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MASSIVE AND MASS-LESS YANG-MILLS AND GRAVITATIONAL FIELDS

H. van DAM * and M. VELTMAN
Institute for Theoretical Physics. University of Utrecht. Utrecht. The Netherlands

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Abstract: Massive and mass-less Yang-Mills and gravitational fields are considered. It is found that there is a discrete difference between the zero-mass theories and the very small, but non-zero mass theories. In the oase of gravitation, comparison of massive and mass-less theories with experiment, in particular the perihelion movement of Mercury, leads to exclusion of the massive theory. It is concluded that the graviton mass must be rigorously zero.

1. INTRODUCTION

Both mass-less [1] and massive [2, 3] Yang-Mills fields [4] and also the zero-mass gravitational field [1] have been the subject of several publications. Feynman rules for mass-less fields have been established using the method of path integrals; Faddeev and Popov [5] have verified unitarity by going over to the Coulomb gauge, while in Mandelstam's formalism unitarity is built in.

For the massive fields, where at least the starting point is unambiguous, work has been done in order to reformulate the Feynman rules in such a way that the limit of zero-mass and also ultra-violet problems could be studied. Superficially seen it appeared that at least for no or one closed loop the limit of zero-mass existed and coincided with the results of the mass-less case. But recently, after a more complete analysis, it has been shown that no such result is valid for diagrams with more than one closed loop [3]. And, as has been observed by Faddeev and Slavnov [6], already the results for one closed loop are different: the closed loops of ghost particles have different coefficients for the cases of zero-mass and the limit of zero-mass respectively.

Actually the solution to the problem is quite simple. The zero-mass case is simply not the limiting case of the finite mass theory, and there is a discrete difference between the theory with zero-mass and a theory with finite mass, no matter how small as compared to all external momenta. The reason is that a finite mass spin 1 particle has three different states

* On leave of absence from the University of North Carolina, Chapel Hill, N.C.

of polarization while a zero-mass particle has only two such states (with helicities ±1). And the "third" state of polarization does not decouple in the limit of small mass. This is quite different from electrodynamics, where indeed the "third" state decouples, and where then the zero-mass theory is indistinguishable from a small-mass theory, at least as far as its Feynman rules are concerned.

Having understood the mechanism one may wonder what happens in the case of the gravitational field. A zero-mass theory has particles of spin 2 with two different states of polarization while finite-mass spin 2 particles have five polarization states. It turns out that also here the limit of zero-mass is discretely different from the zero-mass case, and in fact already at the level of no closed loops. This implies that the bending of light rays near the sun and the perihelion movement of Mercury is distinctly different for zero-mass gravitons as compared to infinitesimally small mass gravitons. Experiment tells us then what theory to take, and the result is of course the zero-mass theory. We may conclude that the graviton has rigorously zero-mass.

In sect. 2 the situation for Yang-Mills fields is clarified by analyzing the simplest diagrams with no or one closed loop. The requirement of unitarity is seen to imply different ghost loop coefficients for the small mass and the zero-mass case respectively. In sect. 3 we will consider massive and mass-less gravitons and establish the form of the graviton propagator for both cases. We then also consider the effect of graviton exchange between two material objects. The coupling constant being fixed by comparison with Newton's law, a different result comes out for the bending of light rays as well as for the perihelion movement of Mercury, the massive theory giving $\frac{3}{4}$ and $\frac{2}{3}$ respectively of the results of the mass-less theory.

Since experiment agrees for the perihelion movement to within 10% with the prediction of the mass-less theory we have the result that the graviton mass is exactly zero.

2. MASSIVE AND MASS-LESS YANG-MILLS FIELDS

A massive spin 1 particle is characterized by its four momentum p_μ and a polarization vector $e_\mu(p)$. There are three independent polarization vectors, and one finds summing over all polarizations

$$\sum_{i=1}^{3} e^i_\mu e^i_\nu = \delta_{\mu\nu} + p_\mu p_\nu / M^2 . \tag{1}$$

It is instructive to consider this formula in the rest system of the particle. There $p_\mu = (0, 0, 0, iM)$ and for the e_μ we may take the orthonormal set (1,0,0,0), (0,1,0,0) and (0,0,1,0). The left-hand side of eq. (1) is a two-tensor that may be given in matrix form:

$$\sum_{i=1}^{3} e^{i}_{\mu} e^{i}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{(massive case)} . \tag{2}$$

One verifies that the right-hand side of eq. (1) has the same form in this system.

For a zero-mass spin 1 particle of momentum p_μ there are two independent polarizations. In the coordinate system where p_μ is aligned along the third axis one has $p_\mu = (0, 0, p, ip)$, with $p = |\boldsymbol{p}|$, and one may take for the e_μ the vectors (1, 0, 0, 0) and (0, 1, 0, 0). We then have

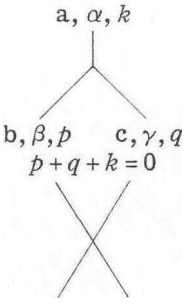
$$\sum_{i=1}^{2} e^{i}_{\mu} e^{i}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{(mass-less case)} . \tag{3}$$

Let now $\bar{p}_\mu$ be the vector obtained from p_μ by applying a space reflection, that is $\bar{p}_\mu = (0, 0, -p, ip)$. A tensor as given in eq. (3) may be constructed out of p_μ and $\bar{p}_\mu$, and we find:

$$\sum_{i=1}^{2} e^{i}_{\mu} e^{i}_{\nu} = \delta_{\mu\nu} - \frac{\bar{p}_\mu p_\nu + p_\mu \bar{p}_\nu}{(p\bar{p})}, \qquad \text{(mass-less case)} . \tag{4}$$

Note that p_μ or $\bar{p}_\mu$ applied to the right-hand side of eq. (4) gives zero, as should be, due to the fact that $(p, p) = (\bar{p}, \bar{p}) = 0$.

Both the massive and the mass-less case have a three and a four-point vertex:



$$-ig\epsilon_{abc} F_{\alpha\beta\gamma}(k, p, q) = -ig\epsilon_{abc}\{\delta_{\alpha\gamma}(k-q)_\beta + \delta_{\beta\gamma}(q-p)_\alpha + \delta_{\alpha\beta}(p-k)_\gamma\} . \tag{5}$$

(6)

We do not need the explicit form of the four-vertex. For the propagators we take

$$\frac{\delta_{\mu\nu}}{p^2 + M^2 - i\epsilon}, \qquad \text{(massive case)} , \tag{7}$$

$$\frac{\delta_{\mu\nu}}{p^2 - i\epsilon}, \qquad \text{(mass-less case)} \ . \tag{8}$$

For further use some formulae are needed. First we note:

$$p_\beta F_{\alpha\beta\gamma}(k,p,q) = \{\delta_{\alpha\gamma}(q^2 - k^2) + k_\alpha k_\gamma - q_\alpha q_\gamma\} \ . \tag{9}$$

Let now $e_\alpha(k)$ be a polarization vector with $e_\alpha k_\alpha = 0$. We find

$$e_\alpha(k)\, p_\beta F_{\alpha\beta\gamma}(k,p,q) = -\, e_\alpha(k)\, q_\alpha q_\gamma \ , \qquad k^2 = q^2 \ . \tag{10}$$

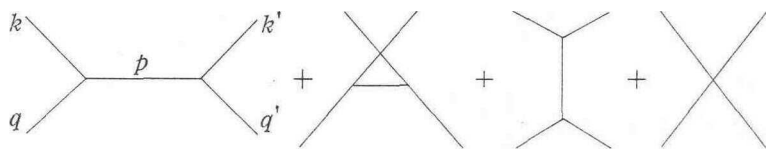
Similarly

$$e_\alpha(k)\, q_\gamma F_{\alpha\beta\gamma}(k,p,q) = e_\alpha(k)\, p_\alpha p_\beta \ , \qquad k^2 = p^2 \ . \tag{11}$$

If $f_\gamma(q)$ is a polarization vector with $f_\gamma q_\gamma = 0$ one has

$$f_\gamma(q)\, e_\alpha(k)\, p_\beta F_{\alpha\beta\gamma} = 0 \ , \qquad k^2 = q^2 \ . \tag{12}$$

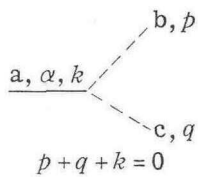
Consider now lowest order two-meson scattering:



(13)

where k and q are the initial, k' and q' the final momenta. To check unitarity we must compute the imaginary part of this set of diagrams. Only the first diagram has a non-zero imaginary part that may be obtained by replacing the propagator (7) or (8) by $i\pi\delta(k^2 + M^2)\,\delta_{\mu\nu}$ or $i\pi\delta(k^2)\,\delta_{\mu\nu}$. The requirement of unitarity is satisfied if we may replace $\delta_{\mu\nu}$ by the respective sums over polarizations (1) and (4). We now note that in diagrams (13) all in and outgoing particles are physical, i.e. on the mass-shell. Thus $k^2 = q^2 = k'^2 = q'^2$ and moreover in these diagrams we have also polarization vectors $e_\alpha(k)$, $f_\beta(q)$ etc. We see that due to eq. (12) no difference arises if we make the replacement $\delta_{\mu\nu} \to$ (1) or (4). Obviously at the no-closed loop level the difference between (1) and (4) does not imply any difference in the Feynman rules for zero-mass and finite, but small mass (i.e. $M^2 \ll -p^2$).

Let us now consider the simplest one closed loop case. With eqs. (5), (6), (7) and (8) we have only one diagram with non-vanishing imaginary part. To this diagram we add another one, involving the vertex

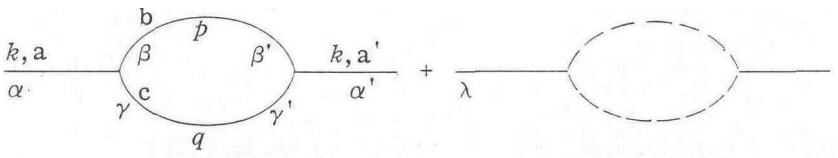


$$-\tfrac{1}{2} i g\, \epsilon_{abc} (p - q)_\alpha \ , \tag{14}$$

and propagator

$$\overset{p}{a \text{ --- } b} \qquad \begin{array}{ll} \delta_{ab}\dfrac{1}{p^2+M^2-i\epsilon}\,, & \text{(massive case)}\,, \\[2ex] \delta_{ab}\dfrac{1}{p^2-i\epsilon}\,, & \text{(mass-less case)}\,, \end{array} \tag{15}$$

but provided with a factor λ. Next we will compute the imaginary part of the two diagrams and then fix λ such that unitarity is satisfied. Thus consider:



(16)

The imaginary part of the first diagram is

$$-2\pi^2 g^2 e_\alpha(k) e'_{\alpha'}(-k)\epsilon_{abc}\epsilon_{a'bc} F_{\alpha\beta\gamma}(k,p,q) F_{\alpha'\beta'\gamma'}(-k,-p,-q) D_{\beta\beta'\gamma\gamma'}\,,$$

$$D_{\beta\beta'\gamma\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'}\delta(p^2+M^2)\delta(q^2+M^2)\,, \qquad \text{(massive case)}\,,$$

$$D_{\beta\beta'\gamma\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'}\delta(p^2)\delta(q^2)\,, \qquad \text{(mass-less case)}\,, \tag{17}$$

Unitarity requires the imaginary part of the second-order diagram to be equal to eq. (17) with now however

$$D_{\beta\beta'\gamma\gamma'} = (\delta_{\beta\beta'} + p_\beta p_{\beta'}/M^2)(\delta_{\gamma\gamma'} + q_\gamma q_{\gamma'}/M^2)\delta(p^2+M^2)\delta(q^2+M^2)\,,$$

(massive case),

$$D_{\beta\beta'\gamma\gamma'} = \left(\delta_{\beta\beta} - \frac{\bar{p}_\beta p_{\beta'} + p_\beta \bar{p}_{\beta'}}{(p\bar{p})}\right)\left(\delta_{\gamma\gamma'} - \frac{\bar{q}_\gamma q_{\gamma'} + q_\gamma \bar{q}_{\gamma'}}{(q\bar{q})}\right)\delta(p^2)\delta(q^2)\,,$$

(mass-less case). (18)

The difference between eqs. (17) and (18) must be provided by the imaginary part of the second diagram of eq. (16). Using eqs. (10) and (11), and noting that $q^2 = p^2 = k^2$, and moreover that $q^2 = p^2 = 0$ in the mass-less case we find after some trivial algebra for the difference between (18) and (17):

$$(18)-(17) = -2\pi^2 g^2 e_\alpha(k) e'_{\alpha'}(-k)\epsilon_{abc}\epsilon_{a'bc} p_\alpha p_{\alpha'} \delta(p^2+M^2)\delta(q^2+M^2) \ ,$$

(massive case) ,

$$= -2\pi^2 g^2 e_\alpha(k) e'_{\alpha'}(-k)\epsilon_{abc}\epsilon_{a'bc} 2 p_\alpha p_{\alpha'} \delta(p^2)\delta(q^2) \ ,$$

(mass-less case) . (19)

These results differ by a factor of two. The imaginary part of the second diagram in (16) is:

$$2\pi^2 g^2 e_\alpha(k) e'_{\alpha'}(-k)\epsilon_{abc}\epsilon_{a'bc} \lambda p_\alpha p_{\alpha'} \begin{cases} \delta(p^2+M^2)\delta(q^2+M^2) \ , \\ \delta(p^2)\delta(q^2) \ . \end{cases} \tag{20}$$

Thus we must choose $\lambda = -1$ for the massive and $\lambda = -2$ for the mass-less case, for then we have that (18) = (17) + (20). Actually the known derivations for the massive and mass-less case give this result.

The above makes clear that the mass-less theory cannot be obtained as a limiting case of the finite-mass theory. The origin of the difference goes back to the difference between the sums over polarizations (1) and (4).

3. MASSIVE AND MASS-LESS GRAVITATION

We now repeat the work of sect. 2 for the case of a spin 2 particle.

A spin 2 particle is characterized by its four-momentum p_μ and a symmetric polarization tensor $e_{\mu\nu}(p)$ such that $e_{\mu\mu} = 0$ and $p_\mu e_{\mu\nu} = 0$. There are five independent polarization tensors, and one finds summing over all polarizations:

$$\sum_{i=1}^{5} e^i_{\mu\nu}(p) e^i_{\alpha\beta}(p) = \tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})$$
$$+ \tfrac{1}{2}\left(\delta_{\mu\alpha}\frac{p_\nu p_\beta}{M^2} + \delta_{\nu\beta}\frac{p_\mu p_\alpha}{M^2} + \delta_{\mu\beta}\frac{p_\nu p_\alpha}{M^2} + \delta_{\nu\alpha}\frac{p_\mu p_\beta}{M^2}\right)$$
$$+ \tfrac{2}{3}\left(\tfrac{1}{2}\delta_{\mu\nu} - \frac{p_\mu p_\nu}{M^2}\right)\left(\tfrac{1}{2}\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{M^2}\right) . \tag{21}$$

In the p-rest frame one may choose for the $e_{\mu\nu}$ the orthonormal set:

$$\sqrt{\frac{2}{3}}\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{22}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

One verifies:

$$\sum_{i=1}^{5} e^{i}_{\mu\nu}(p)e^{i}_{\alpha\beta}(p) = \begin{cases}\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}) - \frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta} & \text{if} \quad \mu,\nu,\alpha,\beta\neq 4\\ 0 \quad \text{otherwise}\,,\end{cases} \tag{23}$$

which coincides with the right-hand side of (21) if $p_\mu=(0,0,0,iM)$. For the mass-less case, when p_μ is aligned along the third axis, $p_\mu=(0,0,p,ip)$ we have two independent polarizations. In this frame we may take

$$e^{1}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad e^{2}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

we note:

$$e^{1}_{\mu\nu}e^{1}_{\alpha\beta}+e^{2}_{\mu\nu}e^{2}_{\alpha\beta} = \begin{cases}\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}-\delta_{\mu\nu}\delta_{\alpha\beta})\,, & \text{if} \quad \mu,\nu,\alpha,\beta=1,2\\ 0 \quad \text{otherwise}\,.\end{cases} \tag{24}$$

Note that eq. (24) is different from eq. (23) even if $\mu,\nu,\alpha,\beta=1,2$. This is due to the first polarization tensor in eq. (22), which corresponds to the state of polarization with zero angular momentum along the third axis.

If, as before, $\bar{p}_\mu$ is the vector obtained from p_μ by space reflection one may write:

$$\sum_{i=1}^{2} e^{i}_{\mu\nu} e^{i}_{\alpha\beta} = \tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})$$
$$- \tfrac{1}{2}\left(\delta_{\mu\alpha}\frac{\bar{p}_\nu p_\beta + p_\nu \bar{p}_\beta}{(p\bar{p})} + \delta_{\nu\beta}\frac{\bar{p}_\mu p_\alpha + p_\mu \bar{p}_\alpha}{(p\bar{p})} + \delta_{\mu\beta}\frac{\bar{p}_\nu p_\alpha + p_\nu \bar{p}_\alpha}{(p\bar{p})} + \delta_{\nu\alpha}\frac{\bar{p}_\mu p_\beta + p_\mu \bar{p}_\beta}{(p\bar{p})}\right)$$
$$+ \tfrac{1}{2}\left(\delta_{\mu\nu}\frac{\bar{p}_\alpha p_\beta + p_\alpha \bar{p}_\beta}{(p\bar{p})} + \delta_{\alpha\beta}\frac{\bar{p}_\mu p_\nu + p_\mu \bar{p}_\nu}{(p\bar{p})}\right)$$
$$+ \tfrac{1}{2}\left(\frac{\bar{p}_\mu p_\alpha + p_\alpha \bar{p}_\mu}{(p\bar{p})}\,\frac{\bar{p}_\nu p_\beta + p_\nu \bar{p}_\beta}{(p\bar{p})} + \frac{\bar{p}_\mu p_\beta + p_\mu \bar{p}_\beta}{(p\bar{p})}\,\frac{\bar{p}_\nu p_\alpha + p_\nu \bar{p}_\alpha}{(p\bar{p})}\right.$$
$$\left.\frac{\bar{p}_\mu p_\nu + p_\mu \bar{p}_\nu}{(p\bar{p})}\,\frac{\bar{p}_\alpha p_\beta + p_\alpha \bar{p}_\beta}{(p\bar{p})}\right) . \tag{25}$$

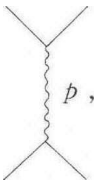
Eq. (25) is obtained from eq. (24) by replacing any tensor $\delta_{\lambda\kappa}$ by

$$\delta_{\lambda\kappa} - \frac{\bar{p}_\lambda p_\kappa + p_\kappa \bar{p}_\lambda}{(p\bar{p})} ,$$

which is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Let us now consider the interaction between two material objects caused by exchange of a gravitation. In lowest order one has one diagram:



p , (26)

where the wiggly line stands for tha graviton. The material objects enter through divergence free symmetric tensors $T_{\mu\nu}$ and $T'_{\alpha\beta}$ respectively, and if p is the four-momentem of the graviton exchanged one has

$$p_\mu T_{\mu\nu} = p_\nu T_{\mu\nu} = p_\alpha T'_{\alpha\beta} = p_\beta T'_{\alpha\beta} = 0 . \tag{27}$$

For the graviton propagator we take:

$$P^{m}_{\mu\nu\alpha\beta} = \frac{\frac{1}{2}\delta_{\mu\alpha}\delta_{\nu\beta} + \frac{1}{2}\delta_{\mu\beta}\delta_{\nu\alpha} - \frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2 + M^2 - i\epsilon}, \qquad \text{(massive case)},$$

$$P_{\mu\nu\alpha\beta} = \frac{\frac{1}{2}\delta_{\mu\alpha}\delta_{\nu\beta} + \frac{1}{2}\delta_{\mu\beta}\delta_{\nu\alpha} - \frac{1}{2}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2 - i\epsilon}, \qquad \text{(mass-less case)}. \tag{28}$$

Noting that eqs. (21) and (25) for $p = 0$ equal to the nominators of the propagators (28) we see that due to eq. (27) unitarity holds in diagram (26) for the choice (28). The diagram (26) leads to:

$$g^{m^2} T_{\mu\nu} P^{m}_{\mu\nu\alpha\beta} T'_{\alpha\beta}, \qquad \text{(massive case)},$$

$$g^2 T_{\mu\nu} P_{\mu\nu\alpha\beta} T'_{\alpha\beta}, \qquad \text{(mass-less case)}. \tag{29}$$

The quantities g^{m^2} and g^2 are fixed by the requirement that (29) contains Newton's law for non-relativistic systems. Then from $T_{\mu\nu}$ and $T'_{\alpha\beta}$ the 44 components are non-zero. For $\mu = \nu = \alpha = \beta = 4$ one finds:

$$\tfrac{2}{3} g^{m^2} T_{44} T'_{44} \frac{1}{p^2 + M^2 - i\epsilon}, \qquad \text{(massive case)},$$

$$\tfrac{1}{2} g^2 T_{44} T'_{44} \frac{1}{p^2 - i\epsilon}, \qquad \text{(mass-less case)}. \tag{30}$$

If g^2 is correctly chosen we see that we must take

$$g^{m^2} = \tfrac{3}{4} g^2 \tag{31}$$

to obtain in both cases the correct result.

Next we consider the case where $T'_{\alpha\beta}$ represents a fixed source (thus only T'_{44} non-zero), like the sun, while $T_{\mu\nu}$ is the energy-momentum tensor associated with a mass-less particle. In that case $T_{\mu\nu}$ has zero trace, $T_{\mu\mu} = 0$. Thus, the last term in the propagators (28) does not contribute. One finds:

$$g^{m^2} T_{44} T'_{44} \frac{1}{p^2 + M^2 - i\epsilon} = \tfrac{3}{4} g^2 T_{44} T'_{44} \frac{1}{p^2 + M^2 - i\epsilon}, \qquad \text{(massive case)},$$

$$g^2 T_{44} T'_{44} \frac{1}{p^2 - i\epsilon}, \qquad \text{(mass-less case)}, \tag{32}$$

Thus in the massive case (but with extremely small mass) the bending of a ray of light passing near the sun is $\frac{3}{4}$ of that predicted in the mass-less case. Experiment is however too vague to decide between the two cases*.

* *Note added in proof.* Recently more precise experiments have been performed [9], agreeing closely with Einstein's theory, thereby excluding the massive theory.

Similarly one may compute the perihelion precession of Mercury [7] in both the massive and mass-less case. For very small mass one obtains $\frac{2}{3}$ of the value for zero mass. This means that the massive case is excluded by experiment, since the experimental value coincides to within 10% with the prediction of the mass-less theory, i.e. the prediction of Einstein's theory.

4. CONCLUSIONS

It appears to us that Feynman rules for massive and mass-less Yang-Mills fields are now well established and understood. The fact that small mass and zero-mass theory are discretely different has been the main source of confusion. In the Yang-Mills theory this difference was rather subtle, at least up to the one closed loop level, but for the gravitational field the difference is hard to miss.

Actually, a theory of massive gravitation may be set up similarly to the massive Yang-Mills theory; in the appendix some details on this rather academic theory are given.

For completeness it must be remarked that both the massive Yang-Mills and massive gravitational theories are singular as the mass goes to zero. As shown in ref. [3], for the Yang-Mills theory these singularities occur for two or more closed loops, and it appears that they take the form of a series in $g^2\Lambda^2/M^2$, where Λ is an ultra-violet cut-off, and g *is* the coupling constant. One might speculate that after summing this series the limit $M \to 0$ exists; however, since this series has an overall factor g^2 (for self-energy diagrams) we cannot expect that the conclusions of this paper are affected. This is because the arguments in the foregoing rely on the imaginary parts of second-order diagrams that are fixed by unitarity alone.

The second author is indebted to Prof. Zumino for a stimulating discussion on this subject.

APPENDIX

The spin-2 field

The Lagrangian for the mass-less spin-2 field has the form:

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\lambda(h_{\lambda\mu}+h_{\mu\lambda})\partial_\mu h_{\nu\nu} - \tfrac{1}{4}\partial_\lambda(h_{\lambda\mu}+h_{\mu\lambda})\partial_\nu(h_{\nu\mu}+h_{\mu\nu}) + \tfrac{1}{8}\partial_\lambda(h_{\mu\nu}+h_{\nu\mu})\partial_\lambda(h_{\mu\nu}+h_{\nu\mu}) - \tfrac{1}{2}\partial_\lambda h_{\mu\mu}\partial_\lambda h_{\nu\nu} . \tag{A.1}$$

This Lagrangian is invariant under the gauge transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \eta_\mu , \tag{A.2}$$

where the $\xi_\nu(x)$ and $\eta_\mu(x)$ are eight arbitrary functions.

To eq. (A.1) we may add mass terms. There are three possibilities, and we write

$$\mathcal{L}_0' = \mathcal{L}_0(h_{\mu\nu}) - a_1 h_{\mu\mu} h_{\nu\nu} - a_2 h_{\mu\nu} h_{\mu\nu} - a_3 h_{\mu\nu} h_{\nu\mu} , \tag{A.3}$$

where a_1 , a_2 and a_3 will be proportional to some mass squared. We must find the form of the propagator of the $h_{\mu\nu}$ field for various choices of the a_i. To this purpose we study

$$\langle 0 | h_{\mu\nu}(x) h_{\alpha\beta}(0) | 0 \rangle = \frac{1}{(2\pi)^3} \int d_4 k \, e^{ikx} F_{\mu\nu\alpha\beta}(k) . \tag{A.4}$$

As much as possible we will try to determine the unknown function $F_{\mu\nu\alpha\beta}$ with the help of the equations of motion that follow from the Lagrangian (A.3). These equations are

$$\partial_\beta \partial_\gamma h_{\nu\nu} + \tfrac{1}{2}\delta_{\beta\gamma}\partial_\nu\partial_\alpha(h_{\nu\alpha} + h_{\alpha\nu}) - \tfrac{1}{2}\partial_\beta\partial_\nu(h_{\nu\gamma} + h_{\gamma\nu}) - \tfrac{1}{2}\partial_\gamma\partial_\nu(h_{\nu\beta} + h_{\beta\nu})$$

$$+ \tfrac{1}{2}\partial^2(h_{\beta\gamma} + h_{\gamma\beta} - 2\delta_{\beta\gamma} h_{\nu\nu}) + 2a_1 \delta_{\beta\gamma} h_{\nu\nu} + 2a_2 h_{\beta\gamma} + 2a_3 h_{\gamma\beta} = 0 . \tag{A.5}$$

Subtracting from eq. (A.5) the equation obtained by interchanging β and γ we get

$$(a_2 - a_3)(h_{\beta\gamma} - h_{\gamma\beta}) = 0 . \tag{A.6}$$

We will not study the case $a_2 = a_3$, and eq. (A.6) therefore has as consequence that $h_{\beta\gamma}$ must be symmetrical:

$$h_{\beta\gamma} = h_{\gamma\beta} . \tag{A.7}$$

Next we apply ∂_ν, $\partial_\gamma\partial_\beta$ and $\delta_{\gamma\beta}$ to eq. (A.5) with the result

(i) $x\,\partial_\gamma$

$$2a_1 \partial_\beta h_{\nu\nu} + 2a_2 \partial_\gamma h_{\beta\gamma} + 2a_3 \partial_\gamma h_{\gamma\beta} = 0 , \tag{A.8}$$

or with eq. (A.7)

$$a_1 \partial_\beta h_{\nu\nu} = -(a_2 + a_3)\partial_\gamma h_{\beta\gamma} ,$$

(ii) $x\,\partial_\gamma\partial_\beta$

$$2a_1 \partial^2 h_{\nu\nu} + 2(a_2 + a_3)\partial_\gamma\partial_\beta h_{\beta\gamma} = 0 , \tag{A.9}$$

(iii) $x\,\delta_{\gamma\beta}$

$$-2\partial^2 h_{\nu\nu} + 2\partial_\nu\partial_\alpha h_{\nu\alpha} + (8a_1 + 2a_2 + 2a_3) h_{\nu\nu} = 0 . \tag{A.10}$$

With the abbrevation

$$b = \frac{a_1}{a_2 + a_3} , \tag{A.11}$$

we have

$$\partial_\gamma h_{\beta\gamma} = \partial_\gamma h_{\gamma\beta} = -b\,\partial_\beta h_{\nu\nu} , \tag{A.12}$$

$$\partial_\gamma \partial_\beta h_{\beta\gamma} = -b\,\partial^2 h_{\nu\nu} , \tag{A.13}$$

$$\partial^2 h_{\nu\nu} - m_0^2 h_{\nu\nu} = 0 , \qquad m_0^2 = a_1 \frac{4+1/b}{1+b} . \tag{A.14}$$

With the help of eqs. (A.11), (A.12), (A.13) and (A.14) the equation of motion (A.5) reduces to

$$(1+2b)\partial_\beta \partial_\gamma h_{\nu\nu} - a_1\left(2+\frac{1}{b}\right)\delta_{\beta\gamma} h_{\nu\nu} + \partial^2 h_{\beta\gamma} + \frac{2a_1}{b} h_{\beta\gamma} = 0 . \tag{A.15}$$

Writing

$$h_{\mu\nu} = H_{\mu\nu} - \frac{(1+4b)}{3m_0^2}\,\partial_\mu \partial_\nu h_{\sigma\sigma} + \tfrac{1}{3}(1+b)\,\delta_{\mu\nu} h_{\sigma\sigma} , \tag{A.16}$$

we have, using eq. (A.14)

$$H_{\mu\mu} = 0 , \tag{A.17}$$

while eq. (A.12) implies

$$\partial_\gamma H_{\beta\gamma} = 0 . \tag{A.18}$$

Note that the divergence of h is non-zero, which implies according to Fierz [8] a negative energy particle. The equation of motion (A.15) reduces to

$$\partial^2 H_{\beta\gamma} + \frac{2a_1}{b} H_{\beta\gamma} = 0 . \tag{A.19}$$

Thus the Lagrangian (A.3) describes a situation with a (spin 0) particle with mass squared

$$m_0^2 = a_1 \frac{4+1/b}{1+b} \tag{A.20}$$

and a (spin 2) particle with mass squared

$$m_2^2 = -\frac{2a_1}{b} . \tag{A.21}$$

This fixes the function $F_{\mu\nu\alpha\beta}$ up to two arbitrary factors that determine the spin 0 resp. spin 2 content. Furthermore, according to Fierz [8] the spin 0 particle has negative energy, while the energy of the spin 2 particle is positive. We find

$$F_{\mu\nu\alpha\beta} = c_2 \Big\{ \tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})$$

$$+ \tfrac{1}{2}\left(\delta_{\mu\alpha}\frac{p_\nu p_\beta}{m_2^2} + \delta_{\nu\beta}\frac{p_\mu p_\alpha}{m_2^2} + \delta_{\mu\beta}\frac{p_\nu p_\alpha}{m_2^2} + \delta_{\nu\alpha}\frac{p_\mu p_\beta}{m_2^2}\right)$$

$$+ \tfrac{2}{3}\left(\tfrac{1}{2}\delta_{\mu\nu} - \frac{p_\mu p_\nu}{m_2^2}\right)\left(\tfrac{1}{2}\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{m_2^2}\right)\Big\}\ \delta(p^2 + m_2^2)\Theta(p_0)$$

$$+ c_0\, \tfrac{2}{3}(1+b)^2\left(\delta_{\mu\nu} + \frac{b}{a_1}p_\mu p_\nu\right)\left(\delta_{\alpha\beta} + \frac{b}{a_1}p_\alpha p_\beta\right)\delta(p^2 + m_0^2)\Theta(-p_0)\ . \qquad \text{(A.22)}$$

Let now $z_{\mu\nu}(x)$ be some arbitrary two tensor. Positive definiteness of the norm of the physical state implies that

$$\int d_4x\, d_4x'\, z_{\mu\nu}(x)\, \langle 0 | h_{\mu\nu}(x) h_{\alpha\beta}(x') | 0 \rangle \bar{z}_{\alpha\beta}(x') \geqslant 0\ , \qquad \text{(A.23)}$$

where $\bar{z}$ is related to the complex conjugate z^* by

$$\bar{z}_{\alpha\beta} = z^*_{\alpha\beta}(-1)^{\delta_{\alpha 4}}(-1)^{\delta_{\beta 4}}\ .$$

We now also insist that $m_0^2 > 0$ and $m_2^2 > 0$. Then by choosing

$$z_{\mu\nu} = \delta_{\mu\nu} f(x)\ ,$$

with some appropriate scalar function *f(x)* we get

$$c_0 \geqslant 0\ . \qquad \text{(A.24)}$$

The **choice**

$$z_{\mu\nu}(x) = \int d_4q\, e^{iqx} f(q)\,(\delta_{\mu\nu} + c\, q_\mu q_\nu)\ ,$$

$$c = -\frac{1}{b\, m_0^2}$$

leads to $c_2 \geqslant 0$.

Actually the Lagrangian (A.3) implies $c_2 = c_0 = 1$.

To see that one may eliminate $h_{\mu\nu}$ from the Lagrangian with the help of eq. (A.16); after some cumbersome algebra one finds

$$\mathcal{L}_0(h_{\mu\nu}) - a_1 h h - a_2 h_{\mu\nu} h_{\mu\nu}$$

$$= \mathcal{L}_0(H_{\mu\nu}) - a_2(H_{\mu\nu}H_{\mu\nu} - H_{\mu\mu}H_{\nu\nu}) - \tfrac{1}{3}(1+b)^2(\partial_\mu h\, \partial_\mu h + m_0^2 h h)\ , \qquad \text{(A.25)}$$

where $h \equiv h_{\sigma\sigma}$. Comparison with the Lagrangian of a scalar field gives the required coefficient c_0.

It is possible to have positive energy for the spin-zero particle provided its metric is made negative. Thus one may have $c_0 = -1$ and $\Theta(p_0)$ instead of $\Theta(-p_0)$ in eq. (A.22). Of course negative energy or indefinite metric are not physically acceptable, so the acceptable case is $b = -1$, i.e. no spin 0 part. Indeed, with $b = -1$, $a_1 = \frac{1}{2}m_2^2$ and $a_3 = 0$ one has $a_2 = -\frac{1}{2}m_2^2$ which is precisely the Lagrangian given by Fierz and Pauli [8].

Gravitational interaction is obtained by introducing terms involving the tensor $h_{\mu\nu}$ in such a way that, apart from the mass term, the Lagrangian is invariant under the infinitesimal transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2g\{\bar{h}_{\mu\nu,\sigma}\xi_\sigma + \bar{h}_{\sigma\mu}\partial_\nu \xi_\sigma + \bar{h}_{\sigma\nu}\partial_\mu \xi_\sigma\} ,$$

$$\bar{h}_{\mu\nu} = \tfrac{1}{2}(h_{\mu\nu} + h_{\nu\mu}) . \tag{A.26}$$

A mass term must be invented such that the transformation (A.26), to first order in ξ, results only in the addition of terms of the form

$$h_{\mu\nu}\partial_\mu \xi_\nu \qquad \text{and} \qquad h_{\mu\mu}\partial_\nu \xi_\nu . \tag{A.27}$$

This is necessary in order to be able to apply the techniques of ref. [3], that is in order to be able to obtain Ward-identities. The solution is

$$\mathcal{L} = \mathcal{L}_{\text{grav}}(h_{\mu\nu}) - \frac{M^2}{2g^2}\sqrt{\det(\delta_{\mu\nu} + 2gh_{\mu\nu})} + \frac{M^2}{2g^2} + \frac{M^2}{2g} h_{\mu\mu} , \tag{A.28}$$

where $\mathcal{L}_{\text{grav}}$ is the well-known Lagrangian for the mass-less gravitational field

$$\mathcal{L}_{\text{grav}}(h_{\mu\nu}) = \frac{1}{4g^2}\sqrt{\det(\delta_{\mu\nu} + 2gh_{\mu\nu})}\, R . \tag{A.29}$$

However, this theory is not acceptable. If we work out the square root up to second order in the $h_{\mu\nu}$ we find

$$-\tfrac{1}{2}M^2(\tfrac{1}{2}h_{\mu\mu}h_{\nu\nu} - h_{\mu\nu}h_{\nu\mu}) , \tag{A.30}$$

which corresponds to $m_0^2 = m_2^2 = M^2$, and $b = -\frac{1}{2}$, i.e. a non-zero amount of spin 0 admixture. One verifies that with the gravitation propagator of this theory, in the limit of small mass for no closed loop, precisely the results of the mass-less theory are obtained. However, the theory would *be* physically unacceptable, containing either negative energy or indefinite metric.

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