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GENERALIZED WARD IDENTITIES AND YANG-MILLS FIELDS

M. VELTMAN *
CERN, Geneva
and
Instituut voor Theoretische Fysica, Utrecht

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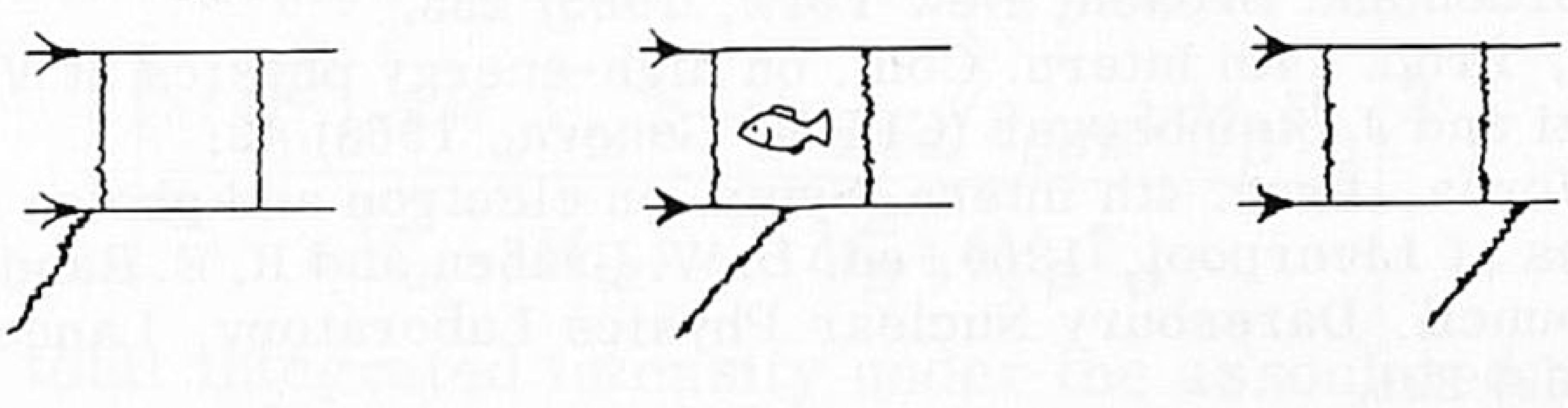
Abstract: Ward identities for non-Abelian gauges are derived within the framework of field theory. Subsequently these identities are used to analyse diagrams of the massive Yang-Mills theory to arbitrary order. The detailed results for two-closed loops are given.

1. INTRODUCTION

In establishing Feynman rules, renormalizability, low-energy theorems, etc., gauge invariance plays a very important role in quantum electrodynamics. In particular it implies that one need not worry about $k_\mu k_\nu$ terms in the photon propagator since it may be shown, with the help of gauge invariance, that such terms do not contribute. Usually, this gauge invariance is implemented by means of the Ward identity; another way, essentially the same, is to consider what happens if the polarization vector of an external photon is replaced by its four-momentum. Provided that one considers all diagrams where this particular photon is connected in all possible ways to a charge carrying line, the result is zero. An example is the following set of diagrams:



In this way one understands the connection between gauge invariance and charge conservation in terms of diagrams. Because the photon itself carries no charge one may connect further photons without spoiling this property, and it is interesting to note that this remains unchanged if the photon is given a non-zero mass.

In case of a non-Abelian gauge an essential difference shows up. For in-

* Permanent address: Institute for Theoretical Physics, Utrecht.

stance, consider gravitation. The graviton is coupled to energy momentum and, therefore, is also coupled to itself. Needless to say that the situation becomes much more complicated, in particular if one wishes to consider the implications of gauge invariance with respect to two or more external lines simultaneously. Thus, if one considers the appropriate group of diagrams having, one, two or more external photon lines, and if one replaces simultaneously the polarization vectors of one, two, etc., photons by their four-momenta, one finds zero. But, in case of a non-Abelian gauge this may be true for one (graviton), however, it is not clear what happens in the general case.

The simplest case of a non-Abelian gauge field is the Yang-Mills [1] field, where the vector particle is coupled to isospin rather than to the z-component of isospin, like the photon. In that model the vector-mesons carry also isospin, and are thus coupled to themselves. This case may be realized in nature, in strong interactions, where the triplet of ρ-mesons could be a Yang-Mills type field with non-zero mass. And the weak interactions may be some broken form of a Yang-Mills theory, in particular because the weak currents are thought to be of the required gauge type.

In this paper we will limit ourselves to massive Yang-Mills fields. In sect. 2 we introduce the "free field" technique in order to be able to establish generalized Ward identities in sect. 3. In sect. 4 we apply these identities to the diagrams of the Yang-Mills theory, with the result that one can omit, or partially omit the $k_\mu k_\nu$ terms in the propagator at the cost of having to add new diagrams constructed according to some well-defined rules. The technique is perfectly general, and can be used for diagrams with an arbitrary number of closed loops. Explicit and detailed constructions are given for the simplest diagrams having zero, one or two closed loops. At the end of sect. 4 the construction rule for two closed loops is precisely formulated. The rule for one closed loop coincides with that given before by many authors for the massless [2] and massive [3] case *.

With respect to renormalization the situation is not yet clear. For one closed loop the theory behaves as a renormalizable theory. Simple power counting reveals a slightly worse situation for two closed loops: irreducible diagrams with seven or more external lines are convergent (for one closed loop the number is five). Calculation of the coefficient of the leading divergence in the lowest order where it occurs does not give obviously zero.

Superficially seen the limit of zero mass does not seem to exist. But this is a highly intricate question which is not settled in this paper.

2. LAGRANGIAN AND FEYNMAN RULES

Consider a massive Yang-Mills vector-boson field coupled to some cur-

* Boulware [3] has studied the massive Yang-Mills theory with the method of path integrals, and without limitation to one closed loop. He established occurrence of vertices of a non-renormalizable type in the multiple loop case. To the extent that he gives results they seem to agree with the results found here. As to the third paper of ref. [3] it must be said that the results quoted are in contradiction with the explicit calculations of ref. [4], as well as with the work of Boulware.

rent, or field, or combination of fields. The Lagrangian is:

$$\mathcal{L} = -\tfrac{1}{4} G'^{a}_{\mu\nu} G'^{a}_{\mu\nu} - \tfrac{1}{2} M^2 W'^{a}_{\mu} W'^{a}_{\mu} + W'^{a}_{\mu} F^{a}_{\mu} ,$$

$$G'^{a}_{\mu\nu} = \partial_{\mu} W'^{a}_{\nu} - \partial_{\nu} W'^{a}_{\mu} + g \epsilon_{ade} W'^{d}_{\mu} W'^{e}_{\nu} . \qquad (1)$$

The (isospin) indices a, d and e take the values 1, 2, 3. The $W'^{a}_{\mu}(x)$ form a triplet of vector mesons. The $F^{a}_{\mu}(x)$ stand for whatever the W'^{a}_{μ} is coupled to.

To the Lagrangian (1) we add a scalar field $\varphi(x)$ with mass m in three distinct ways. That is, we add:

(i) a free field Lagrangian for the field φ;

(ii) an interaction $\lambda M W_{\mu}(x) \partial_{\mu} \varphi(x)$, with λ a dimensionless parameter;

(iii) further interaction terms containing φ, W_{μ} and F_{μ} such that φ obeys up to first order in λ the free field equations of motion.

Both (ii) and (iii) can be achieved in closed form by the replacement

$$W'^{a}_{\mu}(x) = W^{a}_{\mu}(x) - \frac{\lambda g}{M} \epsilon_{abc} W^{b}_{\mu}(x) \varphi^{c}(x) - \frac{\lambda}{M} \partial_{\mu} \varphi^{a} . \qquad (2)$$

The first term of $\mathcal{L}$ is invariant under this transformation. In fact, eq. (2) is an infinitesimal Yang-Mills gauge transformation. However the second and third terms are not invariant, and we obtain to order λ:

$$\mathcal{L} = -\tfrac{1}{2} (\partial_{\mu} \varphi^{a})^2 - \tfrac{1}{2} m^2 (\varphi^{a})^2 - \tfrac{1}{4} G^{a}_{\mu\nu} G^{a}_{\mu\nu} - \tfrac{1}{2} M^2 W^{a}_{\mu} W^{a}_{\mu} + W^{a}_{\mu} F^{a}_{\mu}$$

$$+ \lambda M W^{a}_{\mu} \partial_{\mu} \varphi^{a} + \frac{\lambda g}{M} \epsilon_{abc} W^{a}_{\mu} F^{b}_{\mu} \varphi^{c} - \frac{\lambda}{M} \partial_{\mu} \varphi^{a} F^{a}_{\mu} . \qquad (3)$$

By construction, and as may be verified by actual calculation, the φ-field satisfies up to first order in λ the free equation of motion:

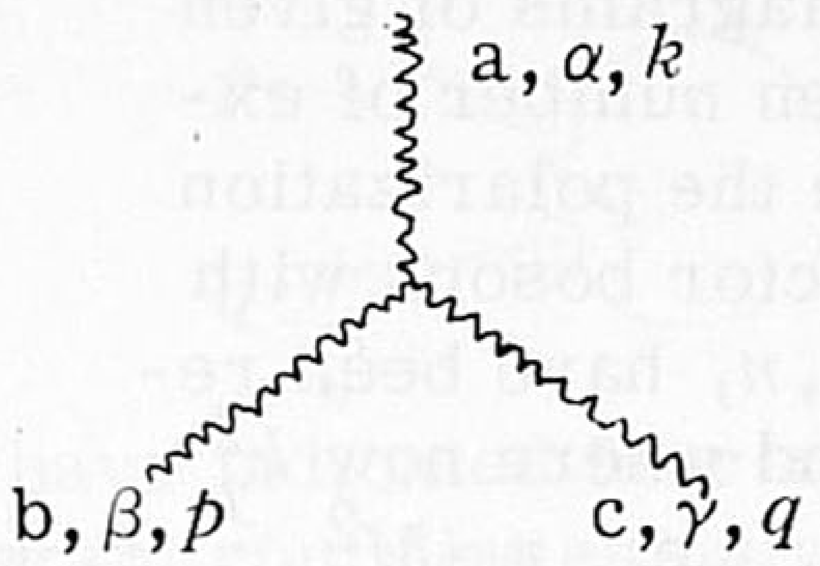
$$(\Box - m^2) \varphi^{a} = O(\lambda^2) . \qquad (4)$$

Thus, the current to which the φ-field is coupled is zero. Therefore, according to the well-known reduction formulae, we have the following result: to first order in λ, but for any given order in g and F_{μ} the set of S-matrix diagrams with a given non-zero number of external φ-lines on or off the mass shell adds up to zero. Since we are dealing with the S-matrix the external W lines must correspond to physical W, i.e., on the mass shell with polarization vectors $e_{\mu}(k)$ that are orthogonal to the four-momentum k_{μ}.

Thus, eq. (4) provides us with a large number of relations between diagrams, which we will investigate, working to arbitrary order in g.

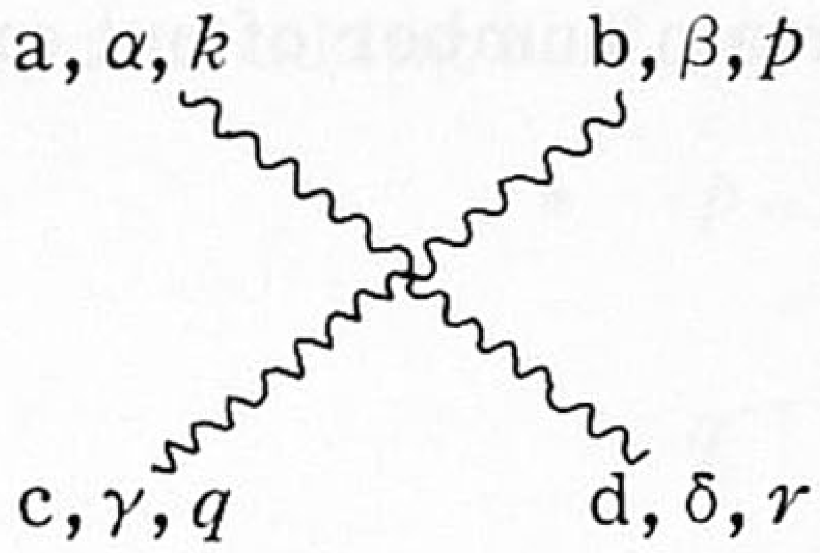
The Feynman rules corresponding to eq. (3) are:

$$\delta_{ab} \left(\frac{\delta_{\mu\nu} + k_{\mu} k_{\nu}/M^2}{k^2 + M^2 - i\epsilon} \right) , \qquad (5)$$



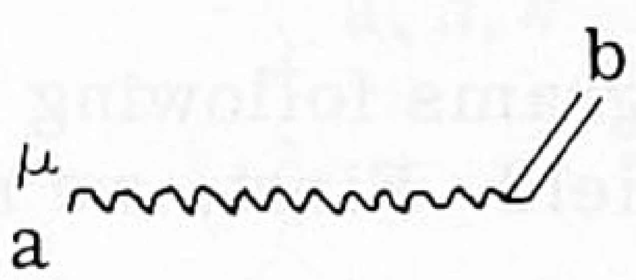
$$-ig\,\epsilon_{abc}\{\delta_{\alpha\gamma}(k-q)_\beta + \delta_{\beta\gamma}(q-p)_\alpha + \delta_{\alpha\beta}(p-k)_\gamma\}, \qquad (6)$$

$k+p+q=0$

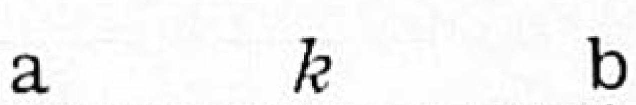


$$-g^2\{\epsilon_{gdc}\,\epsilon_{gba}(2\delta_{\alpha\gamma}\,\delta_{\beta\delta} - \delta_{\alpha\delta}\,\delta_{\beta\gamma} - \delta_{\alpha\beta}\,\delta_{\gamma\delta}) + \epsilon_{gdb}\,\epsilon_{gca}(2\delta_{\alpha\beta}\,\delta_{\gamma\delta} - \delta_{\alpha\delta}\,\delta_{\gamma\beta} - \delta_{\alpha\gamma}\,\delta_{\beta\delta})\}, \qquad (7)$$

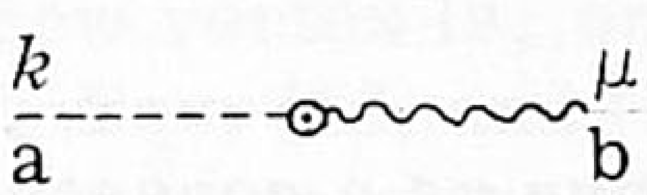
$k+p+q+r=0$



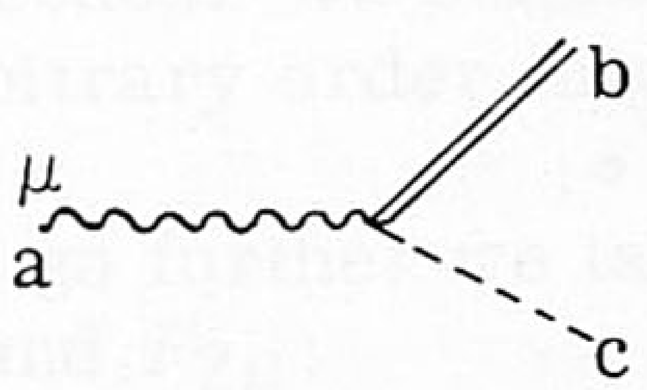
$$\delta_{ab}\,F^b_\mu, \qquad (8)$$



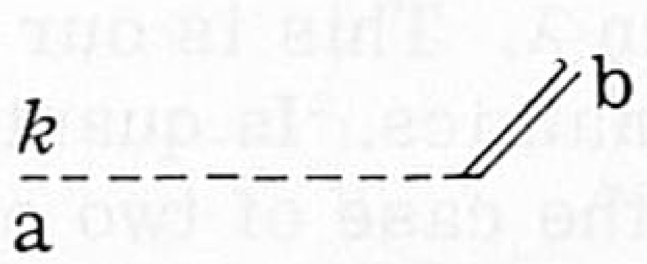
$$-\delta_{ab}\,\frac{1}{k^2+m^2-i\epsilon}, \qquad (9)$$



$$i\lambda M\,\delta_{ab}\,k_\mu, \qquad (10)$$



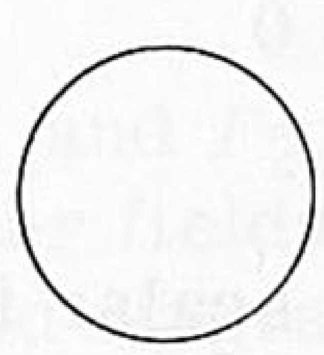
$$\frac{\lambda g}{M}\,\epsilon_{abc}\,F^b_\mu, \qquad (11)$$



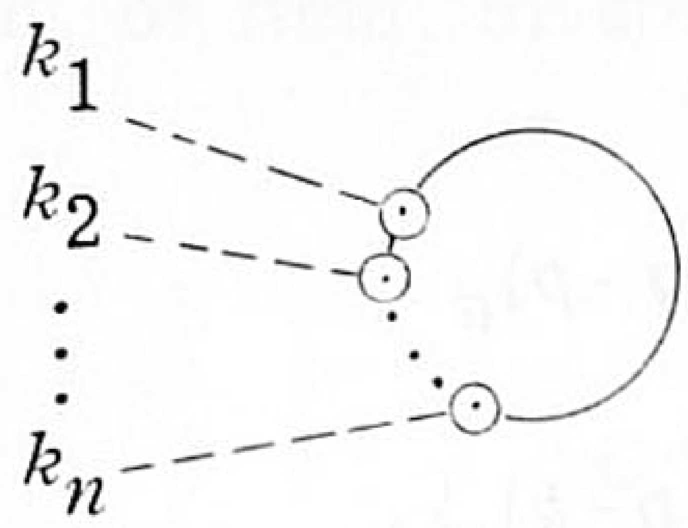
$$-\frac{i\lambda}{M}\,\delta_{ab}\,k_\mu\,F^b_\mu. \qquad (12)$$

For convenience, in future use, we included a minus sign in the propagator (9), in order to have a plus sign in (28).

We now introduce some conventions.

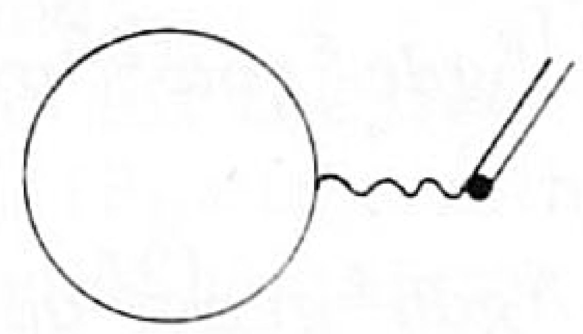


stands for the set of diagrams of order 0 in λ and F_μ, and of given, but arbitrary, order in g, with a given, but arbitrary number of external W-lines. (13)



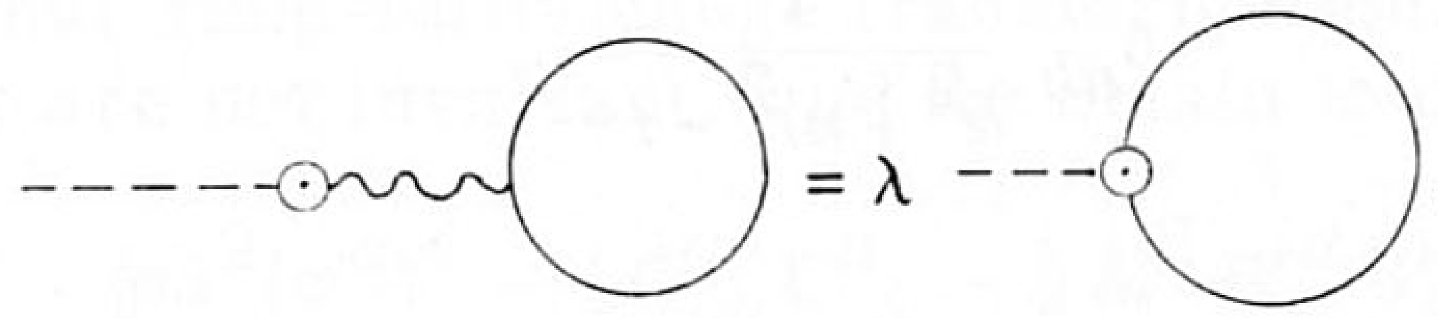
stands for the set of diagrams of given order in g, with a given number of external W-lines, where the polarization vectors $e_\mu(k_j)$ of n vector bosons with momenta k_j, $j = 1, \ldots, n$, have been replaced by $(ik_j)_\mu / M$, and where now k_j^2 may be off mass shell, i.e., $\neq -M^2$. (14)

Eventual vertices containing λ and/or F_μ will be indicated explicitly. Thus, the set of diagrams of certain order in g, with a certain number of out or ingoing W-lines, with one vertex (8) is indicated by



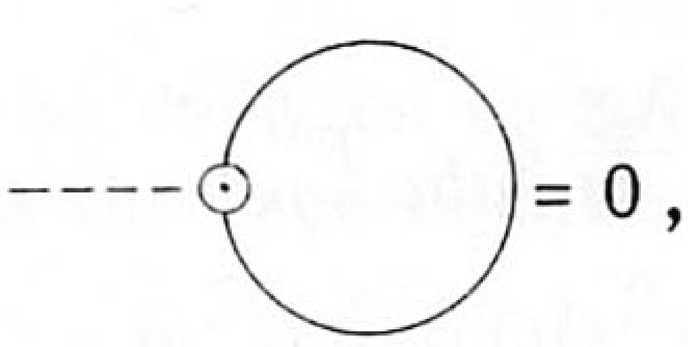
3. GENERALIZED WARD IDENTITIES

In this section we will derive relations between diagrams following from the fact that to first order in λ the φ-field is a free field. First, we note by using rules (5) and (10):



(15)

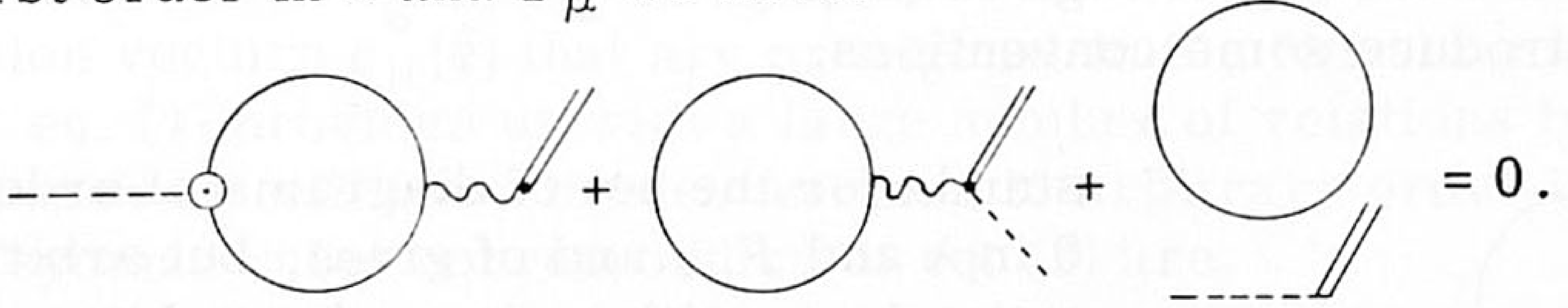
Next, to first order in λ and to zero order in F_μ we have as a consequence of (4) and the equality (15)



(16)

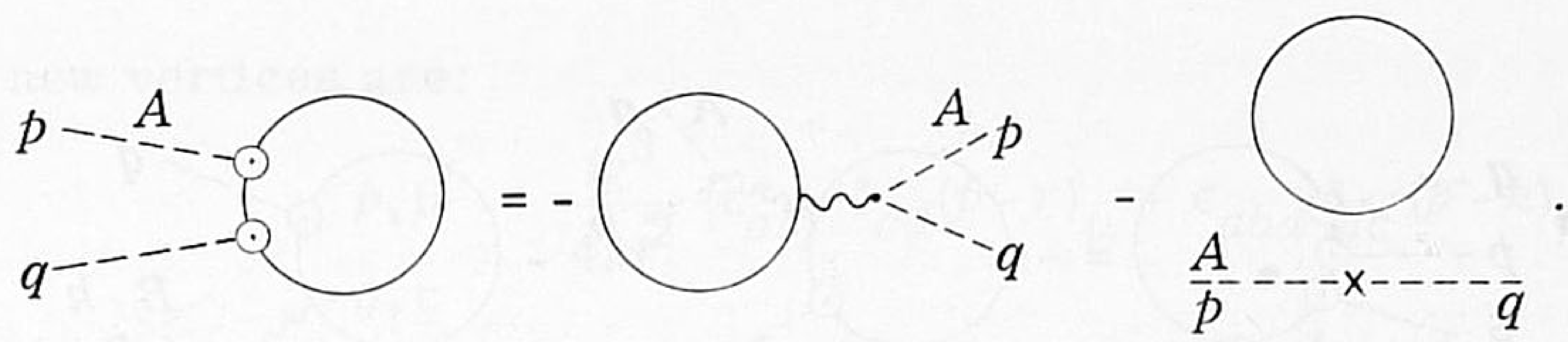
because (10) is the only vertex without F_μ and linear in λ. This is our first Ward identity, which is just like in quantum electrodynamics. In quantum electrodynamics one has that eq. (16) is true also for the case of two or more dotted lines (with circles) instead of one, but this is not true in the case of a non-Abelian gauge such as we are considering here.

To first order in λ and F_μ we have:



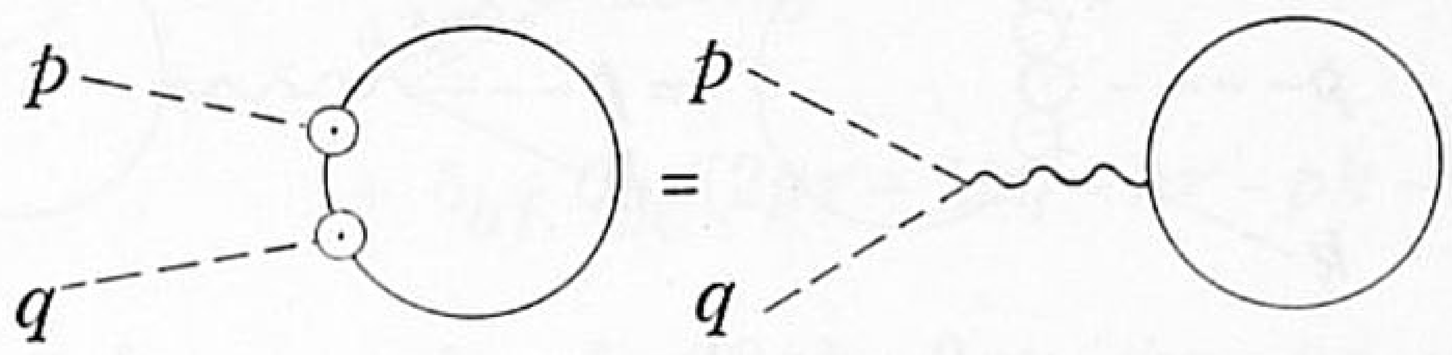
(17)

In particular, this is true if $F_\mu = M \partial_\mu A$, where A is some scalar field with arbitrary mass. Dividing through by λ we find:



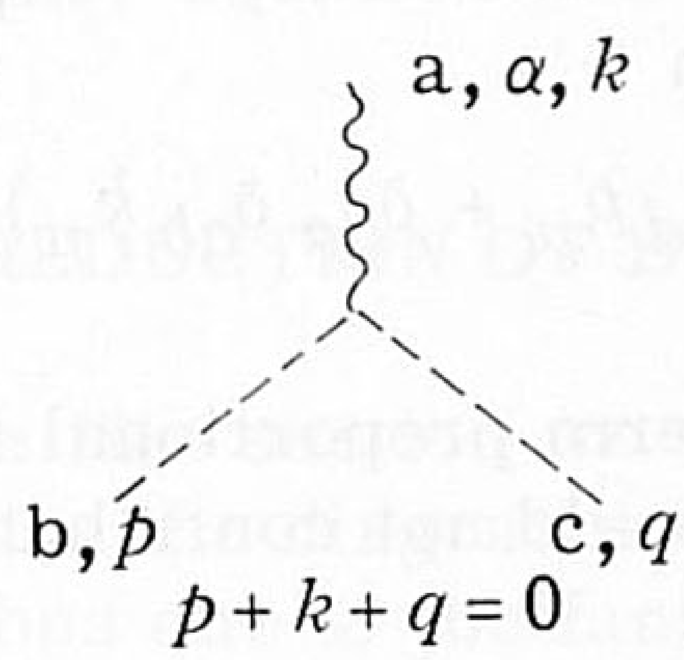
(18)

We have indicated four-momenta p and q, but not yet isospin. Here the last diagram contributes only if the set of diagrams on the left side contains a line without vertices going straight from p to q, which we will suppose not to be the case. By interchanging p and q we get another equation of the type (18), and taking half the sum we obtain



(19)

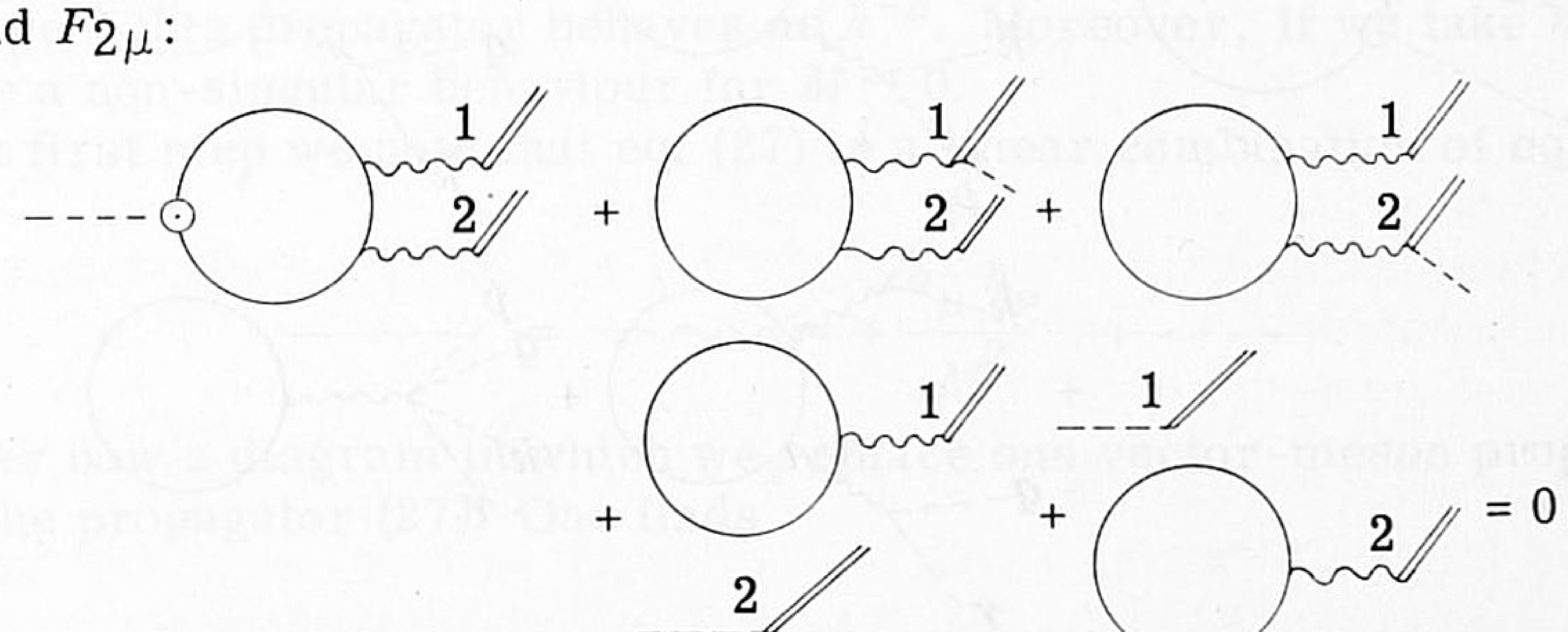
with



$$-\tfrac{1}{2} ig\, \epsilon_{abc} (p - q)_\alpha . \qquad (20)$$

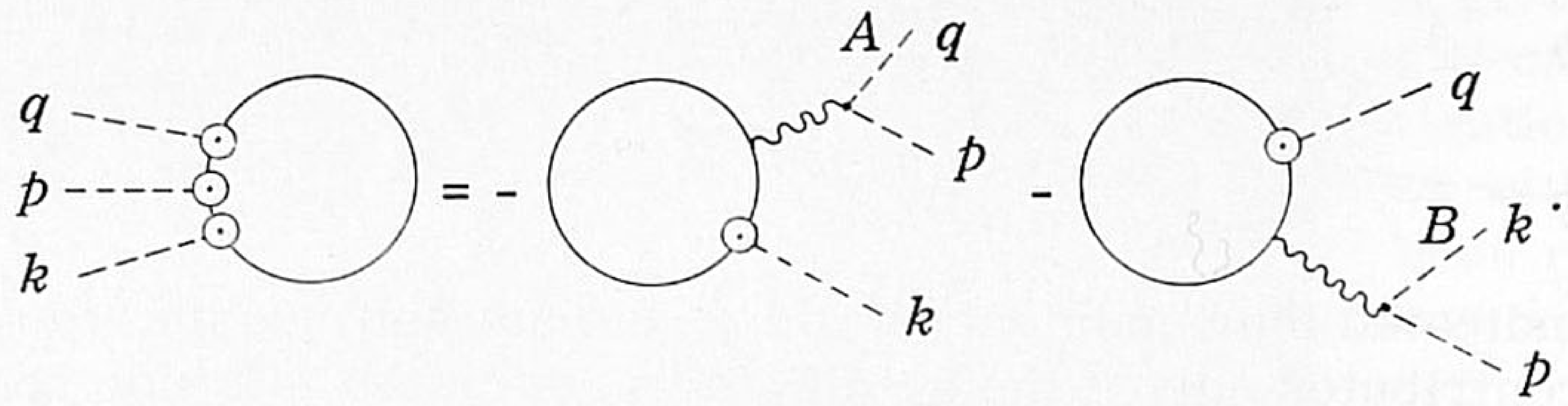
This new vertex is, apart from a sign, just vertex (36) from ref. [3], first paper. Eqs. (16) and (19) are all that is necessary to investigate diagrams with no or one closed loop, but if one wants to go further, more equations are needed. We emphasize that eqs. (16) and (19), by themselves, are true to arbitrary order in g, and are thus not restricted to zero or one closed loop.

To go further we take $F_\mu = F_{1\mu} + F_{2\mu}$, and find to first order in λ, $F_{1\mu}$ and $F_{2\mu}$:

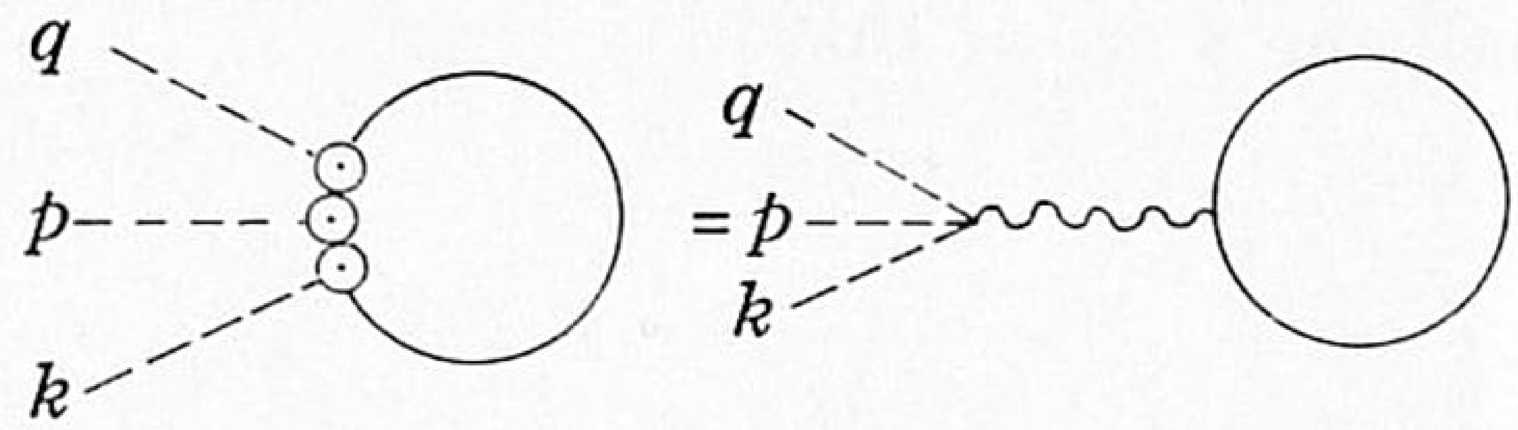


(21)

For $F_{1\mu}$ and $F_{2\mu}$ we take $M\partial_\mu A$ and $M\partial_\mu B$ respectively where A and B are scalar fields with arbitrary mass. After division by λ and omitting disconnected diagrams we get:

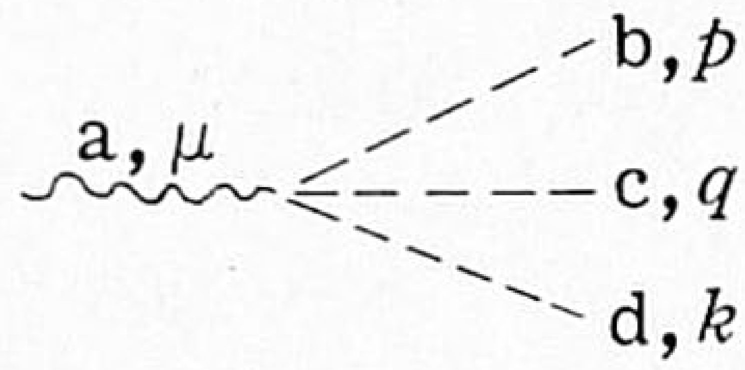


The right-hand side may be worked out further by using eq. (17). The contributions due to the last term of eq. (17) cancel. The result is:



(22)

with



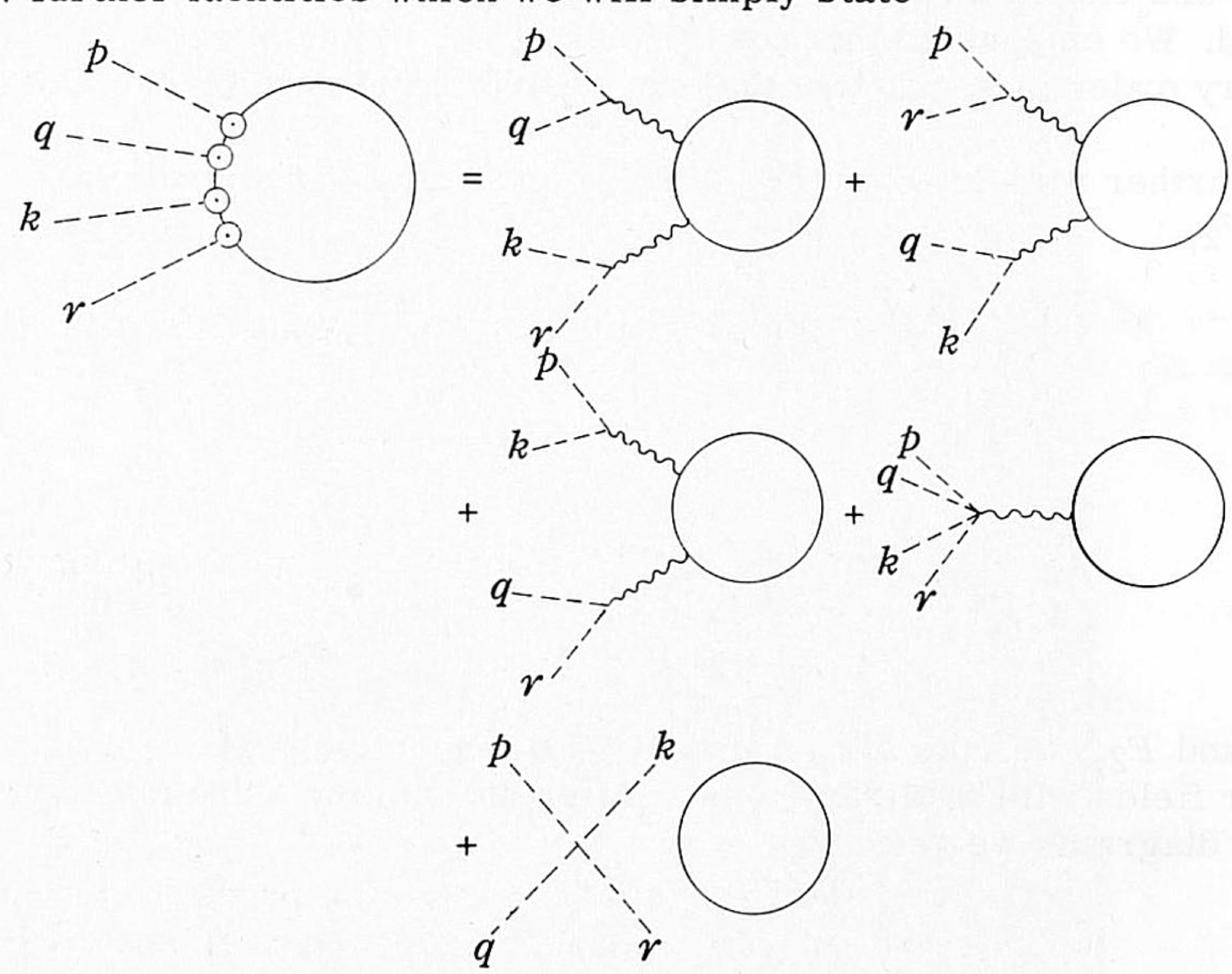
$$\frac{ig^2}{M}(\epsilon_{ibc}\,\epsilon_{aid}\,p_\mu + \epsilon_{idc}\,\epsilon_{aib}\,k_\mu)$$

or, after symmetrization (23)

$$-\frac{ig^2}{M}(\delta_{db}\,\delta_{ca}\,q_\mu + \delta_{ba}\,\delta_{cd}\,p_\mu + \delta_{da}\,\delta_{cb}\,k_\mu).$$

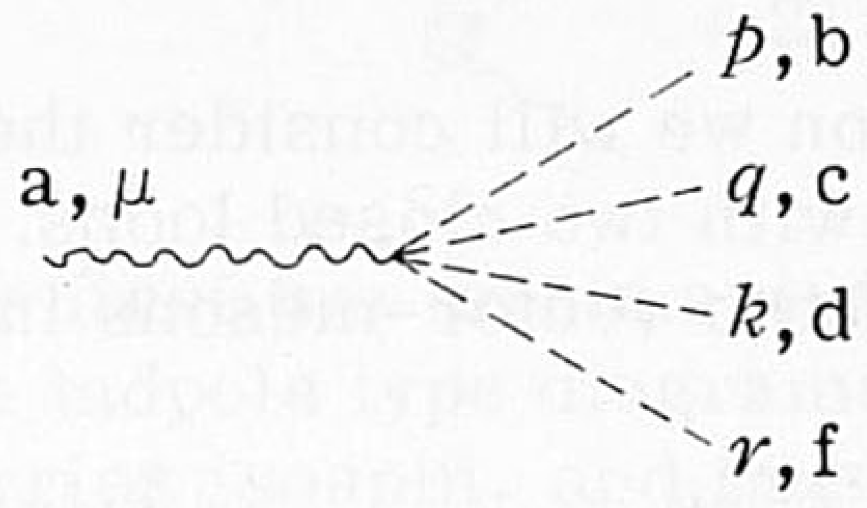
Actually, the last formula differs from the first by a term proportional to the vector-meson momentum, however, such a term would not contribute in the right-hand side of eq. (22) on account of eq. (16).

In this way one may continue. We will confine ourselves in the following section to diagrams with at most two closed loops, and for that we need only a few further identities which we will simply state

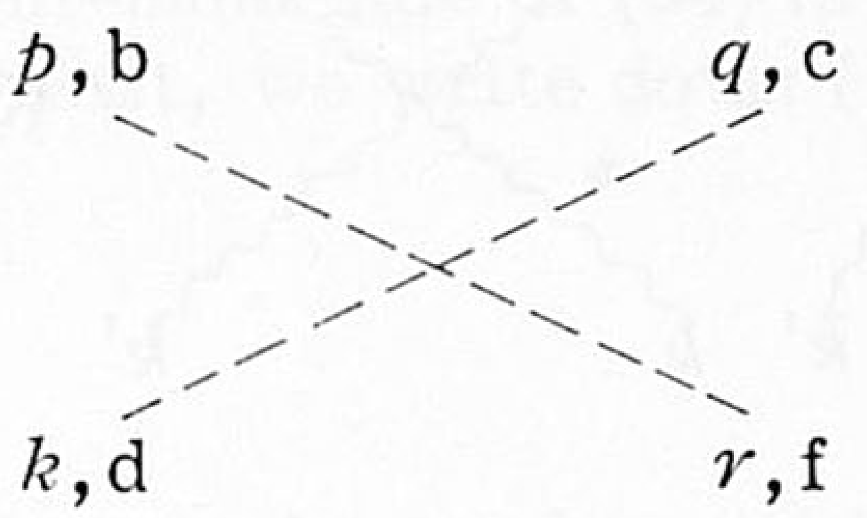


(24)

The two new vertices are:



$$\frac{ig^3}{4M^2}\{\epsilon_{abf}\,\delta_{cd}(p-r)_\mu + \epsilon_{abd}\,\delta_{fc}\,(p-k)_\mu + \epsilon_{abc}\,\delta_{df}(p-q)_\mu + \epsilon_{acf}\,\delta_{bd}(q-r)_\mu + \epsilon_{acd}\,\delta_{bf}(q-k)_\mu + \epsilon_{adf}\,\delta_{bc}(k-r)_\mu\}\,. \tag{25}$$



$$\frac{g^2}{4M^2}\{\delta_{bc}\,\delta_{df}(2pq+2kr-kq-pk-rq-rp) + \delta_{bf}\,\delta_{dc}(2pr+2kq-kr-pk-rq-qp) + \delta_{bd}\,\delta_{fc}\,(2pk+2qr-kq-pq-rk-rp)\}\,. \tag{26}$$

Another equation will be derived in the course of the work in sect. 4.

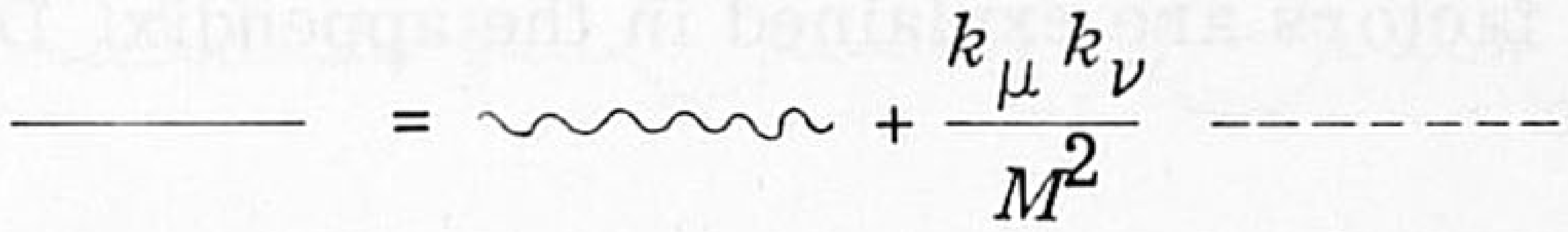
4. REDUCTION OF DIAGRAMS

In this section we will apply the identities of the previous section to reduce the diagrams of the Yang-Mills theory. We will disregard complications due to the fact that most of the diagrams are highly divergent. The aim of the work is to eliminate in diagrams, as much as possible, the $k_\mu k_\nu/M^2$ term in the vector-meson propagator (5). To this purpose we introduce a new propagator (Ω-line):

$$\delta_{ab}\left(\frac{\delta_{\mu\nu} + (m^2-M^2)\,k_\mu k_\nu/M^2(k^2+m^2-i\epsilon)}{k^2+M^2-i\epsilon}\right). \tag{27}$$

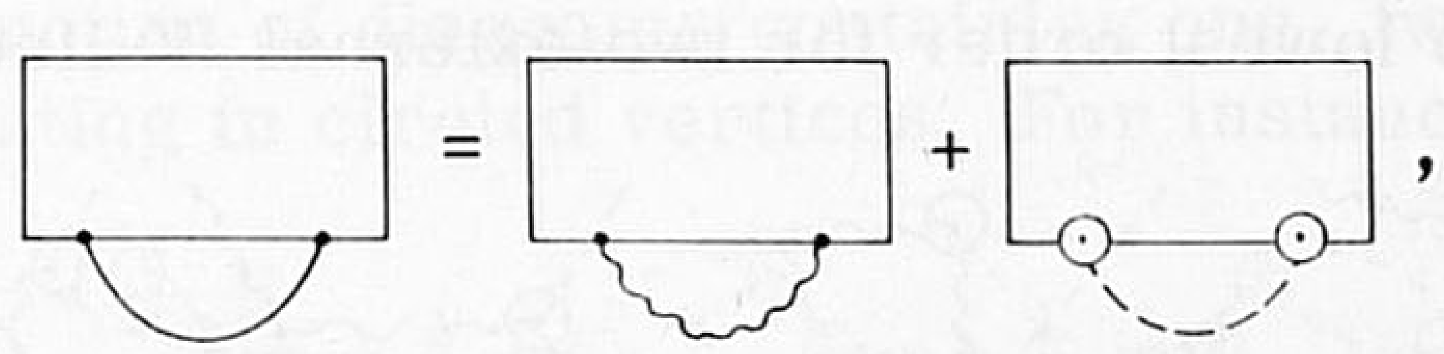
For large k this propagator behaves as k^{-2}. Moreover, if we take $m^2 = \kappa^2 M^2$ we have a non-singular behaviour for $M \to 0$.

As a first step we note that eq. (27) is a linear combination of eqs. (5) and (9)



(28)

Consider now a diagram in which we replace one vector-meson propagator (5) by the propagator (27). One finds

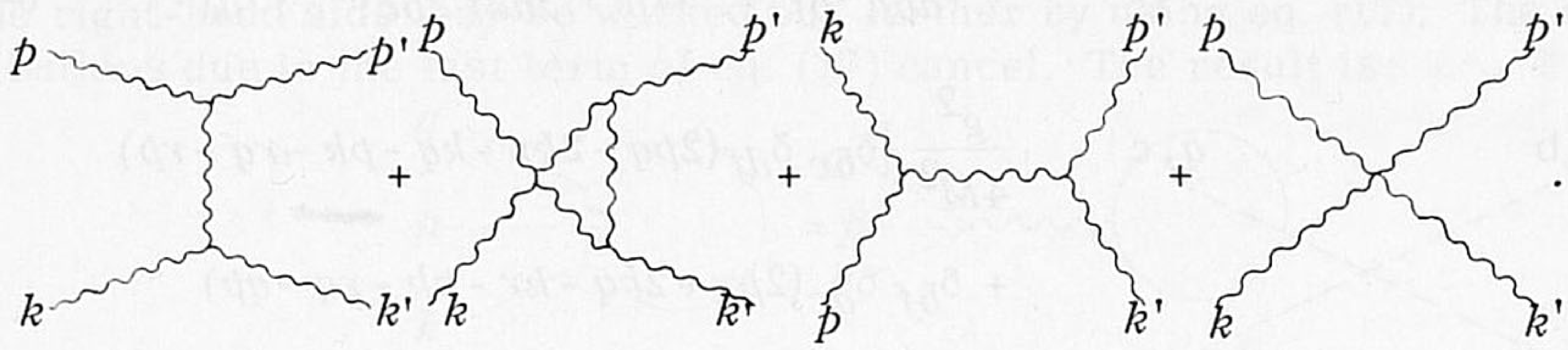


(29)

where the square box stands for any diagram containing propagators and vertices (5), (6) and (7), and external W-lines. Using the previously derived identities the second term in the right-hand side may be simplified and in this way we can eliminate propagators of the type (5), getting in return propagators (9) and (27), and new vertices of the type (20), (22), (25), (26), etc.

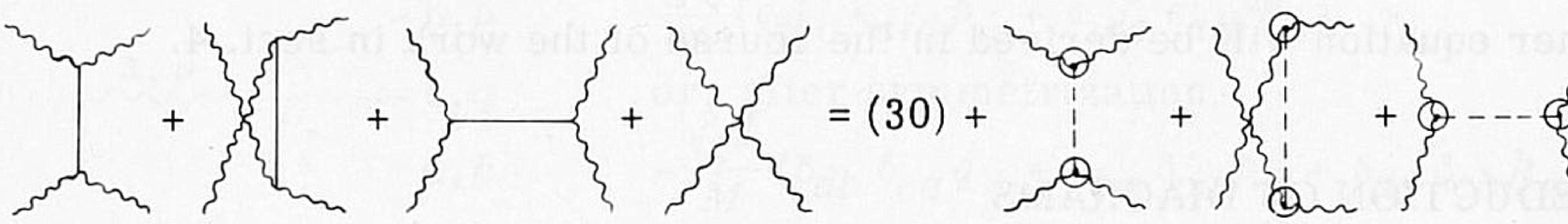
Rather than trying to discuss the general situation we will consider the simplest cases of diagrams without, with one, and with two closed loops.

As a first example we consider the scattering of two vector-mesons in lowest order. There are four diagrams:



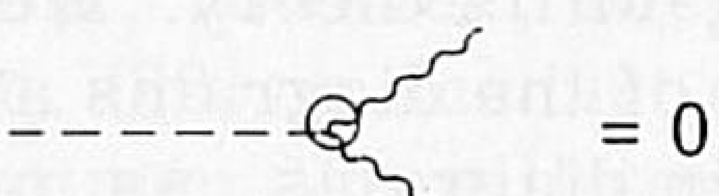
(30)

Performing the replacement (28) we find:



(31)

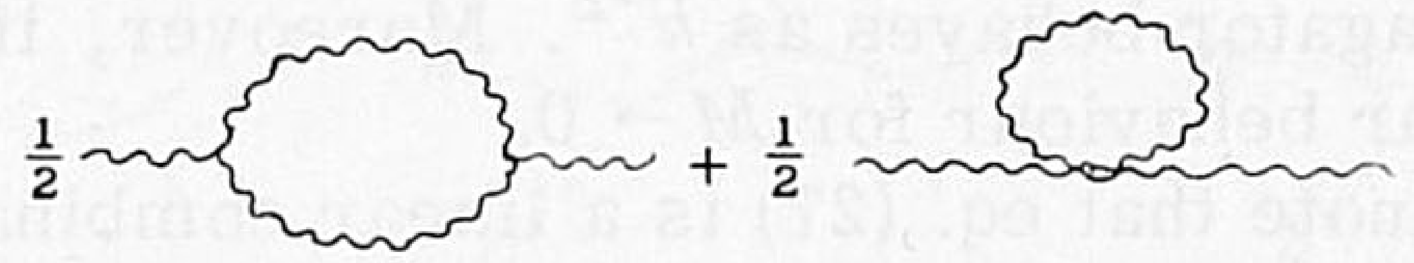
Eq. (15), in order g for two external meson lines tells us



(32)

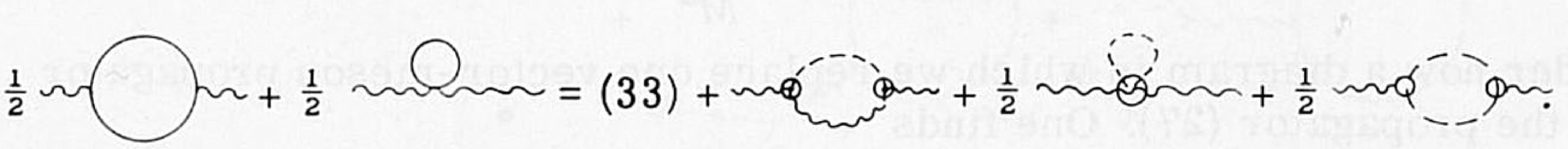
and, therefore, the last three diagrams in (31) are zero. Thus, we have the result that in eq. (30) we may replace all W-lines by Ω-lines (27). Similarly, for higher-order diagrams without closed loops.

Next, we consider the simplest one closed loop case, that is the lowest order self-energy diagrams



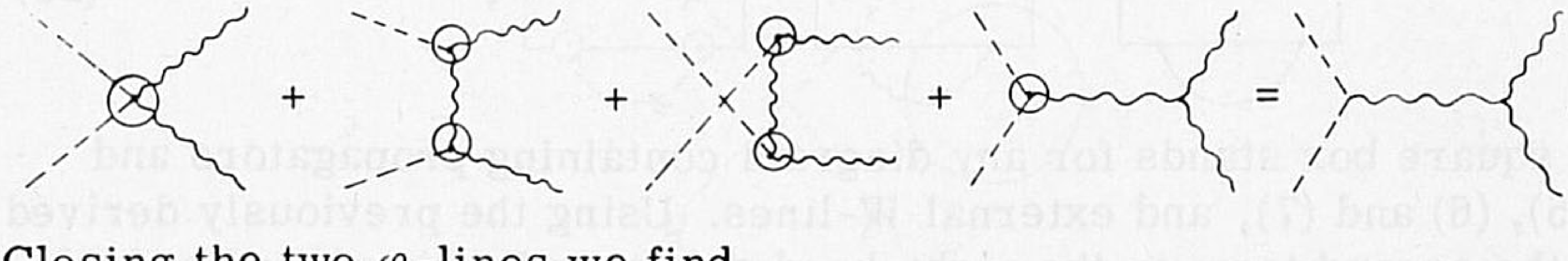
(33)

The combinational factors are explained in the appendix. Doing the replacement (28), we have



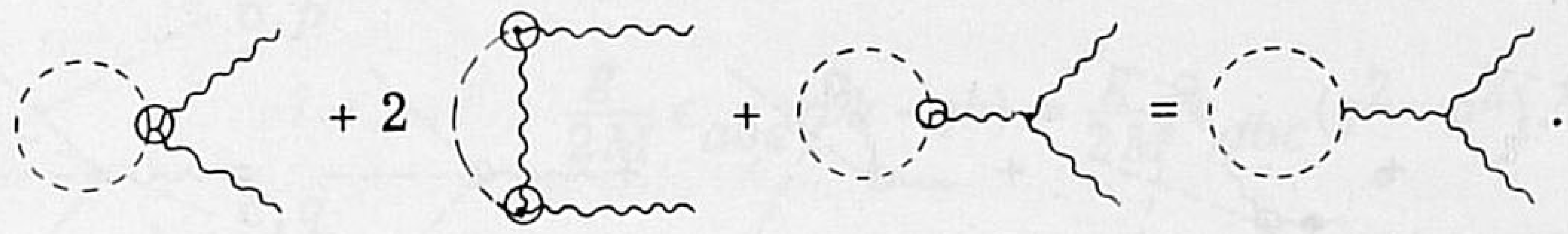
(34)

Using identity (19) to lowest order for two external W-lines



(35)

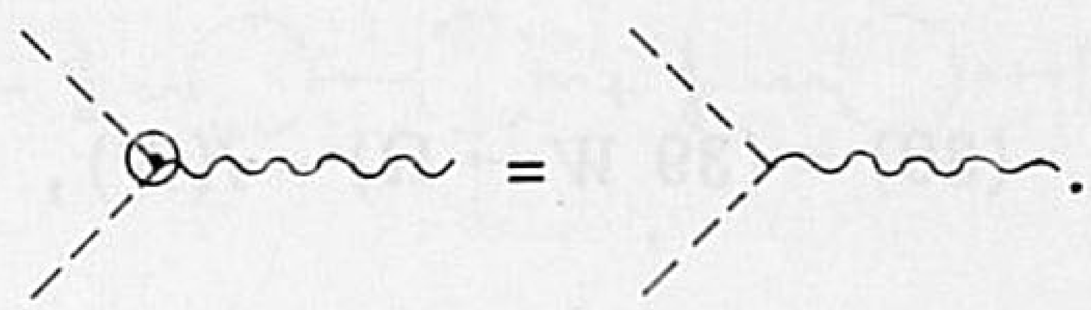
Closing the two φ-lines we find



(36)

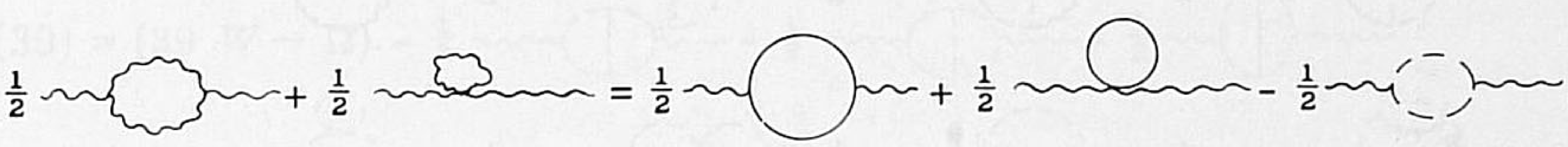
The first two terms are diagrams occurring in (34). The other two diagrams are tadpole type diagrams, and these are always zero here (because the W carries isospin, and thus has not the quantum numbers of the vacuum). Thus, as a consequence of (36) the sum of the second and third diagram in the right-hand side of (34) is zero.

Next, we write down (19) in lowest order, for one external W-line



(37)

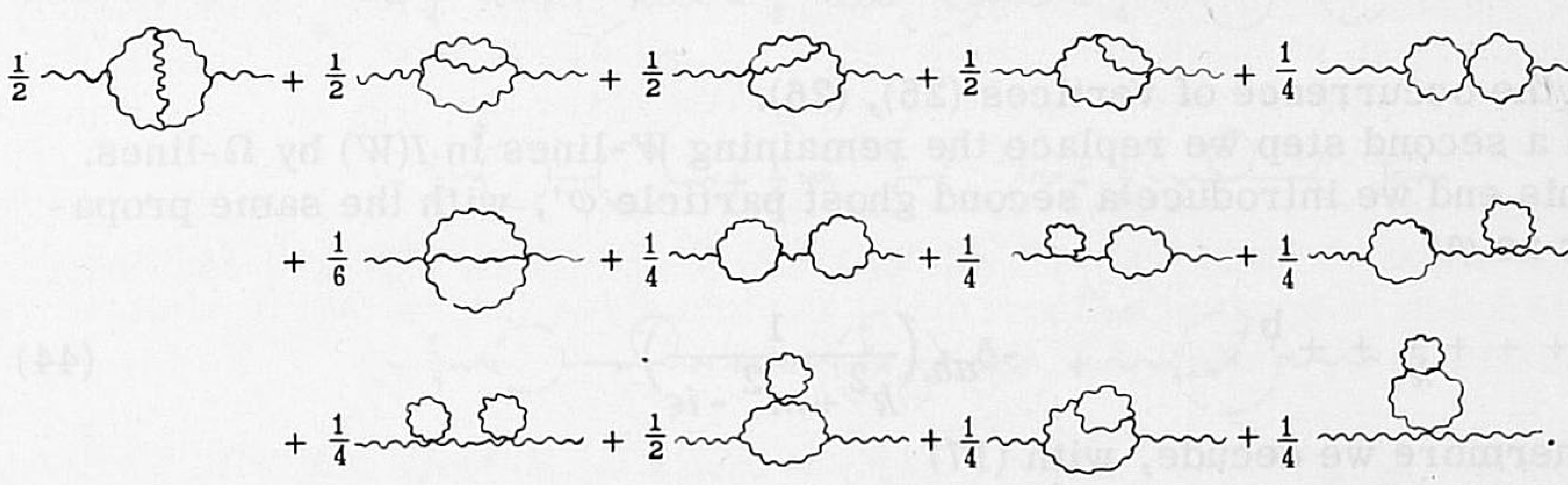
This may be used in the last diagram of (34) and we finally get



(38)

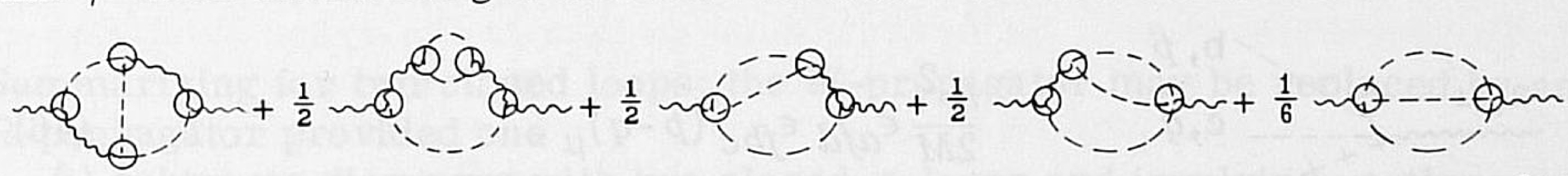
One finds in this way as a general rule: in diagrams with one closed loop the W-propagator may be replaced by the Ω-propagator provided one subtracts diagrams containing closed loops of φ-particles with propagators and vertices (9) and (20). This is the well-known rule derived before for massless [2] and massive [3] Yang-Mills fields.

Next, we consider the simplest two-loop diagrams, i.e., the fourth-order self-energy diagrams



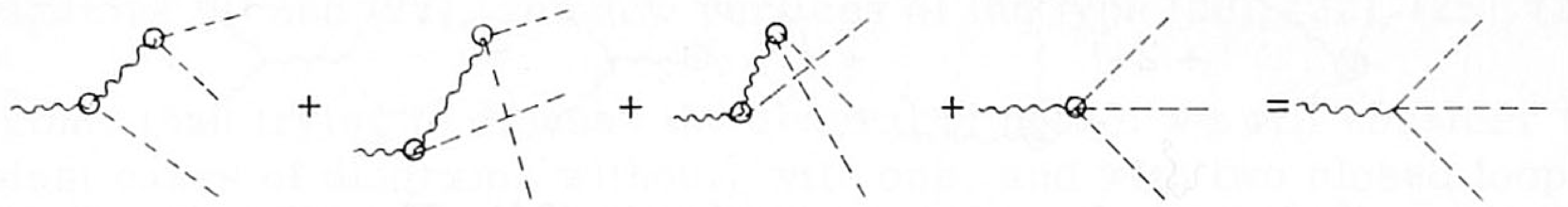
(39)

It becomes a bit cumbersome to do everything in detail, but the procedure is in first instance precisely as above. Thus, we first consider the set of diagrams (39) with Ω-propagators instead. This set will be equal to the set (39) plus a collection of diagrams containing one, two, up to five internal φ-lines terminating in circled vertices. For instance we find



(40)

and eq. (22) gives, in order g^2:



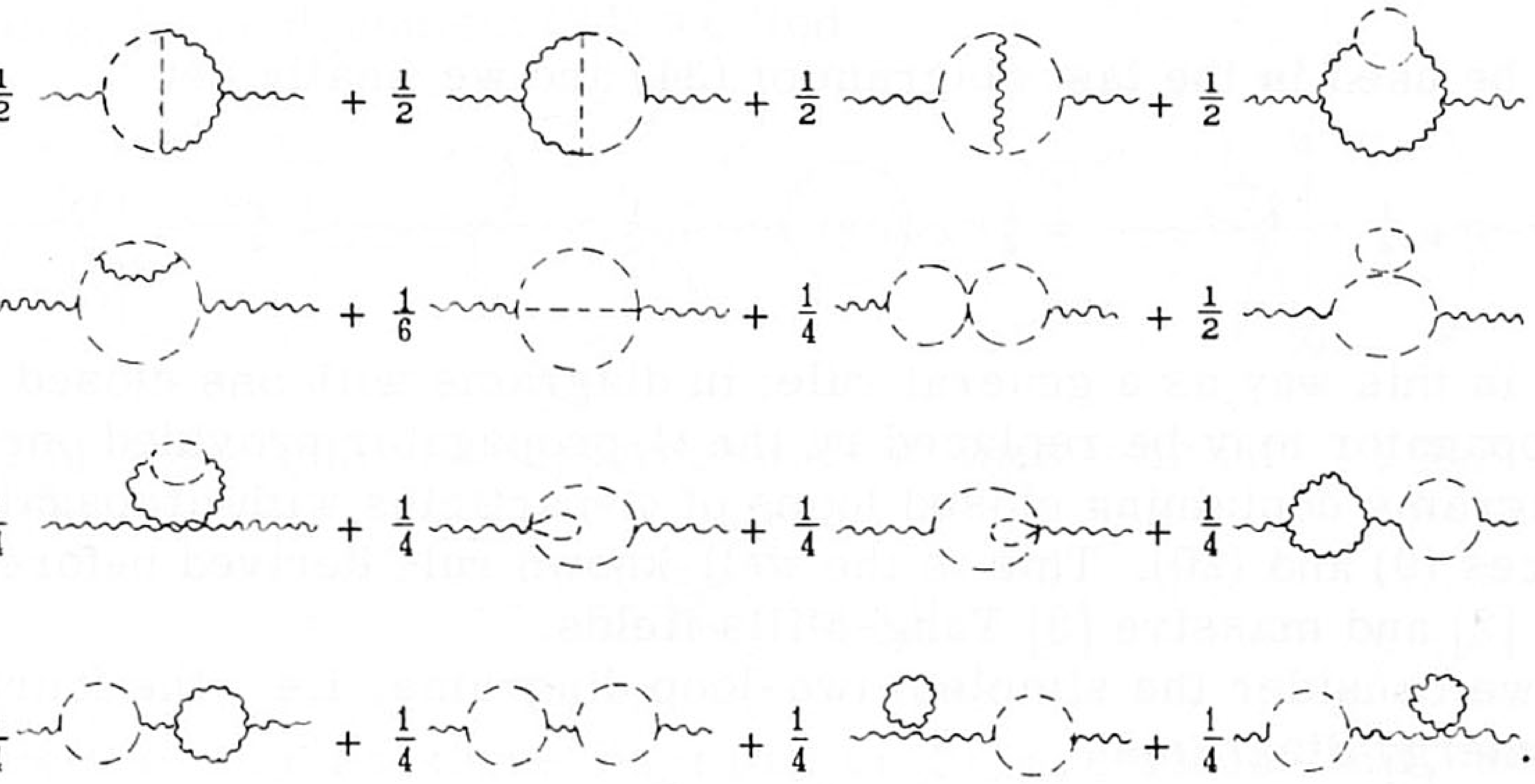
, (41)

which involves the new vertex (23). Thus

$(40) = \frac{1}{6}$  . (42)

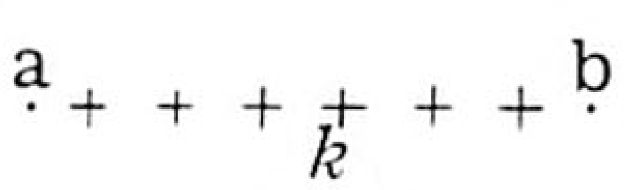
In this way one finds

$$(39) = (39\ W \to \Omega) - I(W),$$

$I(W) =$  . (43)

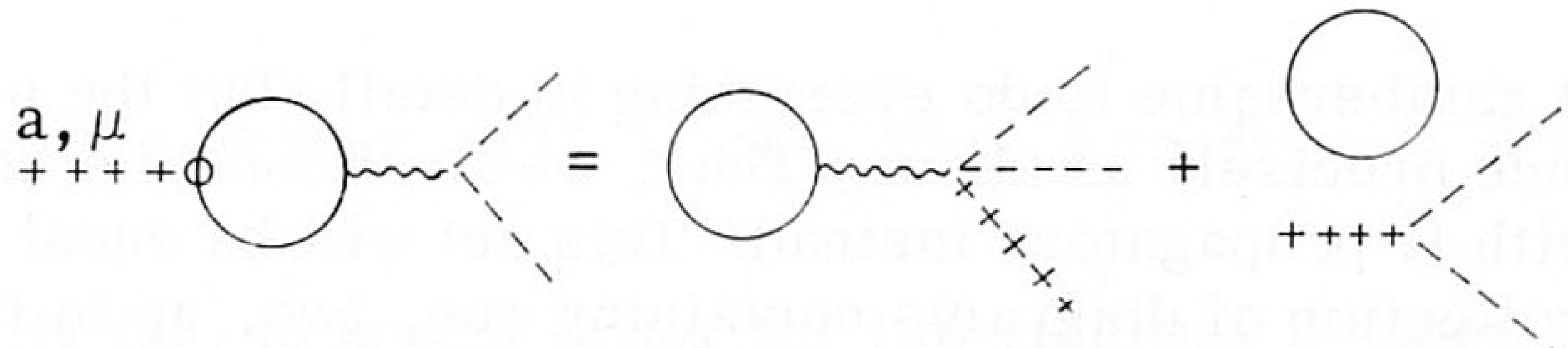
Note the occurrence of vertices (25), (26).

In a second step we replace the remaining W-lines in $I(W)$ by Ω-lines. To this end we introduce a second ghost particle φ', with the same propagator as φ



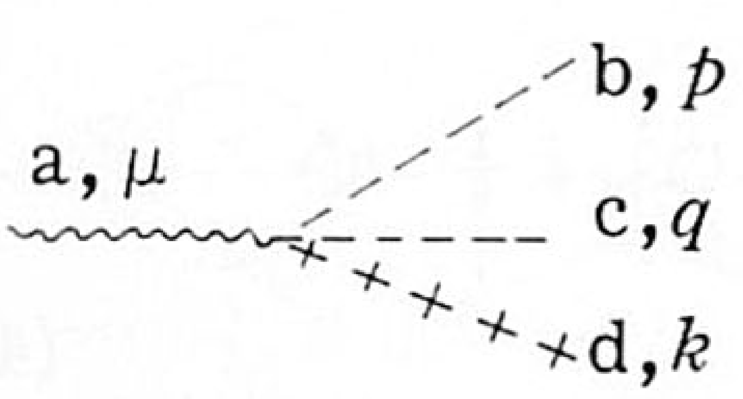
$$-\delta_{ab}\left(\frac{1}{k^2+m^2-i\epsilon}\right). \qquad (44)$$

Furthermore we decude, with (17)

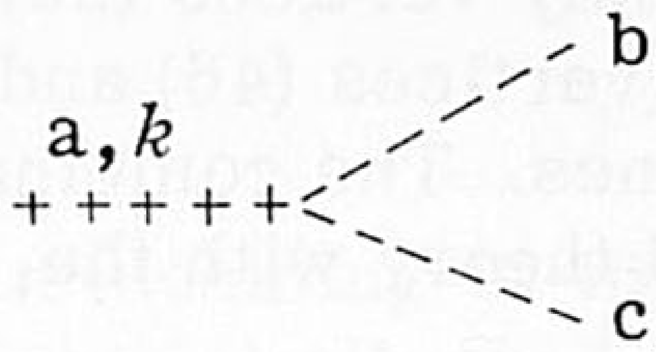


(45)

with

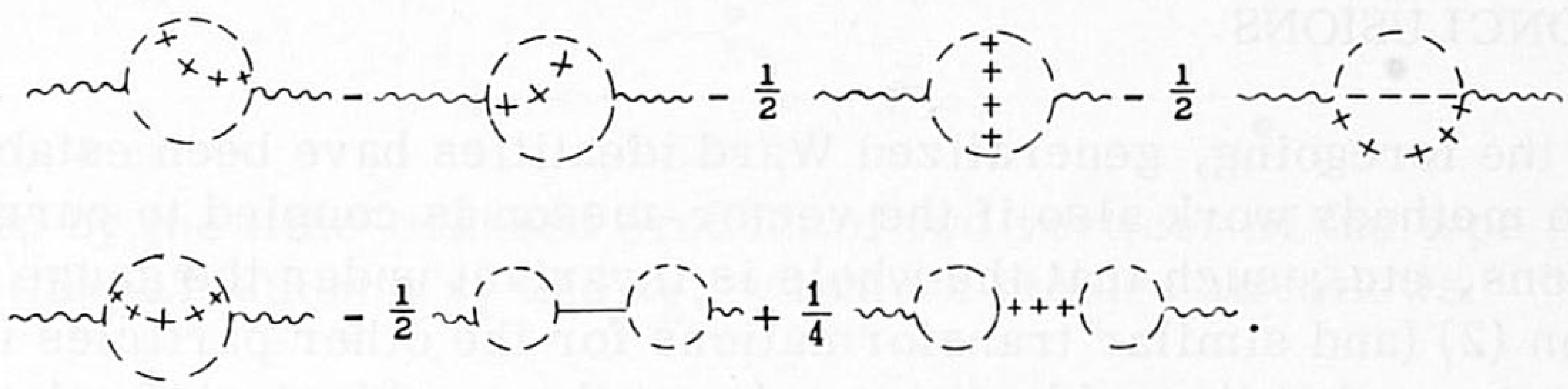


$$\frac{ig^2}{2M}\epsilon_{afd}\,\epsilon_{fbc}\,(p-q)_\mu\,, \qquad (46)$$

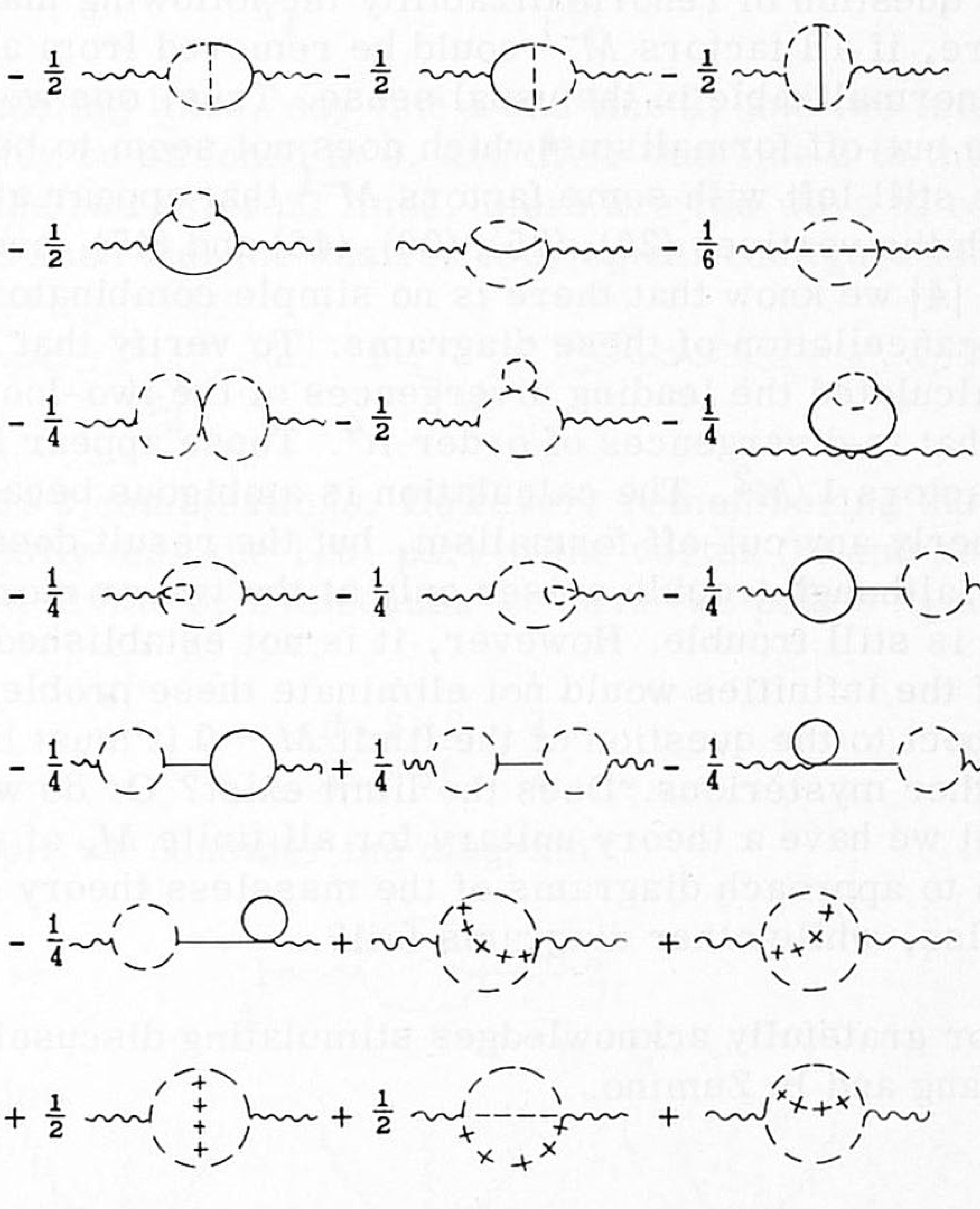


$$\frac{g}{2M}\epsilon_{abc}(pk - qk) = \frac{g}{2M}\epsilon_{abc}(q^2 - p^2). \tag{47}$$

With the help of (45) one establishes

$I(W) = I(\Omega)$  (48)

All together one finds

$(39) = (39\ W \to \Omega)$  (49)

Summarizing for two closed loops: the W-propagator may be replaced by an Ω-propagator provided one

(i) subtracts diagrams with two closed φ-loops and involving vertices (23), (25) and (26), and possibly (20);

(ii) adds diagrams with two closed φ-loops involving only vertices (20);

(iii) subtracts diagrams containing also φ' lines, with vertices (46) and (47), with a minus sign for every closed loop of only φ-lines. The combinatorial factors are just those that one would have in a field theory with the vertices mentioned.

5. CONCLUSIONS

In the foregoing, generalized Ward identities have been established. These methods work also if the vector-meson is coupled to currents of nucleons, etc., such that the whole is invariant under the gauge transformation (2) (and similar transformations for the other particles involved). It seems to us that these identities exhaust the combinatorial relations between the diagrams.

As to the question of renormalizability the following may be noted. As argued before, if all factors M^{-1} could be removed from all diagrams the theory is renormalizable in the usual sense. True, one would have to invent suitable cut-off formalisms which does not seem to be so trivial. Anyway, we are still left with some factors M^{-1} that appear at the two-loop level through the vertices (23), (25), (26), (46) and (47), and because of the work in ref. [4] we know that there is no simple combinatorial relation that establishes cancellation of these diagrams. To verify that once more we have also calculated the leading divergences of the two-loop self-energy diagrams, that is divergences of order Λ^4. These appear in the diagrams containing factors $1/M^2$. The calculation is ambigous because we have not defined properly any cut-off formalism, but the result does not seem to be zero. Thus, although trouble arises only at the two or more closed loop level, there is still trouble. However, it is not established that a careful treatment of the infinities would not eliminate these problems.

With respect to the question of the limit $M \to 0$ it must be said that this becomes rather mysterious. Does the limit exist? Or do we have the strange situation that we have a theory unitary for all finite M, of which some diagrams seem to approach diagrams of the massless theory that is claimed to be unitary also, while other diagrams fail?

The author gratefully acknowledges stimulating discussions with Professors C. N. Yang and B. Zumino.

APPENDIX

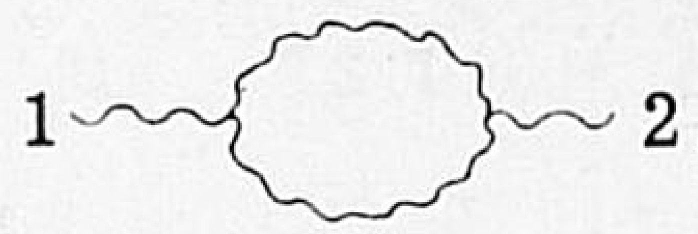
Combinatorial factors

To derive the correct factors one must go back to the Dyson-Wick reduction procedure. We will consider as an example diagrams built up from the three and four-line vertices (6) and (7). The three-line vertex arises from a term in the Lagrangian containing three vector-mesons

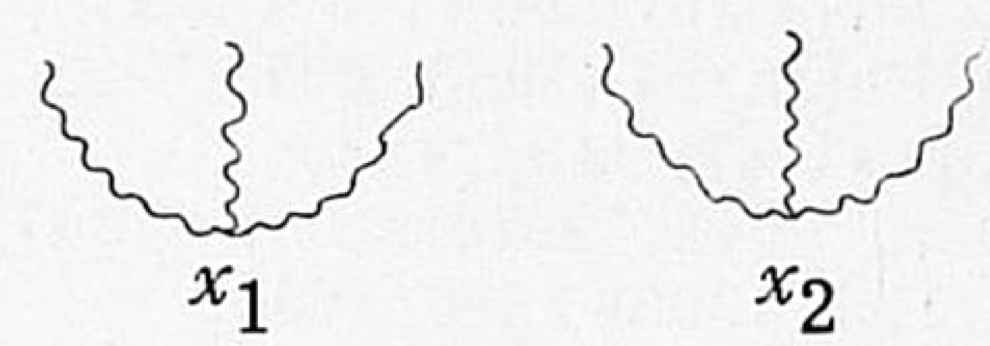
$$\epsilon_{abc}\, W^a_\mu W^b_\nu \partial_\mu W^c_\nu . \tag{A.1}$$

An incoming W-line may be absorbed by the first, the second or third W-field, similarly for the other two lines. The vertex (6) is obtained by summing over all 3! possibilities.

Consider the lowest-order self-energy diagram

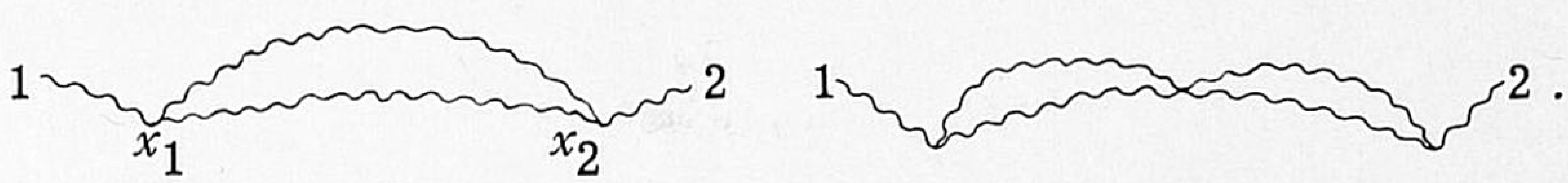


This is given by the time-ordered product of two vertices of the type (A.1), with space-time arguments x_1 and x_2, which we picture as follows



(A.2)

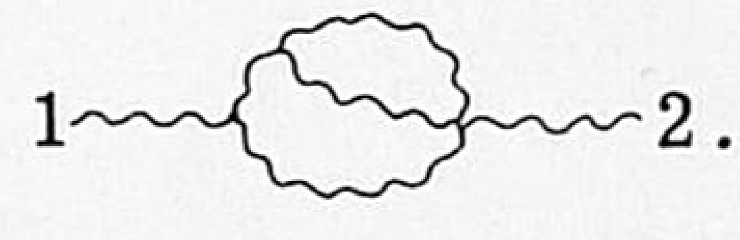
There are two outgoing lines, say line 1 and line 2, and two internal lines. External line 1 may be attached in 6, and after that line 2 in three ways. Having attached the two external lines, there are two ways to connect the two internal lines such that the desired topological configuration results



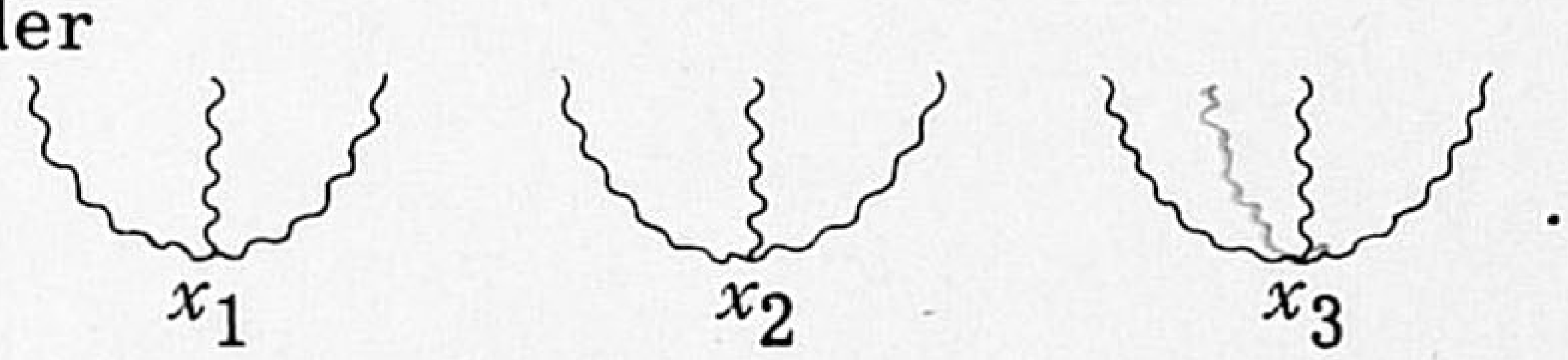
Thus we find $6\times 5\times 2$ combinations. However, remembering that the vertices (A.2) are really only the 1/3! part of the vertex (6) and keeping in mind that we must divide by 2! because we have two points x_1 and x_2, we find as factor

$$\frac{6\times 3\times 2}{2!\,3!\,3!} = \frac{1}{2}.$$

As another example we consider the diagram

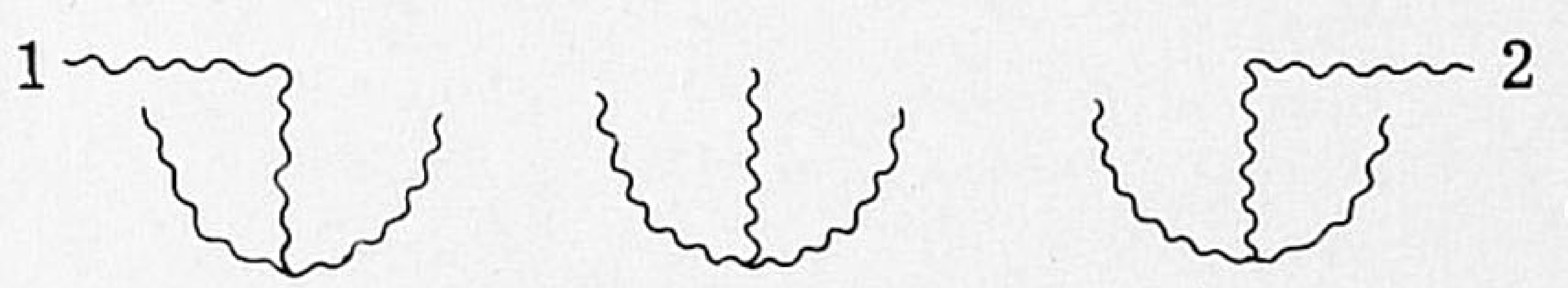


Thus, now consider



(A.3)

Line 1: six ways. Line 2: four ways. After attaching these two one has for instance



There are $6\times3\times2$ ways to connect the rest such as to get the desired diagram. We must divide by $3!\times3!\times4!$ because of the way the vertices (6) and (7) are defined, and furthermore divide by 2!. The latter because we have three points x_1, x_2 and x_3, (this implies $1/3!$), but in addition to (A.3) the four-vertex may occur also at x_1 and x_2, (thus three possibilities). The over-all factor is

$$\frac{6\times4\times6\times3\times2}{3!\,3!\,4!\,2!}=\frac{1}{2}.$$

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