic, scalar, axial vector, axial magnetic and induced pseudoscalar. Thus, we have:

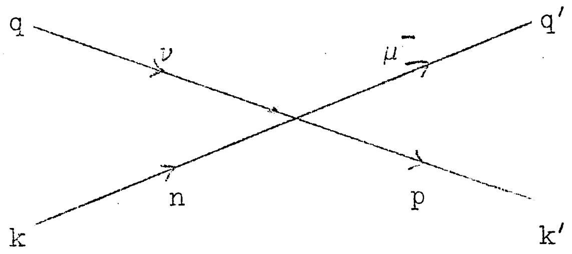
$$\begin{aligned}\mathcal{L}_I = & \left[\bar{\psi}_{\mu}(x)\gamma^{\alpha}(1+\gamma^5)\psi_{\nu}(x)\right]\Big[G_V\left(\bar{\psi}_p(x)\gamma^{\alpha}\psi_n(x)\right) \\ & + \frac{\mu}{4m}\frac{\partial}{\partial x_{\beta}}\left(\bar{\psi}_p(\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha})\psi_n\right) + \frac{A}{m_{\mu}}\frac{\partial}{\partial x_{\alpha}}\left(\bar{\psi}_p\psi_n\right) \\ & + G_A\left(\bar{\psi}_p\gamma^{\alpha}\gamma^5\psi_n\right) + \frac{B}{4m}\frac{\partial}{\partial x_{\beta}}\left(\bar{\psi}_p(\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha})\gamma^5\psi_n\right) \\ & + \frac{b}{m_{\mu}}\frac{\partial}{\partial x_{\alpha}}\left(\bar{\psi}_p\gamma^5\psi_n\right)\Big] + \text{h.c.}\end{aligned} \tag{38}$$

The fields ψ are fermion fields and obey certain anticommutation relations. We do not discuss the detailed questions of proper symmetrization but refer the reader to the literature[5,6]. This Lagrangian density leads immediately to the matrix element factors (neglecting high-order diagrams in the strong interactions):

$$\ell_{\alpha} = \bar{u}_{\mu}(q')\,\gamma^{\alpha}(1+\gamma^5)\,u_{\nu}(q)\,, \tag{39}$$

$$\begin{aligned}L_{\alpha} = \bar{u}_p(k')\Big\{ & G_V\gamma^{\alpha} + \frac{i\mu}{4m}(\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha})Q_{\beta} + \frac{iA}{m_{\mu}}Q_{\alpha} \\ & + G_A\gamma^{\alpha}\gamma^5 + \frac{iB}{4m}(\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha})\gamma^5 Q_{\beta} + \frac{ib}{m_{\mu}}\gamma^5 Q_{\alpha}\Big\}u_n(k)\end{aligned} \tag{40}$$

for the process



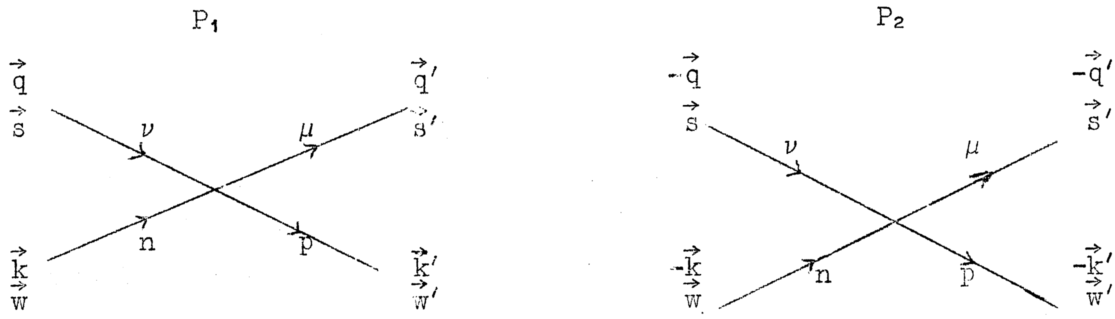
As before, $Q \equiv q' - q$.

1. Parity - P

Under the parity transformation the space co-ordinates are reflected and the time co-ordinate is unchanged:

$$x_\mu = (\vec{x}, i\,t) \xrightarrow{P} x_\mu' = (-\vec{x}, i\,t) \,. \qquad (41)$$

The signs of three-momenta change, while the signs of spins*) are unchanged. Thus, the process P_1, and its space-reflected counterpart P_2 are:



(42)

where, $\vec{q}$, $\vec{q}\,'$, $\vec{k}$, $\vec{k}'$ are three-momenta and $\vec{s}$, $\vec{s}\,'$, $\vec{w}$, $\vec{w}'$ are spins. Invariance under parity means that the transition probabilities for the two processes are identical. Process P_1 is given by Eqs. (39) and (40), while P_2 is given by

$$\ell_\alpha(P) = \bar{u}_\mu(-\vec{q}\,', \vec{s}\,') \left\{ \gamma^\alpha (1 + \gamma^5) \right\} u_\nu(-\vec{q}, \vec{s}) \;, \qquad (43)$$

*) Remember that spin is an axial vector.

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$$L\alpha(P) = \bar{u}_p(-\vec{k}',\vec{w}') \Big\{ G_V \gamma^\alpha \mp \frac{i\mu}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q_\beta \mp \frac{iA}{m_\mu} Q_\alpha$$
$$+ G_A \gamma^\alpha\gamma^5 \mp \frac{iB}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta \mp \frac{ib}{m_\mu} \gamma^5 Q_\alpha \Big\} u_n(-\vec{k},\vec{w}) \quad (44)$$

where the upper sign is for Q_μ, $\mu = 1,2,3$, and the lower sign belongs to Q_4 in accordance with Eq. (41). It is desired to relate Eq. (43) to Eq. (39) and Eq. (44) to Eq. (40) or more exactly $\ell\alpha(P)L\alpha(P)$ to $\ell_\alpha L_\alpha$.

There is a very simple relation between the spinors involved in P_1 and P_2*):

$$u(-\vec{q},\vec{s}) = \gamma^4 u(\vec{q},\vec{s}) \ ,$$
$$\bar{u}(-\vec{q}',\vec{s}') = \bar{u}(\vec{q}',\vec{s}')\gamma^4 \ . \quad (45)$$

This may be verified explicitly by using the explicit form of the spinors (see Appendix A or Ref. 4). Substitution of Eq. (45) into $\ell\alpha(P)$ gives:

$$\ell\alpha(P) = \bar{u}_\mu(\vec{q}',\vec{s}') \Big[\gamma^4\gamma^\alpha(1+\gamma^5)\gamma^4 \Big] u_\nu(\vec{q},\vec{s})$$
$$= \bar{u}_\mu(\vec{q}',\vec{s}') \Big[\gamma^4\gamma^\alpha\gamma^4(1-\gamma^5) \Big] u_\nu(\vec{q},\vec{s}) \quad (46)$$
$$= \mp\, \bar{u}_\mu(\vec{q}',\vec{s}') \Big[\gamma^\alpha(1-\gamma^5) \Big] u_\nu(\vec{q},\vec{s})$$

with (-) for $\alpha = 1,2,3$ and (+) for $\alpha = 4$. (Recall $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta_{\alpha\beta}$.) After the use of Eqs. (45) in expression (44) for $L\alpha(P)$ the result is:

$$L\alpha(P) = \mp\, \bar{u}_p(\vec{k}',\vec{w}') \Big\{ G_V\gamma^\alpha + \frac{i\mu}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q_\beta + \frac{iA}{m_\mu} Q_\alpha$$
$$- G_A\gamma^\alpha\gamma^\beta - \frac{iB}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta - \frac{ib}{m_\mu}\gamma^5 Q_\alpha \Big\} u_n(\vec{k},\vec{w}) \quad (47)$$

where again (-) for $\alpha = 1,2,3$ and (+) for $\alpha = 4$. When the Lorentz scalar $\ell\alpha(P)L\alpha(P)$ is formed, the ($\mp$) signs will cancel; however,

*) Relations similar to Eq. (45) will also be encountered in connection with C and T. They are only true up to a sign for a given spinor. Such signs are cancelled by signs in the transformations of creation and annihilation operators under C.

$\ell_\alpha(P)\,L_\alpha(P)$ will differ from $\ell_\alpha L_\alpha$ in that the sign of every term containing a γ^5 is minus. Thus, it is the mixing of non-γ^5 and γ^5 terms in the factors ℓ_α and L_α which result in the difference between P_1 and P_2. For example, if ℓ_α contained only $\gamma^\alpha\gamma^5$ and L_α contained only γ^α then $\ell_\alpha(P)\,L_\alpha(P) = -\,\ell_\alpha L_\alpha$ and the transition probabilities would be identical. It is clear that a V-A Hamiltonian, which mixes both γ^α and $\gamma^\alpha\gamma^5$ terms, will not conserve parity, i.e., processes P_1 and P_2 will be very different.

Instead of working with the matrix elements it is possible to see the effect of space reflection by using the Lagrangian density, Eq. (38) and noting

$$\psi(-\vec{x},t) = \gamma^4\,\psi(\vec{x},t)\ .$$

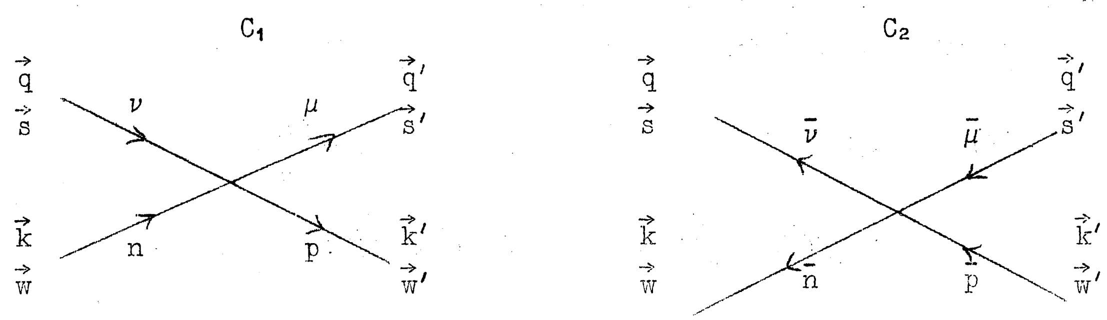
where $\psi(-\vec{x},t)$ satisfies the space-inverted Dirac equation.

2. Charge Conjugation - C

Under the charge-conjugation transformation spins and momenta are unchanged while particles and antiparticles are exchanged:

$$\text{particles} \xrightarrow{C} \text{antiparticles}\ ,$$
$$\text{antiparticles} \xrightarrow{C} \text{particles}\ . \qquad (48)$$

Thus, the process C_1 and the charge-conjugated process C_2 are represented by



(49)

where the arrows for antiparticles are reversed (i.e., an incoming antiparticle has an arrow pointing out) according to the Feynman rules[7].

To write the matrix element for process C_2 it is necessary to use the Hermitian conjugate part of the Lagrangian density, Eq. (38) since it is this part which will destroy $\bar{\nu}$ and $\bar{n}$ and create $\bar{\mu}$ and $\bar{p}$. The Lagrangian density for process C_2 is*)

$$\mathcal{L}_I(\text{h.c.}) = \left[\bar{\psi}_\nu(x)\gamma^\alpha(1+\gamma^5)\psi_\mu(x)\right]\left[G_V^*\left(\bar{\psi}_n(x)\gamma^\alpha\psi_p(x)\right)\right.$$

$$+\frac{\mu^*}{4m}\partial_\beta\left(\bar{\psi}_n(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)\psi_p\right)-\frac{A^*}{m_\mu}\partial_\alpha(\bar{\psi}_n\psi_p) \tag{50}$$

$$+G_A^*\left(\bar{\psi}_n\gamma^\alpha\gamma^5\psi_p\right)-\frac{B^*}{4m}\partial_\beta\left(\bar{\psi}_n(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)\gamma^5\psi_p\right)$$

$$\left.+\frac{b^*}{m_\mu}\partial_\alpha\left(\bar{\psi}_n\gamma^5\psi_p\right)\right].$$

Taking the matrix element of this density between the states $|\bar{\nu},\bar{n}>$ and $|\bar{\mu},\bar{p}>$ gives, for C_2,

$$\ell_\alpha(C) = \bar{u}_{\bar{\nu}}(-\vec{q},\vec{s})\left\{\gamma^\alpha(1+\gamma^5)\right\}\bar{u}_{\bar{\mu}}(-\vec{q}\,',\vec{s}\,')\ , \tag{51}$$

$$L_\alpha(C) = \bar{u}_{\bar{n}}(-\vec{k},\vec{w})\left\{G_V^*\gamma^\alpha+\frac{i\mu^*}{4m}(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)Q_\beta\right.$$

$$-\frac{iA^*}{m_\mu}Q_\alpha+G_A^*\gamma^\alpha\gamma^5-\frac{iB^*}{4m}(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta \tag{52}$$

$$\left.+\frac{ib^*}{m_\mu}\gamma^5 Q_\alpha\right\}u_{\bar{p}}(-\vec{k}\,',\vec{w}\,')$$

where, for example, $u_{\bar{\nu}}(-\vec{q},\vec{s})$ is the spinor for an antineutrino of momentum $\vec{q}$ and spin $\vec{s}$. As in the parity discussion it is now desired

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*) Because the four vectors have an imaginary time component care must be used when taking the Hermitian conjugate of a four-vector operator such as the term $G_V\bar{\psi}_p\gamma^\alpha\psi_n$. Hence, the Hermitian conjugate ($\ddagger$) of such an operator is defined by

Footnote continued.

$$J_\alpha = (\vec{J},\ J_4) = (\vec{J}, i J_0)\ ,$$
$$J_\alpha^{\ddagger} = (\vec{J}^{\dagger}, -J_4^{\dagger}) = (\vec{J}^{\dagger}, i J_0^{\dagger}),$$

where J_0 is "real" and where ($\dagger$) denotes the usual Hermitian conjugate of an operator. Due to the fact that the four vectors always appear in scalar products (e.g., $p_\alpha k_\alpha = \vec{p}\cdot\vec{k} + p_4 k_4 = \vec{p}\cdot\vec{k} - p_0 k_0$) the operation ($\ddagger$) will, for any Lorentz invariant structure, be equal to the usual Hermitian conjugate operation ($\dagger$). An exception arises when $\epsilon_{\mu\nu\alpha\beta}$ occurs, then $\ddagger = -\dagger$. As a simple example, consider the four vector ∂_α:

$$\partial_\alpha = (\vec{\partial},\ \partial_4) = (\vec{\partial}, i\partial_t)\ ,$$
$$\partial_\alpha^{\ddagger} = (\vec{\partial}^{\dagger}, -\partial_4^{\dagger}) = (\vec{\partial}, i\partial_t) = \partial_\alpha .$$

A more interesting example in the present discussion is the vector term in $\mathcal{L}_I(x)$. Its Hermitian conjugate is (note that it is a four vector):

$$\left[G_V \bar{\psi}_p \gamma^\alpha \psi_n\right]^{\ddagger} = \left[G_V \psi_p^{\dagger} \gamma^4 \gamma^\alpha \psi_n\right]^{\ddagger}$$
$$= G_V^* \psi_n^{\dagger} \gamma^{\alpha\ddagger} \gamma^{4\dagger} \psi_p = G_V^* \psi_n^{\dagger} \gamma^{\alpha\ddagger} \gamma^4 \psi_p .$$

According to the definition of ($\ddagger$)

$$\gamma^{\alpha\ddagger} = \gamma^{\alpha\dagger} \qquad \alpha = 1,2,3\ ,$$
$$\gamma^{\alpha\ddagger} = -\gamma^{\alpha\dagger} \qquad \alpha = 4 .$$

Since the γ matrices are Hermitian ($\gamma_\alpha^{\dagger} = \gamma_\alpha$), the result is

$$\gamma^{\alpha\ddagger} = \gamma^{\alpha} \qquad \alpha = 1,2,3\ ,$$
$$\gamma^{\alpha\ddagger} = -\gamma^{\alpha} \qquad \alpha = 4 .$$

Using this result

$$\left[G_V \bar{\psi}_p \gamma^\alpha \psi_n\right]^{\ddagger} = G_V^* \psi_n^{\dagger} \left\{\pm\, \gamma^\alpha \gamma^4\right\} \psi_p , \qquad \begin{matrix}(+) & \alpha = 1,2,3 \\ (-) & \alpha = 4\end{matrix}$$
$$= G_V^* \psi_n^{\dagger} \left\{-\gamma^4 \gamma^\alpha\right\} \psi_p = -\,G_V^* \bar{\psi}_n \gamma^\alpha \psi_p .$$

Similarly, $\mathcal{L}_I$ can be treated term by term to find $\mathcal{L}_I$(h.c.). The reader should derive Eq. (50) from Eq. (38) using the above methods, i.e., compute

$$\mathcal{L}_I(\mathrm{h.c.}) \equiv \mathcal{L}_I^{\ddagger} .$$

to find the connection between C_2 given by Eq. (51) and Eq. (52) and C_1 given by Eq. (39) and Eq. (40) by working on $\ell_\alpha(C)$ and $L_\alpha(C)$. Again, one may establish a relation between the spinors, using the formulae (see Appendix A)*):

$$\begin{aligned} u_{\bar{p}}(-\vec{k}',\vec{w}') &= C\ \bar{u}_p(\vec{k},\vec{w}) \ , \\ \bar{u}_{\bar{n}}(-\vec{k},\vec{w}) &= -\ u_n(\vec{k},\vec{w})\, C^{-1} . \end{aligned} \tag{53}$$

Further,

$$\begin{aligned} C^{-1}\gamma^\alpha C &= -\ \tilde{\gamma}^\alpha \ , \\ C^{-1}\gamma^5 C &= +\ \tilde{\gamma}^5 \ , \\ C &= \gamma^2\gamma^4 \ , \end{aligned} \tag{54}$$

where $(\sim)$ means transpose.

Consider the leptonic factor $\ell_\alpha(C)$:

$$\begin{aligned} \ell_\alpha(C) &= \bar{u}_{\bar{\nu}}(-\vec{q},\vec{s})\left\{\gamma^\alpha(1+\gamma^5)\right\} u_\mu(-\vec{q}',\vec{s}') \\ &= -\ u_\nu(\vec{q},\vec{s})\left\{C^{-1}\gamma^\alpha(1+\gamma^5)\,C\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= -\ u_\nu(\vec{q},\vec{s})\left\{-\ \tilde{\gamma}^\alpha - \tilde{\gamma}^\alpha\tilde{\gamma}^5\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= -\ u_\nu(\vec{q},\vec{s})\left\{-\ \tilde{\gamma}^\alpha - \widetilde{\gamma^5\gamma^\alpha}\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= +\ u_\nu\left\{\widetilde{(1+\gamma^5)\gamma^\alpha}\right\}\bar{u}_\mu \\ &= \bar{u}_\mu(\vec{q}',\vec{s}')\left\{(1+\gamma^5)\gamma^\alpha\right\} u_\nu(\vec{q},\vec{s}) \\ &= \bar{u}_\mu(\vec{q}',\vec{s}')\left\{\gamma^\alpha(1-\gamma^5)\right\} u_\nu(\vec{q},\vec{s}) \ . \end{aligned} \tag{55}$$

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*) See footnote on page 17.

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Now, if exactly the same procedure is used term by term on $L_\alpha(C)$, the result is*):

$$L_\alpha(C) = \bar{u}_p(\vec{k}',\vec{w}')\Big\{G_V^*\gamma^\alpha + \frac{i\mu^*}{4m}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)Q_\beta + \frac{iA^*}{m_\mu}Q_\alpha - G_A^*\gamma^\alpha\gamma^5 - \frac{iB^*}{4m}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta - \frac{ib^*}{m_\mu}\gamma^5 Q_\alpha\Big\}u_n(\vec{k},\vec{w}) \; . \tag{56}$$

Inspection of Eqs. (39), (40), (55) and (56) shows that the result of the charge-conjugation transformation is:

$$G_V \; \ldots \; b \xrightarrow{C} G_V^* \; \ldots \; b^* \; , \qquad \gamma^5 \xrightarrow{C} -\gamma^5 \; . \tag{57}$$

Clearly, if G_V, μ, A are real and G_A, B, b are purely imaginary then $L_\alpha(C) = L_\alpha$; however, the transition probabilities for C_1 and C_2 would still be unequal because of the lepton factor ℓ_α where $\gamma^5 \to -\gamma^5$.

Again, instead of working with the matrix elements, it is possible to use the Lagrangian density, Eq. (50) noting that

$$\psi^c(x) = C\tilde{\bar{\psi}}(x) \; , \qquad \bar{\psi}^c(x) = -\tilde{\psi}(x)C^{-1} \; , \tag{58}$$

where $\psi^c(x)$ satisfies the charge-conjugated Dirac equation.

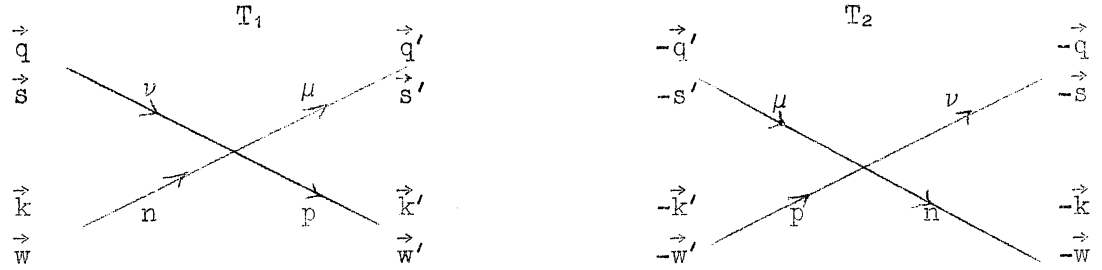
3. Time Reversal - T

Under the time-reversal transformation the sign of the time co-ordinate is changed:

$$x_\mu = (\vec{x}, i\,t) \xrightarrow{T} x'_\mu = (\vec{x}, -i\,t) \; . \tag{59}$$

*) The reader should try several terms.

Consequently, the signs of all three momenta and spins are reversed and, further, incoming particles become outgoing and vice versa. Thus, the process T_1 and its time-reversed counterpart process T_2 are represented by



(60)

where invariance under time reversal means that the transition probabilities of T_1 and T_2 are identical. As in the discussion of charge conjugation, it is necessary to use the Hermitian conjugate part of the Lagrangian density given by Eq. (50), since in T_2 μ and p come in and ν and n go out. Taking the matrix element of $\mathcal{L}_I(\text{h.c.})$ between the time-reversed states of (μ, p) and (ν, n) gives for T_2 *):

$$\ell_\alpha(T) = \bar{u}_\nu(-\vec{q},-\vec{s}) \left\{\gamma^\alpha(1+\gamma^5)\right\} u_\mu(-\vec{q}',-\vec{s}') \ , \quad (61)$$

$$\begin{aligned} L_\alpha(T) = \bar{u}_n(-\vec{k},-\vec{w}) \Big\{ & G_V^* \gamma^\alpha + \frac{i\mu^*}{4m}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q'_\beta \\ & - \frac{iA^*}{m_\mu} Q'_\alpha + G_A^* \gamma^\alpha\gamma^5 - \frac{iB^*}{4m}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q'_\beta \\ & + \frac{ib^*}{m_\mu}\gamma^5 Q'_\alpha \Big\} u_p(-\vec{k}',-\vec{w}') \end{aligned} \quad (62)$$

*) Here, $Q'_\alpha = [-(\vec{k}' - \vec{k}), i(k'_0 - k_0)]$ thus, $Q'_\alpha = (\vec{Q}, -iQ_0)$ where, as before, $Q_\alpha = (\vec{Q}, iQ_0) = (\vec{k} - \vec{k}'), i(k_0 - k'_0)$. Thus, Q' differs from Q through exchange of k and k' and the change of sign of the three momentum.

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where, for example, $u_\nu(-\vec{q},-\vec{s})$ is the Dirac spinor for a neutrino of momentum $-\vec{q}$ and spin $-\vec{s}$. It is desired to find the relationship between T_1 given by Eq. (39) and Eq. (40) and T_2 given above by Eq. (61) and Eq. (62). Now, if $u_p(\vec{k}',\vec{w}')$ and $u_p(-\vec{k}',-\vec{w}')$ are Dirac spinors satisfying, respectively, the Dirac equation and the time-reversed Dirac equation, then (see Appendix A and Ref. 6):

$$\begin{aligned} u_p(-\vec{k}',-\vec{w}') &= -D\tilde{\gamma}^4\,\bar{u}_p(\vec{k}',\vec{w}') \ , \\ \bar{u}_n(-\vec{k},-\vec{w}) &= -u_n(\vec{k},\vec{w})\,\tilde{\gamma}^4 D^{-1} \end{aligned} \tag{63}$$

where D is a 4×4 matrix obeying

$$\begin{aligned} D^{-1}\gamma^\mu D = \tilde{\gamma}^\mu, \quad \mu &= 1,2,3,4,5 \ , \\ D = -C\gamma^5 &= \gamma^2\gamma^4\gamma^5 \ . \end{aligned} \tag{64}$$

Consider the leptonic factor $\ell_\alpha(T)$ and use Eq. (63) and Eq. (64):

$$\begin{aligned} \ell_\alpha(T) &= \bar{u}_\nu(-\vec{q},-\vec{s})\left\{\gamma^\alpha(1+\gamma^5)\right\}u_\mu(-\vec{q}',-\vec{s}') \\ &= u_\nu(\vec{q},\vec{s})\left\{\tilde{\gamma}^4 D^{-1}\gamma^\alpha(1+\gamma^5)D\tilde{\gamma}^4\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= u_\nu(\vec{q},\vec{s})\left\{\tilde{\gamma}^4\tilde{\gamma}^\alpha\widetilde{(1+\gamma^5)}\,\tilde{\gamma}^4\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= u_\nu(\vec{q},\vec{s})\left\{\widetilde{\gamma^4(1+\gamma^5)\gamma^\alpha\gamma^4}\right\}\bar{u}_\mu(\vec{q}',\vec{s}') \\ &= \bar{u}_\mu(\vec{q}',\vec{s}')\left\{\gamma^4(1+\gamma^5)\gamma^\alpha\gamma^4\right\}u_\nu(\vec{q},\vec{s}) \\ &= \bar{u}_\mu(\vec{q}',\vec{s}')\left\{\gamma^4\gamma^\alpha\gamma^4(1+\gamma^5)\right\}u_\nu(\vec{q},\vec{s}) \\ &= \mp\,\bar{u}_\mu(\vec{q}',\vec{s}')\left\{\gamma^\alpha(1+\gamma^5)\right\}u_\nu(\vec{q},\vec{s}) \ , \end{aligned} \tag{65}$$

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where (-) is for $\alpha = 1,2,3$ and (+) for $\alpha = 4$. Applying the same treatment to $L_\alpha(T)$, term by term, yields*):

$$L_\alpha(T) = \mp \bar{u}_p(\vec{k}',\vec{w}') \Big\{ G_V^* \gamma^\alpha + \frac{i\mu^*}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) Q_\beta + \frac{iA^*}{m_\mu} Q_\alpha + G_A^* \gamma^\alpha\gamma^5 + \frac{iB}{4m} (\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^5 Q_\beta + \frac{ib}{m_\mu} \gamma^5 Q_\alpha \Big\} u_n(\vec{k},\vec{w}) \,, \tag{66}$$

where (-) is for $\alpha = 1,2,3$ and (+) is for $\alpha = 4$, just as in Eq. (65) for $\ell_\alpha(T)$. When the Lorentz scalar $L_\alpha(T)\,\ell_\alpha(T)$ is formed, these signs will cancel and, hence, the only difference between $\ell_\alpha L_\alpha$ and $\ell_\alpha(T)\,L_\alpha(T)$ is that the coefficients of the latter are complex conjugated. Therefore, under time reversal

$$G_V \;\ldots\; b \xrightarrow{\;T\;} G_V^* \;\ldots\; b^* \,. \tag{67}$$

If the $G_V \ldots b$ are real, the transition probabilities of T_1 and T_2 are equal, and time reversal invariance holds.

Instead of considering the matrix elements it is also possible to obtain the same results with the Lagrangian density, Eq. (50) and suitable transformations of the Dirac wave functions, $\psi(x)$.

4. The PCT Theorem

In summary, the effect of the three transformations may be represented symbolically by:

$$\begin{aligned} P:&\quad G_V \ldots \longrightarrow G_V \ldots \\ &\quad \gamma^5 \longrightarrow -\gamma^5 \,, \\ C:&\quad G_V \ldots \longrightarrow G_V^* \ldots \\ &\quad \gamma^5 \longrightarrow -\gamma^5 \,, \\ T:&\quad G_V \ldots \longrightarrow G_V^* \ldots \\ &\quad \gamma^5 \longrightarrow +\gamma^5 \,, \end{aligned}$$

*) The reader should try a few of these terms also!

from which it follows immediately that the combined operation PCT should leave a system invariant since

$$\text{PCT}: \quad G_V \ldots \longrightarrow G_V \ldots$$
$$\gamma^5 \longrightarrow +\gamma^5 \ .$$

Of course, the foregoing discussions do not constitute a proof, however, such a theorem has been proved[8]. It is the PCT theorem and states that (under certain very general assumptions such as Lorentz invariance, and hermiticity of the Lagrangian), a theory will be invariant under the combined application of P, C and T.

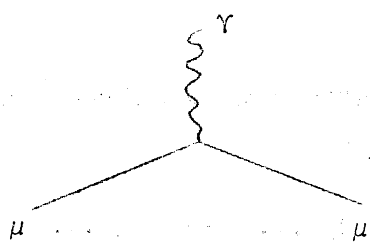
One should note that for matrix elements, PCT changes an incoming particle with momentum q and spin $\vec{s}$ into an outgoing anti-particle with momentum +q and spin $-\vec{s}$, etc.

V. G PARITY; FIRST AND SECOND CLASS CURRENTS

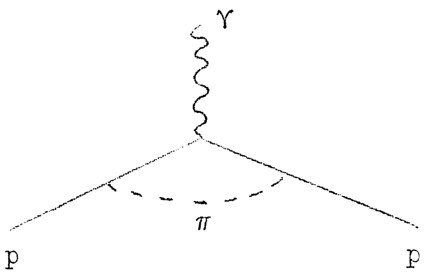
It is well known that the correctly antisymmetrized[5,6] electromagnetic current of the leptons is:

$$J_\mu(\text{e.m.}) = \bar{\psi}\gamma^\mu \psi - \psi\tilde{\gamma}^\mu \bar{\psi} \qquad (68)$$

which describes, for example, the process



The ψ describes the muon. As the muon does not participate in strong interactions, the γ-μ-μ matrix element will be very close to the interaction, Eq. (68). The question is, how must a matrix element arising from the current, Eq. (68), be modified when the muons are replaced by strongly interacting particles, such as protons, since then one can add diagrams like



Aside from the current, Eq. (68), we may consider the C properties of

$$J'_\mu = \partial_\mu(\bar{\psi}\psi - \psi\bar{\psi}) \ . \qquad (69)$$

To decide about the specific current J'_μ, it is useful to study the behaviour of J_μ and J'_μ under charge conjugation since such properties are supposedly invariant under the strong interactions.

Consider first the original current:

$$\begin{aligned} J_\mu \longrightarrow J_\mu^c &= \bar{\psi}^c \gamma^\mu \psi^c - \psi^c \tilde{\gamma}^\mu \bar{\psi}^c \\ &= -\psi C^{-1} \gamma^\mu C \bar{\psi} + (C\bar{\psi}) \tilde{\gamma}^\mu (\psi C^{-1}) \\ &= \psi \tilde{\gamma}^\mu \bar{\psi} + \bar{\psi} \tilde{C} \tilde{\gamma}^\mu \tilde{C}^{-1} \psi \\ &= \psi \tilde{\gamma}^\mu \bar{\psi} - \bar{\psi}\gamma^\mu \psi \\ &= - J_\mu \ , \end{aligned}$$

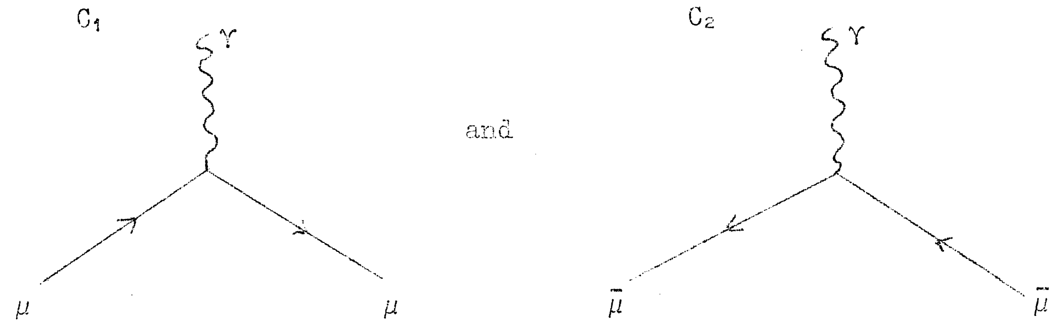
where use was made of $\psi^c = C\bar{\psi}$, $\bar{\psi}^c = -\psi C^{-1}$, $C^{-1}\gamma^\mu C = -\tilde{\gamma}^\mu$. Hence,

$$J_\mu \xrightarrow{C} J_\mu^c = - J_\mu \ . \qquad (70)$$

A similar calculation shows for J'_μ:

$$J'_\mu \xrightarrow{C} J'^c_\mu = + J'_\mu \ . \qquad (71)$$

Thus, the currents J'_μ and J_μ have opposite charge conjugation properties. The same will be true for the matrix elements of these currents, as can be seen by comparing the matrix elements of:



But if we take the Lagrangian to be of the form, Eq. (68), the strong interactions cannot induce a matrix element with opposite C properties. Thus, the proton-photon matrix element does not contain terms of the form:

$$\left(\bar{u}_p Q_\mu u_p\right) e_\mu ,$$

where e_μ is the polarization vector of the electromagnetic field. The same type of argument can also be applied to the weak interactions, however, the charge conjugation operation will have to be extended since the weak current involves n and p instead of simply p-p or n-n.

The weak interaction is assumed to have the form (again anti-symmetrized):

$$J_\mu = J_\mu^V + J_\mu^A$$

$$= G_V\left(\bar{\psi}_p \gamma^\mu \psi_n - \psi_n \tilde{\gamma}^\mu \bar{\psi}_p\right) + G_A\left(\bar{\psi}_p \gamma^\mu \gamma^5 \psi_n - \psi_n \widetilde{\gamma^\mu \gamma^5} \bar{\psi}_p\right) . \qquad (72)$$

Under the charge-conjugation operation the currents transform as:

$$J_\mu^V \longrightarrow J_\mu^{V,c} = - G_V\left[\bar{\psi}_n \gamma^\mu \psi_p - \psi_p \tilde{\gamma}^\mu \bar{\psi}_n\right] , \qquad (73)$$

$$J_\mu^A \longrightarrow J_\mu^{A,c} = + G_A\left[\bar{\psi}_n \gamma^\mu \gamma^5 \psi_p - \psi_p \widetilde{\gamma^\mu \gamma^5} \bar{\psi}_n\right] , \qquad (74)$$

where time-reversal invariance has been assumed so that the coefficients G_V, G_A are real. The currents, Eq. (73) and Eq. (74) must be transformed further to relate them to their original forms, Eq. (72). The operation C interchanged n and p and, hence, another n,p interchange is desired. The operation which will do this is a rotation by 180° about the second axis in isospace, which may be labelled T_2. As the τ matrices behave as a vector, we have:

$$\tau_2 \xrightarrow{T_2} +\tau_2 \ , \ \tau_{1,3} \xrightarrow{T_2} -\tau_{1,3} \tag{75}$$

and

$$\tau_\pm = 1/2\ (\tau_1 \pm i\tau_2) \longrightarrow 1/2(-\tau_1 \pm i\tau_2) = -1/2(\tau_1 \mp i\tau_2)$$

hence,

$$\tau_+ \xrightarrow{T_2} -\tau_- \ , \ \tau_- \xrightarrow{T_2} -\tau_+ \ . \tag{76}$$

Consider the $\bar{\psi}_n\psi_p$ terms [recall $\psi = \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}$]:

$$\bar{\psi}_n\psi_p = \bar{\psi}\tau_-\psi \xrightarrow{T_2} -\bar{\psi}\tau_+\psi = -\bar{\psi}_p\psi_n \ . \tag{77}$$

The transpose terms transform similarly. Hence, using Eq. (77) on the charge-conjugated currents, Eq. (73) and Eq. (74) yields:

$$J_\mu^{V,c} \xrightarrow{T_2} +G_V\left[\bar{\psi}_p\gamma^\mu\psi_n - \psi_n\tilde{\gamma}^\mu\bar{\psi}_p\right] = +J_\mu^V \ ,$$

$$J_\mu^{A,c} \xrightarrow{T_2} -G_A\left[\bar{\psi}_p\gamma^\mu\gamma^5\psi_n - \psi_n\widetilde{\gamma^\mu\gamma^5}\bar{\psi}_p\right] = -J_\mu^A \ .$$

The combined operation of charge conjugation C, and rotation by 180° about the second axis of isospace is called G parity. Thus, under G the weak vector and axial-vector currents transform as:

$$J_\mu^V \xrightarrow{G} + J_\mu^V \ ,$$
$$J_\mu^A \xrightarrow{G} - J_\mu^A \ . \tag{78}$$

Currents which transform as Eq. (78) are called currents of the first class[9] while those that transform with an opposite sign are called second class.

Up to this point the strong interactions have not been turned on. Since the strong interactions are invariant under C and transformations in I-spin space, and hence under G, the transformation properties of currents under G will be unchanged when the strong interactions are introduced. The G parity may be used to exclude certain currents exactly as C was used in the electromagnetic discussion.

Turning on the strong interactions is expected to add only first class currents to the Lagrangian, i.e., only proper Lorentz vectors (without γ^5) with G parity $+1$, and axial Lorentz vectors (with γ^5) with G parity -1 in accordance with the bare currents, Eq. (78). For example, the second class terms

$$\frac{A}{m_\mu}\, \partial_\mu(\bar\psi_p \psi_n) \xrightarrow{G} -\frac{A}{m_\mu}\, \partial_\mu(\bar\psi_p \psi_n) \ ,$$

$$\frac{B}{4m}\, \partial_\beta \left[\bar\psi_p(\gamma^\mu\gamma^\beta - \gamma^\beta\gamma^\mu)\gamma^5 \psi_n\right] \xrightarrow{G} + \frac{B}{4m}\, \partial_\beta \left[\bar\psi_p(\gamma^\mu\gamma^\beta - \gamma^\beta\gamma^\mu)\gamma^5 \psi_n\right]$$

are excluded.

The crucial point of the reasoning is the assumption of the form of the weak current without the strong interactions. For example, if the original current contained a small axial part with G parity $+1$, then the current with strong interactions might have a relatively large term of this form, since the effect of the strong interactions is difficult to calculate. There is no *a priori* reason to expect one class of current and not the other.

The Lagrangian which results is

$$\mathcal{L}_I = \left[\bar{\psi}_\mu \gamma^\alpha (1+\gamma^5)\psi_\nu\right]\left[G_V \bar{\psi}_p \gamma^\alpha \psi_n + \frac{\mu}{4m}\,\partial_\beta\left(\bar{\psi}_p(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\psi_n\right) + G_A \bar{\psi}_p \gamma^\alpha\gamma^5 \psi_n + \frac{b}{m_\mu}\,\partial_\alpha\left(\bar{\psi}_p \gamma^5 \psi_n\right)\right] + \text{h.c.} \tag{79}$$

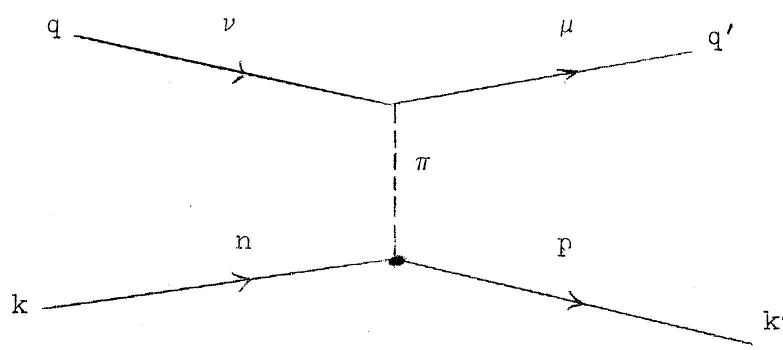
The second (μ) and fourth (b) terms are considered to be induced by the action of the strong interactions.

VI. THE GOLDBERGER-TREIMAN RELATION

The conserved vector-current theory relates the weak vector coefficients G_V and μ to the corresponding electromagnetic form factors. It is desired to see if some relation can be established for the axial vector-current coefficients G_A and b. To begin, an attempt is made to estimate the magnitude of the induced pseudoscalar term

$$\left[\bar{u}_\mu \gamma^\alpha (1+\gamma^5) u_\nu\right]\left[\bar{u}_p \left(\frac{ib}{m_\mu}\,\gamma^5 Q\alpha\right) u_n\right]. \tag{80}$$

This term has G parity -1 and apart from the factor $Q\alpha$ looks like the pion-nucleon matrix element $\bar{u}\gamma^5 u$. Hence, consider the diagram:



(81)

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For the upper vertex the pion-lepton matrix element has the form:

$$\pm i\, g_{\pi\ell}\, Q\alpha \left(\bar{u}_\mu \gamma^\alpha (1+\gamma^5)\, u_\nu\right), \tag{82}$$

where $g_{\pi\ell} = g_{\pi\ell}(k_\pi^2)$. From π decay one knows the value for $k_\pi^2 = -m_\pi^2$:

$$g_{\pi\ell}(-m_\pi^2) = (1.48\times 10^{-7})/m_\pi .$$

The pion-nucleon vertex in the diagram gives rise to the matrix-element factor*)

$$\pm\, g_{\pi NN}\, (\bar{u}_p \gamma^5 u_n) , \tag{83}$$

where $g_{\pi NN} = 19.5$ for charged-pion coupling. The internal pion line gives rise to a factor $(Q^2 + m_\pi^2)^{-1}$. Thus, we get

$$\left[\bar{u}_\mu \gamma^\alpha (1+\gamma^5)\, u_\nu\right] \left[\bar{u}_p \left(i\, g_{\pi NN}\, g_{\pi\ell}\, \gamma^5 \frac{Q\alpha}{Q^2 + m_\pi^2}\right) u_n\right] . \tag{84}$$

If it is assumed that the one-pion-exchange graph makes the dominant contribution to the induced pseudoscalar term in weak interactions**) then equating expressions (80) and (84) gives

$$b = \pm\, g_{\pi NN}\, g_{\pi\ell} \frac{m_\mu}{Q^2 + m_\pi^2} , \tag{85}$$

which relates the weak interaction induced pseudoscalar coefficient to the pion-nucleon and pion-lepton couplings. This relation has been checked in mu-capture and the agreement with experiment is reasonably good, indicating that the one-pion-exchange picture is a good approximation. Experimentally $g_{\pi NN}$ is determined by studying the pion-nucleon

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*) Gradient-type couplings will be considered later.

**) The peripheral model states that the one-pion-exchange graph is the most important contribution to this process.

vertex and $g_{\pi\ell}$ is known from pi-decay. Note that we need $g_{\pi\ell}$ for $k_\pi^2 \cong 0$, so that we also use the assumption that $g_{\pi\ell}(-m_\pi^2) \cong g_{\pi\ell}(0)$.

The axial part of the baryonic matrix-element factor is therefore

$$L_\alpha^A = \bar{u}(k') \left\{ G_A \gamma^\alpha \gamma^5 + \frac{ib}{m_\mu} \gamma^5 Q\alpha \right\} u(k)$$

or (86)

$$L_\alpha^A = \bar{u}(k') \left\{ G_A \gamma^\alpha \gamma^5 \pm i\, g_{\pi NN}\, g_{\pi\ell} \frac{Q\alpha}{Q^2 + m_\pi^2} \gamma^5 \right\} u(k) \;.$$

Numerically the factors in this expression seem very unrelated. But let us now form the divergence of this current. Recall that L_α^A is, by definition, the matrix element of the axial current $J_\alpha^A(x)$, i.e.,

$$L_\alpha^A \equiv \langle k' \,|\, J_\alpha^A \,|\, k \rangle \;, \tag{87}$$

where $|k\rangle$ is a nucleon state of momentum k. The matrix element of the divergence of J_α^A may be written in terms of the matrix element of J_α^A using

$$Q \equiv q' - q = k - k' \;.$$

Thus

$$\begin{aligned} \langle k' \,|\, \partial_\alpha J_\alpha^A \,|k\rangle &= i\, Q_\alpha \langle k' \,|\, J_\alpha^A \,|k\rangle \\ &= i\, Q_\alpha L_\alpha^A \;. \end{aligned} \tag{88}$$

Using Eqs. (86) and (88) gives

$$\langle k' |\, \partial_\alpha\, J_\alpha^A |k\rangle = \bar{u}(k') \left\{ G_A\, i\, Q_\alpha \gamma^\alpha \mp g_{\pi NN}\, g_{\pi\ell} \frac{Q^2}{Q^2 + m_\pi^2} \right\} \gamma^5 u(k) \;, \tag{89}$$

where the first term may be rewritten using the Dirac equation:

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$$\bar{u}(k') \, i \, \gamma^{\alpha} (k_{\alpha} - k'_{\alpha}) \gamma^{5} u(k)$$

$$= -\bar{u}(k') \gamma^{5} (i \gamma^{\alpha} k_{\alpha}) \, u(k) - \bar{u}(k') (i \gamma^{\alpha} k'_{\alpha}) \gamma^{5} u(k) \tag{90}$$

$$= 2m_N \, \bar{u}(k') \gamma^{5} u(k) \; ,$$

where m_N is the nucleon mass. The second term of Eq. (89) may be rewritten by noting

$$\frac{Q^2}{Q^2 + m_\pi^2} = \left(1 - \frac{m_\pi^2}{Q^2 + m_\pi^2} \right) . \tag{91}$$

The use of Eqs. (90) and (91) in Eq. (89) gives directly

$$\langle k' | \partial_\alpha J_\alpha^A | k \rangle = 2m_N \, \bar{u}(k') \gamma^5 u(k) \times$$

$$\left\{ G_A \mp \frac{g_{\pi NN} \, g_{\pi \ell}}{2m_N} \pm \frac{g_{\pi NN} \, g_{\pi \ell}}{2m_N} \, \frac{m_\pi^2}{Q^2 + m_\pi^2} \right\} , \tag{92}$$

where all factors may be functions of Q^2. It is now found experimentally that for small Q^2 the first two terms of the r.h.s. of Eq. (92) approximately cancel when the minus sign is used. More precisely, for $Q^2 = 0$

$$G_A = \frac{1.15}{m_N^2} \times \frac{10^{-5}}{\sqrt{2}} \; ,$$

while, using the above values for $g_{\pi NN}$ and $g_{\pi \ell}$, gives

$$\frac{g_{\pi NN} \, g_{\pi \ell}}{2m_N} = \frac{1.3}{m_N^2} \times \frac{10^{-5}}{\sqrt{2}} \; .$$

There are two interpretations of this cancellation.

The dispersion relation view assumes that the matrix element of the divergence of the axial current, Eq. (92), vanishes for large Q^2 and can therefore be represented by an unsubtracted dispersion relation of the form

$$\langle k' | \partial_\alpha J_\alpha^A | k \rangle = \frac{\text{constant}}{Q^2 + m_\pi^2} + \int \frac{\rho(Q'^2)\, Q'^2\, dQ'^2}{Q^2 + Q'^2 - i\epsilon} \quad . \tag{93}$$

For $Q^2 \cong 0$ it is assumed[10] that the contribution of the integral may be neglected. Hence, comparison of Eqs. (92) and (93) gives

$$G_A(0) = \frac{g_{\pi NN}\, g_{\pi \ell}}{2m_N} \quad . \tag{94}$$

This is the Goldberger-Treiman relation[11] between the axial vector-current coefficient G_A and the pion-nucleon and pion-lepton couplings.

The Feynman graph approach can be used to derive the Goldberger-Treiman relation by assuming that the divergence of the axial vector current vanishes in the limit of zero pion mass. Here we merely note that for $m_\pi^2 = 0$ this requirement in Eq. (92) leads at once to the result Eq. (94). See the Erice notes (1964) by R.P. Feynman for details.

In deriving the Goldberger-Treiman relation use was made of the Dirac equation to rewrite the first term in the matrix element. However, the derivation would still work with the initial and final nucleons not free if the pion-nucleon coupling was of the so-called gradient type:

$$\frac{g_{\pi NN}}{2m_N} \frac{\partial \pi}{\partial x_\beta} \left(\bar{\psi}_p \gamma^\beta \gamma^5 \psi_n \right)$$

which leads to an induced pseudoscalar term:

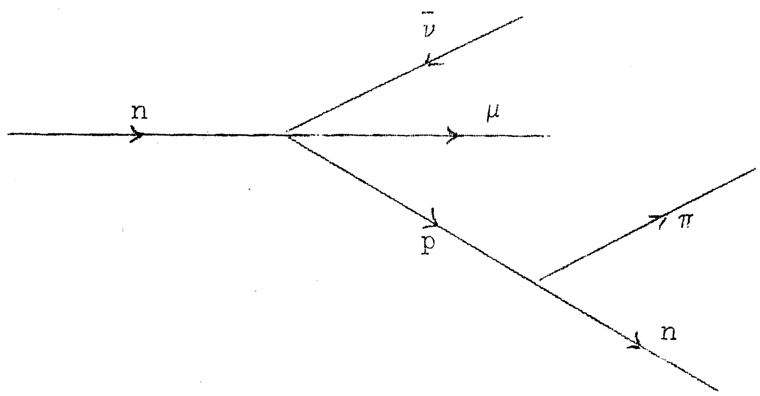
$$\left[\bar{u}_\mu \gamma^\alpha (1 + \gamma^5)\, u_\nu \right] \left[\bar{u}_p \left(\pm \frac{g_{\pi NN}\, g_{\pi \ell}}{2m_N} \frac{Q_\alpha Q_\beta}{Q^2 + m_\pi^2} \gamma^\beta \gamma^5 \right) u_n \right] \quad . \tag{95}$$

The matrix element of the divergence of the current is then

$$\langle k' | \partial_\alpha J_\alpha^A | k \rangle = i Q_\alpha L_\alpha^A$$

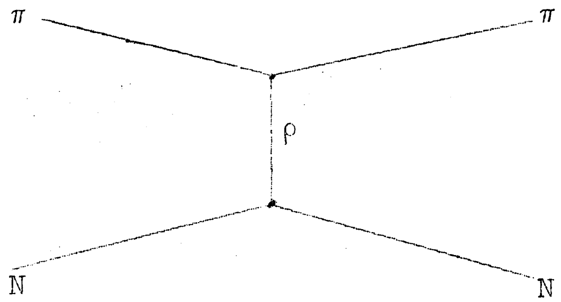
$$= \bar{u}(k') \left\{ G_A i Q_\alpha \gamma^\alpha \gamma^5 \pm \frac{g_{\pi NN} \, g_{\pi \ell}}{2m_N} \, \frac{Q^2}{Q^2 + m_\pi^2} \, i Q_\beta \gamma^\beta \gamma^5 \right\} u(k) \tag{96}$$

which again leads to the relation (94) by comparison of coefficients. This time the Dirac equation has not been used and, thus, the derivation would go through even for virtual initial and final nucleons. Such a situation may occur in pion production by neutrinos:



where, clearly, the proton is not on the mass shell.

If the gradient coupling is the one governing strong interactions then the amplitudes are expected to vanish when the pion four-momentum is zero. Thus, the s-wave in pion-nucleon scattering should be small as proportionality to the pion momentum implies p-waves or higher angular momentum states. This seems to be, more or less, experimentally true. However, there is some s-wave, usually attributed to ρ exchange:



Thus, π-ρ coupling seems not to satisfy this requirement.

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VII. TIME-REVERSAL INVARIANCE

As shown in the section on the C, P and T transformations, time-reversal invariance requires that the coefficients of the weak baryonic current G_V, μ, A, G_A, B and b be real. A simple theorem can be proved that relates time-reversal violating effects to the second-class currents.

Theorem: If the lepton and antilepton currents,

$$\left[\bar{\psi}_\mu \gamma^\alpha (1+\gamma^5)\psi_\nu\right] \text{ and } \left[\bar{\psi}_\nu \gamma^\alpha (1+\gamma^5)\psi_\mu\right],$$

are coupled to baryonic currents that are members of the same isospin multiplet, then time-reversal violating effects occur only through currents of the second class.

Proof: For simplicity, only one first-class term and one second-class term are considered. The Lagrangian with the Hermitian conjugate part written explicitly is

$$\mathcal{L} = \left[\bar{\psi}_\mu \gamma^\alpha (1+\gamma^5)\psi_\nu\right]\left[G_V\left(\bar{\psi}_p \gamma^\alpha \psi_n\right) + \frac{A}{m_\mu}\partial\alpha\left(\bar{\psi}_p \psi_n\right) + \dots\right]$$

$$+ \left[\bar{\psi}_\nu \gamma^\alpha (1+\gamma^5)\psi_\mu\right]\left[G_V^*\left(\bar{\psi}_n \gamma^\alpha \psi_p\right) - \frac{A^*}{m_\mu}\partial\alpha\left(\bar{\psi}_n \psi_p\right) + \dots\right],$$

where the G_V term is first class and the A term is second class*). Now if the baryonic currents, $G_V(\bar{\psi}_p \gamma^\alpha \psi_n)$ and $G_V^*(\bar{\psi}_n \gamma^\alpha \psi_p)$, are the τ_+ and τ_- components of the same isospin multiplet**),

$$G_V\left(\bar{\psi}\gamma^\alpha \vec{\tau}\psi\right),$$

---

*) See section V for a discussion of first and second-class currents.

**) Recall

$$\psi = \begin{pmatrix}\psi_p \\ \psi_n\end{pmatrix}, \quad \bar{\psi}_p \psi_n = \bar{\psi}\tau_+\psi, \quad \bar{\psi}_n \psi_p = \bar{\psi}\tau_-\psi .$$

then clearly, $G_V^* = G_V$. Likewise, if the baryonic currents, $(A/m_\mu)\ \partial\alpha(\bar{\psi}_p \psi_n)$ and $(-A^*/m_\mu)\ \partial\alpha(\bar{\psi}_n \psi_p)$, are members of the same isospin multiplet,

$$\frac{A}{m_\mu}\ \partial_\alpha(\bar{\psi}\vec{\tau}\psi) \ ,$$

then $-A^* = A$. The requirement that $G_V^* = G_V$ is in accordance with time-reversal invariance, however, the restriction $-A^* = A$ makes A purely imaginary and, hence, the second-class current, if non-zero, will lead to time-reversal violating effects. The same reasoning applies to the other terms in the Lagrangian. This completes the proof of the theorem.

The above theorem may be generalized in the SU_3 context. The result is Cabibbo's[12]) form of time-reversal violation in weak interactions: assuming that the lepton and antilepton currents couple to members of the same octet, implies again, that time-reversal violations proceed via currents of the second class.

VIII. FINAL STATE INTERACTIONS

It has been shown in section IV that time-reversal invariance is simply related to the reality of the coefficients in the baryonic current. This has been demonstrated using one part of the Lagrangian for a process T_1 and the Hermitian conjugate part of the Lagrangian for the time-reversed process T_2. Thus, the reality properties of the matrix elements for T_1 and T_2 are those of the Lagrangian itself if we neglect higher order effects. We will now consider the effect of strong interactions in this respect.

Consider a scattering process with initial state A and final state B which proceeds via a scattering matrix S. The transition probability is given by:

$$w_{A\to B} = |\langle B|S|A\rangle|^2 = \langle A|S^\dagger|B\rangle\langle B|S|A\rangle \ , \qquad (97)$$

where $\langle B|S|A\rangle^* = \langle A|S^\dagger|B\rangle$. The requirement of conservation of probability is for any complete set of states B:

$$\sum_B w_{A\to B} = 1 \ ,$$

and hence by Eq. (97)

$$\sum_B \langle A|S^\dagger|B\rangle \langle B|S|A\rangle = \langle A|S^\dagger S|A\rangle = 1 \ . \qquad (98)$$

As this must hold for <u>any</u> A, we find that $S^\dagger S$ must be the unit matrix:

$$S^\dagger S = 1 \ ,$$

or

$$S^\dagger = S^{-1} \ , \qquad (99)$$

in other words, S is a <u>unitary</u> matrix. If there is no interaction, S acting on $|A\rangle$, merely gives back $|A\rangle$. This part is separated off by the division:

$$S = 1 + iT \ , \ S^\dagger = 1 - iT^\dagger \ , \qquad (100)$$

which, using the unitarity condition Eq. (99), gives

$$-i(T - T^\dagger) = T^\dagger T \ . \qquad (101)$$

This equation, when taken between initial and final states, A and B, gives

$$-i\left[\langle B|T|A\rangle - \langle B|T^\dagger|A\rangle\right] = \langle B|T^\dagger T|A\rangle \ ,$$

or

$$-i\left[\langle B|T|A\rangle - \langle A|T|B\rangle^*\right] = \sum_C \langle B|T^\dagger|C\rangle \langle C|T|A\rangle \ ,$$

or

$$-i\left[\langle B|T|A\rangle^* - \langle A|T|B\rangle\right]^* = \sum_C \langle C|T|B\rangle^* \langle C|T|A\rangle \ . \qquad (102)$$

If there exists no state $|C>$ such that the transitions $A \to C$ and $B \to C$ are possible, then the r.h.s. of Eq. (102) is zero and consequently

$$<A|T|B> = <B|T|A>^* . \qquad (103)$$

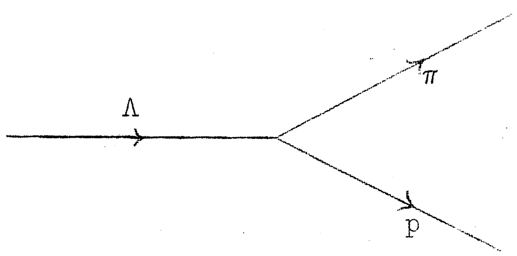
If the time-reversed states A', B' are used instead of A, B, Eq. (103) is*)

$$<A'|T|B'> = <B'|T|A'>^* . \qquad (104)$$

The quantity $<A'|T|B'>$ is the amplitude for the time-reversed process T_2, where T_1, the original process, has the amplitude $<B|T|A>$. By using the spinor transformations which relate A' and A, B' and B, the r.h.s. of Eq. (104) can be related to $<B|T|A>$, i.e., to process T_1. Thus, the original process T_1 ($<B|T|A>$) and the time-reversed process T_2 ($<A'|T|B'>$) can be directly related through Eq. (104).

If, however, an intermediate state $|C>$ does exist, such that the r.h.s. of Eq. (102) is <u>not</u> zero, then Eqs. (103) and (104) are not valid and there is no simple relation between the amplitudes $<B|T|A>$ and $<A'|T|B'>$.

To make these ideas clearer, consider the weak decay of the lambda into a proton and a pion. With no intermediate state the diagram for this process is:



However, the proton and pion can rescatter such that the process has the diagram

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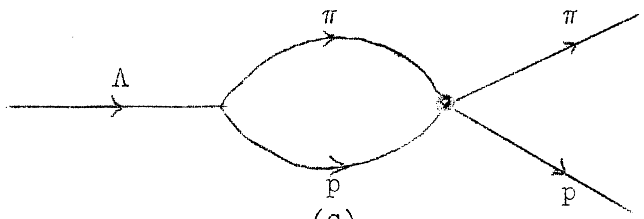
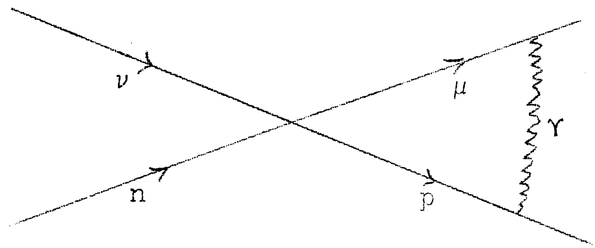
*) A' is the state A with the signs of all three-momenta and spins reversed. The same is true for B.



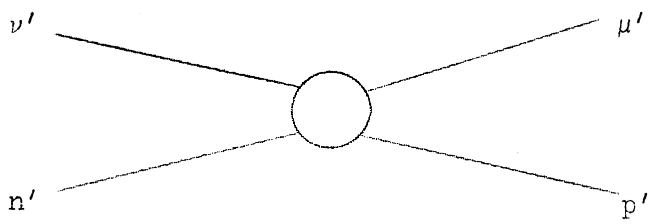
(C)

where the intermediate state C is a pion-proton state with <u>different</u> momenta than the final pion-proton state. Thus, in studying the time-reversal invariance properties of a reaction, it is important to know whether or not such rescattering processes occur, and if they do occur what is the magnitude of their effect. For obvious reasons, the above effects are called final-state interactions.

Another example of a final state interaction is given by the electromagnetic interaction between the final states (μ^- and p) of a high-energy neutrino reaction. A diagram is, for instance:

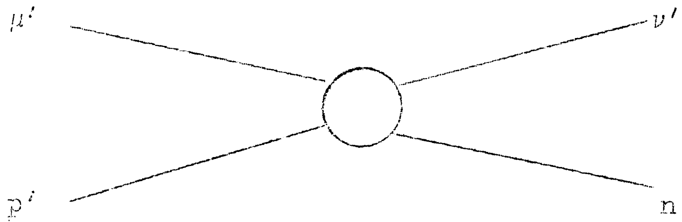


The electromagnetic interaction of the p and the μ^- may cause effects in the process that forbid any simple relation to the process



(105)

where the primed states are the states with three-momenta and spins reversed. Recall that time-reversal invariance requires that the $\nu + n \to \mu + p$ process equals



(106)

The manipulations with unitarity serve to connect processes (105) and (106), so that time-reversal invariance gives a constraint on the process itself, rather than connecting it to a process with different particles in and different particles out.

IX. REVIEW OF UNITARY SYMMETRY

The reader who desires an outline of the theory of groups is referred to the textbook of Roman[13] and to the Racah lectures[14]. An elementary introduction to the groups SU_2 and SU_3 is given in Appendix B, while the Gell-Mann paper The Eightfold Way[15] provides a discussion of the role of SU_3 symmetry in physics. We begin with a discussion of a group familiar to physicists: SU_2. This is, in fact, the group of I-spin rotations and the "covering group" of space rotations.

1. SU_2 Symmetry

The group SU_2 is the set (u) of unitary, unimodular linear transformations in a complex space of two dimensions, i.e., the group of two by two matrices u, such that

$$u^+u = uu^+ = 1 \; ,$$
$$\det u = 1 \; . \tag{107a}$$

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To say that nature is invariant under the group SU_2 means that there is a group of operations on the Hilbert space of physical states which are in complete correspondence to the elements of SU_2. That is, to each matrix u there will correspond a unitary operator U(u) such that

$$U(u)U(v) = U(uv) \,, \qquad U^\dagger(u)U(u) = U(u)U^\dagger(u) = 1 \,. \tag{107b}$$

The matrices u can be described in terms of three real parameters (we have five conditions among 2 × 2 = 4 complex numbers). The choice of the three parameters is quite arbitrary, but suppose we have chosen a particular set (a_1, a_2, a_3). Then we can write

$$U(u) = U(a_1, a_2, a_3) \,. \tag{107c}$$

U transforms a state $|\psi>$ into a new state $|\psi'>$:

$$|\psi'> = U(a_1, a_2, a_3)\,|\psi> \,, \tag{107d}$$

and because of this the operators U will frequently be referred to as transformations.

The identity transformation is defined to have $a_1 = a_2 = a_3 = 0$:*)

$$U(0)U(a) = U(a)U(0) = U(a) \,,$$

or

$$U(0) = 1 \,. \tag{108}$$

The inverse to the transformation U(a) is defined so that

$$U^{-1}(a)U(a) = U(a)U^{-1}(a) = U(0) = 1 \,. \tag{109}$$

*) The notation U(a) means $U(a_1, a_2, a_3)$.

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Operators (as opposed to the states $|\psi\rangle$) transform under SU_2 by

$$O' = U(a)\, O\, U^{-1}(a) \; . \qquad (110)$$

The unitary property of the transformations,

$$U^{-1}(a) = U^{\dagger}(a) \; , \qquad (111)$$

means that matrix elements are invariant under transformations:

$$\begin{aligned} \langle\varphi'|O'|\psi'\rangle &= \langle\varphi|U^{-1}U\, O U^{-1}\, U|\psi\rangle \\ &= \langle\varphi|O|\psi\rangle \; , \end{aligned} \qquad (112)$$

as is the norm of a state:

$$\begin{aligned} \langle\psi'|\psi'\rangle &= \langle\psi|U^{-1}\, U|\psi\rangle \\ &= \langle\psi|\psi\rangle \; . \end{aligned} \qquad (113)$$

Since the successive application of transformations yields a new transformation*)

$$U(a'') = U(a')\, U(a) \; , \qquad (114)$$

it is possible to build up a finite transformation by repeated application of infinitesimal transformations:

$$U(a) = U(\epsilon''' \cdots) \ldots U(\epsilon'')\, U(\epsilon')\, U(\epsilon) \; ,$$

where $\epsilon = \epsilon_1, \epsilon_2, \epsilon_3$ means a set of very small a_i. Consequently, it is sufficient to study the properties of the infinitesimal transformations and using Eq. (108) we expand such a transformation in powers of ϵ_i **):

---

*) Successive application is the composition law for the group.

**) The Einstein summation convention is used: $\epsilon_\alpha F_\alpha = \sum_{\alpha=1}^{3} \epsilon_\alpha F_\alpha$

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$$U(\epsilon_1, \epsilon_2, \epsilon_3) = U(0,0,0) + i\,\epsilon_\alpha F_\alpha + \ldots$$

$$= 1 + i\,\epsilon_\alpha F_\alpha + \ldots , \qquad (115)$$

where

$$i F_\alpha \equiv \frac{\partial U}{\partial a_\alpha}\Big]_{a_1 = a_2 = a_3 = 0} , \quad \alpha = 1,2,3 .$$

The operators F_α are called the infinitesimal generators of the group. The unitarity condition

$$U^\dagger(\epsilon)U(\epsilon) = 1 = [1 - i\,\epsilon_\alpha F_\alpha^\dagger + \ldots]\,[1 + i\,\epsilon_\alpha F_\alpha + \ldots]$$

$$= 1 + i\,\epsilon_\alpha\,[F_\alpha - F_\alpha^\dagger] + \ldots ,$$

implies that the infinitesimal generators are Hermitian:

$$F_\alpha^\dagger = F_\alpha . \qquad (116)$$

In order that successive application of infinitesimal transformations yield a finite transformation, the generators must obey commutation relations of the form

$$[F_i, F_j] = c_{ijk} F_k , \qquad (117)$$

c_{ijk} being constants which define the structure of the group.

A representation of a group is a group of matrices of finite dimension which can be put in correspondence with the elements of the group. In the case of SU_2 we know certainly at least one representation, namely, the one by 2×2 matrices (since, of course, the group was defined that way!). The generators of the group F_1, F_2, F_3, will be represented by three independent, Hermitian, 2×2 matrices which we call $\frac{1}{2}\tau_1$, $\frac{1}{2}\tau_2$, $\frac{1}{2}\tau_3$. These matrices should also be traceless, because for infinitesimal ϵ_i

$$\det u = 1 = \det(1 + i/2\,\epsilon_\alpha \tau_\alpha + \ldots) = 1 + i/2\,\epsilon_\alpha \mathrm{Tr}(\tau_\alpha) + 0(\epsilon^2) . \qquad (118)$$

A possible choice of three Hermitian, traceless, 2 × 2 matrices is given by the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{119}$$

which obey the commutation relations

$$[\tau_1, \tau_2] = 2i\,\tau_3, \quad \text{and cyclically,}$$

or (120)

$$[\tau_i, \tau_j] = 2i\,\epsilon_{ijk}\,\tau_k,$$

where ϵ_{ijk} is the Ricci symbol:

$\epsilon_{123} = 1$, $\epsilon_{213} = -1$, and cyclically, and
$\epsilon_{ijk} = 0$ for any two indices the same.

Since the correspondence is $F_i \rightarrow \frac{1}{2}\tau_i$, the F_i obey the commutation relations

$$[F_i, F_j] = i\,\epsilon_{ijk}\,F_k\,. \tag{121}$$

The fact that the Pauli matrices must be independent, Hermitian, and traceless is sufficient to determine their form*). Thus, we have determined the commutation relations of the infinitesimal generators of the group SU_2 by finding a convenient representation of the group (in this case, the 2 × 2 representation) and then directly calculating the commutators of the matrices $\frac{1}{2}\tau_i$ which represent the infinitesimal generators F_i. We shall employ this useful technique again for the group SU_3.

The Pauli matrices satisfy the further relations

$$\{\tau_i, \tau_j\} = 2\,\delta_{ij}\,1\,,$$

$$\mathrm{Tr}(\tau_i\,\tau_j) = 2\,\delta_{ij}\,. \tag{122}$$

*) Actually, the form of the three matrices is determined only up to a similarity transformation. Thus, other possible representations are related to the Pauli matrices by $\tau_i' = S\,\tau_i\,S^{-1}$.

These relations, together with the condition $Tr(\tau_i) = 0$, are peculiar to the 2 × 2 representations and are not, in general, true for the infinitesimal generators F_i.

The space-time symmetries are those associated with rotations in real three-dimensional space and Lorentz transformations (with generators denoted by the operator $M_{\mu\nu}$), and with translations in space and time (generator denoted by the momentum operator P_μ). The internal symmetries are those associated with an internal degree of freedom such as isotopic spin or hypercharge, and these have as operators the infinitesimal generators F_i. The space-time symmetries are assumed to commute with the internal symmetries:

$$[M_{\mu\nu}, F_i] = [P_\mu, F_i] = 0 \, , \qquad (123)$$

so that if

$$P_\mu |\psi> = p_\mu |\psi> \, ,$$

then

$$P_\mu |\psi'> = p_\mu |\psi'> \, ,$$

where

$$|\psi'> = U(a)|\psi> \, .$$

2. SU_3 Symmetry

The group SU_3 was used by Sakata as an extension of the isospin concept. He attempted to construct particles using the p, n, and Λ as fundamental building blocks. It was years before Gell-Mann discovered that the $J^P = \frac{1}{2}^+$ baryons fitted neatly into the octet representation of the group SU_3.

The set (u) of unitary, unimodular transformations in a complex space of three dimensions is the group SU_3. Hence, u is a 3 × 3 unimodular unitary matrix and U(u) is the corresponding transformation in Hilbert space. u depends on eight real parameters (nine complex numbers with ten conditions). As in the case of SU_2, the general infinitesimal transformation is written

$$U(\epsilon_1 \ldots \epsilon_8) = 1 + i\,\epsilon_\alpha F_\alpha + \ldots \quad , \tag{124}$$

where now $\alpha = 1 \ldots 8$ as the group elements of SU_3 are functions of eight real parameters. As before,

$$F_i^\dagger = F_i \ . \tag{125}$$

To find the commutation relations of the infinitesimal generators we use the technique discussed above, i.e., we find a convenient representation (in this case 3×3) of the group and then calculate the commutators directly. Thus we write in the three-dimensional representation

$$u(\epsilon) = 1 + i\,\epsilon_\alpha \frac{\lambda_\alpha}{2} + \ldots \quad , \tag{126}$$

where the λ_i are eight independent, Hermitian, and traceless 3×3 matrices. A possible choice is the following[15]:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{127}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

Using this representation the commutation relations are calculated:

$$\begin{aligned} [\lambda_i, \lambda_j] &= 2i\, f_{ij\alpha} \lambda_\alpha \ , \\ \{\lambda_i, \lambda_j\} &= 2 d_{ij\alpha} \lambda_\alpha + \tfrac{4}{3}\, \delta_{ij}\, 1 \ , \\ \mathrm{Tr}(\lambda_i \lambda_j) &= 2\, \delta_{ij} \ . \end{aligned} \tag{128}$$

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where the last relation may be calculated from the first two using $Tr(A+B) = TrA + TrB$. Equations (128) imply that

$$Tr(\lambda_i \lambda_j \lambda_k - \lambda_j \lambda_i \lambda_k) = 4i f_{ijk} ,$$
$$Tr(\lambda_i \lambda_j \lambda_k + \lambda_j \lambda_i \lambda_k) = 4 d_{ijk} . \qquad (129)$$

The non-zero elements of the tensors f_{ijk} and d_{ijk} are:

ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	-1/2	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	-1/2
345	1/2	256	1/2
367	-1/2	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	-1/2
		377	-1/2
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/\sqrt{3}$

(130)

where f_{ijk} changes sign when two indices are interchanged and d_{ijk} does not. The numbers f_{ijk} are called the structure constants of the group SU_3 and they are the same for all representations. Thus, the commutation relations for the infinitesimal generators of the group SU_3 are:

$$[F_i, F_j] = i f_{ij\alpha} F_\alpha , \qquad (131)$$

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since $F_i \to \lambda_i/2$. As in the case of SU_2, the anticommutation rules are characteristic of the particular (three-dimensional) representation we have chosen, not of the generators F_i.

3. The Eight-Dimensional Representation of SU_3; Isospin, hypercharge and the baryon octet

In anticipation of being able to identify the eight $\frac{1}{2}^+$ baryons ($\Sigma^0, \Sigma^\pm, \Lambda, p, n, \Xi^-, \Xi^0$) with the basis states of an eight representation of the group SU_3, we now construct such a representation. To this end consider the combination of three transformations:

$$U(\epsilon)U(\eta)U^{-1}(\epsilon) = 1 + i\,\eta_\alpha\, U(\epsilon) F_\alpha\, U^{-1}(\epsilon) + \ldots$$

$$= 1 + i\,\eta_\alpha\, F'_\alpha + \ldots \;,$$

where

$$F'_i \equiv U(\epsilon) F_i\, U^{-1}(\epsilon) \;,$$

and where $U(\eta)$ is near the identity transformation. Since $U(\epsilon)U(\eta)U^{-1}(\epsilon)$ is itself an infinitesimal transformation it must be expressible in terms of the infinitesimal generators F_i. Therefore, the "transformed" generators F'_i must be expressible in terms of the F_i, and we write

$$F'_i = u_{i\alpha}(\epsilon) F_\alpha \;, \quad i,\alpha = 1 \ldots 8 \;. \qquad (132)$$

Up to this point the transformation $U(\epsilon)$ has not been considered infinitesimal, i.e., near the identity. In order to explore the meaning of the $u_{ij}(\epsilon)$ however, it is convenient to assume that the ϵ_i are very small also, and write

$$F'_i = U(\epsilon)F_i\,U^{-1}(\epsilon) = (1 + i\epsilon_\alpha F_\alpha + \ldots)\, F_i\, (1 - i\epsilon_\alpha F_\alpha + \ldots)$$

$$= F_i + i\epsilon_\alpha [F_\alpha, F_i] + \ldots$$

$$= F_i + i\epsilon_\alpha\; i f_{\alpha i\beta} F_\beta + \ldots$$

$$= (\delta_{i\beta} + i\epsilon_\alpha\, i\, f_{\alpha i \beta}) F_\beta + \ldots$$

$$\equiv u_{i\beta}(\epsilon) F_\beta \;,$$

where use has been made of the commutation relations (131). Thus, to every infinitesimal SU_3 transformation

$$U(\epsilon) = 1 + i\epsilon_\alpha F_\alpha + \dots , \tag{133}$$

there corresponds an 8×8 matrix

$$u_{ij}(\epsilon) = \delta_{ij} + i\epsilon_\alpha \, if_{\alpha ij} + \dots . \tag{134}$$

In other words, we have established a mapping of the set $U(\epsilon)$ onto the set $u_{ij}(\epsilon)$ *):

$$\begin{aligned} U(0) = 1 &\longrightarrow u_{ij}(0) = \delta_{ij} , \\ U(\epsilon) &\longrightarrow u_{ij}(\epsilon) , \\ U(\epsilon'') = U(\epsilon')U(\epsilon) &\longrightarrow u_{ij}(\epsilon'') = u_{i\alpha}(\epsilon')u_{\alpha j}(\epsilon) , \end{aligned} \tag{135}$$

and hence the set $u_{ij}(\epsilon)$ is an eight-dimensional representation of the group SU_3. In particular, in this representation (135) the structure constants f_{ijk} correspond to the infinitesimal generators [compare Eqs. (133) and (134)]:

$$F_\alpha \longrightarrow if_{\alpha ij} .$$

It can be shown[14] that the if_{ijk} satisfy the correct commutation relations (131), and further, that they are Hermitian and traceless. This representation is the regular or adjoint representation.

The eight basis states $|B_i\rangle$ of the eight representation transform as

$$\begin{aligned} |B_i'\rangle = U|B_i\rangle &= u_{i\alpha}|B_\alpha\rangle , \\ \langle B_i'| = \langle B_i|U^{-1} &= u_{i\alpha}\langle B_\alpha| , \end{aligned} \tag{136}$$

*) The reader may prove the last of Eqs. (135).

since the u_{ij} are real. Further, we consider operators in the space of the states $|B_i>$ that transform as the infinitesimal generators F_i:

$$O_i' = U O_i U^{-1} = u_{i\alpha} O_\alpha \ . \qquad (137)$$

Hence, they should obey the commutation relations

$$[F_i, O_j] = i f_{ij\alpha} O_\alpha \ . \qquad (138)$$

Now, using Eqs. (134) and (136) it is possible to find the effect of the infinitesimal generators operating on the basis states:

$$(1 + i\epsilon_\alpha F_\alpha + \dots)|B_i> = (\delta_{i\beta} + i\epsilon_\alpha\, i f_{\alpha i\beta} + \dots)|B_\beta> \ ,$$

or since the ϵ_i are arbitrary

$$F_\alpha |B_i> = i f_{\alpha i\beta} |B_\beta> \ . \qquad (139)$$

Noting that the λ_1, λ_2 and λ_3 are closely related to the Pauli matrices τ_1, τ_2 and τ_3 and hence to the isotopic spin, and further, noting that λ_8 commutes with $\lambda_1, \lambda_2, \lambda_3$ we make the identification:

$$\text{isospin:} \quad I_1 = F_1 \ , \quad I_2 = F_2 \ , \quad I_3 = F_3 \ , \qquad (140)$$

and tentatively put (the constant α to be determined later):

$$\text{hypercharge:} \quad Y = \alpha F_8 \ .$$

The baryon states are constructed of linear combinations of the eight basis states of the eight representation. The identification is done by looking for the combinations which have the right I^2, I_3 eigenvalues and are eigenvectors of F_8 [making use of Eq. (139)]:

$$
\begin{aligned}
|\Sigma^0\rangle &= |B_3\rangle \, , \\
|\Sigma^\pm\rangle &= \frac{i}{\sqrt{2}} \left[|B_1\rangle \pm i |B_2\rangle \right] \, , \\
|\Lambda\rangle &= |B_8\rangle \, , \\
|p\rangle &= \frac{1}{\sqrt{2}} \left[|B_4\rangle + i |B_5\rangle \right] \, , \\
|\Xi^-\rangle &= \frac{1}{\sqrt{2}} \left[|B_4\rangle - i |B_5\rangle \right] \, , \\
|n\rangle &= \frac{1}{\sqrt{2}} \left[|B_6\rangle + i |B_7\rangle \right] \, , \\
|\Xi^0\rangle &= \frac{1}{\sqrt{2}} \left[|B_6\rangle - i |B_7\rangle \right] \, .
\end{aligned} \tag{141}
$$

We can then identify

$$
Y = \frac{2}{\sqrt{3}} F_8 \, . \tag{142}
$$

To verify that the operator assignments (140) and (142) indeed give the correct results for the above baryon states it is useful to apply Eq. (139). For example,

$$
\begin{aligned}
I_3 |\Sigma^0\rangle &= F_3 |B_3\rangle = i f_{33\alpha} |B\alpha\rangle = 0 \, ; \\
I_3 |\Sigma^\pm\rangle &= F_3 \frac{i}{\sqrt{2}} \left[|B_1\rangle \pm i |B_2\rangle \right] = \frac{i}{\sqrt{2}} \left[i f_{31\alpha} |B\alpha\rangle \pm i i f_{32\alpha} |B\alpha\rangle \right] \\
&= \frac{i}{\sqrt{2}} \left[i |B_2\rangle \pm i i (-|B_1\rangle) \right] = \pm \frac{i}{\sqrt{2}} \left[|B_1\rangle \pm i |B_2\rangle \right] = \pm |\Sigma^\pm\rangle \, ; \\
I_3 |p\rangle &= F_3 \frac{1}{\sqrt{2}} \left[|B_4\rangle + i |B_5\rangle \right] = \tfrac{1}{2} |p\rangle \, ; \\
Y |p\rangle &= \frac{2}{\sqrt{3}} F_8 \frac{1}{\sqrt{2}} \left[|B_4\rangle + i |B_5\rangle \right] \\
&= \sqrt{\frac{2}{3}} \left[i f_{84\alpha} |B\alpha\rangle + i i f_{85\alpha} |B\alpha\rangle \right] \\
&= \sqrt{\frac{2}{3}} \frac{\sqrt{3}}{2} \left[i |B_5\rangle + |B_4\rangle \right] = |p\rangle \, .
\end{aligned}
$$

The reader may continue the verification using the table (130).

At this point, the eight $\frac{1}{2}^+$ baryon states have been identified with linear combinations of the eight basis states of an 8×8 matrix representation of the group SU_3, and the isospin and hypercharge operators have been identified with certain of the infinitesimal generators of the group.

Now, to each of the quantities F_i may be assigned a current*)

$$F_i \longrightarrow j_\lambda^{(i)} , \qquad (143)$$

such that

$$F_i = \int d^3x \; j_0^{(i)}(x) . \qquad (144)$$

In particular, corresponding to the third component of isospin and to the hypercharge are the currents

$$I_3 \longrightarrow j_\lambda^{(3)} , \qquad Y \longrightarrow j_\lambda^{(Y)} = \frac{2}{\sqrt{3}} \, j_\lambda^{(8)} , \qquad (145)$$

where

$$I_3 = \int d^3x \; j_0^{(3)} , \quad Y = \int d^3x \, \frac{2}{\sqrt{3}} \, j_0^{(8)} . \qquad (146)$$

Since the charge Q is the integral of the fourth Lorentz component of the electromagnetic current, and since

$$Q = I_3 + \tfrac{1}{2} Y$$

or

$$Q = F_3 + \frac{1}{\sqrt{3}} F_8 , \qquad (147)$$

we find

$$j_\lambda(\text{e.m.}) = j_\lambda^{(3)} + \frac{1}{\sqrt{3}} \, j_\lambda^{(8)} . \qquad (148)$$

---

*) λ = Lorentz index = 1,2,3,4, and $j_4 = i\, j_0$.

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The above currents satisfy the commutation relations (138) and hence transform according to Eq. (137). The fact that j_λ(e.m.) is conserved implies that all the currents (143) are conserved in the limit of SU_3 symmetry, since the currents can be transformed into one another by SU_3 rotations.

So far, nothing startling has occurred because of the introduction of SU_3 symmetry. The next section bears the first fruit of our labours.

4. The Wigner-Eckart Theorem and the Anomalous Magnetic Moments

A convenient mathematical relation in the theory of angular momentum (i.e. three-dimensional real rotations) is the Wigner-Eckart theorem, sometimes written

$$\langle j\, m | T(j_2 m_2) | j_1\, m_1 \rangle = \langle j\, m\, j_1\, j_2 | j_1 m_1\; j_2 m_2 \rangle \langle j || T(j_2) || j_1 \rangle \,, \qquad (149)$$

where T is a tensor operator, $\langle \| \; \| \rangle$ is a reduced matrix element, and $\langle \; | \rangle$ is a Clebsch-Gordan coefficient. It is seen that the m dependence "factors out" of the matrix element. A similar theorem holds for the eight representation of SU_3 in which the matrix elements factor into "C-G coefficients" and two reduced matrix elements.

The matrix elements of interest are those of the form

$$\langle B_i | O_j | B_k \rangle$$

which transform by Eqs. (136) and (137):

$$\begin{aligned}\langle B_i' | O_j' | B_k' \rangle &= \langle B_i | U^{-1}\, U O_j\, U^{-1} U | B_k \rangle \\ &= u_{i\alpha} u_{j\beta} u_{k\nu} \langle B_\alpha | O_\beta | B_\nu \rangle \,,\end{aligned}$$

and since

$$U^{-1} U = 1$$

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$$<B_i|O_j|B_k> = u_{i\alpha} u_{j\beta} u_{k\nu} <B_\alpha|O_\beta|B_\nu> \quad . \tag{150}$$

Thus, the matrix element is an invariant tensor in the sense that its value does not change under SU_3 transformations. The quantity $Tr(\lambda_i \lambda_j \lambda_k)$ also transforms according to Eq. (150):

$$Tr(\lambda_i \lambda_j \lambda_k) = Tr\left(u^{-1} u\lambda_i u^{-1} u\lambda_j u^{-1} u\lambda_k\right)$$

$$= Tr\left(u\lambda_i u^{-1} u\lambda_j u^{-1} u\lambda_k u^{-1}\right) ,$$

or

$$Tr(\lambda_i \lambda_j \lambda_k) = u_{i\alpha} u_{j\beta} u_{k\nu} Tr(\lambda_\alpha \lambda_\beta \lambda_\nu) \quad . \tag{151}$$

If we take the sum and difference of Eq. (151) with itself (changing indices) and use Eqs. (129) we have

$$\begin{aligned} f_{ijk} &= u_{i\alpha} u_{j\beta} u_{k\nu} f_{\alpha\beta\nu} \quad , \\ d_{ijk} &= u_{i\alpha} u_{j\beta} u_{k\nu} d_{\alpha\beta\nu} \quad , \end{aligned} \tag{152}$$

which are the <u>only</u> three index quantities that can be found which transform as does the matrix element in Eq. (150) *). Thus, the matrix element may be expanded in terms of f_{ijk} and d_{ijk}, and the result is a Wigner-Eckart theorem[16]:

$$<B_i|O_j|B_k> = i f_{ijk} F + d_{ijk} D \quad , \tag{153}$$

---

*) The transformation properties of f_{ijk} and d_{ijk} are analogous to those of δ_{ij} and ϵ_{ijk} in three-dimensional real rotations. There, for example, $a_{i\alpha} a_{j\beta} \delta_{\alpha\beta} = a_{i\alpha} a_{j\alpha} = \delta_{ij}$.

where f and d are the equivalent of C - G coefficients and F and D are the reduced matrix elements. The form of F and D depend upon the nature of the operator 0_j. For the current operators $j_\lambda^{(i)}$ we have

$$<B_i|j_\lambda^{(i)}|B_k> = if_{ijk}F + d_{ijk}D \quad , \tag{154}$$

where

$$F = \bar{u}(k'')\left[F_1^F(k^2)\,\gamma^\lambda + F_2^F(k^2)\,\frac{i}{2M}\,\sigma_{\lambda\alpha}k_\alpha\right]u(k') \quad ,$$

$$D = \bar{u}(k'')\left[F_1^D(k^2)\,\gamma^\lambda + F_2^D(k^2)\,\frac{i}{2M}\,\sigma_{\lambda\alpha}k_\alpha\right]u(k') \quad , \tag{155}$$

and where $k = k'' - k'$, γ^λ are the usual Dirac matrices, $\sigma_{ij} = (\gamma^i\gamma^j - \gamma^j\gamma^i)$, the $F_{1,2}^{F,D}$ are form factors, M is a common baryon mass, and $\bar{u}(k'')$ and $u(k')$ are free particle Dirac spinors.

For the specific case of the electromagnetic current (148), Eq. (154) becomes

$$<B_i|j_\lambda(e.m.)|B_k> = i\left[f_{i3k} + \frac{1}{\sqrt{3}}\,f_{i8k}\right]F + \left[d_{i3k} + \frac{1}{\sqrt{3}}\,d_{i8k}\right]D \quad , \tag{156}$$

which, with the states (141), table (130) and Eq. (139), may be used to derive

$$\begin{aligned} <\Sigma^0|j|\Sigma^0> &= -<\Lambda^0|j|\Lambda^0> = \tfrac{1}{3}\,D \quad , \\ <n|j|n> &= <\Xi^0|j|\Xi^0> = -\tfrac{2}{3}\,D \quad , \\ <\Sigma^\pm|j|\Sigma^\pm> &= F \pm \tfrac{1}{3}\,D \quad , \\ <p|j|p> &= F + \tfrac{1}{3}\,D \quad , \\ <\Xi^-|j|\Xi^-> &= F - \tfrac{1}{3}\,D \quad . \end{aligned} \tag{157}$$

Now, since by definition of F_1^p and F_2^p

$$\langle p | j_\lambda(\text{e.m.}) | p \rangle = \bar{u}(k'') \left[F_1^p(k^2)\, \gamma^\lambda + F_2^p(k^2)\, \frac{i}{2M}\, \sigma_{\lambda\alpha} k_\alpha \right] u(k')$$

(and similarly for the other states), we may use Eq. (157) to write

$$\begin{aligned} F_2^{\Sigma^o}(0) &= -\, F_2^{\Lambda^o}(0) = \tfrac{1}{3}\, F_2^D(0) \ , \\ F_2^n(0) &= F_2^{\Xi^o}(0) = -\tfrac{2}{3}\, F_2^D(0) \ , \\ F_2^p(0) &= F_2^F(0) + \tfrac{1}{3}\, F_2^D(0) \ , \end{aligned} \qquad (158)$$

and similarly for the other states. Thus, for example,

$$F_2^n(0) = 2\, F_2^{\Lambda^o}(0) \ ,$$

or in terms of magnetic moments*)

$$\mu_n = 2\, \mu_{\Lambda^o} \ . \qquad (159)$$

Thus, the introduction of SU_3 symmetry as a means of describing the $\tfrac{1}{2}^+$ baryons leads to definite and experimentally verifiable predictions concerning their electromagnetic properties. SU_3 symmetry also proves useful in interpreting some of the weak interaction properties of the baryons as we shall see in the next section.

X. UNITARY SYMMETRY AND THE WEAK INTERACTIONS

The following discussion of the use of SU_3 symmetry in the description of the weak interactions is divided into three sections:

1) a brief review of the conserved vector current theory and the Goldberger-Treiman relations with emphasis upon the parts of these theories that suggest generalization to an SU_3 model;

2) a statement of the hypotheses on the SU_3 structure of the weak vector and axial vector currents with remarks why these hypotheses are reasonable;

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*) These relations hold exactly only in the limit of SU_3 symmetry. Corrections must be applied when the symmetry-breaking interactions are turned on.

3) a comparison of the consequences of these hypotheses with the experimental data.

Only leptonic weak interactions will be discussed since the SU_3 approach has been most successful in this field.

1. Review of the Conserved Vector Current Theory and the Goldberger-Treiman Relations

As discussed in earlier sections, the Lagrangian density in the V-A theory of the weak interactions has the current-current form:

$$\mathcal{L} = \frac{G}{\sqrt{2}}\left[J_\lambda \ell_\lambda + J_\lambda^\dagger \ell_\lambda^\dagger + \dots \right] ,$$

where the weak current of the leptons (ℓ_λ) is known to be

$$\ell_\lambda = \bar{\psi}_\mu \gamma^\lambda (1+\gamma^5)\psi_\nu + \bar{\psi}_e \gamma^\lambda (1+\gamma^5)\psi_\nu \quad .$$

It is impossible to give a similar explicit description of the weak current of the strongly interacting particles (J_λ) because of the difficulties with a field theory of the strongly interacting particles. To characterize J_λ we use properties which are meaningful because of the exact or approximate symmetries of the strong interactions.

The current J_λ may be divided into vector (V_λ) and axial vector (A_λ) parts:

$$J_\lambda = V_\lambda + A_\lambda \quad , \tag{160}$$

which may be further subdivided according to strangeness changing character:

$$V_\lambda = V_\lambda(\Delta S = 0) + V_\lambda(\Delta S = 1) + \dots \quad , \tag{161}$$

$$A_\lambda = A_\lambda(\Delta S = 0) + A_\lambda(\Delta S = 1) + \dots \quad . \tag{162}$$

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For example, the piece $V_\lambda(\Delta S = 1)$ is responsible for the decay $K^- \to \pi^0 + \mu^- + \tilde{\nu}$, while $A_\lambda(\Delta S = 1)$ acts in the decay $K^- \to \mu^- + \tilde{\nu}$. The four parts of the current shown in Eqs. (161) and (162) are the only parts firmly established experimentally. The conserved vector current theory and the Goldberger-Treiman relations contain information about $V_\lambda(\Delta S = 0)$ and $A_\lambda(\Delta S = 0)$, respectively.

Specifically, the conserved vector current theory states that the non-strangeness changing piece of the vector current (161) is a conserved current and is proportional to the current which carries the I^+ component of isotopic spin. The constant of proportionality is obtained by comparing the vector coupling constant (G_V) in beta decay and the coupling constant in muon decay (G). Thus,

$$V_\lambda(\Delta S = 0) = \frac{G_V}{G}\, j_\lambda^{(+)} = \frac{G_V}{G}\left[j_\lambda^{(1)} + i\, j_\lambda^{(2)} \right], \tag{163}$$

where the indices (1) and (2) refer either to isospin or to SU_3 since they are the same. In other words, $j_\lambda^{(1)}$ and $j_\lambda^{(2)}$ are the currents defined by Eqs. (143) and (144).

The Goldberger-Treiman relations for the axial current (162) are best understood by assuming an intimate relation between the axial current and the source of the pion field (φ_{π^-}), for example, in the form proposed by Gell-Mann and Lévy[10] *):

$$\partial_\lambda A_\lambda(\Delta S = 0) \propto \varphi_{\pi^-} \quad . \tag{164}$$

2. Hypotheses on the SU_3 Structure of the Weak Currents V_λ and A_λ [17]

Since SU_3 symmetry connects strange and non-strange particles, its validity in strong interactions suggests that the properties of $\Delta S \neq 0$ weak currents and $\Delta S = 0$ weak currents should be similar. In particular, Eq. (163) shows that the I spin current $j_\lambda^{(+)}$ is a member of the octet of conserved currents $j_\lambda^{(i)}$ required by SU_3 invariance,

*) Assumption (164) boils down to the statement that the divergence of the current is given by the one-pion pole, viz. Eq. (93).

while it is known that the pion field in Eq. (164) is a member of an octet of pseudoscalar mesons. Thus, we have the first hypothesis.

A) The weak currents V_λ and A_λ are members of SU_3 octets. The vector current V_λ is a member of the octet $j_\lambda^{(i)}$ to which the I spin current and the electromagnetic current belong. The axial current A_λ is a member of an octet of axial currents $g_\lambda^{(i)}$ whose divergence is proportional to the (renormalized) fields of the pseudoscalar octet $\varphi^{(i)}$:

$$\partial_\lambda g_\lambda^{(i)} \propto \varphi^{(i)} \quad . \tag{165}$$

In other words, we expand V_λ and A_λ in members of octets:

$$V_\lambda = a\, j_\lambda(\Delta S = 0) + b\, j_\lambda(\Delta S = \pm 1) \quad , \tag{166}$$

$$A_\lambda = a' g_\lambda(\Delta S = 0) + b' g_\lambda(\Delta S = \pm 1) \, , \tag{167}$$

where

$$j_\lambda(\Delta S = 0) = j_\lambda^{(1)} \pm i\, j_\lambda^{(2)} \quad , \tag{168}$$

$$j_\lambda(\Delta S = \pm 1) = j_\lambda^{(4)} \pm i\, j_\lambda^{(5)} \quad , \tag{169}$$

$$g_\lambda(\Delta S = 0) = g_\lambda^{(1)} \pm i\, g_\lambda^{(2)} \quad , \tag{170}$$

$$g_\lambda(\Delta S = \pm 1) = g_\lambda^{(4)} \pm i\, g_\lambda^{(5)} \quad . \tag{171}$$

The selection rules associated with these currents follow from the commutation relations (131). For example, the isospin raising (or lowering) operator $I^\pm = I_1 \pm i I_2$ associated with the current (168) has the commutation relations

$$[I^\pm, Y] = 0, \; [I^\pm, Q] = \mp I^\pm, \; [I^\pm, I_3] = \mp I^\pm \quad , \tag{172}$$

and thus the current (168) has the properties $\Delta Y = \Delta S = 0$ *), $\Delta Q = \pm 1$, $\Delta I_3 = \pm 1$. The operator $I_{45}^\pm \equiv F_4 \pm i F_5$ has the commutation relations

*) Since the entire octet has baryon number equal to one, changes in hypercharge are equivalent to changes in strangeness.

$$[I_{45}^{\pm},Y] = \mp I_{45}^{\pm},\ [I_{45}^{\pm},Q] = \mp I_{45}^{\pm},\ [I_{45}^{\pm},I_3] = \mp \tfrac{1}{2} I_{45}^{\pm}\ , \tag{173}$$

and hence the current (169) associated with this operator has the properties $\Delta Y = \Delta S = \Delta Q = \pm 1$, $\Delta I_3 = \pm \frac{1}{2}$. A current with the properties $\Delta S = -\Delta Q$ cannot be constructed. Thus, the first consequences of hypothesis A) are: the absence of $\Delta S = -\Delta Q$ currents, the $\Delta S = \Delta Q = 1$ currents have $\Delta I = \frac{1}{2}$, and the $\Delta S = 0$, $\Delta Q = 1$ currents have $\Delta I = 1$. These rules are in accord with the present experimental evidence.

The next hypothesis concerns the relative direction in SU_3 space of the two SU_3 "vectors" V_λ and A_λ.

B) The vector current V_λ and axial current A_λ are "parallel" members of the respective octets $j_\lambda^{(i)}$ and $g_\lambda^{(i)}$. This means that Eqs. (166) and (167) become

$$V_\lambda = a\, j_\lambda(\Delta S = 0) + b\, j_\lambda(\Delta S = 1)\ , \tag{174}$$

$$A_\lambda = a\, g_\lambda(\Delta S = 0) + b\, g_\lambda(\Delta S = 1)\ . \tag{175}$$

That the a' and a of Eqs. (166) and (167) can be set equal is clear because, although the normalization of $j_\lambda^{(i)}$ is fixed by the normalization of the electromagnetic current, no such restriction applies to the octet $g_\lambda^{(i)}$ and therefore we set $a' = a$. With $a' = a$ the hypothesis that $b' = b$ becomes extremely strong and is motivated by the following reasoning.

It is usually hypothesized that the form of the weak current of the strongly interacting particles before the strong interactions are turned on is*)

$$\begin{aligned} & \qquad\qquad (\Delta S = 0) \qquad\qquad\qquad\qquad (\Delta S = 1) \\ J_\lambda &= a\left[\bar{\psi}_p \gamma^\lambda (1+\gamma^5)\psi_n\right] + b\left[\bar{\psi}_p \gamma^\lambda (1+\gamma^5)\psi_\Lambda\right] \\ &= a\bar{\psi}_p \gamma^\lambda \psi_n + b\bar{\psi}_p \gamma^\lambda \psi_\Lambda \longleftarrow V_\lambda \\ &\quad + a\bar{\psi}_p \gamma^\lambda\gamma^5\psi_n + b\bar{\psi}_p \gamma^\lambda \gamma^5 \psi_\Lambda \longleftarrow A_\lambda\ . \end{aligned} \tag{176}$$

*) We use the Sakata model for definiteness: p,n,Λ.

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Hence the "bare" weak interaction has the currents V_λ and A_λ parallel, which suggests letting the renormalized V_λ and A_λ be parallel, since this form of parallelism is preserved by SU_3 conserving strong interactions.

The last hypothesis concerns the "strength" of the current J_λ, i.e., the magnitude of a and b.

C) The SU_3 form of universality is obtained by letting

$$a^2 + b^2 = 1 ,$$

or (177)

$$a = \cos \vartheta , \ b = \sin \vartheta ,$$

where ϑ is to be determined experimentally. This form of the universality of the weak interactions is suggested by the following considerations.

In the limit of exact SU_3 symmetry the definition of strangeness becomes completely arbitrary, since unitary transformations of the form $\exp(i 2\vartheta F_7)$ are physically allowed. With such a transformation the ratio b/a can be made to assume any value; for example, with $b/a = \tan \vartheta$, we find:

$$a\, j_\lambda(\Delta S = 0) + b\, j_\lambda(\Delta S = 1) = (a^2 + b^2)^{1/2}\ e^{i2\vartheta F_7}\ j_\lambda(\Delta S = 0)\, e^{-i2\vartheta F_7} . \tag{178}$$

Thus, in the SU_3 limit, the direction of V_λ (and J_λ) in the $\Delta Q = 1$ subspace of the octet space is devoid of any physical meaning. Equation (177) assures that, whatever the direction of J_λ, its "length" remains constant. The angle ϑ can be considered as the angle between the direction in the $\Delta Q = 1$ plane defined by the weak interactions and the direction chosen by the symmetry-breaking interactions which define states of definite charge, hypercharge, and isospin.

In conclusion, then, hypotheses A), B), and C) state that the weak current of the strongly interacting particles ($J_\lambda = V_\lambda + A_\lambda$) is composed of currents which are members of SU_3 octets in the form

$$V_\lambda = \cos\vartheta \; j_\lambda(\Delta S = 0) + \sin\vartheta \; j_\lambda(\Delta S = 1) \; , \qquad (179)$$

$$A_\lambda = \cos\vartheta \; g_\lambda(\Delta S = 0) + \sin\vartheta \; g_\lambda(\Delta S = 1) \; , \qquad (180)$$

where ϑ is a parameter to be determined experimentally.

3. Consequences of the SU₃ Structure of the Weak Currents V_λ and A_λ and Comparison with Experiment

An immediate consequence of hypothesis (179 - 180) is that the angle ϑ determined from pure vector decays and the angle ϑ determined from pure axial vector decays must be equal. From the axial decays $K^+ \to \mu^+ + \nu$, $\pi^+ \to \mu^+ + \nu$ the angle ϑ may be found, while the vector decays $K^+ \to \pi^0 + e^+ + \nu$, $\pi^+ \to \pi^0 + e^+ + \nu$ furnish an independent determination of ϑ. Consider first the axial decays.

The decays $K^+ \to \mu^+ + \nu$ and $\pi^+ \to \mu^+ + \nu$ are induced only by the axial current (180) and hence, the matrix element of interest is $<0|A_\lambda|K^+$ or $\pi^+>$. The only Lorentz vector available in the K^+ matrix element is

$$k_\lambda(K) = k_\lambda(\mu) + k_\lambda(\nu)$$

and we write

$$<0|A_\lambda|K^+> = F(K\mu\nu) \; k_\lambda(K) \; ,$$

which may be rewritten using Eq. (180) as

$$\sin\vartheta <0|g_\lambda(\Delta S = 1)|K^+> = F(K\mu\nu) \; k_\lambda(K) \; . \qquad (181)$$

The same procedure for the $\pi^+ \to \mu^+ + \nu$ process yields

$$\cos\vartheta <0|g_\lambda(\Delta S = 0)|\pi^+> = F(\pi\mu\nu) \; k_\lambda(\pi) \; . \qquad (182)$$

Now the analogue of theorem (153) for matrix elements with two indices [such as (181) and (182)] is

$$<0|g^{(i)}|\pi_j> = \delta_{ij}\ G \ , \tag{183}$$

where π_j is one of the eight states of the pseudoscalar meson octet, G is the reduced matrix element, and δ_{ij} is the "C - G coefficient". Since $g_\lambda(\Delta S = 1)$ and K^+ transform alike, i.e., are the same members of their respective octets, and since $g_\lambda(\Delta S = 0)$ and π^+ transform alike, we have by theorem (183)

$$<0|g_\lambda(\Delta S = 1)|K^+> = <0|g_\lambda(\Delta S = 0)|\pi^+> \ . \tag{184}$$

In the limit of exact SU_3 symmetry the K^+ and π^+ masses are equal and, thus, the four-momenta $k_\lambda(K)$ and $k_\lambda(\pi)$ are equal. Hence, by Eqs. (181) and (182)

$$\frac{F(K\mu\nu)}{F(\pi\mu\nu)} = \tan\vartheta \ , \tag{185}$$

and the ratio of the decay rates is thus[18])

$$\frac{\Gamma(K^+ \to \mu\nu)}{\Gamma(\pi^+ \to \mu\nu)} = \tan^2\vartheta\ \frac{m_K}{m_\pi}\ \frac{(1 - m_\mu^2/m_K^2)}{(1 - m_\mu^2/m_\pi^2)} \ , \tag{186}$$

which may be used to calculate ϑ from the $K^+_{\mu 2}$ branching ratio and the K^+ and π^+ lifetimes. The result is

$$|\vartheta| = 0.266 \pm 0.005 \text{ (axial)} \ . \tag{187}$$

An independent calculation of ϑ can be made using the decays $K^+ \to \pi^0 + e^+ + \nu$ and $\pi^+ \to \pi^0 + e^+ + \nu$ which are $0^- \to 0^-$ and, hence, involve only V_λ. We give only the result[18])

$$|\vartheta| = 0.241 \pm 0.008 \text{ (vector)} \ . \tag{188}$$

The agreement between the two values of ϑ is remarkable. Some difference is to be expected due to SU_3 breaking interactions which can affect in different ways the vector and axial vector parts of J_λ.

Another consequence of the hypothesis (179 - 180) is that the vector coupling constant in beta decay is given by Eqs. (163) and (179) as

$$G_V = G \cos \vartheta \, , \tag{189}$$

so that using the value of ϑ in Eq. (188)

$$\frac{G^2 - G_V^2}{G^2} = \sin^2 \vartheta = 0.056 \, . \tag{190}$$

The same quantity can be measured experimentally by comparing G, as determined by the muon lifetime, with G_V as determined from pure Fermi beta decays like $O^{14} \to N^{14}$, after applying the necessary radiative corrections to both cases. The experimental number is[19)]

$$\frac{G^2 - G_V^2}{G^2} = 0.044 \pm 0.003 \tag{191}$$

in satisfactory agreement with the theoretical value (190). (The error in Eq. (191) does not include possible "theoretical" errors in the evaluation of the radiative corrections.)

4. Leptonic Decays of the Baryons[17)]

The SU_3 structure of the weak currents V_λ and A_λ makes specific predictions for the branching ratios for the baryon leptonic decays

$$B_k \to B_i + e + \nu \, .$$

By defining the strangeness non-changing and strangeness-changing weak baryonic currents as [see Eqs. (166) - (171)]:

$$J_\lambda(\Delta S = 0) \equiv j_\lambda(\Delta S = 0) + g_\lambda(\Delta S = 0)$$
$$= \left(j_\lambda^{(1)} + g_\lambda^{(1)}\right) + i\left(j_\lambda^{(2)} + g_\lambda^{(2)}\right) \qquad (192)$$

$$J_\lambda(\Delta S = 1) \equiv j_\lambda(\Delta S = 1) + g_\lambda(\Delta S = 1)$$
$$= \left(j_\lambda^{(4)} + g_\lambda^{(4)}\right) + i\left(j_\lambda^{(5)} + g_\lambda^{(5)}\right), \qquad (193)$$

we may state a Wigner-Echart theorem in analogy to theorem (156) for the e.m. current:

$$< B_i | J_\lambda(\Delta S = 0) | B_k > = i\left(f_{i1k} + if_{i2k}\right) F_\lambda + \left(d_{i1k} + id_{i2k}\right) D_\lambda \, , \qquad (194)$$

$$< B_i | J_\lambda(\Delta S = 1) | B_k > = i\left(f_{i4k} + if_{i5k}\right) F_\lambda + \left(d_{i4k} + id_{i5k}\right) D_\lambda \, . \qquad (195)$$

For the decay processes in question where the momentum transfer is quite small, with respect to the baryon masses, we may neglect everything in the current except the γ^λ and $\gamma^\lambda\gamma^5$ terms. Thus, for the decay of the baryons with four momentum k' into a baryon with four momentum k'' and leptons, we have ($k = k' - k''$ = mom. transfer):

$$F_\lambda = \bar{u}(k'') \left[F_V^F(k^2)\, \gamma^\lambda + F_A^F(k^2)\, \gamma^\lambda\gamma^5 \right] u(k') \, , \qquad (196)$$

$$D_\lambda = \bar{u}(k'') \left[F_V^D(k^2)\, \gamma^\lambda + F_A^D(k^2)\, \gamma^\lambda\gamma^5 \right] u(k') \, . \qquad (197)$$

By the conserved vector-current hypothesis, which connects the isovector part of j_λ(e.m.) and the baryonic vector current ($\Delta S = 0$), we get

$$F_V^F(0) = 1 \qquad (198)$$

$$F_V^D(0) = 0 \, . \qquad (199)$$

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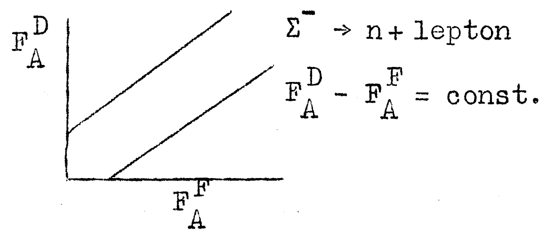
We are left with the unknown parameters F_A^F and F_A^D. In terms of these parameters the currents are tabulated below:

	vector	axial vector
$n \rightarrow p$ + leptons	$\cos\Theta$	$\cos\Theta \left(F_A^F + F_A^D\right)$
$\Sigma^- \rightarrow \Lambda$ + leptons	0	$\frac{2}{\sqrt{6}} \cos\Theta \left(F_A^D\right)$
$\Lambda \rightarrow p$ + leptons	$-\frac{3}{\sqrt{6}} \sin\Theta$	$\frac{1}{\sqrt{6}} \sin\Theta \left(-3F_A^F - F_A^D\right)$
$\Sigma^- \rightarrow n$ + leptons	$-\sin\Theta$	$\sin\Theta \left(-F_A^F + F_A^D\right)$
$\Xi^- \rightarrow \Lambda$ + leptons	$\frac{3}{\sqrt{6}} \sin\Theta$	$\frac{1}{\sqrt{6}} \sin\Theta \left(3F_A^F - F_A^D\right)$
$\Xi^- \rightarrow \Sigma^0$ + leptons	$\frac{1}{\sqrt{2}} \sin\Theta$	$\frac{1}{\sqrt{2}} \sin\Theta \left(F_A^F + F_A^D\right)$
$\Xi^0 \rightarrow \Sigma^+$ + leptons	$\sin\Theta$	$\sin\Theta \left(F_A^F + F_A^D\right)$

In Appendix C the total decay rate for an interaction of the form $G_\nu \gamma^\mu + G_A \gamma^\mu \gamma^5$ is calculated.

It is seen that the total decay rate depends on G_V^2 and G_A^2 and not on $G_V G_A$. As we know F_V^F and therefore G_V, for any process we may determine G_A^2 from the total decay rate. For any process this gives a a linear relation between F_A^F and F_A^D which may be represented as a straight line in an F-D plot. The above theory requires all lines to go through one point, and the F_A^F and F_A^D values belonging to that point are the values which were to be determined. Each process gives, in fact, two lines ($G_A = \pm\sqrt{G_A^2}$), but with perfect experimental data only one point in the graph should exist through which a line for each process goes. Full details of this work may be found in the publication of H. Willis et al., Phys.Rev. Letters 13, 293 (1964). See also N. Brene et al., Physics Letters 11, 344 (1964).



APPENDIX A

CONVENTIONS FOR THE DIRAC EQUATION

The Dirac equation is:

$$\left(\gamma^\mu \partial_\mu + m\right)\psi(x) = 0 \ , \ \bar{\psi}(x)\left(-\gamma^\mu \overleftarrow{\partial}_\mu + m\right) = 0$$

where $\bar{\psi}(x) = \psi^\dagger(x)\gamma^4$. For $\psi(x) = u(\vec{p}) \exp\left[i p_\mu x_\mu\right]$

$$\left(i\gamma^\mu p_\mu + m\right)u(\vec{p}) = 0 \ , \ \bar{u}(\vec{p})\left(i\gamma^\mu p_\mu + m\right) = 0 \ ,$$

where the Dirac representation of the γ matrices is:

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also $\gamma^\dagger = \gamma$ and

$$\begin{aligned} \tilde{\gamma}^1 &= -\gamma^1 , & \tilde{\gamma}^4 &= +\gamma^4 , & \tilde{\sigma}_1 &= +\sigma_1 , \\ \tilde{\gamma}^2 &= +\gamma^2 , & \tilde{\gamma}^5 &= +\gamma^5 , & \tilde{\sigma}_2 &= -\sigma_2 , \\ \tilde{\gamma}^3 &= -\gamma^3 , & & & \tilde{\sigma}_3 &= +\sigma_3 . \end{aligned}$$

The Dirac spinors u^i, solution of the above equations, are:

u^1	u^2	u^3	u^4
1	0	$+k_3/(m+k_c)$	$-(k_1 - ik_2)/(m+k_0)$
0	1	$+(k_1 + ik_2)/(m+k_0)$	$k_3/(m+k_0)$
$k_3/(m+k_c)$	$(k_1 - ik_2)/(m+k_0)$	-1	0
$(k_1 + ik_2)/(m+k_0)$	$-k_3/(m+k_0)$	0	1

with an overall factor $\sqrt{(m+k_0)/2k_0}$.

The transformation matrices are:

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

where also

$$C = \gamma^2 \gamma^4 , \quad C^{-1} = -C , \quad \tilde{C} = -C ,$$

$$D = -\gamma^5 C .$$

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APPENDIX B

ELEMENTARY REVIEW OF THE GROUPS SU_2 and SU_3

The group SU_2 is the set (u) of unitary, unimodular transformations in a complex space of two dimensions. Such a transformation may be written

$$\psi' = u\psi \quad ,$$

or

$$\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} ,$$

where a,b,c and d are complex numbers. The requirement that u be unitary

$$u^{\dagger} = u^{-1} \quad ,$$

and unimodular

$$\det u = 1 \quad ,$$

immediately restricts the allowed transformations to those with the form

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} ,$$

$$|a|^2 + |b|^2 = 1 \quad ,$$

where (*) means complex conjugation. Since a and b are complex numbers subject to one condition ($|a|^2 + |b|^2 = 1$), the transformations u are characterized by three real parameters a_1, a_2, a_3: $u(a_1, a_2, a_3)$. These parameters can vary continuously and, hence, the group SU_2 is called a continuous or Lie group. The identity element is defined so that $a_i = 0$:

$$u(0,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \quad ,$$

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while the inverse of the group element $u(a_1,a_2,a_3)$ is*)

$$u^{-1}(a) = u^{\dagger}(a) = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} .$$

The composition law for the group is successive application of transformations:

$$u(a_1'',a_2'',a_3'') = u(a_1',a_2',a_3')\, u(a_1,a_2,a_3) \quad ,$$

or

$$\begin{pmatrix} a'' & b'' \\ -b''^* & a''^* \end{pmatrix} = \begin{pmatrix} a' & b' \\ -b'^* & a'^* \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} .$$

The reader may verify that the transformations u form a group, i.e., that they satisfy the definition of a group:

1) the composition of two group elements yields another group element:

$$u(a'') = u(a')\,u(a) \quad ;$$

2) the composition obeys the associative rule:

$$u(a'')[u(a')u(a)] = [u(a'')u(a')]\,u(a) \quad ;$$

3) there exists a unit element such that

$$u(a)\,u(0) = u(0)\,u(a) = u(a) \quad ;$$

4) for every element of the group there exists an inverse element such that

$$u(a)\,u^{-1}(a) = u^{-1}(a)\,u(a) = u(0) \quad .$$

---

*) The notation $u(a)$ means $u(a_1,a_2,a_3)$.

If there exists another group w which is related to the group u by a mapping

$$u(0) \longrightarrow w(0) \ ,$$
$$u(a) \longrightarrow w(a) \ ,$$

such that

$$u(a')\,u(a) \longrightarrow w(a')\,w(a) \ ,$$

then the group w is said to be homomorphic to the group u. If the group w is a group of linear transformations in a linear space of n dimensions then w is called an n-dimensional representation of the group u. Frequently it is convenient to study a group by investigating the properties of one of its representations.

The above discussion of the group SU_2 holds in all its details for the group SU_3 with the exception that the complex space now has three dimensions:

$$\psi' = u\psi \ ,$$

or

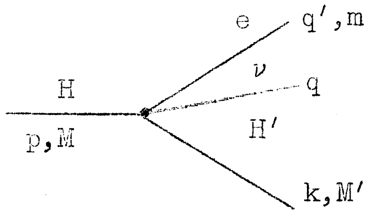
$$\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} .$$

The conditions of unitarity and unimodularity require that the group elements u be functions of eight real parameters $a_1 \ldots a_8$: $u(a_1 \ldots a_8)$.

APPENDIX C

CALCULATION OF HYPERON LEPTONIC DECAY

Four momenta and masses of the particles are denoted in the figure.



Decay $H \to H' + e + \bar{\nu}$

$Q = q + q' = p - k$

The total decay rate (= inverse lifetime) in $\sec^{-1}$ is given by ($h = 6.582 \cdot 10^{-22}$ MeV sec):

$$\frac{1}{\tau} = \Gamma = \frac{1}{h(2\pi)^{13}} \int d_3q \; d_3q' \; d_3k \; \delta_4(p-k-q-q') \sum_{\text{spins}} \frac{(M)^2}{2} \; ,$$

where averaging over the initial spin states leads to the factor $1/2$. All quantities have to be expressed in MeV. We have:

$$M = (2\pi)^4 G \left\{ \bar{u}(k) \left(G_V \gamma^\mu + G_A \gamma^\mu \gamma^5 \right) u(p) \right\} \left\{ \bar{u}(q') \gamma^\mu (1+\gamma^5) u(-q) \right\}$$

with

$$G = \frac{1.02}{\sqrt{2}} \cdot \frac{10^{-5}}{m_p^2} \; ,$$

m_p = mass of the proton. Thus, from this

$$\sum \frac{|M|^2}{2} = \frac{(2\pi)^8 G^2}{2^5 p_0 k_0 q_0 q_0'} \mathrm{Tr} \left\{ (-i\gamma k + M') \left(G_V \gamma^\mu + G_A \gamma^\mu \gamma^5 \right) \times \right.$$

$$\left. (-i\gamma p + M) \times \left(-G_V \gamma^\nu - G_A \gamma^\nu \gamma^5 \right) \right\} \times$$

$$\mathrm{Tr} \left\{ (-i\gamma q' + m) \gamma^\mu (1+\gamma^5)(-i\gamma q)(-\gamma^\nu)(1+\gamma^5) \right\}$$

$$\equiv \frac{(2\pi)^8 G^2}{2^5 p_0 k_0 q_0' q_0} F(Q^2, (pq)) \; ,$$

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where we exploited the fact that all invariants in this problem can be expressed in Q^2 and (pq), apart from the masses of course. For example,

$$Q^2 = (p-k)^2 = p^2 + k^2 - 2pk = -M^2 - M'^2 - 2\,(pk)$$

thus,

$$(pk) = -\frac{Q^2 + M^2 + M'^2}{2} .$$

The integration over $\vec{q}\,'$ may be done at once, giving $\vec{q}\,' = \vec{k} + \vec{q}$ and we get:

$$\Gamma = \frac{G^2}{2^5 (2\pi)^5 p_0 h} \int \frac{d_3 k}{k_0} \int \frac{d_3 q}{q_0 q_0'} \delta(p_0 - k_0 - q_0 - q_0') \; F(Q^2, (pq)).$$

Note that $q_0' = \sqrt{\vec{q}\,'^2 + m^2} = \sqrt{(\vec{k} + \vec{q})^2 + m^2}$, $Q^2 = -M^2 - M'^2 + 2Mk_0$ and $(pq) = -Mq_0$ if we go to the rest system of the initial hyperon, i.e. $\vec{p} = 0$. Thus, the function F does not depend on any angle. We may now perform the $\vec{q}$ or the $\vec{k}$ integration. Doing the $\vec{q}$ integration in the system where $\vec{k}$ is along the z axis, we have

$$J \equiv \int \frac{d_3 q}{q_0 q_0'} \delta(p_0 - k_0 - q_0 - q_0') \; F(Q^2, pq)$$

$$= 2\pi \int \frac{|\vec{q}|^2 d|\vec{q}|}{q_0 q_0'} \int_{-1}^{1} dx \; \delta\left(p_0 - k_0 - q_0 - \sqrt{|\vec{q}|^2 + |\vec{k}|^2 + 2|\vec{q}|\,|\vec{k}|\,x + m^2}\right) F(Q^2, pq) .$$

For the moment we will assume a non-zero neutrino rest mass m_ν. The integration over x may be performed, giving non-zero if the argument of the delta function is zero for some x in between -1 and +1. We find:

$$x_0 = \frac{(M - k_0 - q_0)^2 - m^2 - |\vec{k}|^2 - |\vec{p}|^2}{2|\vec{p}|\;|\vec{k}|}$$

$$J = 2\pi \int \frac{|\vec{q}|^2 d|\vec{q}|}{q_0 q_0'} \cdot \frac{q_0'}{|\vec{q}|\,|\vec{k}|} \; \Theta(1 - x_0^2) ,$$

where we used

$$\int dx\ \delta(f(x)) = \frac{1}{|f'(x_0)|} ,$$

with $f(x_0) = 0$.

The condition $x_0^2 \leq 1$ may be solved to give:

$$a_+ \leq q_0 \leq a_-$$

$$a_{\pm} = \frac{(M^2 + M'^2 + m_\nu^2 - m^2 - 2Mk_0)\ (M - k_0)\ \pm\ |\vec{k}| \cdot \sqrt{(M^2 + M'^2 - m_\nu^2 - m^2 - 2Mk_0)^2 - 4m_\nu^2 m^2}}{2(M^2 + M'^2 - 2Mk_0)} .$$

The constraint that the argument of the root is positive gives:

$$a_0 \equiv \frac{1}{2M}\left(M^2 + M'^2 - (m + m_\nu)^2\right) \geq k_0 .$$

From $q_0 = \sqrt{\vec{q}^{\,2} + m_\nu^2}$ we find $|\vec{q}|d|\vec{q}| = q_0\, dq_0$ and, consequently,

$$J = 2\pi \int_{a_-}^{a_+} \frac{dq_0}{|\vec{k}|}\ \Theta\left(\frac{1}{2M}\left(M^2 + M'^2 - (m + m_\nu)^2\right) - k_0\right) F(Q^2, pq) .$$

The integration over directions of $\vec{k}$ is trivial, we find:

$$\Gamma = \frac{G^2}{2^7 \pi^3 M h} \int_{M'}^{a_0} dk_0 \int_{a_-}^{a_+} dq_0\ F(Q^2, pq) , \qquad (1)$$

with a_0, a_- and a_+ as defined above.

If we had chosen instead to perform the $\vec{k}$ integration first, we would have obtained:

$$\Gamma = \frac{G^2}{2^7 \pi^3 M h} \int_{m_\nu}^{a_0'} dq_0 \int_{a_-'}^{a_+'} dk_0\ F(Q^2, pq) , \qquad (2)$$

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where, with $m_\nu = 0$,

$$a'_{\pm} = \frac{(M^2 + M'^2 - m^2 - 2Mq_0)\,(M - q_0) \pm |\vec{q}|\,\sqrt{(M^2 - M'^2 - m^2 - 2Mq_0)^2 - 4M'^2 m^2}}{2(M^2 - 2Mq_0)}$$

$$a'_0 = \frac{1}{2M}\left(M^2 - (m + M')^2\right).$$

The function $F(Q^2, (pq))$ may be determined. One finds (setting $m_\nu = 0$):

$$\begin{aligned} F(Q^2,(pq)) = {} & (G_A - G_V)^2 \,\{16\,(Q^2 + m^2)(M^2 - M'^2) - 16Q^2(Q^2 + m^2) + 64Q^2(pq)\} \\ & - 32\,(G_A{}^2 - G_V{}^2)(Q^2 + m^2)MM' \\ & - 16\,(G_A{}^2 + G_V{}^2)\{8(pq)^2 + 4(pq)(M^2 - M'^2 - m^2)\}\,. \end{aligned}$$

If we use formula (1) we see that

$$\int_{a_-}^{a_+} dq_0\, \{q_0\} = \int_{a_-}^{a_+} dq_0 \left\{\frac{a_+ + a_-}{2}\right\}\left(= \frac{a_+^2 - a_-^2}{2}\right),$$

where $a_\pm$ may be expressed in terms of $Q^2 = -M^2 - M'^2 + 2Mk_0$:

$$a_\pm = \frac{(Q^2 + m^2)}{2Q^2}\left\{\frac{M^2 - M'^2 - Q^2}{2M} \pm |\vec{k}|\right\}.$$

Thus, we may replace q_0 by $\frac{1}{2}(a_+ + a_-)$ or

$$\int dq_0\, q_0 = \int dq_0\, \frac{(Q^2 + m^2)(M^2 - M'^2 - Q^2)}{4MQ^2}$$

and, therefore,

$$-64Q^2(pq) = 64Q^2 Mq_0 = 16(Q^2 + m^2)(M^2 - M'^2) - 16Q^4 - 16m^2 Q^2$$

which makes the coefficient of $(G_V - G_A)^2$ exactly zero, and no interference term $G_V G_A$ is left. There is another way to see this: the matrix element squared is the product of a lepton factor and a hyperon factor:

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$$|M|^2 \propto \ell_{\mu\nu} H_{\mu\nu} ,$$

where $\ell_{\mu\nu} = \mathrm{Tr}\{(-i\gamma q' + m)\gamma^{\mu} \ldots\}$, $H_{\mu\nu} = \mathrm{Tr}\{(-i\gamma k + M')\ldots\}$. $\ell_{\mu\nu}$ may depend on q' ($= Q - q$) and q, $H_{\mu\nu}$ on k (or Q) and p. Interference terms between vector and axial vector terms have the parity minus, and must, therefore, contain an ϵ symbol, together with the two available vectors:

$$\epsilon_{\mu\nu\alpha\beta} p_\alpha Q_\beta .$$

Now, $\ell_{\mu\nu}$ may contain terms of the form

$$q_\mu q_\nu, \; q_\mu' q_\nu', \; q_\mu' q_\nu, \; q_\mu q_\nu', \; \delta_{\mu\nu}, \; \epsilon_{\mu\nu\lambda\kappa} q_\lambda q_\kappa' = \epsilon_{\mu\nu\lambda\kappa} q_\lambda Q_\kappa .$$

In the product $\ell_{\mu\nu} H_{\mu\nu}$ the only non-vanishing contribution of the interference term comes from the products of the epsilons:

$$\epsilon_{\mu\nu\alpha\beta} p_\alpha Q_\beta \epsilon_{\mu\nu\lambda\kappa} q_\lambda Q_\kappa = 2\{(pq)Q^2 - (pQ)(qQ)\}$$

[which may be seen to be equal to the coefficient of $(G_V - G_A)^2$ above]. But the Lorentz invariant expression

$$\int \frac{d_3 q}{q_0} \int \frac{d_3 q'}{q_0'} \delta_4(Q - q - q') \, \epsilon_{\mu\nu\lambda\kappa} q_\lambda q_\kappa'$$

depends on Q only thus, it cannot give a result containing an ϵ with two free indices, and is thus zero. In other words, integrating over all lepton configurations leads to vanishing of the interference terms.

Thus, we may omit the $(G_V - G_A)^2$ term. We are left with:

$$F(Q^2,(pq)) = -32 \; (G_A^2 - G_V^2)(Q^2 + m^2)MM'$$
$$-16 \; (G_A^2 + G_V^2)\{8(pq)^2 + 4(pq)(M^2 - M'^2 - m^2)\} .$$

We now use formula (2). Going over onto Q^2 as the integration variable, we have:

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$$\int_{a'_-}^{a'_+} dk_0 = \frac{1}{2M} \int_{b_-}^{b_+} dQ^2$$

where, using $Q^2 = -M^2 - M'^2 + 2Mk_0$ we find $b_\pm = -M^2 - M'^2 + 2Ma'_\pm$:

$$b_\pm = \frac{2M^2 q_0^2 + q_0 M (M'^2 - M^2 + m^2) - m^2 M^2 \pm M q_0 \sqrt{B}}{(M^2 - 2M q_0)}$$

$$B = (M^2 - M'^2 - m^2 - 2M q_0)^2 - 4M'^2 m^2 .$$

Replacing Q^2 in the $G_A^2 - G_V^2$ term by $\frac{1}{2}(b_+ + b_-)$ and, further, pq by $-Mq_0$, gives:

$$F(Q^2,(pq)) = -8(G_A^2 - G_V^2)\{8M^2 q_0^2 - 4q_0 M (M^2 - M'^2 - m^2) - 4m^2 M^2\} \frac{MM'}{M^2 - 2Mq_0}$$

$$-16(G_A^2 + G_V^2)\{8M^2 q_0^2 - 4q_0 M (M^2 - M'^2 - m^2)\}$$

$$-32(G_A^2 - G_V^2)\, m^2\, MM'$$

or

$$F(Q^2,(pq)) = 8(3\, G_A^2 + G_V^2)\{-8M^2 q_0^2 + 4q_0 M (M^2 - M'^2 - m^2)\}$$

$$+ 8(G_A^2 - G_V^2)\{-8M^2 q_0^2 + 4q_0 M (M^2 - M'^2 - m^2)\} \frac{MM' - M^2 + 2Mq_0}{M^2 - 2Mq_0}$$

$$+ 32(G_A^2 - G_V^2)\, m^2 MM' \frac{2Mq_0}{M^2 - 2Mq_0} .$$

For all interesting processes $\delta = M - M'$ is small with respect to M. Furthermore,

q_0 is of order δ
Q^2 is of order δ^2
m^2 is of order δ^2 ,

or even $m^2 \ll \delta^2$ for Λ, Σ and Ξ electronic decay. To lowest order in δ only the term with $3G_A^2 + G_V^2$ survives. This is good enough for neutron decay where $\delta/M \sim 10^{-3}$ but, on the other hand, the annoying presence of the term $-4M'^2 m^2$ in the root in $b_\pm$ makes the final integration quite cumbersome.

We will now consider Λ, Σ and Ξ electronic decays, and set $m^2 = 0$, but we must be more accurate in δ because $\delta/M \sim 0.2$. This means, in fact, that we should also consider magnetic and pseudoscalar terms, but we will not enter into these complications here.

To zero[th] and first order in δ we have, with $\Delta = (M+M')/2M$,

$$F(Q^2,(pq)) = 8(3G_A^2 + G_V^2)\{-8M^2 q_o^2 + 8 q_o M^2 \Delta\delta\} + 8(G_A^2 - G_V^2)\{-8M^2 q_o^2 + 8 q_o M^2 \delta\} \frac{2q_o - \delta}{M} .$$

The integral over Q^2 may now be performed:

$$\Gamma = \frac{G^2}{2^7 \pi^3 M h} \int_0^{\Delta\delta} dq_o \; F(Q^2,(pq)) \frac{-2M q_o^2 + q_o(M^2 - M'^2)}{M^2 - 2M q_o} .$$

We write:

$$\frac{-2M q_o^2 + q_o(M^2 - M'^2)}{M^2 - 2M q_o} = \frac{-2M q_o^2 + q_o(M^2 - M'^2)}{M^2}\left(1 + \frac{2q_o}{M}\right)$$

$$= \frac{(-2M^2 q_o^2 + 2 q_o M^2 \Delta\delta)(1 + 2q_o/M)}{M^3} .$$

Thus

$$\Gamma \simeq \frac{G^2}{2^7 \pi^3 M^4 h} \int_0^{\Delta\delta} dq_o \Big[128(3G_A^2 + G_V^2)(-M^2 q_o^2 + q_o M^2 \Delta\delta)^2 + 128(3G_A^2 + G_V^2)(-M^2 q_o^2 + q_o M^2 \delta)^2 \frac{2q_o}{M} + 64(G_A^2 - G_V^2)(-M^2 q_o^2 + q_o M^2 \delta)^2 \frac{2q_o - \delta}{M} \Big] .$$

The $G_A^2 - G_V^2$ term turns out to be zero, and one finds:

$$\Gamma = \frac{G^2}{30\pi^3 h}(3G_A^2 + G_V^2)\,\delta^5 \left[1 - \frac{3\delta}{2M}\right] ,$$

where we developed $\Delta = (M + M')/2M = 1 - \delta/2M$. We remark that in the literature usually $G' = 10^{-5} \cdot 1.02/m_p^2$ is used instead of $G = 10^{-5} \cdot 1.02/m_p^2\sqrt{2}$ (as intended here) so that $G^2 = \frac{1}{2}\, G'^2$. For the neutron $G_V = \cos\Theta$, etc.

We repeat that the last formula is not accurate for neutron decay, where one cannot neglect m^2. For other decays it is valid up to terms of the order $[(M - M')/M]^2$, neglecting magnetic etc. contributions.

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