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WEAK AND E.M. RADIATIVE CORRECTIONS TO LOW-ENERGY PROCESSES

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Radiative corrections to μ-decay, the electric charge and neutrino–electron scattering are computed. The standard model with 6 quarks, and three leptons with their neutrinos is employed. Using μ-decay, the electric charge and the ratio of $\bar{\nu}_\mu$e to ν_μe scattering cross sections as input, a correction of +1 percent to the ν_μe total cross section is computed.

1. Introduction

Recent years have shown increasing experimental support for the renormalizable [1] standard* model [2] of weak and e.m. interactions. Even the relation $M_0 = M/c_\theta$ between the masses of the neutral and charged vector bosons holds remarkably well [3]. Here c_θ is the cosine of the weak mixing angle. This relation is a consequence of the use of the simplest Higgs system, containing just one isodoublet, and we may speak of the Higgs $\Delta I = \frac{1}{2}$ rule [4].

As has been noted before [5], this relation provides us with a window on the mass spectrum above 100 GeV. Thus, careful measurements of neutral current reactions provides information concerning new "flagpoles", that is new sets of quarks and leptons [5]. Radiative corrections involving such new flagpoles may give rise to deviations from the Higgs $\Delta I = \frac{1}{2}$ rule.

The increasing experimental accuracy [3] in the measurement of the weak mixing angle has now reached the point where the standard radiative corrections become important. In this paper we compute these corrections, with the result that they are of the order of 1 percent, in the same direction as that which would arise from mass differences within new flagpoles. Thus, this correction decreases the margin for such possibilities.

*By standard model we mean Weinberg's model of leptons enlarged according to the GIM mechanism, and made renormalizable (i.e., anomaly free) through the threefold colour degeneracy of the quarks. Furthermore, the designation standard model implies the inclusion of all known quarks and leptons, and in addition the hypothesised top quark.

In this article we compute the weak and e.m. radiative corrections to the following processes and quantities:

muon decay;

e-μ scattering for zero energy and momentum transfer;

the ratio of $\bar{\nu}_\mu$e to ν_μe scattering;

the total cross section for ν_μe scattering.

The first three calculations, together with the experimental knowledge of the Fermi coupling constant, the electric charge and the neutrino data provide us with the input information needed to fix the parameters of the standard model. After this we are able to compute the radiative corrections to other processes, notably ν_μe scattering. Since the magnitude of the ν_μe total cross section is fixed by the Higgs $\Delta I = \frac{1}{2}$ rule we find in this way the corrections to this rule at low energies. We wish to point out that these corrections are not the corrections to the actual vector boson masses; they are simply corrections to the low-energy processes used to determine experimentally the parameters of the theory. Actually, it is also quite simple to compute the radiative corrections to the masses of the charged and neutral vector bosons, and one obtains the rather large mass shifts of 3080 and 3310 MeV respectively, within the standard model. This will not be discussed further in this article.

Here we must add a few words concerning earlier literature. The important paper in this context is by Salomonson and Ueda [12], where many details on radiative corrections to νe scattering are given. Our approach is technically somewhat different, and also we are slightly more ambitious, including hard bremsstrahlung as well.

The paper is organized as follows. In sects. 2 and 3 we discuss the weak and e.m. radiative corrections to μ-decay and the electric charge. In sect. 4 the purely e.m. corrections, including bremsstrahlung, to ν_μe and $\bar{\nu}_\mu$e scattering, are treated. In sect. 5 further radiative corrections to these latter processes are evaluated. Sect. 6 contains details of the renormalization procedure, and the evaluation of the radiative corrections to ν_μe scattering. In doing the work reported here we relied heavily on methods and equations introduced before, see refs. [6–9].

2. Muon decay

The parameters in the lagrangian, to be fixed by comparison with experiment, are g, M and s_θ^2, denoting the SU(2) gauge coupling constant, the charged vector boson mass and the sine squared of the weak mixing angle. The μ-lifetime is one of the data points, and we must find the equation giving the μ-lifetime as a function of g, M and $s_\theta^2 = \sin^2 \theta_w$ up to order g^6, thus including one-loop corrections. In deriving our results we will make the approximation of large vector boson mass, neglecting terms such as m_e^2/M^2, where m_e is the electron mass, provided they do not contain infinities.

The expression for the μ-lifetime can be written in the form (for numerical values see sect. 6):

$$\frac{1}{\tau_\mu} = \frac{m_\mu^5}{192\pi^3} \frac{g^4}{32M^4}(1+\delta_\mu), \tag{2.1}$$

where δ_μ is of order g^2. Terms proportional to the electron mass have been neglected. A very convenient fact is that the purely e.m. radiative corrections are finite by themselves, and this fact allows us to use the results of previous calculations [10] for these corrections. However, some care is required here, and we will discuss this point first.

The first question to be settled is this: are the results for the pure e.m. corrections also finite if the dimensional regularization method is used? And do the results, using the latter method, then agree with the usual results? The relevant diagrams are given in fig. 1. The $\mu\nu_\mu$-$e\nu_e$ vertex is taken to be pointlike. Furthermore, after a Fiertz transformation, the $\bar{\nu}\nu$ part may be separated off, and one essentially deals with a situation closely resembling radiative corrections to the eeγ vertex, the differences being that one of the lines is now a μ-line, and the vertex is $\gamma^\mu(1+\gamma^5)$ instead of γ^μ.

To compare the results of the previous calculations with the results obtained using dimensional regularization we introduce another cut-off method. From the above diagrams we subtract an identical set of diagrams with the difference that the photon is given a very large mass Λ. The combination of both sets is finite, and for $\Lambda \to \infty$ the conventional result obtains. Note that this cut-off method respects the e.m. Ward identities.

The dimensional regularization method will give the same results if in that scheme the contribution of the heavy photon diagrams tends to zero for large Λ. The point is to verify that there is no residual finite contribution from these heavy photon diagrams.

In terms of the B and C functions defined in refs. [6, 7] we find that the pure e.m. corrections to an electron vertex of the form $\gamma^\mu(a+b\gamma^5)$ are obtained by the replacement

$$\gamma^\mu(a_0+b_0\gamma^5) \to \gamma^\mu(a_1+b_1\gamma^5) + imP_\mu(c_1\gamma^5+d_1) + imp_\mu(e_1\gamma^5+f_1),$$

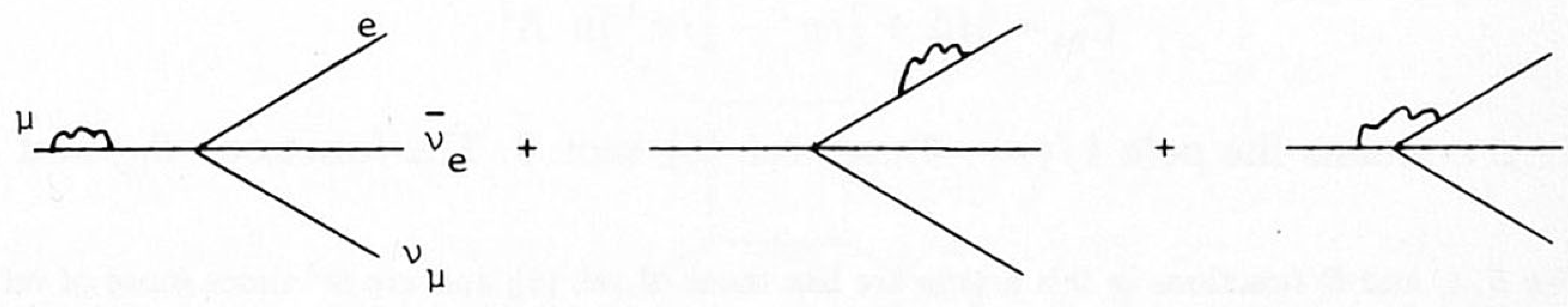

Fig. 1. Purely e.m. corrections to μ-decay, excluding bremsstrahlung.

where P and p are the momenta of the electrons (taken to be ingoing). The terms contributing to c_1, d_1, e_1 and f_1 all go to zero by themselves for large Λ, and we do not consider them any further. The quantities a_1 and b_1 are given by

$$a_1 = a_0\{1 + 2W_V^e + \delta_a\}, \qquad b_1 = b_0\{1 + 2W_V^e + \delta_b\},$$

where W_V^e arises from wave-function factors, and δ_a and δ_b from the triangle diagrams. In terms of the B and C functions* one has

$$W_V^e = \frac{e^2}{2(2\pi)^4 i}\{2B_1(p^2, m, \Lambda) + 4m^2 B_{1p}(p^2, m, \Lambda) + 8m^2 B_{0p}(p^2, m, \Lambda) + i\pi^2\},$$

where B_{1p} and B_{0p} are equal to $-\partial B_1/\partial p^2$ and $-\partial B_0/\partial p^2$ respectively. Furthermore m is the fermion mass, and the whole is to be evaluated at the point $p^2 = -m^2$**. B_0 and B_1 are defined by

$$B_0;\ B_1 p_\mu = \int \mathrm{d}_n q \frac{1;\ q_\mu}{(q^2+m^2)((q+p)^2+\Lambda^2)}.$$

The quantities δ_a and δ_b are

$$\delta_a = \frac{4e^2}{(2\pi)^4 i}\{C_{24} - \tfrac{1}{2}i\pi^2 - (P, p)(C_{23} + C_{11}) - \tfrac{1}{2}m^2(C_{21} + C_{22} + 2C_{12} - C_0)\},$$

$$\delta_b = \delta_a - \frac{4e^2}{(2\pi)^4 i} m^2(C_{11} - C_{12} + C_0).$$

All C functions except C_{24} have the dimension of an inverse mass squared, and go to zero as Λ becomes large. The function C_{24}, relating to the $\delta_{\mu\nu}$ part of the integral

$$\int \mathrm{d}_n q \frac{q_\mu q_\nu}{(q^2+m^2)((q+p)^2+\Lambda^2)((q+p+P)^2+m^2)},$$

behaves for large Λ as

$$C_{24} = \tfrac{1}{4}i\Delta + \tfrac{3}{8}i\pi^2 - \tfrac{1}{4}i\pi^2 \ln \Lambda^2,$$

where Δ contains the pole $1/(n-4)$, see ref. [6], sect. 3. The functions B_{0p} and B_{1p}

*The B, C and D functions in this article are like those of ref. [6], and are $i\pi^2$ times those of ref. [7].
**Our metric is such that a timelike momentum squared is negative. A physical four-momentum has imaginary fourth component.

tend to zero for large Λ, but the functions B_0 and B_1 become

$$B_0 = i\Delta + i\pi^2 - i\pi^2 \ln \Lambda^2 , \qquad B_1 = -\tfrac{1}{2} i\Delta - \tfrac{1}{4} i\pi^2 + \tfrac{1}{2} i\pi^2 \ln \Lambda^2 .$$

For large Λ the behaviour of W_V^e is

$$W_V^e \sim \frac{e^2}{(2\pi)^4 i} \left\{ -\tfrac{1}{2} i\Delta + \tfrac{1}{4} i\pi^2 + \tfrac{1}{2} i\pi^2 \ln \Lambda^2 \right\} ,$$

while we find for δ_a and δ_b

$$\delta_a \sim \delta_b \sim \frac{e^2}{(2\pi)^4 i} \left\{ i\Delta - \tfrac{1}{2} i\pi^2 - i\pi^2 \ln \Lambda^2 \right\} ,$$

and we see that in a_1 and b_1 not only the infinite parts (i.e., Δ) but also the finite parts cancel in the limit of infinite Λ.

It is noteworthy that the constant $-\frac{1}{2} i\pi^2$ in the equation for δ_a arises from terms proportional to $n-4$ in the γ-algebra combining with the pole $1/(n-4)$ in C_{24} to give a finite contribution. In other words, if all traces are done in four dimensions, then the correct results are obtained if one replaces C_{24} by $C_{24} - \frac{1}{2} i\pi^2$.

To separate off the purely e.m. part from the various contributions in the standard model a number of steps must be taken. The wave-function factors offer no problem, but in the standard model the remaining e.m. contribution is a box graph (see fig. 2). This diagram is finite, and we therefore may do all γ-algebra as in four dimensions. If q denotes the four-momentum flowing through the photon line, and $q+k$ the momentum through the vector boson line then we may in the limit of large vector boson mass, M, use the following approximations:

$$\frac{1}{q^2((q+k)^2 + M^2)} = \frac{1}{q^2(q^2+M^2)} \left\{ 1 + \frac{2qk + k^2}{q^2+M^2} \right\}^{-1}$$

$$= \frac{1}{q^2(q^2+M^2)} \left\{ 1 - \frac{2qk + k^2}{q^2+M^2} + \cdots \right\}$$

$$\simeq \frac{1}{q^2(q^2+M^2)} = \frac{1}{M^2} \left(\frac{1}{q^2} - \frac{1}{q^2+M^2} \right) .$$

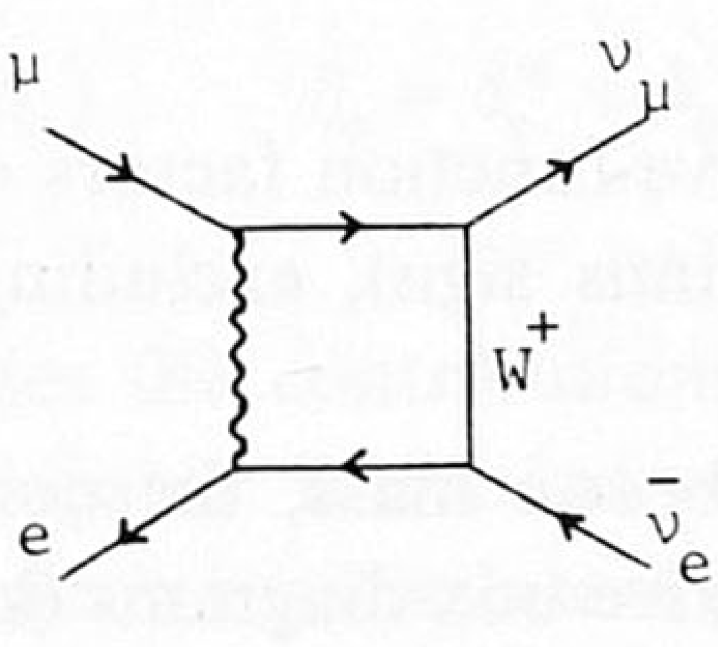

Fig. 2.

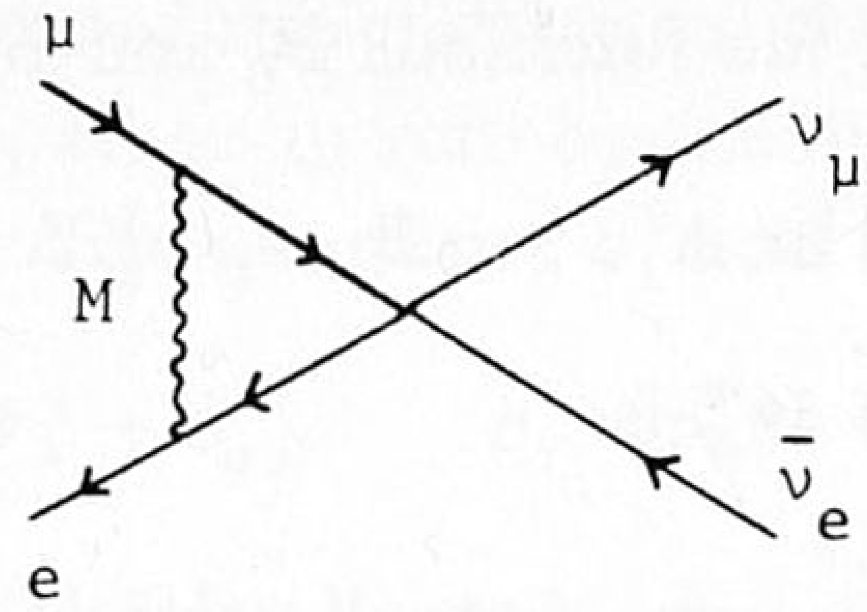

Fig. 3.

This amounts to (i) shrinking the W-line to a point, i.e., replacing the W-propagator by $1/M^2$, and (ii) taking the difference of two diagrams, one with a photon exchanged, the other with exchange of a heavy photon with mass M. If for both diagrams we use the prescription $C_{24} \to C_{24} - \frac{1}{2} i\pi^2$ then we have isolated precisely the purely e.m. contribution as given in the textbooks [10]. And this separation amounts to the replacement of the box diagram of fig. 2 by the triangle diagram of fig. 3, with the prescription $C_{24} \to C_{24} - \frac{1}{2} i\pi^2$, and with a minus sign. The γ-algebra is to be done as in four dimensions.

The remainder of the calculation is straightforward. The quantity δ_μ may be written as a sum of contributions

$$\delta_\mu = \delta_\mu^{\mathrm{em}} + \delta_\mu^{\mathrm{w}} + \delta_\mu^{\mathrm{v}} + \delta_\mu^{\mathrm{b}} + \delta_\mu^{\mathrm{p}} .$$

The various terms arise from the purely e.m. contribution, and from wave function renormalization, vertex diagrams, box diagrams, and propagator diagrams (self-energy insertions in the W-propagator). The e.m. correction as computed in ancient times is [10]

$$\delta_\mu^{\mathrm{em}} = -\frac{\alpha}{2\pi}\left(\pi^2 - \tfrac{25}{4}\right), \qquad \alpha = \frac{e^2}{4\pi} \sim \tfrac{1}{137} .$$

This includes bremsstrahlung, $\mu \to \mathrm{e} + \bar{\nu}_\mathrm{e} + \nu_\mu + \gamma$. Further,

$$\delta_\mu^{\mathrm{w}} = 2(W_\mathrm{V}^\mathrm{e} + W_\mathrm{V}^\mu + W_\mathrm{A}^\mathrm{e} + W_\mathrm{A}^\mu + 2W_\mathrm{V}^{\nu\mu} + 2W_\mathrm{V}^{\nu\mathrm{e}}) ,$$

where the various terms are wave-function factors of the various lines (see ref. [7], sect. 6; our W_A differ by a minus sign), excluding e.m. contributions. They are explicitly given in appendix A.

In the limit of large vector boson mass, the contributions of the four triangle (equal for $\mu\nu_\mu$W and $\mathrm{e}\nu_\mathrm{e}$W) and five box diagrams (keeping in mind the prescription for the box diagram of fig. 2) are

$$\delta_\mu^{\rm v} = -\frac{g^2}{2\pi^2}\left[-\frac{11}{8}+\frac{1}{16c_\theta^2}+\left(\frac{7}{4}-\frac{3}{2s_\theta^2}-\frac{1}{8c_\theta^2}\right)\ln c_\theta^2\right.$$

$$\left.+\left\{\frac{5}{4}+\frac{1}{8c_\theta^2}+\frac{1}{16M^2}\left(m_\mu^2+m_{\rm e}^2\right)\right\}\left(\ln M^2-\frac{\Delta}{\pi^2}\right)\right]$$

$$\delta_\mu^{\rm b} = -\frac{g^2}{\pi^2}\left[-\tfrac{1}{16}s_\theta^2+\tfrac{1}{8}\left(s_\theta^2-5+\frac{5}{2s_\theta^2}\right)\ln c_\theta^2-\tfrac{1}{8}s_\theta^2\left(\ln M^2-\frac{\Delta}{\pi^2}\right)\right],$$

where $s_\theta = \sin\theta_{\rm w}$ and $c_\theta = \cos\theta_{\rm w}$. These expressions include the infinite parts from two diagrams involving Higgs scalars and ghosts. Finally,

$$\delta_\mu^{\rm p} = 2S_+(0)/M^2$$

where $S_+(0)$ denotes the collection of charged vector boson self-energy diagrams for zero four-momentum squared, and is given in appendix B.

Unlike the purely e.m. contribution there is no cancellation of infinities. In the standard model μ-decay is not free of ultraviolet divergences. However, the infinities containing m_μ^2 or $m_{\rm e}^2$ as a factor cancel out.

3. Electric charge

The second data point that we take is the electric charge. We define the electric charge through the coefficient of the pole at zero momentum transfer of electron-muon scattering. In lowest order this coefficient is e^4 (apart from irrelevant other factors), and the inclusion of one-loop diagrams amounts to the replacement

$$e^4 \to e^4(1+\delta_{\rm e}), \qquad \delta_{\rm e} = \delta_{\rm e}^{\rm w}+\delta_{\rm e}^{\rm v}+\delta_{\rm e}^{\rm p}+\delta_{\rm e}^{\rm 0p}.$$

The latter decomposition denotes the contributions of wave-function factors, two vertex diagrams, photon propagator insertions, and of the $\gamma - {\rm W}^0$ transition diagrams. The purely e.m. contributions, exclusive of photon propagator diagrams, cancel. One finds (ignoring γ^5 terms):

$$\delta_e^w = 4(W_V^e + W_V^\mu),$$

$$\delta_e^V = \frac{g^2}{4\pi^2}\left[\tfrac{1}{8} - \frac{5}{16c_\theta^2} + \tfrac{1}{2}s_\theta^2 + \left(\frac{5}{8c_\theta^2} - \tfrac{1}{2} - s_\theta^2\right)\ln c_\theta^2\right.$$

$$\left. + \left\{s_\theta^2 - \frac{5}{8c_\theta^2} - \tfrac{1}{4} - \frac{3}{16M^2}(m_\mu^2 + m_e^2)\right\}\left(\ln M^2 - \frac{\Delta}{\pi^2}\right)\right],$$

$$\delta_e^P = \lim_{k^2=0} \frac{2S_\gamma}{k^2},$$

$$\delta_e^{0P} = -\frac{v_\theta c_\theta}{s_\theta M^2} S_{\gamma 0}, \qquad v_\theta = 4s_\theta^2 - 1.$$

The functions W_V, S_γ and $S_{\gamma 0}$ are given in the appendices A and B. The factors in the last mentioned correction can be understood if one remembers that the lowest-order photon-electron coupling involves the factor $-ie = -igs_\theta$, while the vector part of the W^0 electron coupling contains the factor $igv_\theta/4c_\theta$. Note further that $M^2 = M_0^2c_\theta^2$, where M_0 is the mass of the neutral vector boson.

Finally it should be mentioned that the infinities containing m_μ^2 or m_e^2 cancel, otherwise our definition of electric charge would depend on the kind of particle considered.

4. E.m. corrections to neutrino-electron scattering

As our third data point we take the ratio of $\bar{\nu}_\mu$-e to ν_μ-e scattering. Here we must deal with the complications of the pure e.m. radiative corrections since these have not been calculated before. In actual fact they very much resemble the μ-decay e.m. corrections: they are finite by themselves and also of the same order of magnitude, i.e., a fraction of a percent. In this section we will treat these e.m. corrections.

Besides the corrections due to one-loop diagrams we must also consider bremsstrahlung. Without this the corrections would be infrared divergent. Moreover, experiment is very unlikely to detect the extra photon, and therefore the experimental results will include bremsstrahlung.

The reaction is best considered in the c.m.s. We will give the photon a small mass λ, and compute the corrections in the limit of zero λ. Technically bremsstrahlung can be quite complicated, and we will describe things in some detail. The main problem in the evaluation is to avoid infrared-divergent integrals that diverge like k^{-n} (where k is the photon momentum) and have a factor λ^{n-1}. Such terms are involved if one sets the photon energy k_0 equal to k carelessly during the evaluation.

To avoid such problems we consider three different expressions:

(i) the exact bremsstrahlung cross section as a function of k, but integrated over all phase-space variables;

(ii) an approximate expression, equal to the exact expression for small k, and evaluated along precisely the same lines;

(iii) the same approximate expression as in (ii), but now evaluated in another, in fact the standard, way.

The complete radiative corrections are then made up from two parts, each of them infrared finite; namely, the difference of expressions (i) and (iii), and the sum of expression (iii) and the loop corrections. Usually the term hard bremsstrahlung is used to designate the first part. See ref. [11] for treatment of hard bremsstrahlung in other cases.

In the lab system the total cross section for the process $\nu_\mu + \mathrm{e} \to \nu_\mu + \mathrm{e} + \gamma$ is given by

$$\sigma_{\text{lab}} = \frac{1}{4mq_0}\sigma_{\text{inv}}, \tag{4.1}$$

$$\sigma_{\text{inv}} = \int \frac{\mathrm{d}_3 k}{2k_0} \int \frac{\mathrm{d}_3 p}{2p_0} \int \frac{\mathrm{d}_3 r}{2r_0} \overline{A^2} \delta_4(q + P - r - k - p) \tag{4.2}$$

$$\overline{A^2} = \tfrac{1}{2} \sum_{\text{spin}} |A|^2/(2\pi)^{13}\,.$$

The momenta q, P, r, p, k refer in this order to the momenta of ν_μ, $\mathrm{e} \to \nu_\mu$, e, γ. The amplitude A is given by

$$A = \frac{(2\pi)^4 i \cdot i g^3 s_\theta}{16 M_0^2 c_\theta^2} e_\alpha(k)\{\bar{u}(r)\gamma^\mu(1+\gamma^5)u(q)\}$$

$$\times \left[\bar{u}(p) \left\{ \gamma^\alpha \frac{-i\gamma(k+p)+m}{(k+p)^2+m^2} \gamma^\mu(v_\theta - \gamma^5) \right.\right.$$

$$\left.\left. + \gamma^\mu(v_\theta - \gamma^5) \frac{-i\gamma(P-k)+m}{(P-k)^2+m^2} \gamma^\alpha \right\} u(P) \right]. \tag{4.3}$$

The electron mass is denoted by m. The two terms correspond to the photon radiating from the outgoing and incoming electrons, respectively. As before $s_\theta = \sin\theta_{\mathrm{w}}$, $c_\theta = \cos\theta_{\mathrm{w}}$ and $v_\theta = 4s_\theta^2 - 1$. Further $M_0 = M/c_\theta$ is the neutral vector boson mass.

Since σ_{inv} is Lorentz invariant we may evaluate it in the c.m.s. In actual experiments the energy in the c.m.s. ranges from 0 to about 500 MeV for lab energies up to 200 GeV, and the phase space is quite comparable to the case of μ-decay.

Eqs. (4.1)–(4.3) define the exact expression (i). Expression (ii) [and (iii)] is obtained by neglecting the k dependence in the δ-function, and by neglecting the terms γk in the electron propagators.

Arranging for the k integration to be the last we may write

$$\sigma_{\text{inv}} = 2\pi \int_{\lambda}^{\omega} \frac{k k_0 \, \mathrm{d}k_0}{2k_0} \int_{-1}^{1} \mathrm{d}x \int_{p_a}^{p_b} \frac{p p_0 \, \mathrm{d}p_0}{2p_0} \int_0^{2\pi} \mathrm{d}\phi$$
$$\times \int_{-1}^{1} \frac{\mathrm{d}z}{2r_0} \overline{A}^2 \delta(q_0 + P_0 - k_0 - p_0 - r_0) \,. \tag{4.4}$$

Here k and p now denote $|\boldsymbol{k}|$ and $|\boldsymbol{p}|$, respectively. Our coordinate system is such that $\boldsymbol{q}$ and $\boldsymbol{k}$ are in the 1-3 plane, with $\boldsymbol{k}$ along the third axis. The cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$ is denoted by x, and z is the cosine of the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$. Finally ϕ is the azimuthal angle of $\boldsymbol{p}$ with respect to the $\boldsymbol{q}$, $\boldsymbol{k}$ plane. Note that $r_0 = |\boldsymbol{r}| = |\boldsymbol{k} + \boldsymbol{p}|$ in the c.m.s.

Doing the z-integral we obtain

$$\sigma_{\text{inv}} = \tfrac{1}{4}\pi \int_{\lambda}^{\omega} \mathrm{d}k_0 \int_{y_a}^{y_b} \mathrm{d}y \int_{-1}^{1} \mathrm{d}x \int_0^{2\pi} \mathrm{d}\phi \frac{kq}{E} \overline{A}^2 \,, \tag{4.5}$$

where E is the c.m. energy. The requirements that $z_0^2 \leqslant 1$, where z_0 is the value of z for which the argument of the δ-function is zero, and that $E - k_0 - p_0 \geqslant 0$ lead to the limits on the p_0 and k_0 integrations:

$$p_{a,b} = \frac{\kappa(E - k_0) \pm k(\kappa - 2m^2)}{2(\kappa - m^2)} \,, \tag{4.6}$$

$$\kappa = E^2 + m^2 + \lambda^2 - 2Ek_0 = (E - k_0)^2 + m^2 - k^2 \,, \tag{4.7}$$

$$\omega = \frac{E^2 + \lambda^2 - m^2}{2E} \,. \tag{4.8}$$

Also we introduced the variable y instead of p_0:

$$p_0 = \bar{p}_0 + \frac{kq}{E} y \,, \qquad \bar{p}_0 = \frac{\kappa}{2(E - k_0)} \,. \tag{4.9}$$

Note that $\kappa = 2m^2$ if $k_0 = \omega$. Furthermore, for $k = 0$ the quantity $\bar{p}_0$ equals the energy of the electron in the non-bremsstrahlung process $\nu_\mu + e \to \nu_\mu + e$, i.e., $\bar{p}_0$ is the value obtained for p_0 when neglecting k in the δ-function. From (4.6) and (4.9) one deduces:

$$y_{a,b} = \frac{E}{2q(\kappa - m^2)} \left\{ \frac{\kappa k}{E - k_0} \pm (\kappa - 2m^2) \right\}. \tag{4.10}$$

For $k = \lambda = 0$ we have $y_{a,b} = \pm 1$, because in the c.m.s. $q = (E^2 - m^2)/2E$.

In terms of these variables the various scalar products are:

$$(q, k) = qkx - qk_0 \qquad (P, k) = -qkx - P_0 k_0, \qquad (p, k) = -kqy - \bar{p}_0 k_0,$$

$$(P, p) = -qpxz_0 - qp\sqrt{1 - x^2}\sqrt{1 - z_0^2}\cos\phi - P_0 p_0, \tag{4.11}$$

with $pz_0 = -qy + qk_0 y/E$ to be substituted in (P, p).

The propagator denominators are $(p + k)^2 + m^2 = 2(p, k)$ and $(P - k)^2 + m^2 = -2(P, k)$ respectively. Thus $\bar{A}^2$ will be of the form

$$\bar{A}^2 = \frac{f_1}{(P, k)^2} + \frac{f_2}{(p, k)^2} + \frac{f_3}{(P, k)(p, k)},$$

where f_1, f_2 and f_3 may contain (P, k) or (p, k). Clearly the integration over ϕ is trivial. The x and y integrations are much more difficult, for example

$$\int \mathrm{d}x \int \mathrm{d}y \int \frac{\mathrm{d}\phi}{2\pi} \frac{(P, p)}{(P, k)(p, k)} = \int \mathrm{d}x \int \mathrm{d}y \frac{q^2xy - q^2kxy/E - P_0 p_0}{(qkx + P_0 k_0)(kqy + \bar{p}_0 k_0)}$$

$$= \int \mathrm{d}x \int \mathrm{d}y \left[\frac{1 - k/E}{k^2} \left\{ 1 - \frac{\bar{p}_0 k_0}{qkx + P_0 k_0} - \frac{P_0 k_0}{kqy + \bar{p}_0 k_0} \right\} + \frac{P_0}{k^2} \frac{(\bar{p}_0 k_0^2 - p_0 k^2 - \bar{p}_0 k_0^2 k/E)}{(qkx + P_0 k_0)(kqy + \bar{p}_0 k_0)} \right].$$

The numerator of the last term behaves as λ^2 for small λ, but contributes a finite amount in the limit of small λ due to the extra infrared divergences arising from the factor $1/k^2$. The evaluation of such terms is cumbersome, and we would like to put $k_0 = k$ in these terms. In order to achieve this we subtract the approximate expression, to be considered now.

The approximate expression is obtained from eqs. (4.1)–(4.3) by omitting the k dependence in the δ-function, as well as the γk terms in the propagator numerators.

Denoting the resulting matrix element by A_a we again obtain an expression of the form (4.4), but now $r_0 = |\boldsymbol{r}| = |\boldsymbol{p}|$, and there is no z-dependence in the δ-function. Integrating over p_0 we find:

$$\sigma^a_{\text{inv}} = \tfrac{1}{4}\pi \int_\lambda^\omega \mathrm{d}k_0 \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \frac{kq}{E} \overline{A_a^2} \tag{4.12}$$

where now $p = q$ inside $\overline{A_a^2}$. The various scalar products are now as in eq. (4.11), however with z instead of z_0, and $p = q = (E^2 - m^2)/2E$, with $p_0 = P_0 = (E^2 + m^2)/2E$. Moreover, now $(p, k) = -kqz - p_0 k_0$.

Evidently eqs. (4.5) and (4.12) are very similar, with z in eq. (4.12) playing the role of y in eq. (4.5). For small k the integrands become identical. For larger k the approximate expression is generally larger.

The important result is the difference $\sigma_{\text{inv}} - \sigma^a_{\text{inv}}$. The difference of the integrands is infrared finite, term by term, and we may set the photon mass to zero. Next we may compute the difference as a function of the lab energy; the result is shown in table 1 for both the neutrino and antineutrino case. The result was obtained by doing all integrations except the k integral analytically; the k integration was done numerically. Relative to the non-bremsstrahlung cross sections the results are found to be of order 1.5 to 4 percent.

Next we must evaluate the loop corrections. The diagram of fig. 4 together with the e.m. wave-function factors for the electron and the lowest-order amplitude give rise to the amplitude

$$\frac{(2\pi)^4 i \cdot ig}{4c_\theta} g_0\left[\gamma^\mu(\bar{v}_\theta - \gamma^5) + iP_\mu m\gamma^5 H + iP_\mu m v_\theta K\right],$$

with

$$\bar{v}_\theta = (1 + F - G)v_\theta, \qquad g_0 = 1 + 2W^e_V + G,$$

$$F = \frac{4g^2 s_\theta^2}{(2\pi)^4 i}\left\{C_{24} - \tfrac{1}{2}i\pi^2 - (P, p)(C_{23} + C_{11}) - \tfrac{1}{2}m^2(C_{21} + C_{22} + 2C_{12} + C_0)\right\},$$

$$G = F - \frac{4g^2 s_\theta^2}{(2\pi)^4 i} m^2(C_{11} - C_{12}),$$

$$H = -\frac{4g^2 s_\theta^2}{(2\pi)^4 i}(C_{22} - C_{21} - C_{11} + C_{12}),$$

$$K = -\frac{4g^2 s_\theta^2}{(2\pi)^4 i}(C_{22} + C_{21} - 2C_{23} + C_{11} - C_{12}),$$

TABLE 1
e.m. corrections to ν_μe and $\bar{\nu}_\mu$e scattering

(E_{lab}(GeV))	(E_{cm}(MeV))	σ_ν^h	σ_ν^s	σ_ν^t	$\sigma_{\bar\nu}^h$	$\sigma_{\bar\nu}^s$	$\sigma_{\bar\nu}^t$
0.1	10	−1.5	1.0	−0.5	−1.5	1.0	−0.5
0.5	23	−2.0	1.6	−0.4	−1.9	1.5	−0.4
1	32	−2.2	1.8	−0.35	−2.1	1.8	−0.35
2	45	−2.3	2.0	−0.31	−2.3	2.0	−0.31
5	71	−2.6	2.4	−0.26	−2.6	2.3	−0.25
10	101	−2.8	2.6	−0.21	−2.8	2.6	−0.20
20	142	−3.0	2.8	−0.17	−3.0	2.8	−0.16
50	226	−3.3	3.2	−0.11	−3.2	3.1	−0.10
100	320	−3.5	3.4	−0.07	−3.4	3.4	−0.06
200	452	−3.7	3.6	−0.03	−3.6	3.6	−0.01
500	715	−3.9	4.0	+0.03	−3.9	4.0	+0.04
1000	1011	−4.1	4.2	0.07	−4.1	4.2	0.09
10 000	3197	−4.8	5.0	0.22	−4.8	5.0	0.24

The values are percentages relative to the lowest order cross-section. σ^h refers to the difference of exact and approximate bremsstrahlung, σ^s to the sum of the approximate bremsstrahlung and internal corrections, and σ^t is the total e.m. correction. We used $\sin^2\theta = 0.238$ [3].

and W_V^e is given in appendix A. The form factors C depend on the Lorentz-invariant variables as follows:

$$C = C(-m^2, -m^2, (P-p)^2, m^2, \lambda^2, m^2)\,.$$

The first three variables denote the external four-momenta squared, the other three are the internal masses.

Finally we must compute the approximate bremsstrahlung cross section in a manner convenient for addition to the loop corrections. Going back to eqs. (4.1)–(4.3), with the approximation of small k and applying the Dirac equation we obtain

$$\sigma^a_{\text{inv}} = \frac{g^2 s_\theta^2}{(2\pi)^3}\int_{-1}^{1} \mathrm{d}x \int \frac{\mathrm{d}_3 k}{2k_0}\left\{\frac{(p,p)}{(p,k)^2} + \frac{(P,P)}{(P,k)^2} - 2\frac{(p,P)}{(p,k)(P,k)}\right\}\sigma_0(x)\,,$$

where $\sigma_0(x)$ is the lowest order non-bremsstrahlung cross section as a function of x, the cosine of the angle between p and P in the c.m.s. In terms of the bremsstrahlung integrals of ref. [6] we have

$$\sigma^a_{\text{inv}} = \frac{g^2 s_\theta^2}{2(2\pi)^3}\int_{-1}^{1} \mathrm{d}x\left[-m^2\{\mathfrak{L}(p,p) + \mathfrak{L}(P,P)\} - 2(P,p)\mathfrak{L}(p,P)\right]\sigma_0(x)\,.$$

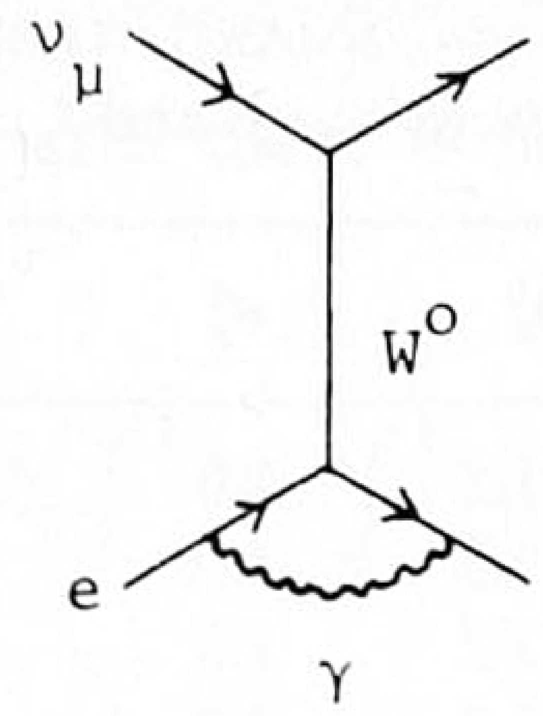

Fig. 4. e.m. vertex corrections.

Table 1 gives the final results. The total e.m. corrections are quite small, of the order of 0.3 percent or less relative to the lowest-order cross sections.

5. Weak corrections to neutrino-electron scattering

The evaluation of the remaining radiative corrections to ν_μe and $\bar{\nu}_\mu$e scattering is straightforward. We may write

$$\sigma_{\nu e} = \sigma^0_{\nu e}\{1 + \delta_{\mathrm{em}} + \delta_{\mathrm{wf}} + \delta_{\mathrm{v}} + \delta_{\mathrm{b}} + \delta_{\mathrm{w}} + \delta_\gamma\}\ .$$

In this equation the various corrections are:

δ_{em}, the purely e.m. corrections;
δ_{wf}, corrections due to wave function factors;
δ_{v}, corrections due to vertex diagrams;
δ_{b}, corrections due to box diagrams;
δ_{w}, corrections to the W-propagator;
δ_γ, corrections involving photon exchange.

The e.m. corrections have been computed in sect. 4, and are in fact quite small, of the order of 0.2 percent. All other corrections are best expressed in terms of corrections to the vector and axial vector coefficients of the electron vertex. Let this vertex be given by

$$(2\pi)^4 i \frac{ig}{4c_\theta}\{\bar{e}\gamma^\mu(a + b\gamma^5)e\}\ .$$

In lowest order $a \equiv a_0 = v_\theta = 4s^2_\theta - 1$ and $b \equiv b_0 = -1$. The total cross section is a quadratic function of a and b:

$$\sigma_{\nu e} = \frac{g^4}{4\pi}\frac{1}{(16c^2_\theta M^2_0)^2 4m_e q_0}\left(\frac{|\boldsymbol{q}|}{2E}\right)_{\mathrm{cm}} F(a, b)\,,$$

where q_0 is the energy of the neutrino in the lab system. The expression $F(a, b)$ is

$$F(a, b) = 64q^2(2qE + \tfrac{4}{3}q^2 + m_e^2)a^2 + 256q^2(qE - \tfrac{2}{3}q^2)ab$$
$$+ 64q^2(2qE + \tfrac{4}{3}q^2 + 3m_e^2)b^2 ,$$

where $q = |\boldsymbol{q}|$ and E are the neutrino momentum and total energy in the c.m.s., thus $E = \sqrt{2m_e E_{\rm lab} + m_e^2}$, and $q = m_e E_{\rm lab}/E$. The quantity $E_{\rm lab}$ is the total energy in the lab system, $E_{\rm lab} = q_0 + m_e$. To a good approximation one may set $q = \frac{1}{2}E$ and neglect terms proportional to the electron mass in $F(a, b)$, at least for high-energy experiments where E is of the order of 100 MeV (for 10 GeV neutrinos).

The various corrections to a_0 and b_0 may now be written down:

$$\delta a_{\rm wf} = a_0(2W_{\rm V}^{\nu\mu} + 2W_{\rm A}^{\nu\mu} + 2W_{\rm V}^{\rm e}) + 2b_0 W_A^{\rm e} ,$$

$$\delta b_{\rm wf} = b_0(2W_{\rm V}^{\nu\mu} + 2W_{\rm A}^{\nu\mu} + 2W_{\rm V}^{\rm e}) + 2a_0 W_{\rm A}^{\rm e} .$$

Further, the vertex contributions are

$$\delta a_{\rm v} = A\left[a_0\delta_{\nu c} + \tfrac{1}{2}s_\theta^2 c_\theta + \tfrac{1}{2}c_\theta + \frac{9}{32c_\theta^3} - \frac{7}{8c_\theta} + \left(\frac{15}{8c_\theta} - \frac{9}{16c_\theta^3} - \frac{5c_\theta}{4} - c_\theta s_\theta^2\right)\ln c_\theta^2 \right.$$
$$\left. + \left(\frac{12s_\theta^2 - 2}{32c_\theta}\frac{m_e^2}{M^2} - \frac{7}{4c_\theta} + \frac{9}{16c_\theta^3} + \tfrac{1}{2}c_\theta + c_\theta s_\theta^2\right)\left(\ln M^2 - \frac{\Delta}{\pi^2}\right)\right] ,$$

$$\delta b_{\rm v} = A\left[b_0\delta_{\nu c} - \tfrac{1}{2}c_\theta - \frac{7}{32c_\theta^3} + \frac{5}{8c_\theta} + \left(-\frac{9}{8c_\theta} + \frac{7}{16c_\theta^3} + \tfrac{3}{4}c_\theta\right)\ln c_\theta^2 \right.$$
$$\left. + \left(-\frac{4s_\theta^2 + 2}{32c_\theta}\frac{m_e^2}{M^2} + \frac{5}{4c_\theta} - \frac{7}{16c_\theta^3} - \tfrac{3}{2}c_\theta\right)\left(\ln M^2 - \frac{\Delta}{\pi^2}\right)\right] .$$

The coefficient $A = -g^2 c_\theta/(4\pi^2)$. The quantities a_0 and b_0 are as before, $a_0 = v_\theta = 4s_\theta^2 - 1$ and $b_0 = -1$. The quantity $\delta_{\nu c}$ arises from three $\bar{\nu}\nu W^0$ vertex diagrams:

$$\delta_{\nu c} = \frac{1}{32c_\theta^3} + \frac{1}{16c_\theta} - \frac{1}{16c_\theta^3}\ln c_\theta^2 + \left(\frac{m_\mu^2}{16M^2 c_\theta} + \tfrac{1}{2}c_\theta + \frac{1}{16c_\theta^3} + \frac{1}{8c_\theta}\right)\left(\ln M^2 - \frac{\Delta}{\pi^2}\right) .$$

The contributions of three $\bar{\rm e}{\rm e}W^0$ diagrams are contained in the above. The terms proportional to the electron or muon mass are the infinite parts of diagrams involving Higgs particles; two such diagrams for the $\nu\nu W^0$ vertex, and six for the eeW^0 vertex.

There are three box diagrams, giving rise to

$$\delta a_{\rm b} = -\frac{g^2}{\pi^2}\left(\frac{7}{16} - \frac{9}{64 c_\theta^2}\right), \qquad \delta b_{\rm b} = -\frac{g^2}{\pi^2}\left(\tfrac{1}{16} - \tfrac{3}{8}s_\theta^2 + \frac{15}{64 c_\theta^2}\right).$$

The propagator corrections arise from the W^0 self-energy diagrams:

$$\delta a_{\rm w} = a_0 \frac{S_0(0)}{M_0^2}, \qquad \delta b_{\rm w} = b_0 \frac{S_0(0)}{M_0^2},$$

where $S_0(0)$ is the collection of W^0 self-energy diagrams $S_0(k^2)$ computed for zero k^2, see appendix B.

The corrections due to γ-exchange arise in two ways: first due to vertex diagrams contributing to $\nu\nu\gamma$ (two diagrams), and secondly as a consequence of a γ-W^0 self-energy type transition. One obtains:

$$\delta a_\gamma = -\frac{M^2 g^2 s_\theta^2}{\pi^2 k^2} F_1(k^2) - \frac{4 c_\theta s_\theta M_0^2}{k^2(k^2 + M_0^2)} S_{0\gamma}(k^2),$$

computed in the limit $k^2 = 0$. The k^2 dependence in the W^0 propagator must not be neglected. The quantity $S_{0\gamma}(k^2)$ is the W^0-γ transition amplitude, see appendix B, and F_1 arises from the $\nu\nu\gamma$ diagrams:

$$F_1(k^2) = 3C_{24}^b - C_{24}^a - \tfrac{1}{2}m_\mu^2 C_0^a - \tfrac{1}{2}k^2\left(C_0^b + C_{11}^b + C_{23}^b - C_{11}^a - C_{23}^a\right)$$
$$+ m_\mu^2\left[\tfrac{1}{2}C_0^b + \frac{1}{4M^2}\left\{\tfrac{1}{2} - 2\left(C_{24}^a + \tfrac{1}{2}m_\mu^2 C_0^a - C_{24}^b\right)\right\}\right].$$

The functions C^a and C^b are 3-point form factors with the arguments

$$C^a = C^a(0, 0, k^2, m_\mu^2, M^2, m_\mu^2), \qquad C^b = C^b(0, 0, k^2, M^2, m_\mu^2, M^2).$$

Also, contributions of four diagrams involving Higgs exchange have been included. For the functions C^a and C^b approximations have been used; for C^b we used

$$\frac{1}{i\pi^2}C_{11}^b = -\bar{C} + \frac{1}{4\delta M} - \frac{m_\mu^2}{2\delta M}\bar{C}, \qquad \frac{1}{i\pi^2}C_{12}^b = -\frac{1}{4\delta M} + \frac{m_\mu^2}{2\delta M}\bar{C},$$

with

$$\delta M = M^2 - m_\mu^2 , \qquad \bar{C} = \frac{1}{\delta M}\left[1 - \frac{m_\mu^2}{\delta M}\ln\frac{M^2}{m_\mu^2}\right] .$$

Further,

$$C_0^b = \left\{ -k^2 C_{11}^b + B_0(0, m^2, M^2) - B_0(k^2, M^2, M^2)\right\}(\delta M + k^2)^{-1} ,$$

$$\frac{1}{i\pi^2} C_{21}^b = \frac{11}{18M^2} , \qquad \frac{1}{i\pi^2} C_{22}^b = \frac{1}{9M^2} , \qquad \frac{1}{i\pi^2} C_{23}^b = \frac{7}{36M^2} ,$$

$$\frac{1}{i\pi^2} C_{24}^b = \frac{\Delta}{4\pi^2} + \tfrac{1}{8} - \tfrac{1}{4}\ln M^2 - \frac{m_\mu^2}{4\delta M}\left[1 - \frac{m_\mu^2}{\delta M}\ln\frac{M^2}{m_\mu^2}\right] - \frac{k^2}{36M^2} .$$

The refinements with respect to the approximations quoted in appendix E of ref. [7] are needed to make sure that no terms proportional to m_μ^2 survive in the limit $k^2 = 0$. For the function C^a we used the approximations of ref. [7], appendix E, but added the infinite parts of the terms neglected there. This amounts to the additions

$$C_{11} \to C_{11} - \tfrac{2}{3} i\Delta y_2^2 (y_1 + 2p_1^2) ,$$

$$C_{12} \to C_{12} - \tfrac{1}{3} i\Delta y_2^2 p_1^2 ,$$

$$C_{21} \to C_{21} - \tfrac{1}{3} i\Delta y_2^2 \left\{4m^2 + p_2^2 - p_1^2 - y_2(p_1^2 - m^2)^2\right\} ,$$

$$C_{22} \to C_{22} , \qquad C_{23} \to C_{23} - \tfrac{1}{6} i\Delta y_2^2 \left\{5m^2 + p_2^2 - p_1^2\right\} ,$$

$$C_{24} \to C_{24} - \tfrac{1}{6} i\Delta y_2^2 \left\{ y_1(5m^2 + p_2^2 - p_1^2) - m^2 p_1^2 - y_2 p_1^2 (p_1^2 - m^2)^2 \right\} .$$

The y, etc., are as in ref. [7], appendix E. Of course, in our case $p_1^2 = p_2^2 = 0$ and $m = m_\mu$. Note that $(p_1 p_2)$ of ref. [7], appendix E is defined as $\frac{1}{2}\{(p_1 + p_2)^2 - p_1^2 - p_2^2\}$, while for M_0 the charged W mass M must be substituted.

It is to be noted that neither F_1 nor $S_{0\gamma}$ are zero in the limit $k^2 = 0$, but in the expression for δa_γ the finite parts cancel in that limit. So, as should be, δa_γ has no term of the form $1/k^2$. A term of that form would imply a finite charge for the neutrino.

This completes the discussion of the corrections to ν_μe scattering. Those for $\bar{\nu}_\mu$e scattering follow trivially, in the expression for the cross section one simply replaces a by $-a$.

6. Renormalization

The parameters g^2, M^2 and s_θ^2 must now be determined by comparison with the data. As a first step we determine a first approximation by comparing the lowest-order expressions, thus not including any radiative corrections, with the experimental results for the electric charge, neutral currents and μ-decay. We get

$$g^2 = e^2/s_\theta^2 , \qquad \text{with} \qquad e^2/4\pi = \alpha = \tfrac{1}{137} , \qquad s_\theta^2 = 0.238 \; [3] ,$$

$$M^2 = \frac{g^2}{8G_F} ,$$

$$G_F = 8.2297 \times 10^{-12}\ \text{MeV}^{-2} = \frac{1.0246}{m_p^2\sqrt{2}} \times 10^{-5} , \qquad m_p = \text{proton mass} .$$

These values for g^2, s_θ^2 and M^2, correct up to first order in g, can now be used in the expressions for the radiative corrections. In the lowest-order expressions we subsequently replace g^2 by $g^2(1+\delta g^2)$, M^2 by $M^2(1+\delta M^2)$ and s_θ^2 by $s_\theta^2(1+\delta s_\theta^2)$, and δg^2, δM^2 and δs_θ^2 must be chosen such that they precisely compensate the radiative corrections for μ-decay, e-μ scattering and the ratio $\sigma_{\bar{\nu}e}/\sigma_{\nu e}$. Having then determined these quantities, we may calculate the radiative corrections to, say, $\sigma_{\nu e}$ by taking the lowest-order expression, using the corrected values $g^2(1+\delta g^2)$, etc., and adding the radiative corrections. In practice this goes as follows. The ratio $\sigma_{\bar{\nu}e}/\sigma_{\nu e} = R$ is a function of s_θ^2. At zero energy and momentum transfer it does not depend on g^2 or M^2 at lowest order. Thus, first compute

$$R_0(s_\theta^2 + s_\theta^2 \delta s_\theta^2) = R_0(s_\theta^2) + R_0' s_\theta^2 \delta s_\theta^2 ,$$

where R_0' is the derivative of R_0 with respect to s_θ^2. Now compute the radiative corrections for some value of Δ, representing the infinite part. Let the result be R_1:

$$R = R_0 + R_1 .$$

Then δs_θ^2 is fixed by $\delta s_\theta^2 = -R_1/(R_0' s_\theta^2)$.

Next compute the radiative correction to the electric charge:

$$e^4 \to e^4(1+\delta e^4) .$$

Since $e^2 = g^2 s_\theta^2$ we have then $\delta g^2 = -\frac{1}{2}\delta e^4 - \delta s_\theta^2$. Finally the μ-decay correction is

$$\Gamma_\mu \to \Gamma_\mu(1+\delta_\mu) .$$

Since Γ_μ depends only on g^2/M^2 we have $\delta M^2 = \frac{1}{2}\delta_\mu + \delta g^2$. In this way δs_θ^2, δg^2 and δM^2 are determined.

Now consider the ν_μe scattering amplitude. In lowest order it is, apart from irrelevant factors,

$$\frac{g^2}{c^2}\left\{\frac{1}{k^2+M_0^2}\right\}(v-\gamma^5)\,,\qquad c=c_\theta\,,\qquad v=v_\theta=4s_\theta^2-1\,.$$

Including now the corrections, this becomes

$$\frac{g^2}{c^2M_0^2}\left\{\frac{M_0^2}{k^2+M_0^2}\right\}$$

$$\times\left\{1+\delta g^2+\frac{s^2}{c^2}\delta s^2-\frac{M_0^2}{k^2+M_0^2}\left(\delta M^2+\frac{s^2}{c^2}\delta s^2\right)\right\}(v+4s^2\delta s^2-\gamma^5)$$

$$=\frac{g^2}{c^2M_0^2}\left\{\frac{M_0^2}{k^2+M_0^2}\right\}\left\{1+\delta g^2-\frac{M_0^2}{k^2+M_0^2}\delta M^2+\frac{k^2}{k^2+M_0^2}\frac{s^2}{c^2}\delta s^2\right\}$$

$$\times(v+4s^2\delta s^2-\gamma^5)\,.$$

In the last three equations we have suppressed the subscript θ for c_θ, s_θ, v_θ and δs_θ^2.

Together with the radiative corrections this defines the complete amplitude correct up to and including terms of order g^4. In the above expressions we have explicitly kept the k^2 dependence arising from the W^0 propagator; to obtain the correct cancellation of the infinities the same must be done in the radiative corrections. This may seem somewhat excessive, since k^2 in actual experiments is at most of the order of (1000 MeV)^2, compared to $M_0^2 \sim (90\,000)^2$, but actually this is needed to obtain complete stability against variations in Δ, the infinite part.

7. Results and conclusions

Carrying through the complete calculation along the lines described in sect. 6 the following results are obtained.

For the ratio $\sigma_{\bar{\nu}e}/\sigma_{\nu e}$ the e.m. corrections as determined from those in sect. 4 are practically zero (of the order of 0.02 percent). After renormalization the correction to $\sigma_{\nu e}$ may be computed, and they turn out to be only slightly energy dependent. If the standard model is correct then one should observe slightly larger cross sections than computed in lowest order. The percentage increase is given in table 2, for both $\sigma_{\nu e}$ and $\sigma_{\bar{\nu}e}$. The difference between these two represents the influence of the energy dependence of the cross section; the calculation that we did automatically fixed the corrections to be equal for an energy of 5000 MeV and the ratio $\sigma_{\bar{\nu}e}/\sigma_{\nu e}$ at this energy defines the weak mixing angle. In practice, of course, the experiment will

TABLE 2

Radiative corrections in percent with respect to the lowest-order result for the total cross sections $\nu_\mu e \to \nu_\mu e$ and $\bar{\nu}_\mu e \to \bar{\nu}_\mu e$

E_{lab}(GeV)	E_{cm}(MeV)	$\delta\sigma_{\nu e}$	$\delta\sigma_{\bar{\nu} e}$
0.1	10	0.65	0.69
0.5	23	0.76	0.78
1	32	0.80	0.82
2	45	0.85	0.87
5	71	0.91	0.92
10	101	0.96	0.96
20	142	1.01	1.00
50	226	1.08	1.05
100	320	1.14	1.07
200	452	1.21	1.09
500	715	1.33	1.09
1000	1011	1.44	1.08
10 000	3197	1.73	0.90

The calculational scheme defines equal $\delta\sigma_{\nu e}$ and $\delta\sigma_{\nu e}$ at 5 GeV for the non-e.m. part. The numbers include the radiative corrections of table 1. The Higgs mass m has been taken to be 200 GeV; for a light Higgs of 10 GeV add 0.1 percent to $\delta\sigma$.

observe some average over some energy range, but it is clear that this will change the result only by a small fraction of a percent. Table 2 includes the radiative corrections found in sect. 4. Most of the energy dependence relates to e^+e^- or $\nu\bar{\nu}$ intermediate states in the t-channel through self-energy type diagrams.

Also, the mass shifts to the neutral and charged vector boson mass may now be computed. They are found to be 3080 and 3310 MeV respectively, but this will not be discussed here any further.

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Appendix A

WAVE-FUNCTION FACTORS

The wave-function factors can be computed from the fermion self-energy diagrams. Consider a fermion with a propagator $(\not{p}+m)^{-1}=(i\gamma p+m)^{-1}$. The self-energy diagrams lead to a corrected propagator. Let the self-energy be represented by $\not{A}$, with

$$\not{A}=(2\pi)^4 i\{a_1+a_2\gamma^5+a_3 i\gamma p+a_4 i\gamma p\gamma^5\}\,.$$

The corrected propagator will be:

$$(\not{p}+m)^{-1}+(\not{p}+m)^{-1}\frac{\not{A}}{(2\pi)^4 i}(\not{p}+m)^{-1}+\cdots=\left(i\gamma p+m-\frac{\not{A}}{(2\pi)^4 i}\right)^{-1}$$

$$=\frac{1}{1-2a_3-2m^2a_{3p}+2ma_{1p}}\,\frac{-i\gamma p(1-a_3-a_4\gamma^5)+m-a_1+a_2\gamma^5}{p^2+(m+ma_3-a_1)^2},$$

where $a_p=-da/dp^2$. From this one obtains the following wave-function factors:

$$1+W_{\mathrm{V}}+W_{\mathrm{A}}\gamma^5,\qquad \text{incoming particle},$$

$$1+W_{\mathrm{V}}-W_{\mathrm{A}}^{*}\gamma^5,\qquad \text{outgoing particle},$$

$$1+W_{\mathrm{V}}+W_{\mathrm{A}}\gamma^5,\qquad \text{outgoing antiparticle},$$

$$1+W_{\mathrm{V}}-W_{\mathrm{A}}^{*}\gamma^5,\qquad \text{incoming antiparticle},$$

with $W_{\mathrm{V}}=\frac{1}{2}a_3+m^2a_{3p}-ma_{1p}$ and $W_{\mathrm{A}}=(1/2m)a_2+\frac{1}{2}a_4$. Our W_{A} differs by a minus sign from W_{A} as defined in ref. [7].

In the standard model the expressions found for the various W are (omitting the subscript θ for s, c and v, and e for the mass of the electron m_{e})

$$W_{\mathrm{e}}^{\mathrm{V}}=\frac{g^2}{(2\pi)^4 i}\,\frac{1}{2}\Bigg[2s^2\{B_1(k,m,\lambda)+\tfrac{1}{2}i\pi^2\}+\frac{v^2+1}{8c^2}\{B_1(k,m,M_0)+\tfrac{1}{2}i\pi^2\}$$

$$+\tfrac{1}{2}\{B_1(k,0,M)+\tfrac{1}{2}i\pi^2\}+4m^2s^2B_{1p}(k,m,\lambda)$$

$$+\frac{m^2(v^2+1)}{4c^2}B_{1p}(k,m,M_0)+m^2B_{1p}(k,0,M)$$

$$+8s^2m^2B_{0p}(k,m,\lambda)+\frac{v^2-1}{2c^2}+m^2B_{0p}(k,m,M_0)$$

$$+\frac{m^2}{4M^2}\{B_1(k,0,M)+B_1(k,m,m_H)+B_1(k,m,M_0)\}\Bigg].$$

For k^2 the value $-m^2$ must be used; λ is the photon mass. The last term arises from three diagrams involving Higgs particles; only the infinite parts of those are considered:

$$W_{\rm e}^{\rm A}=\frac{g^2}{(2\pi)^4 i}\frac{1}{2}\left[-\frac{v}{4c^2}\left\{B_1(k,m,M_0)+\tfrac{1}{2}i\pi^2\right\}+\tfrac{1}{2}\left\{B_1(k,0,M)+\tfrac{1}{2}i\pi^2\right\}\right.$$
$$\left.-\frac{m^2}{4M^2}B_1(k,0,M)\right].$$

These are, of course, also the expressions for the muon, using $m=m_\mu$. For the electron neutrino one has $W^{\rm A}=W^{\rm V}$, and:

$$W_\nu^{\rm V}=\frac{g^2}{(2\pi)^4 i}\frac{1}{2}\left[\frac{1}{4c^2}\left\{B_1(k,0,M_0)+\tfrac{1}{2}i\pi^2\right\}+\tfrac{1}{2}\left\{B_1(k,m,M)+\tfrac{1}{2}i\pi^2\right\}\right.$$
$$\left.+\frac{m^2}{4M^2}B_1(k,m,M)\right].$$

Again, the last term arises from one Higgs diagram. For the muon neutrino the same expression can be used, with $m=m_\mu$.

Appendix B

EXPRESSIONS FOR VECTOR-BOSON AND PHOTON SELF-ENERGIES

Notation: $s_\theta=\sin\theta_{\rm w}$, $c_\theta=\cos\theta_{\rm w}$, $e=gs_\theta$.

Masses:	W^+:	M,	up-quark:	$m_{\rm u}$
	W^0:	M_0,	down-quark:	$m_{\rm d}$
	Higgs:	m,	neutrino:	m_ν
	photon:	λ,	electron:	$m_{\rm e}$.

Only the contributions of one flagpole, containing up and down quarks, electron and neutrino have been included. The others give analogous contributions. See appendix C for a discussion of the two-point form factors.

Through self-energy insertions the photon propagator is modified:

$$\frac{1}{k^2}\rightarrow\frac{1}{k^2-S_\gamma},$$

$$S_\gamma=\frac{e^2}{(2\pi)^4 i}\left[-k^2\left\{12B_{21}(k^2,M,M)-7B_0(k^2,M,M)-\tfrac{2}{3}i\pi^2\right\}\right.$$

$$+k^2\{8B_{21}(k^2, m_e, m_e) - 4B_0(k^2, m_e, m_e)\}$$

$$+\tfrac{4}{3}k^2\{8B_{21}(k^2, m_u, m_u) - 4B_0(k^2, m_u, m_u)\}$$

$$+\tfrac{1}{3}k^2\{8B_{21}(k^2, m_d, m_d) - 4B_0(k^2, m_d, m_d)\}\Big] .$$

Similarly, for the W^0 propagator

$$(k^2 + M_0^2)^{-1} \to (k^2 + M_0^2 - S_0)^{-1} ,$$

$$S_0 = \frac{g^2}{(2\pi)^4 i}\left[k^2\left(4 - \frac{1}{c_\theta^2} - 12c_\theta^2\right)B_{21}(k^2, M, M) + \tfrac{2}{3}i\pi^2 k^2 c_\theta^2\right.$$

$$+\left\{k^2\left(7c_\theta^2 - 1 + \frac{1}{4c_\theta^2}\right) + M^2\left(\frac{2}{c_\theta^2} - 4\right)\right\}B_0(k^2, M, M)$$

$$-\frac{k^2}{c_\theta^2}B_{21}(k^2, M_0, m) + \frac{M_0^2 - m^2 - 2k^2}{2c_\theta^2}B_1(k^2, M_0, m)$$

$$+\frac{5M_0^2 - m^2 - k^2}{4c_\theta^2}B_0(k^2, M_0, m)$$

$$+\frac{k^2}{16c_\theta^2}\left\{(4s_\theta^2 - 1)^2 + 1\right\}\{8B_{21}(k^2, m_e, m_e) - 4B_0(k^2, m_e, m_e)\}$$

$$+\frac{k^2}{8c_\theta^2}\{8B_{21}(k^2, m_\nu, m_\nu) - 4B_0(k^2, m_\nu, m_\nu)\}$$

$$+\frac{3k^2}{16c_\theta^2}\left\{\left(1 - \tfrac{8}{3}s_\theta^2\right)^2 + 1\right\}\{8B_{21}(k^2, m_u, m_u) - 4B_0(k^2, m_u, m_u)\}$$

$$+\frac{3k^2}{16c_\theta^2}\left\{\left(\tfrac{4}{3}s_\theta^2 - 1\right)^2 + 1\right\}\{8B_{21}(k^2, m_d, m_d) - 4B_0(k^2, m_d, m_d)\}$$

$$-\frac{1}{2c_\theta^2}\{m_\nu^2 B_0(k^2, m_\nu, m_\nu) + m_e^2 B_0(k^2, m_e, m_e)$$

$$\left. + 3m_u^2 B_0(k^2, m_u, m_u) + 3m_d^2 B_0(k^2, m_d, m_d)\}\right] .$$

Similarly, for the W^+ propagator

$$(k^2+M^2)^{-1}\to(k^2+M^2-S_+)^{-1},$$

$$\begin{aligned}S_+=\frac{g^2}{(2\pi)^4 i}\Bigg[&(8c_\theta^2+1)\{-k^2B_{21}(k^2,M_0,M)+\tfrac{1}{4}k^2B_0(k^2,M_0,M)\\&+\tfrac{1}{2}(M_0^2-M^2-2k^2)(B_1(k^2,M_0,M)+\tfrac{1}{2}B_0(k^2,M_0,M))\}\\&+\left\{M^2\left(\frac{1}{c_\theta^2}-3\right)+4k^2c_\theta^2\right\}B_0(k^2,M_0,M)+\tfrac{2}{3}i\pi^2k^2\\&-8s_\theta^2k^2\{B_{21}(k^2,\lambda,M)+B_1(k^2,\lambda,M)-\tfrac{1}{4}B_0(k^2,\lambda,M)\}\\&-2s_\theta^2M^2\{2B_1(k^2,\lambda,M)+B_0(k^2,\lambda,M)\}\\&-k^2B_{21}(k^2,M,m)+\tfrac{1}{2}(M^2-m^2-2k^2)B_1(k^2,M,m)\\&+\tfrac{1}{4}(5M^2-m^2-k^2)B_0(k^2,M,m)\\&+2k^2\{B_{21}(k^2,m_\nu,m_e)+B_1(k^2,m_\nu,m_e)\}\\&+(m_e^2-m_\nu^2)B_1(k^2,m_\nu,m_e)-m_\nu^2B_0(k^2,m_\nu,m_e)\\&+6k^2\{B_{21}(k^2,m_u,m_d)+B_1(k^2,m_u,m_d)\}\\&+3(m_d^2-m_u^2)B_1(k^2,m_u,m_d)-3m_u^2B_0(k^2,m_u,m_d)\Bigg].\end{aligned}$$

Finally, the γ-W^0 transition gives rise to a propagator

$$k^{-2}S_{0\gamma}(k^2+M_0^2)^{-1},$$

$$\begin{aligned}S_{0\gamma}=\frac{g^2}{(2\pi)^4 i}\Bigg[&k^2s_\theta\left(\frac{2}{c_\theta}-12c_\theta\right)B_{21}(k^2,M,M)+\tfrac{2}{3}i\pi^2k^2s_\theta c_\theta\\&+s_\theta\left(7k^2c_\theta-\frac{k^2}{2c_\theta}-\frac{2M^2}{c_\theta}\right)B_0(k^2,M,M)\end{aligned}$$

$$- k^2 \frac{s_\theta}{c_\theta} (4 s_\theta^2 - 1) \{ 2 B_{21}(k^2, m_e, m_e) - B_0(k^2, m_e, m_e) \}$$

$$+ k^2 \frac{2 s_\theta}{c_\theta} (1 - \tfrac{8}{3} s_\theta^2) \{ 2 B_{21}(k^2, m_u, m_u) - B_0(k^2, m_u, m_u) \}$$

$$\left. - k^2 \frac{s_\theta}{c_\theta} (\tfrac{4}{3} s_\theta^2 - 1) \{ 2 B_{21}(k^2, m_d, m_d) - B_0(k^2, m_d, m_d) \} \right] .$$

Appendix C

THE TWO-POINT FUNCTION

Only the two-point function is needed for the calculation reported in this paper, and we will therefore describe these in some detail. The function $\chi(x)$ is defined by

$$\chi(x) = -k^2 x^2 + x(k^2 + m_2^2 - m_1^2) + m_1^2 .$$

In terms of this function we have

$$B_0(k, m_1, m_2) = \int \mathrm{d}_n q \frac{1}{(q^2 + m_1^2)((q+k)^2 + m_2^2)} = i\Delta - i\pi^2 \int_0^1 \ln \chi \, \mathrm{d}x .$$

The quantity Δ has been defined before, and contains the pole factor at $n = 4$. The integration limits will be left understood in the following.

The functions B_1, B_2, etc., are obtained when we consider integrals as above with factors q in the numerator:

$$k_\mu B_1 ;\ k_\mu k_\nu B_{21} + \delta_{\mu\nu} B_{22} = \int \mathrm{d}_n q \frac{q_\mu ;\ q_\mu q_\nu}{(q^2 + m_1^2)((q+k)^2 + m_2^2)} ,$$

and we have

$$B_1 = -\tfrac{1}{2} i\Delta + i\pi^2 \int x \ln \chi ,$$

$$B_{21} = \tfrac{1}{3} i\Delta - i\pi^2 \int x^2 \ln \chi ,$$

$$B_{22} = -\tfrac{1}{2} i(\Delta + \pi^2) \int \chi + \tfrac{1}{2} i\pi^2 \int \chi \ln \chi .$$

Furthermore the derivatives of the B's are:

$$-\frac{\mathrm{d}B_0}{\mathrm{d}k^2}=i\pi^2\int\frac{x(1-x)}{\chi}\,,\qquad -\frac{\mathrm{d}B_1}{\mathrm{d}k^2}=-i\pi^2\int\frac{x^2(1-x)}{\chi}\,,$$

$$-\frac{\mathrm{d}B_{21}}{\mathrm{d}k^2}=i\pi^2\int\frac{x^3(1-x)}{\chi}\,,\qquad -\frac{\mathrm{d}B_{22}}{\mathrm{d}k^2}=\tfrac{1}{12}i\Delta-\tfrac{1}{2}i\pi^2\int x(1-x)\ln\chi\,.$$

The integrals are not particularly difficult to work out, but the resulting expressions are often very difficult to use because of strong cancellations. For numerical evaluation other methods are needed, which we will describe here schematically.

The functions $G_n(y)$ and $F_n(y)$ are defined by

$$G_n(y)=\int_0^1 x^{n-1}\ln(x-y)\,\mathrm{d}x\,,\qquad F_n(y)=-\int_0^1\frac{x^n}{x-y}\,\mathrm{d}x\,.$$

The expressions for the B's can be rewritten in terms of the F and G. One easily establishes

$$F_n(y)=-y^n\ln\frac{y-1}{y}-y^{n-1}-\tfrac{1}{2}y^{n-2}-\tfrac{1}{3}y^{n-3}\ldots-\frac{1}{n}\,.$$

For large y the logarithm may be series developed:

$$F_n(y)=\frac{1}{n+1}\frac{1}{y}+\frac{1}{n+2}\frac{1}{y^2}+\frac{1}{n+3}\frac{1}{y^3}+\ldots\,,$$

showing that $F_n(y)$ behaves like y^{-1} for large y rather than like y^n or y^{n-1} as the other equation suggests.

The following relations are easily established:

$$F_n(y)=\frac{1}{y}\left\{\frac{1}{n+1}+F_{n+1}(y)\right\},\qquad F_{n+1}(y)=-\frac{1}{n+1}+yF_n(y)\,,$$

$$G_n(y)=\frac{1}{n}\{\ln(1-y)+F_n(y)\}\,,$$

$$G_{n+1}(y)=\frac{1}{n+1}\left\{nyG_n(y)+(1-y)\ln(1-y)-\frac{1}{n+1}\right\}.$$

The recursion formulae can be used either up or down, depending on the magnitude of y. For small y (for example $|y|<4$) one may start from

$$F_0=-\ln\frac{y-1}{y}\,,\qquad G_1(y)=(1-y)\ln(1-y)+y\ln(-y)-1\,.$$

For large y (say $|y| \geqslant 4$) it is better to compute the highest needed F_n and G_n, and to work downwards by recursion. For such cases a series expansion is needed, and a convenient parameter is $z = \ln(1 - y^{-1})$, i.e., $y^{-1} = 1 - \exp(z)$. Note that $F_0 = -z$. We will give the expansion coefficients for $F_4(y)$:

$$F_4(y) = c_n z^n , \qquad z = \ln \frac{1-y}{y} ,$$

$$c_1 = -\tfrac{1}{5} , \qquad c_2 = \tfrac{1}{15} , \qquad c_3 = -\tfrac{1}{105} , \qquad c_4 = -\tfrac{1}{2520} ,$$

$$c_5 = \tfrac{1}{3150} , \qquad c_6 = -\tfrac{1}{75600} , \qquad c_7 = -\tfrac{1}{103950} ,$$

$$c_8 = \tfrac{1}{950400} , \qquad c_9 = 2.6648836172645 \times 10^{-7} ,$$

$$c_{10} = -4.88745528428 \times 10^{-8} , \qquad c_{11} = -6.75397500794 \times 10^{-9} ,$$

$$c_{12} = 1.90720263471 \times 10^{-9} , \qquad c_{13} = 1.5366300769 \times 10^{-10} ,$$

$$c_{14} = -6.7969790579 \times 10^{-11} , \qquad c_{15} = -2.93683556 \times 10^{-12} ,$$

$$c_{16} = 2.28836696 \times 10^{-12} .$$

We have chosen to write simply the numbers rather than, for example, $c_9 = 11/(9^2 \times 8 \times 7^2 \times 1300)$. To get the expansion for higher n one may use the recursion equation to compute the coefficients. Another difficulty that appears in the calculation is the case of almost coincident roots in the expressions given for the derivative B's. One encounters expressions of the form

$$\int_0^1 \frac{x^m(1-x)}{(x-x_1)(x-x_2)} \, \mathrm{d}x .$$

Depending on the actual value of the roots, and the difference $x_1 - x_2$, one may use recursion and the appropriate expansions; for example, if $|x_1|, |x_2| \geqslant 4$,

$$\int_0^1 \frac{x^n}{(x-x_1)(x-x_2)} = -\frac{1}{\delta}\{F_n(x_1) - F_n(x_2)\}$$

$$= -\frac{1}{\delta} \sum c_j (z_1^j - z_2^j) , \qquad z_i = \ln\left(1 - \frac{1}{x_i}\right) ,$$

where $\delta = x_1 - x_2$. For small δ the difference $z_1^j - z_2^j$ must first be worked out to first order in δ.

For $k^2 = 0$ the expressions for the B-function become simple:

$$B_0(0, m_1, m_2) = i\Delta - i\pi^2\{\ln m_2^2 + F_1(y)\},$$

$$B_1(0, m_1, m_2) = -\tfrac{1}{2}i\Delta + \tfrac{1}{2}i\pi^2\{\ln m_2^2 + F_2(y)\},$$

$$B_{21}(0, m_1, m_2) = \tfrac{1}{3}i\Delta - \tfrac{1}{3}i\pi^2\{\ln m_2^2 + F_3(y)\},$$

$$B_{22}(0, m_1, m_2) = -\tfrac{1}{4}i\Delta(m_1^2 + m_2^2) - \tfrac{1}{2}i\pi^2 m_2^2(1 - \ln m_2^2) + \tfrac{1}{4}i\pi^2(m_1^2 - m_2^2)\{\ln m_2^2 + F_2(y)\},$$

with

$$y = \frac{m_1^2}{m_1^2 - m_2^2}, \qquad i\Delta = -\frac{2i\pi^2}{n-4} - i\pi^2(\gamma + \ln \pi),$$

γ = Eulers constant and, for instance,

$$F_3(y) = -y^3 \ln\frac{y-1}{y} - y^2 - \tfrac{1}{2}y - \tfrac{1}{3}.$$

References

[1] G. 't Hooft, Nucl. Phys. B35 (1971) 167

[2] S. Bludman, Nuovo Cim. 9 (1958) 433;
S. Glashow, Nucl. Phys. 22 (1961) 579;
S. Weinberg, discussion remark at the 1967 Solvay Conf., Brussels, talk of H.P. Dürr; Phys. Rev. Lett. 19 (1967) 1264;
S. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D2 (1970) 1285;
C. Bouchiat, J. Iliopoulos and Ph. Meyer, Phys. Lett. 42B (1972) 91

[3] I. Liede and M. Roos, University of Helsinki preprint (Dec, 1979)

[4] D. Ross and M. Veltman, Nucl. Phys. B95 (1975) 135

[5] M. Veltman, Nucl. Phys. B123 (1977) 89;
M. Chanowitz, M. Furman and I. Hinchliffe, Phys. Lett. 78B (1978) 285;
M. Veltman, Proc. 1979 Int. Symp. on Lepton and photon interactions, p. 529, ed. T. Kirk and H. Abarbanel.

[6] G. 't Hooft and M. Veltman, Nucl. Phys. B153 (1979) 365

[7] G. Passarino and M. Veltman, Nucl. Phys. B160 (1979) 151

[8] M. Consoli, Nucl. Phys. B160 (1979) 208

[9] M. Lemoine and M. Veltman, Nucl. Phys., to be published

[10] G. Kallen, Radiative corrections to weak processes, Springer tracts in Modern Physics, vol. 46 (1968)

[11] F. Berends, R. Gastmans and T.T. Wu, Leuven preprint (June, 1979)

[12] P. Salomonson and Y. Ueda, Phys. Rev. D11 (1975) 2606;
W. Marciano, Nucl. Phys. B84 (1975) 132
S. Sakakibera, Dortmund preprint (1980).