

EXAMPLE OF A GAUGE FIELD THEORY

G. 't Hooft and M. Veltman^{*)}

CERN - Geneva

ABSTRACT

The techniques and results of a previous paper are demonstrated on a simple non-Abelian model.

To be published in the Proceedings of the
Marseille Conference, June 1972

^{*)} On leave from the University of Utrecht, Netherlands.

1. Introduction

The combinatorics of renormalizable massive vector field theories is quite complicated, mainly because so many vertices have to be included in the considerations. A way around this difficulty is the use of composite lines and very general identities between diagrams, and in a previous publication¹⁾ we have used such methods to give a general combinatorial proof of the equivalence of the S-matrix in different gauges. Also renormalization problems were studied in this way. In these circumstances an explicit example may be very welcome, and also inspire more confidence in the method. In this note we will consider such an example in full detail. The case we take is case 2 of ref. 2, resembling most closely pure massive Yang-Mills fields. In that example there are, as physical particles, three massive vector mesons of equal mass forming an isospin triplet and one massive scalar particle of isospin 0. There are several sets of Feynman rules all giving rise to the same S-matrix. One of the gauges is the physical gauge where the only internal lines are those that correspond to physical particles, another gauge shows that the theory is renormalizable. Equivalence of gauges implies that we have a unitary renormalizable theory.

In this note we will more or less follow the general treatment of ref. 1. The idea is to give an explicit example of the various equations. The renormalization procedure will be carried through up to one closed loop. The model contains three parameters: the vector meson mass, the mass of the (physical) Higgs-Kibble particle and the coupling constant. These constants must be renormalized, however in a gauge independent way.

3. Massive vector fields

Consider a massless Yang-Mills isospin 1 triplet of vector mesons coupled to an isospin $\frac{1}{2}$ field K

$$\mathcal{L}_{inv} = -\frac{1}{4} (G_{\mu\nu}^a)^2 - (D_\mu K)^\dagger D_\mu K - \mu K^\dagger K - \frac{1}{2} \lambda (K^\dagger K)^2 \quad (2.1)$$

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon_{abc} W_\mu^b W_\nu^c \quad (2.2)$$

$$D_\mu = \partial_\mu - \frac{i}{2} g W_\mu^a \tau^a \quad (2.3)$$

This Lagrangian is invariant, to first order in the functions $\Lambda^a(x)$, for the replacement

$$W_\mu^a \rightarrow W_\mu^a + g \epsilon_{abc} \Lambda^b W_\mu^c - \partial_\mu \Lambda^a \quad (2.4)$$

$$K \rightarrow \left(1 - \frac{i}{2} g \Lambda^a \tau^a\right) K \quad (2.5)$$

Since (2.5) is an infinitesimal rotation in I-spin space the last and last but one term of (2.1) are obviously invariant; one verifies that under (2.4), (2.5)

$$D_\mu K \rightarrow \left(1 - \frac{i}{2} g \Lambda^a \tau^a\right) D_\mu K \quad (2.6)$$

due to the commutation rules of the SU(2) generators $-\frac{i}{2} \tau^a$:

$$\left[\frac{-i\tau^a}{2}, \frac{-i\tau^b}{2} \right] = \epsilon_{abc} \frac{-i\tau^c}{2}$$

so that also the second term of (2.1) is invariant. This demonstrates the general rule; if we were to study SU(3) rather than SU(2) we would have 8 vector mesons. The commutation rules would have been:

$$\left[\frac{-i\lambda^a}{2}, \frac{-i\lambda^b}{2} \right] = f_{abc} \frac{-i\lambda^c}{2}, \quad a = 1, \dots, 8$$

where the λ and f are the well-known $SU(3)$ matrices and structure constants. Then one would have had f_{abc} instead of ϵ_{abc} in (2.2) and (2.4) and $-i\lambda/2$ instead of $-i\tau/2$ in (2.3), (2.5).

Next we must give the W 's a mass by means of the Higgs-Kibble formalism :

$$\begin{aligned}
 K &= \frac{1}{\sqrt{2}} (Z + \sqrt{2} F + i\psi^a \tau^a) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \frac{1}{\sqrt{2}} \begin{pmatrix} Z + \sqrt{2} F + i\psi^3 \\ -\psi^2 + i\psi^1 \end{pmatrix} \quad (2.7)
 \end{aligned}$$

where Z , ψ^1 , ψ^2 and ψ^3 are real fields. The factor $1/\sqrt{2}$ is chosen such that

$$-(\partial_\mu K)^2 = -\frac{1}{2} (\partial_\mu Z)^2 - \frac{1}{2} (\partial_\mu \psi^a)^2 .$$

The constant F is supposed to be chosen such that the field Z has no vacuum expectation value. It is not a new parameter, because given the Lagrangian (2.1) one may compute the vacuum expectation value of the Z -field. To see this we make the replacement (2.7) in the Lagrangian (2.1). The terms involving the Z -fields are

$$-\frac{1}{2} (\partial_\mu Z)^2 - \frac{1}{2} \mu (Z + F\sqrt{2})^2 - \frac{1}{8} \lambda (Z + F\sqrt{2})^4 .$$

In here μ must be positive, else there is no state of lowest energy. These terms generate vertices that imply absorption of Z in the vacuum, namely those linear in Z :

$$-\sqrt{2} \mu Z F - \sqrt{2} \lambda Z F^3$$

corresponding to vertex of fig. 1

$$\begin{array}{c}
 \text{~~~~~} \bullet \\
 - \sqrt{2} \mu F - \sqrt{2} \lambda F^3
 \end{array}$$

fig. 1

This vertex would give a non-zero value to the diagrams with one ingoing Z-line, and thus a non-zero vacuum expectation value. The only consistent solution is that this vertex is zero, i.e.

$$F = \sqrt{-\frac{\mu}{\lambda}} \quad \text{or} \quad F = 0 . \quad (2.8)$$

In higher orders of perturbation theory there will be more complicated diagrams that give a non-zero contribution to the vacuum expectation value of Z. Then one must readjust F such as to make the total zero. In practice we will keep F fixed and readjust μ such that $\langle 0 | Z | 0 \rangle = 0$. It may be noted that only one of the two solutions (2.8) is physically acceptable. If μ is positive then the solution $F = 0$ is acceptable. The other solution would give the W-particles an imaginary mass. If μ is negative then the solution $F = 0$ gives an imaginary mass to the ψ and Z particles.

Let us now take μ negative and substitute (2.7) into the Lagrangian (2.1). Instead of the parameters F, λ and μ we use

$$M = \frac{g}{\sqrt{2}} F; \quad \alpha = \frac{\lambda}{g^2}; \quad \beta = \mu + \lambda F^2 \quad (2.9)$$

and moreover

$$m^2 = 4\alpha M^2.$$

One finds

$$\begin{aligned} \mathcal{L}_{inv} = & -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{2} M^2 W^2 - \frac{1}{2} (\partial_\mu Z)^2 - \frac{1}{2} m^2 Z^2 - \frac{1}{2} (\partial_\mu \psi^a) (D_\mu^a \psi) \\ & + \frac{1}{2} g W_\mu^a (Z \partial_\mu \psi^a - \psi^a \partial_\mu Z) - \frac{1}{8} g^2 W_\mu^2 (\psi^2 + Z^2) - \frac{1}{2} g M W^2 Z \\ & - \alpha M g Z (\psi^2 + Z^2) - \frac{1}{8} \alpha g^2 (\psi^2 + Z^2)^2 \\ & - \beta \left[\frac{1}{2} (Z^2 + \psi^2) + \frac{2M}{g} Z \right] - M \psi^a \partial_\mu W_\mu^a \end{aligned} \quad (2.10)$$

where

$$W_\mu^2 = W_\mu^a W_\mu^a, \quad \psi^2 = \psi^a \psi^a$$

$$D_\mu^a \psi = \partial_\mu \psi^a + g \epsilon_{abc} W_\mu^b \psi^c$$

(Note that if $(K^b)_{ac} = \epsilon_{abc}$ then $[K^a, K^b] = \epsilon_{abc} K^c$).

The requirement that Z has no vacuum expectation value implies that $\beta = 0$ in zero'th order of g . In higher order β is to be readjusted such that $\langle 0 | Z | 0 \rangle$ remains zero (that is such that the sum of all diagrams with one external Z -line, no other external lines, is zero). It is of no importance if one takes another value for β ; that just means that in the diagrams one must allow also for Z tadpoles. These tadpole diagrams add up to the same thing as if one had started with the adjusted value of β and a different value for M . All this is just like in the σ -model, see the work of B. Lee and Gervais and Lee .

The Lagrangian \mathcal{L}_{inv} describes vector merons with mass M , a Z particle with mass $m = 2M\sqrt{\alpha}$ and mass-less ψ particles which through the last term of (2.10) are coupled directly to the W -field. This, term is of zero'th order in g , and its contributions must be summed up completely if one wants to do perturbation theory with respect to g . There one discovers the singularity of the Lagrangian (2.10), because this series does not converge. In fact for the Lagrangian (2.10) one cannot write down Feynman rules, This singular behaviour comes about as follows. The W -propagator, ψ propagator and ψ - W coupling of the Lagrangian (2.10) are as in fig. 2

$$\text{---} \cdot \frac{\delta_{\mu\nu} + k_\mu k_\nu / M^2}{k^2 + M^2 - i\epsilon}; \quad \text{---} \cdot \frac{1}{k^2 - i\epsilon}; \quad \text{---} \cdot -iM k_\mu.$$

fig. 2

The complete W-propagator is obtained by summing up. see fig. 3

$$\begin{aligned} & \text{---} + \text{---} \text{---} + \text{---} \text{---} \text{---} + \dots \\ + \dots & = \text{---} \cdot \left(1 + \text{---} + \text{---} \text{---} + \dots \right) \\ & = \text{---} \cdot \left(1 - \text{---} \right)^{-1} \end{aligned}$$

fig. 3

The expression inbetween brackets in fig. 3 is:

$$\begin{aligned} & \delta_{\mu\nu} - (iMk_\mu) \frac{1}{k^2} (-iMk_\nu) \left(\frac{\delta_{\alpha\beta} + k_\alpha k_\beta / M^2}{k^2 + M^2 - i\epsilon} \right) \\ & = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} = P_{\mu\nu}. \end{aligned}$$

This Last expression has no inverse, i.e. there is no P^{-1} such that $P_{\mu\nu} P_{\nu\alpha}^{-1} = \delta_{\mu\alpha}$. In fact, $P_{\mu\nu} k_\nu = 0$ which shows that $P_{\mu\nu}$ has zero eigenvalues. Thus the complete W-propagator does not exist here.

This is a very specific property of a Lagrangian that obeys some local symmetry. It goes back to the fact that such Lagrangians possess no extreme, to be discussed below.

We now must investigate the symmetry of the Lagrangian. Very misleading one often speaks in connection with the Higgs-Kibble mechanism of a spontaneously broken symmetry. However, the symmetry is still there.

The transformation of the K-field, eq. (2.5) implies a transformation of the Z and ψ fields as defined by (2.7).

Precisely: let

$$K = S \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then

$$(1 - \frac{i}{2} g \Lambda^2 \tau^a) K = S' \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with

$$S' = S - \frac{i}{2} g \Lambda^a \tau^a S.$$

With $S = \frac{1}{\sqrt{2}} (Z + F \sqrt{2} + i \psi^a \tau^a)$ one obtains:

$$S' = \frac{1}{\sqrt{2}} \left[Z + F \sqrt{2} + \frac{1}{2} g \Lambda^a \psi^a + \frac{i g}{2} \left\{ - (Z + F \sqrt{2}) \Lambda^a + \epsilon_{abc} \Lambda^b \psi^c \right\} \frac{\tau^a}{2} \right].$$

Thus the transformation (2.5) is also obtained if one transforms:

$$Z \rightarrow Z + \frac{1}{2} g \Lambda^a \psi^a \quad (2.11)$$

$$\psi^a \rightarrow \psi^a + \frac{1}{2} g \epsilon_{abc} \Lambda^b \psi^c - \frac{1}{2} g Z \Lambda^a - M \Lambda^a. \quad (2.12)$$

It is important to realize that the Lagrangian (2.10) is strictly invariant under the replacements (2.4), (2.11) and (2.12). The symmetry is what one could call distorted (the "undistorted" symmetry being the one when $M = 0$). It is in fact very hard to say which symmetry has been broken. It just looks different, and

in general the relation between the transformation properties of the physical fields and those that appear in the original Lagrangian will not be very transparent.

The fact that the Lagrangian (2.10) remains unchanged (to first order in the functions $\Lambda^a(x)$) under the replacements (2.4), (2.11) implies that there are relations between the various terms in \mathcal{L} , that is that there are relations between vertices. These relations are the Ward identities, and we must find them. The reason is that the Lagrangian (2.10) cannot be used directly in the usual way to set up diagrams, because one cannot apply the usual canonical formalism. Within that formalism equations of motion are derived by finding the extremum of that Lagrangian; however that is not a meaningful procedure if the Lagrangian remains unchanged under a large class of transformations of the various fields. Thus we cannot trust the usual arguments, and have to follow another way.

The method is as follows. First one adds a piece to the Lagrangian (called \mathcal{L}_c in the following) that breaks the gauge invariance. This removes the above mentioned difficulties. \mathcal{L}_c will be chosen in a special way, namely such that there exists for any value of the fields a gauge transformation that makes \mathcal{L}_c zero. Intuitively this would mean that the new Lagrangian is equal to the old one in a certain gauge. This must be true up to any order in perturbation theory; and where it is easy to do things as far as \mathcal{L}_c is concerned up to zero'th order in perturbation theory, it is less easy to construct \mathcal{L}_c such that it may be gauged to zero up to arbitrary order of

perturbation theory. It is at this point that one must introduce the so-called Faddeev-Popov ghosts. The nice thing about gauge transformations is that they have a group structure, which means that it is sufficient to know the infinitesimal transformations. As is clear from the work of ref. 1, appendix A the fact that the gauge transformations form a group plays indeed a crucial role in the argument.

In itself the question whether the Lagrangian without \mathcal{L}_c is the same in some sense as the one with \mathcal{L}_c is not very relevant since the former Lagrangian is meaningless. From this point of view it is amusing to look in the literature on quantum-electrodynamics. Much of what has been and will be said in this paper applies also to q.e.d., in particular the point concerning the meaninglessness of a gauge invariant Lagrangian. Actually, it is a nice exercise to consider q.e.d. with $\mathcal{L}_c = -\frac{1}{2} C^2$, $C = \partial_\mu A_\mu - e A_\mu^2$, as shown in appendix C, ref. 1.

3. Choice of gauge

Following ref. 1 we now specify a non singular gauge, in fact a set of gauges. We take

$$C_a = -\kappa \partial_\mu W_\mu^a + \lambda M \psi^a \quad (3.1)$$

and add $\mathcal{L}_c = -\frac{1}{2} (C_a)^2$ to the Lagrangian (2.10). This gauge depends on two parameters κ and λ ; we have a renormalizable gauge if $\kappa \neq 0$, and the physical gauge obtains for $\kappa = 1/\lambda$, $\lambda \rightarrow \infty$.

In order to define the Faddeev-Popov ghost Lagrangian we must subject C_a to a gauge transformation. One finds

$$C_a \rightarrow C_a + \kappa \Lambda^a - \lambda M^2 \Lambda^a - \kappa g \epsilon_{abc} \partial_\mu (\Lambda^b W_\mu^c) + \frac{1}{2} \lambda M g \epsilon_{abc} \Lambda^b \psi^c - \frac{1}{2} \lambda M g Z \Lambda^a. \quad (3.2)$$

In the notation of ref. 1, eq. (2.2) we have:

$$\hat{l}_{ab} = -\kappa \epsilon_{abc} (\partial_\mu W_\mu^c + W_\mu^c \partial_\mu) + \frac{1}{2} \lambda M \epsilon_{abc} \psi^c - \frac{1}{2} \lambda M Z \delta_{ab} \quad (3.3)$$

$$\hat{m}_{ab} = (\kappa \Lambda - \lambda M^2) \delta_{ab}. \quad (3.4)$$

If κ and λ are non-zero \hat{m} has an inverse, being $-(\kappa \Lambda^2 + \lambda M^2)^{-1}$ in momentum space, and we thus have a permissible gauge.

According to ref. 1, eq. (2.4) the ghost Lagrangian is:

$$\begin{aligned} \mathcal{L}_\varphi &= \varphi_a^* (\hat{m}_{ab} + g \hat{l}_{ab}) \varphi_b \\ &= \varphi_a^* (\kappa \Lambda - \lambda M^2) \varphi_a + \kappa g \epsilon_{abc} \partial_\mu \varphi_a^* \varphi_b W_\mu^c + \\ &\quad \frac{1}{2} g \lambda M \epsilon_{abc} \varphi_a^* \varphi_b \psi^c - \frac{1}{2} g \lambda M \varphi_a^* \varphi_a Z. \end{aligned} \quad (3.5)$$

We now have the complete Lagrangian necessary to obtain the Feynman rules. It is given by

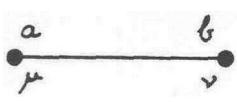
$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \frac{1}{2} (C_a)^2 + \mathcal{L}_\varphi \quad (3.6)$$

with \mathcal{L}_{inv} given by (2.10), C_a by (3.1) and \mathcal{L}_φ by (3.5).

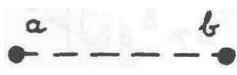
This Lagrangian contains the W - ψ mixing term $M(1 - \kappa\lambda) W_\mu^a \psi^a$; this term must be taken into account to all orders to obtain the W and ψ propagator as well as the ψW transition propagator.

The propagators and vertices following from (3.6) are

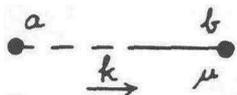
given in figs, 4 and 5 respectively.



$$\frac{\delta_{ab}}{k^2 + M^2 - i\epsilon} \left[\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \left\{ \frac{1/\kappa^2 - 1}{k^2 + \lambda M^2/\kappa - i\epsilon} + \frac{M^2(1 + \lambda^2 - \lambda/\kappa - \lambda\kappa)/\kappa^2}{(k^2 + \lambda M^2/\kappa - i\epsilon)^2} \right\} \right] \quad (a)$$



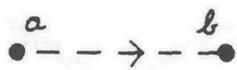
$$\delta_{ab} \left\{ \frac{k^2 + M^2/\kappa^2}{k^2 + \lambda M^2/\kappa - i\epsilon} \right\}; \quad \text{- propagator} \quad (b)$$



$$\left(\frac{1}{\kappa^2} - \frac{\lambda}{\kappa} \right) i M \kappa_\alpha \frac{1}{(k^2 + \lambda M^2/\kappa - i\epsilon)^2}; \quad \psi - W \text{ transition} \quad (c)$$



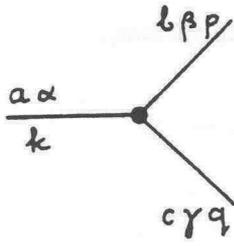
$$\frac{1}{k^2 + m^2 - i\epsilon}; \quad m^2 = 40 M^2 \quad ; \quad Z\text{-propagator} \quad (d)$$



$$\frac{1}{\kappa} \delta_{ab} \frac{1}{k^2 + \lambda M^2/\kappa - i\epsilon} \quad \begin{array}{l} \text{F - P ghost propagator} \\ -1 \text{ for every closed} \\ \text{loop.} \end{array} \quad (e)$$

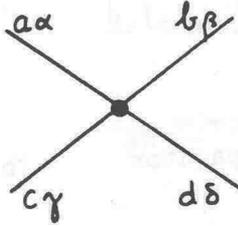
fig.4

The propagators of fig. 4 are quite involved; in most calculations it is best to make the choice $\lambda = \kappa = 1$, already the choice $\kappa = 1/\lambda$ simplifies things considerably. In the following we will mainly use the rules with $\lambda = \kappa = 1$, except for the calculations concerning diagrams with one closed loop where we will keep the full $\lambda - \kappa$ dependence. This is necessary in order to exhibit explicitly the gauge dependence (or independence) of the renormalization counterterms.



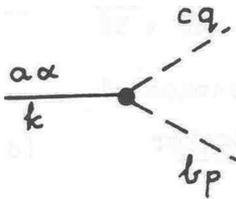
$$-i g \epsilon_{abc} \left\{ \delta_{\alpha\gamma} (k-q)_\beta + \delta_{\beta\gamma} (q-p)_\alpha + \delta_{\alpha\beta} (p-k)_\gamma \right\} \quad (a)$$

Yang-Mills three W vertex

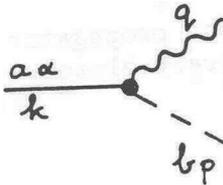


$$-g^2 \left\{ \epsilon_{gdc} \epsilon_{gba} (2 \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\gamma\delta}) + \epsilon_{gdb} \epsilon_{gca} (2 \delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta} - \delta_{\alpha\gamma} \delta_{\beta\delta}) \right\} \quad (b)$$

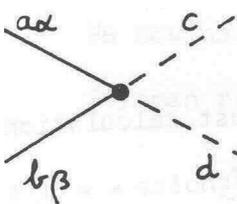
Yang-Mills four W-vertex



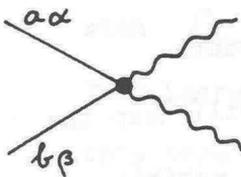
$$\frac{i}{2} g \epsilon_{abc} (p-q)_\alpha \quad \text{from} \quad \frac{g}{2} \epsilon_{abc} W_\mu^a (\partial_\mu \psi^b) \psi^c \quad (c)$$



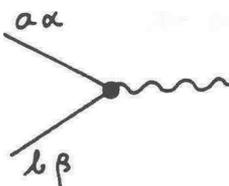
$$\frac{i}{2} g \delta_{ab} (p-q)_\alpha \quad \frac{g}{2} W_\mu^a (Z \partial_\mu \psi^a - \psi^a \partial_\mu Z) \quad (d)$$



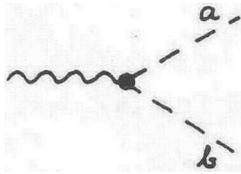
$$-\frac{1}{2} g^2 \delta_{ab} \delta_{cd} \delta_{\alpha\beta} \quad -\frac{1}{8} g^2 W_\mu^a W_\mu^a \psi^b \psi^b \quad (e)$$



$$-\frac{1}{2} g^2 \delta_{ab} \delta_{\alpha\beta} \quad -\frac{1}{8} g^2 W_\mu^a W_\mu^a Z^2 \quad (f)$$



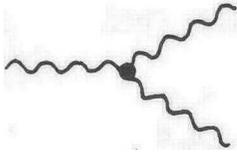
$$-g M \delta_{ab} \delta_{\alpha\beta} \quad -\frac{1}{2} g M W_\mu^a W_\mu^a Z \quad (g)$$



$$- 2 \alpha M g \delta_{ab}$$

$$- \alpha M g Z \psi^2$$

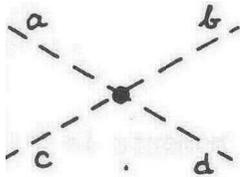
(h)



$$- 6 \alpha M g$$

$$- \alpha M g Z^3$$

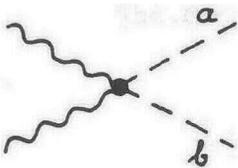
(i)



$$- \alpha g^2 (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc})$$

$$- \frac{1}{8} \alpha g^2 \psi^a \psi^a \psi^b \psi^b$$

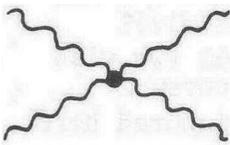
(j)



$$- \alpha g^2 \delta_{ab}$$

$$- \frac{1}{4} \alpha g^2 \psi^a \psi^a Z^2$$

(k)



$$- 3 \alpha g^2$$

$$- \frac{1}{8} \alpha g^2 Z^4$$

(l)



$$- \beta$$

$$- \frac{1}{2} \beta Z^2$$

(m)



$$- \beta \delta_{aa'}$$

$$- \frac{1}{2} \beta \psi^2$$

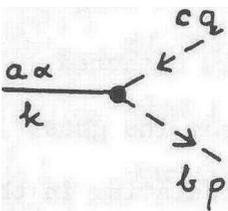
(n)



$$- \frac{2M}{g} \beta$$

$$- \frac{2M}{g} \beta Z$$

(p)



$$+ i \kappa g \epsilon_{abc} p_\alpha$$

$$- \kappa g \epsilon_{abc} \varphi^{*a} \partial_\mu (\varphi^b W_\mu^c)$$

(s)

$$\begin{aligned}
 & -\frac{1}{2} \lambda M g \delta_{ab} & -\frac{1}{2} \lambda M g \varphi^{*a} \varphi^a Z & \quad (t) \\
 & +\frac{1}{2} \lambda M g \epsilon_{abc} & +\frac{1}{2} \lambda M g \epsilon_{abc} \varphi^{*a} \varphi^b \psi^c & \quad (U)
 \end{aligned}$$

fig. 5

The convention for the direction of the momenta is that they are all considered to be ingoing. Thus for the three point vertices one has $k+p+q=0$.

The physical gauge is obtained by choosing $\kappa = 1/\lambda$ and taking the limit $\lambda \rightarrow \infty$. If we ignore divergencies (we wish to emphasize that these divergencies cannot be ignored here) the following happens. The W-propagator reduces to

$$\frac{\delta_{\mu\nu} + k_{\mu} k_{\nu} / M^2}{k^2 + M^2} .$$

The ψ and ghost propagator behave as $1/\lambda^2$ and $1/\lambda$ respectively, and all diagrams containing ψ or φ internal lines go to zero except when the Z-ghost vertex occurs. This vertex behaves as λ . A contribution remains in the limit $\lambda = \infty$ from those diagrams that contain closed ghost loops with only Z-lines attached. Since the momentum dependence has disappeared from the ghost propagator the integration over the momentum circulating in the ghost loop reduces to the integral over a polynomial, which is

zero according to the continuous dimension prescription.

To exhibit the dangers involved here we consider a ghost-loop with two outgoing Z-lines (fig. 6).

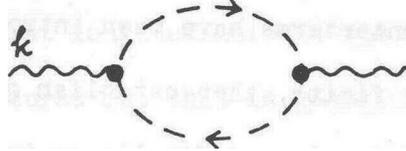


fig. 6

The corresponding expression is, with $\kappa = 1/\lambda$:

$$\frac{3}{4} \lambda^2 M^2 g^2 \int d_n p \frac{\lambda^2}{(p^2 + \lambda^2 M^2)((p-k)^2 + \lambda^2 M^2)} .$$

Performing first the limit $\lambda \rightarrow \infty$ one obtains indeed zero.

However, the integral may be calculated:

$$\frac{3}{4} \lambda^2 M^2 g^2 i\pi^{n/2} \Gamma(2 - \frac{n}{2}) \int_0^1 dx \frac{1}{(\lambda^2 M^2 + k^2 x(1-x))^{2-n/2}} .$$

This behaves as $\lambda^{n/2}$, that is as λ^2 for $n = 4$, and the limit $\lambda \rightarrow \infty$ does not exist. Similarly for diagrams with one closed loop and arbitrarily many outgoing Z-lines.

Now, if we evaluate everything for $n < 0$ then we may take the limit $\lambda \rightarrow \infty$ and the quoted result follows. The independence of the physical sector of the S-matrix of λ and K guarantees that no difficulties arise. Stated differently, if we know that for certain values of λ and K the S-matrix has a certain singularity structure in n then this singularity structure persists for any value of λ and K . We may thus evaluate the

S-matrix for $n < 0$ and then continue to $n = 4$.

However, if one introduces counter terms to cancel divergencies then the situation becomes different, because these counter terms are in general gauge dependent. It is really not very sensible to try to do everything in the physical gauge, that is to take the limit $\lambda \rightarrow \infty$ before counterterms have been introduced. First one must make the S-matrix finite, then establish gauge invariance for the renormalized S-matrix and finally go to the physical gauge to establish unitarity. Of course, for practical calculations it is not necessary to go to the physical gauge as long as one knows that the theory is gauge invariant and that there exists a physical gauge.

4. Physical particles

Consider now sources $J_{a\mu}^W$, J^Z and J_a^ψ coupled to the fields W_μ^a , Z and ψ^a :

$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \frac{1}{2} C_a^2 + \mathcal{L}_\varphi + J_{a\mu}^W W_\mu^a + J^Z Z + J_a^\psi \psi^a. \quad (4.1)$$

According to ref. 1, eq. (2.6) a source, or set of sources is physical if the product of the source and the field-independent part in the transformation law of the fields is zero. In this case if

$$- J_{a\mu}^W \delta_\mu - M J_a^\psi = \partial_\mu J_{a\mu}^W - M J_a^\psi = 0. \quad (4.2)$$

Any set of sources that satisfies $\partial_\mu J_{a\mu}^W = 0$ and $J_a^\psi = 0$ separately satisfies this equation. There is no restriction on J^Z . Thus transversal W's (emitted by a source J_μ with $\partial_\mu J_\mu = 0$) and the Z

are physical particles. The general solution of (4.2) is to take any J^Z and $J_{a\mu}^W$ and to take $J_a^\psi = \partial_\mu J_{a\mu}^W / M$. Thus the physical sources appear in \mathcal{L} as follows:

$$J_{a\mu}^W (W_\mu^a - \frac{1}{M} \partial_\mu \psi^a) + J^Z Z . \quad (4.3)$$

It appears that longitudinal W's can be emitted by these sources; however, it turns out that in going on mass-shell this possibility eliminates. To see this consider a longitudinal source $J_{a\mu}^W = \partial_\mu J_a^W$. The source J_a^W is then coupled to the field combination:

$$\partial_\mu W_\mu^a - \frac{1}{M} \partial^2 \psi^a \equiv X^a . \quad (4.5)$$

The propagator of this field combination X^a has no pole; according to the Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta \psi^a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi^a} = 0$$

and the equation $C_a = 0$ (to be discussed later, in fact it will be shown that C_a is a free field) we find

$$X^a = \text{combination of quadratic and linear terms.} \quad (4.6)$$

In other words, the equation of motion for the X-combination is like a Klein Gordon equation without derivatives. The associated propagator is simply a constant, and has no pole. In passing to the S-matrix the external lines obtained from the longitudinal part of $J_{a\mu}^W$ vanish. So finally in considering physical particles one may put

$$\partial_\mu J_{a\mu}^W = J_a^\psi = 0. \quad (4.7)$$

The above may be illustrated in detail. For simplicity we will take $\lambda \cdot \kappa = 1$ in the propagators and vertices of figs. 4 and 5.

Consider a source coupled to the combination X^a of (4.5). With $C_a = -\frac{1}{\lambda} \partial_\mu w_\mu^a + \lambda M \psi^a$ being a free field we may add a coupling

$$-\frac{1}{\lambda M^2} \partial_\mu J_{a\mu}^W(C_a)$$

without affecting the S-matrix between physical states. We get the coupling:

$$J_{a\mu}^W(w_\mu^a - \frac{1}{\lambda^2 M^2} \partial_\mu \partial_\nu w_\nu^a). \quad (4.8)$$

This coupling is represented by a vertex of the form

$$J_{a\mu}^W(k) (\delta_{\mu\nu} + \frac{1}{\lambda^2 M^2} k_\mu k_\nu). \quad (4.9)$$

Let now $J_{a\mu}^W(k)$ be a longitudinal source:

$$J_{a\mu}^W(k) = k_\mu J_a^W(k). \quad (4.10)$$

To the combination (4.9) will be attached a W-propagator. With $\kappa = 1/\lambda$ this propagator has the form

$$\frac{\delta_{ab}}{k^2 + M^2 - i\epsilon} \left[\delta_{\nu\alpha} + k_\nu k_\alpha \frac{\lambda^2 - 1}{k^2 + \lambda^2 M^2} \right]. \quad (4.11)$$

Substituting (4.10) into (4.9) and multiplying this with the propagator (4.11) we obtain

$$J_b^W \cdot \frac{k_\alpha}{M^2} \quad (4.12)$$

and indeed the poles in the propagator have been cancelled out.

In conclusion we see that the physical sources satisfying (4.2) with $J_a^W \neq 0$ do not contribute to the S-matrix when going

on mass-shell (that is multiply by $k^2 + m^2$ and take the limit $k^2 = -m^2$ where m may be any mass). The physical particles are W 's emitted by sources J_μ satisfying $\partial_\mu J = 0$ and furthermore the Z -particles.

In higher order of perturbation theory nothing changes in this particular model. A physical transversal W is emitted by a source that obeys the equation of fig. 7:

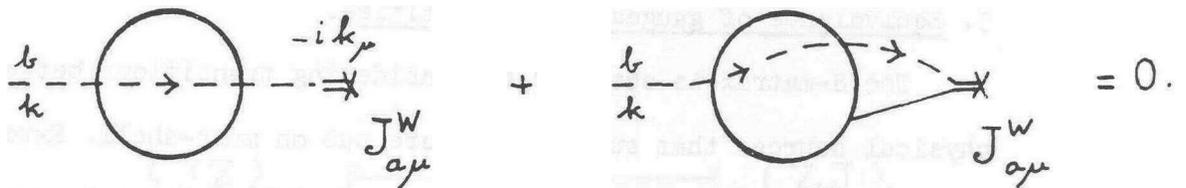


fig. 7

The shown vertex in fig. 7 is obtained from the W transformation law (2.4)

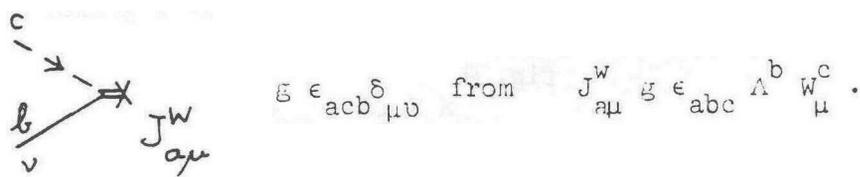


fig. 8

For the very simple reason that our theory is Lorentz and I -spin invariant we know that the second type of diagrams shown in fig. 7 must be proportional to $k_\mu \delta_{ab}$ there being no other external momenta. The first kind is also proportional to $\epsilon_{sb \mu}$, and the condition that J^W is physical remains as

before, i.e. $k_\mu J_{a\mu}^W = 0$. Similarly for the Z and ψ fields.

Of course this is not a general situation. One could have some more scalar particles with I-spin 1, like the ψ 's, and there may be some mixing effects. Or, if I-spin is no more conserved one could have some Z- ψ mixing. A certain linear combination of Z and ψ fields would become the physical field, the remaining Z, ψ field combinations would become the unphysical fields.

5. Equivalence of gauges. Ward identities.

The S-matrix is obtained by considering transitions between physical sources that subsequently are put on mass-shell. External line factors Z_e must be included. These factors are obtained by considering self-energy diagrams. Consider for instance irreducible W self energy diagrams, fig. 9

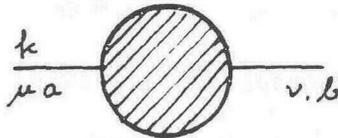


fig. 9

The corresponding expression will be of the form

$$\delta_{ab} \left(A(k^2) \delta_{\mu\nu} + B(k^2) k_\mu k_\nu \right) \quad (5.1)$$

where A and B are functions of k^2 . The bare W-propagator in the gauge with $\lambda = \kappa = 1$ is of the form

$$\frac{\delta_{ab} \delta_{\mu\nu}}{k^2 + M^2 - i\epsilon} \quad (5.2)$$

Repeated insertion of the diagrams of fig. 9 leads to the W-propagator:

$$\delta_{ab} \frac{\delta_{\mu\alpha}}{k^2 + M^2} \left\{ \left(1 - \frac{A}{k^2 + M^2} \right) \delta_{\alpha\nu} - \frac{B}{k^2 + M^2} k_\alpha k_\nu \right\}^{-1} =$$

$$= \delta_{ab} \left\{ \frac{\delta_{\mu\nu}}{k^2 + M^2 - A} + k_\mu k_\nu \frac{B}{(k^2 + M^2 - A)(k^2 + M^2 - A - Bk^2)} \right\}. \quad (5.3)$$

Consider now a simple source-source transition, with factors Z in the external lines. See fig. 10.

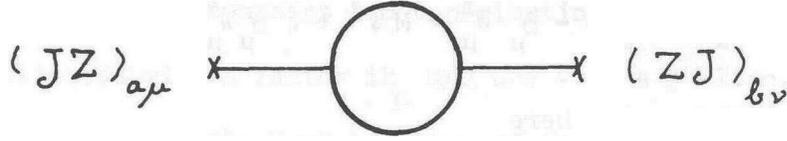


fig. 10

These factors Z can in general be matrices and must be chosen such that one obtains the residue 1 when going on mass-shell. Taking Z to be of the form

$$Z_1 \delta_{\mu\nu} + Z_2 k_\mu k_\nu$$

and remembering that we need to consider physical sources only we see that Z_2 and B do not contribute. The pole of the propagator (5.3) will be at some point $k^2 = -M^2 + A_0$, and we may develop:

$$A(k^2) = A_0 + (k^2 + M^2 - A_0) A_1 + (k^2 + M^2 - A_0)^2 A_{\text{rest}}(k^2).$$

Near the pole $k^2 + M^2 - A_0$ the diagram of fig. 10 behaves as

$$\frac{Z_1^2}{(1 - A_1)(k^2 + M^2 - A_0)}$$

showing that the external line factor Z for the physical W 's is given by $\sqrt{1 - A_1} \delta_{\mu\nu}$.

We have considered the external line factor in probably a little to excessive detail, but mainly we want to emphasize that this factor is really a matrix that is best defined on the basis of fig. 10.

Consider now the gauge (3.1) with $\lambda = \kappa = 1$. Let us choose now an infinitesimally different gauge, specified by:

$$C_a + \epsilon R_a = - \partial_\mu W_\mu^a + M \psi^a + \epsilon \partial_\mu W_\mu^a \quad (5.4)$$

with small ϵ . Thus here

$$R_a = \partial_\mu W_\mu^a \quad (5.5)$$

and after a gauge transformation, in the notation of ref. 1:

$$R'_a = R_a + g \hat{\rho}_{ab} \Lambda_b + \hat{r}_{ab} \Lambda_b \quad (5.6)$$

with

$$\hat{r}_{ab} = - \delta_{ab} L \quad (5.7)$$

$$\hat{\rho}_{ab} = \epsilon_{abc} [(\partial_\mu W_\mu^c) + W_\mu^c \partial_\mu] = - \epsilon_{abc} \overleftarrow{\partial}_\mu W_\mu^c. \quad (5.8)$$

The quantities C , R , $\hat{\rho}$ and \hat{r} are precisely the quantities occurring in fig. 2 of ref. 1. Fig. 11 gives the details.

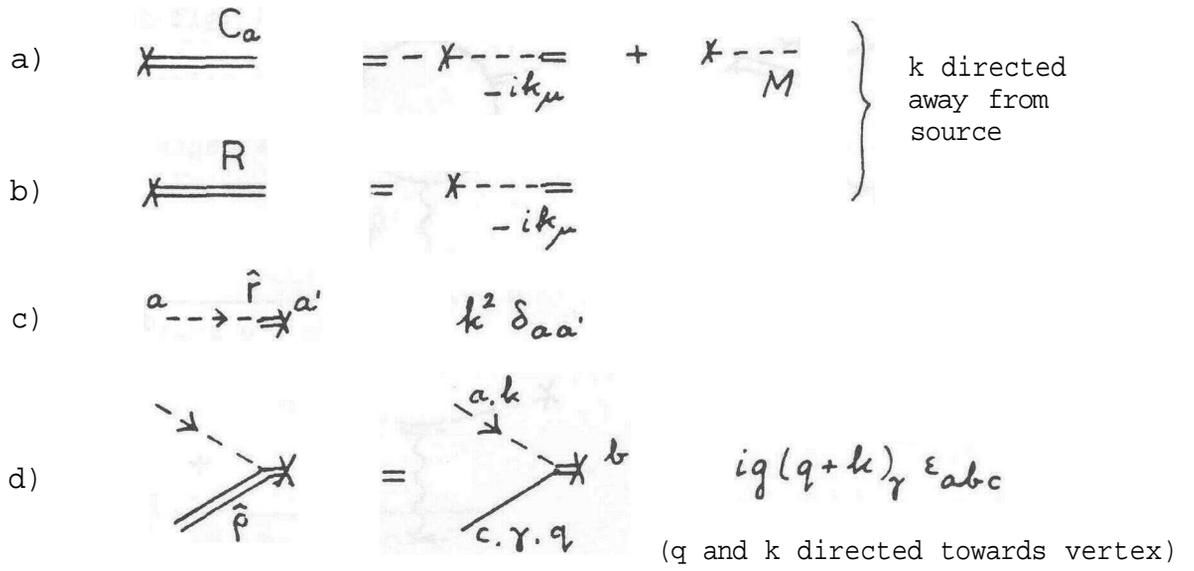
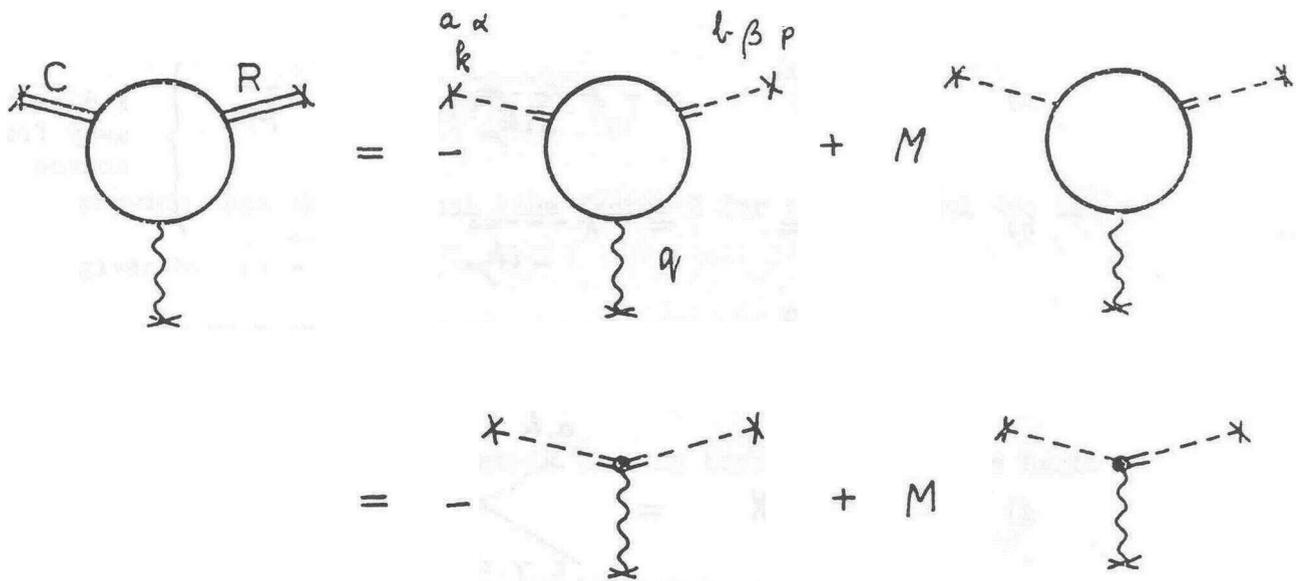


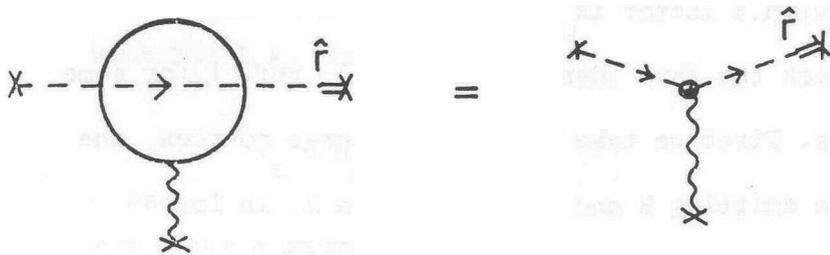
fig. 11

Note that C_a contains two contributions, one where a W-line is contracted with a factor ik_μ and one with a ψ -line.

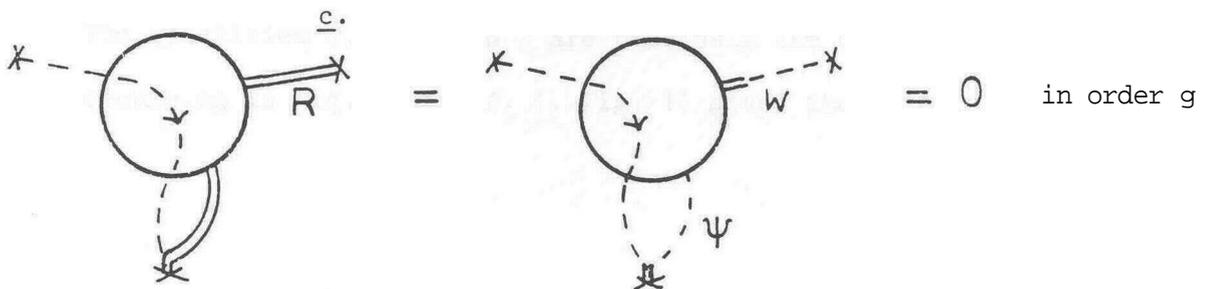
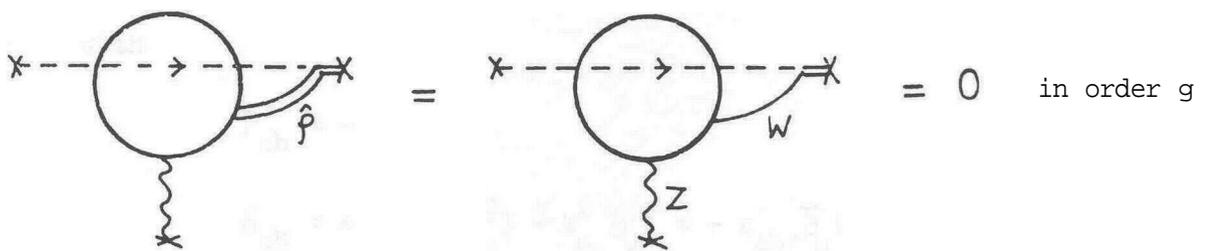
Let us check the Ward identity of fig. 3, ref. 1 for some simple diagrams. First we take the case of three sources, one emitting C, one emitting R and one emitting a Z. In lowest order one has fig. 12



a. Involves vertices (g) and (d) of fig. 5.



b. Involves vertex (t) of fig. 5.



d.

fig. 12

Note that fig. 12d contains the vertex related to a gauge transformation of Z.

The algebraic expressions corresponding to figs. 12a and 12b are:

$$-gM\delta_{ab} \frac{(pk)}{(p^2+M^2)(k^2+M^2)} + M\frac{g}{2}\delta_{ab}(k-q,p) \frac{1}{(p^2+M^2)(k^2+M^2)}$$

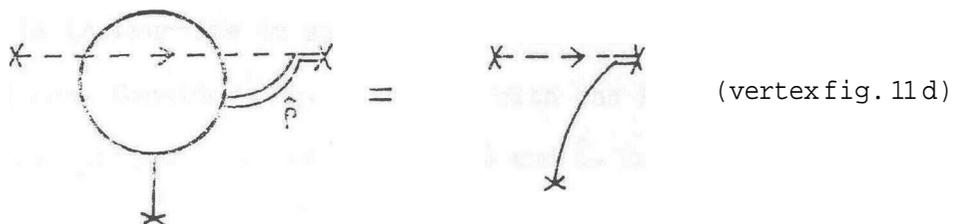
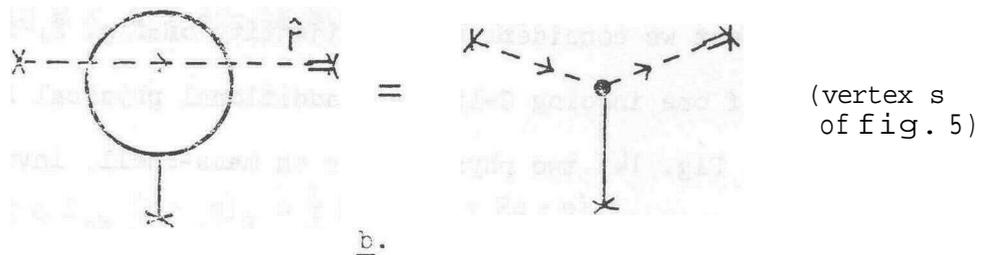
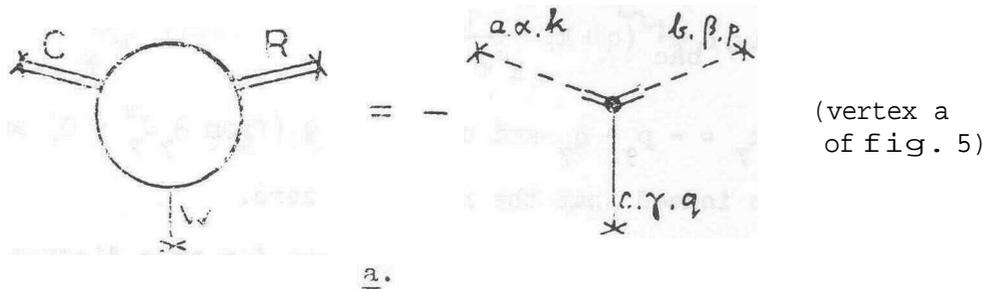
and

$$-\frac{Mg}{2}\delta_{ab} \frac{p^2}{(p^2+M^2)(k^2+M^2)}$$

respectively. With $q = -k-p$ it is seen that they add up to zero.

A slightly more complicated case, when also the vertex of fig. 11d contributes is the case of one external W-line, order g.

We obtain the diagrams of fig. 13 (note that there is no ψWW vertex):



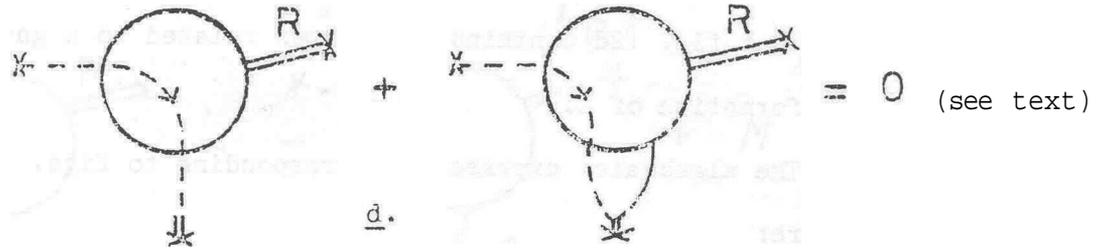


fig. 13

In this Ward identity we make the W physical, that is the first diagram of fig. 13d is zero because $\partial_\gamma J_\gamma^W = 0$. Furthermore the W is taken on mass-shell, i.e. we multiply with $q^2 + M^2$ and take the limit $q^2 + M^2 = 0$. Then also the second diagram of fig. 13d becomes zero. Diagrams 13a, b and c give:

$$\begin{aligned}
 & -ig \epsilon_{abc} \left\{ -q^2 p_\gamma - (kp - p^2)q_\gamma \right\} \frac{1}{(k^2 + M^2)(p^2 + M^2)} \\
 & + ig \epsilon_{cba} p_\gamma \frac{p^2}{(p^2 + M^2)(k^2 + M^2)} \\
 & + ig \epsilon_{bac} (q+k) \frac{1}{k^2 + M^2}
 \end{aligned}$$

With $k_\gamma = -p_\gamma - q_\gamma$ and using $q_\gamma = 0$ (from $\partial_\gamma J_\gamma^W = 0$) and $q^2 = -M^2$ we see indeed that the result is zero.

From the above it is, seen that for tree diagrams physical sources on mass-shell do not give contributions of the type fig. 13d.

Next we consider the Ward identity of fig. 3, ref. 1 for the case of one ingoing C -line and additional. physical sources on mass-shell. Fig. 14. two physical W 's on mass-shell, involving vertex (a) of fig. 5.

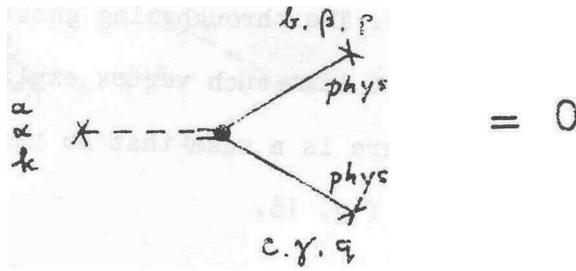


Fig. 14

Indeed:

$$\exists \epsilon_{abc} \left\{ \delta_{\beta\gamma} (p^2 - q^2) + q_\beta q_\gamma - p_\beta p_\gamma \right\} = 0 \quad \text{if } p_\beta = q_\gamma = 0 \text{ and} \\ p^2 + M^2 = q^2 + M^2 = 0.$$

Fig. 15 gives, the case of a physical W and a physical Z line
In addition to C. This involves vertices (g) and (d) of fig. 5.

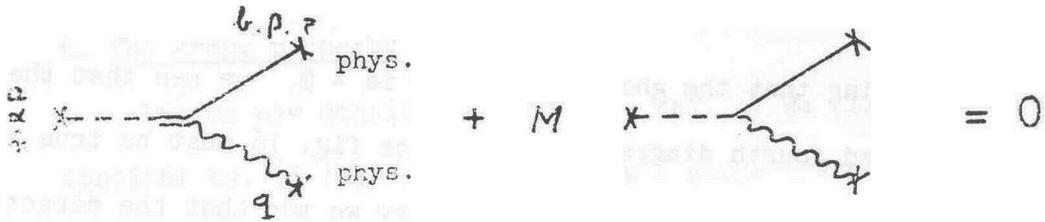


fig. 15

The first diagram of fig. 15 gives

$$-ig M k_\beta \delta_{ab} = -ig M (q + p)_\beta \delta_{ab}.$$

The second

$$M \frac{i}{2} g \delta_{ab} (k - q)_\beta = \frac{i}{2} Mg \delta_{ab} (-2q - p)_\beta.$$

With $p_\beta = 0$ they add up to zero.

Finally it is instructive to see what happens if there are several C-lines. Consider fig. 3, ref. 1 with one R-line and take

... instead of \hat{p} and \hat{q} we get \hat{m} and \hat{l} . Consider the

second diagram of fig. 3, ref. 1 . The throughgoing ghost line has \hat{l} lines attached. Showing the last such vertex explicitly, and taking into account that there is a case that no vertex occurs on the ghost line we get fig. 16.

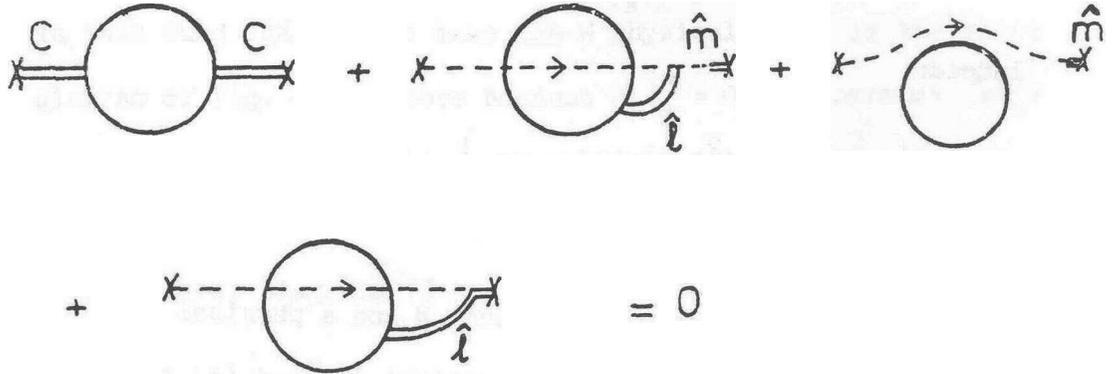


fig. 16

Remembering that the ghost propagator is $-\hat{m}^{-1}$ we see that the second and fourth diagram cancel. Since fig. 16 must be true also in zero'th order of perturbation theory we see that the direct contraction of two C-lines must be one (this gives a nice check on the propagators a,b and c of fig. 4). In general then, the third diagram of fig. 16 cancels that term in the first where the C's are connected directly. We therefore get as result that the diagrams with two C's, excluding the directly connected case add up to zero. Similarly for many C's. This is the property referred to in sect. 4, and it can be expressed by the equation of notion $C_a = 0$. For tree diagrams this remains true if one adds sources that emit physical particles on mass-shell. One example is given in fig. 17, two C-lines, one physical Z on mass-shell. This involves vertices (g), (h), and twice (d) of fig. 5.

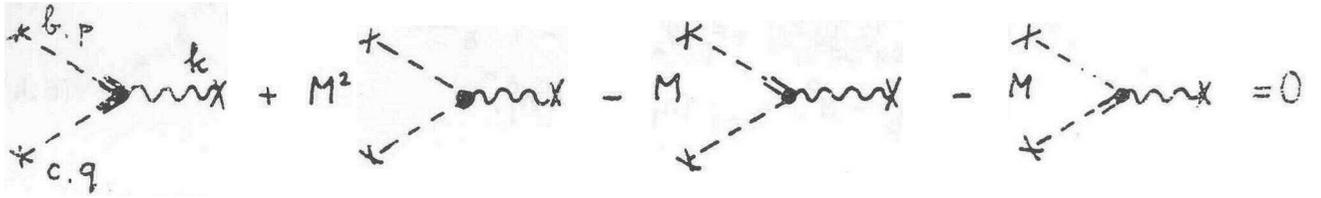


fig.17

The first, third and fourth diagram give:

$$\begin{aligned}
 & g M (pq) \varepsilon_{bc} - \frac{1}{2} M g (pq - pk) - \frac{1}{2} M g (qp - qk) = \\
 & = \frac{1}{2} g M (p + q, k) = - \frac{1}{2} g M k^2 = \frac{1}{2} g M m^2 = 2g \alpha M^3
 \end{aligned}$$

which indeed cancels the second diagram.

6. The group property

Let us now consider Appendix A, ref. 1. We also need to consider eq. (5.1), ref. 1. Applying a gauge transformation to eq. (3.3) with $\lambda = \kappa = 1$ we find:

$$\hat{l}_{ab} \rightarrow \hat{l}_{ab} + g \hat{d}_{abc} \Lambda_c + \hat{e}_{abc} \Lambda_c \quad (6.1)$$

with

$$\hat{e}_{abc} \Lambda_c = \varepsilon_{abc} \overleftrightarrow{\partial}_\mu \partial_\mu \Lambda^c + \frac{1}{2} M^2 \varepsilon_{abc} \Lambda^c \quad (6.2)$$

$$\begin{aligned}
 g \hat{d}_{abc} \Lambda_c = & \left\{ -\delta_{ac} \overleftrightarrow{\partial}_\mu W_\mu^b + \delta_{bc} \overleftrightarrow{\partial}_\mu W_\mu^a - \frac{1}{4} M \delta_{ac} \psi^b \right. \\
 & \left. + \frac{1}{4} M \delta_{bc} \psi^a + \frac{1}{4} M Z \varepsilon_{abc} + \frac{1}{4} M \delta_{ab} \psi^c \right\} \Lambda^c.
 \end{aligned} \quad (6.3)$$

The first order part of eq. (A6), ref. 1 reads:

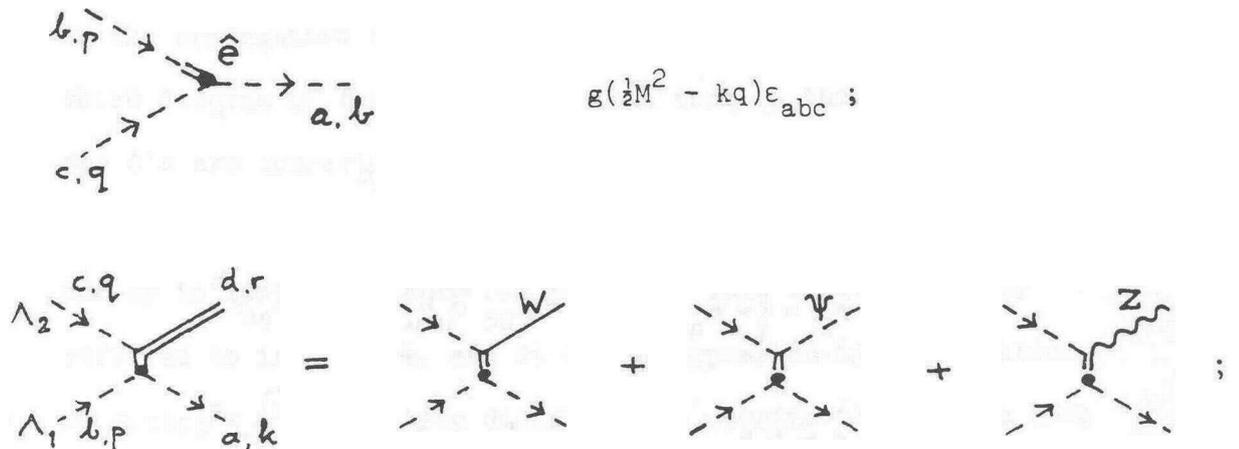
$$\begin{aligned}
& g \epsilon_{abc} \overleftarrow{\partial}_\mu (\Lambda_1^b \partial_\mu \Lambda_2^c - \Lambda_2^b \partial_\mu \Lambda_1^c) + \frac{1}{2} g M^2 \epsilon_{abc} (\Lambda_1^b \Lambda_2^c - \Lambda_2^b \Lambda_1^c) \\
& = \frac{1}{2} g (\partial_\mu^2 - M^2) \epsilon_{acd} (\Lambda_1^c \Lambda_2^d - \Lambda_2^c \Lambda_1^d). \quad (6.4)
\end{aligned}$$

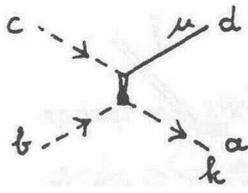
With $\overleftarrow{\partial}_\mu = -\overrightarrow{\partial}_\mu$ one observes, working out the differentiations, that this equation holds. The second order part of eq. (A6), ref. 1 is:

$$\begin{aligned}
& g^2 \left\{ -\delta_{ac} \overleftarrow{\partial}_\mu W_\mu^b + \delta_{bc} \overleftarrow{\partial}_\mu W_\mu^a - \frac{1}{4} M \delta_{ac} \psi^b + \frac{1}{4} M \delta_{bc} \psi^a \right. \\
& \quad \left. + \frac{1}{4} M Z \epsilon_{abc} + \frac{1}{4} M \delta_{ab} \psi^c \right\} (\Lambda_1^b \Lambda_2^c - \Lambda_2^b \Lambda_1^c) = \\
& = -\frac{1}{2} g^2 (\epsilon_{abd} \partial_\mu W_\mu^d + \frac{1}{2} M \epsilon_{abd} \psi^d - \frac{1}{2} M Z \delta_{ab}) \epsilon_{bce} (\Lambda_1^c \Lambda_2^e - \Lambda_2^c \Lambda_1^e). \quad (6.5)
\end{aligned}$$

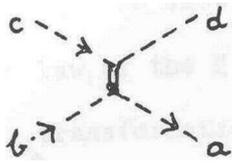
Antisymmetrizing the expression in curly brackets on the left hand side of (6.5) one will have no trouble verifying this equation.

The vertices corresponding to (6.2) and (6.3) are given in fig. 18.

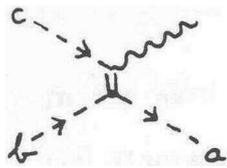




$$g^2 (- ik_{\mu} \delta_{ac} \delta_{bd} + ik_{\mu} \delta_{bc} \delta_{ad})$$



$$g^2 (-\frac{1}{4} M \delta_{ac} \delta_{bd} + \frac{1}{4} M \delta_{bc} \delta_{ad} + \frac{1}{4} M \delta_{ab} \delta_{cd})$$



$$\frac{1}{4} g^2 M \epsilon_{abc}$$

fig. 18

From fig. 18 we deduce fig. 19, with the definition of fig. 20

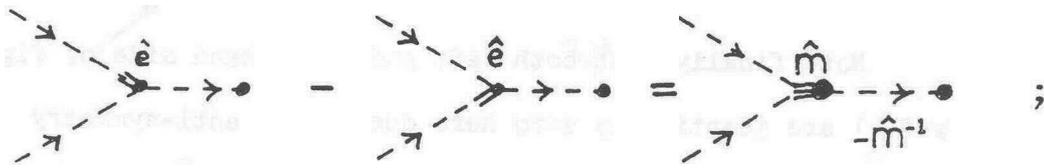
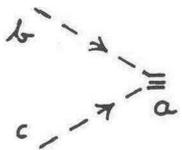


fig. 19



$$- g \epsilon_{abc}$$

fig. 20

Indeed, the left hand side becomes

$$\left\{ g \left(\frac{1}{2} M^2 - kq \right) \epsilon_{abc} - g \left(\frac{1}{2} M^2 - kp \right) \epsilon_{acb} \right\} \cdot \frac{1}{k^2 + M^2} = g \epsilon_{abc}.$$

Furthermore we have fig. 21:

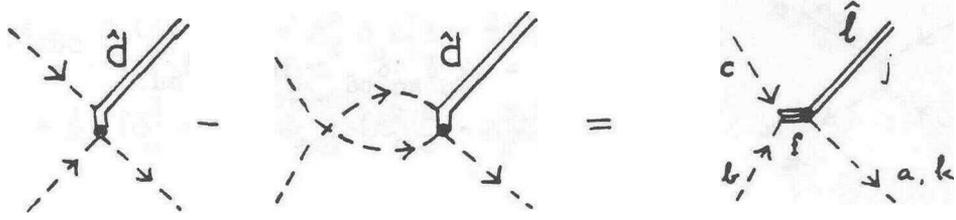


fig. 21

the left hand side becomes, if we first take the W part in \hat{d} and \hat{l} :

$$-ig^2 k_\mu (\delta_{ac} \delta_{bj} - \delta_{ab} \delta_{cj}) = g^2 \epsilon_{fcb} \epsilon_{jaf} (ik_\mu)$$

which corresponds to the right hand side of fig. 21 if we use the vertices (s), fig. 5 and fig. 20. Similarly for the last two vertices of fig. 18. In this way one convinces oneself of the truth of fig. A2, ref. 1 in our model.

Note finally that both left and right hand side of fig. A3, ref. 1 are identically zero here due to the anti-symmetry properties of the various vertices.

7. Renormalization

In this section we will consider one closed loop infinities. In the framework of the continuous dimension method these take the form of simple poles for $n = 4$. Subtracting the poles and their residues makes the theory finite for $n = 4$. In the Ward identities, if they are to be true in the finite theory, we must also subtract the pole terms arising from loops that involve the vertices that are related to the gauge transformations but do not

appear in the Lagrangian. In this way one obtains renormalized gauge transformations.

To show the method we compute the renormalized transformation law of the Z field which is the easiest case. The unrenormalized transformation law is:

$$Z' = Z + \frac{1}{2} g \Lambda^a \psi^a . \quad (7.1)$$

In the Ward identities there may occur, up to one closed loop, two diagrams involving the vertex implied by this law. See fig, 22

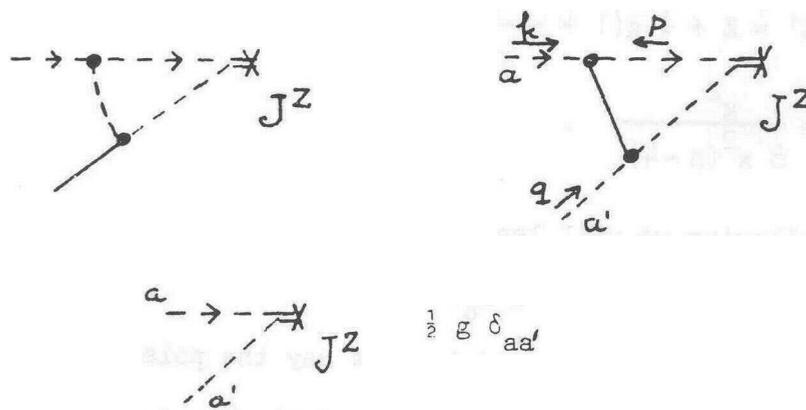


fig. 22

Apart from the transformation vertex shown these diagrams involve the vertices c, s and u of fig. 5. The first diagram of fig. 22 is convergent. The second is logarithmically divergent which implies that the value of the masses is irrelevant and all may be taken to be M. One obtains

$$= \frac{g^3}{4} \epsilon_{dba} \epsilon_{dba'} \frac{1}{k^2} \int d_n p \frac{-p^2}{(p^2 + M^2)^3} \Rightarrow$$

$$\frac{g^3}{2} \delta_{aa'} \frac{1}{\kappa^2} i\pi^{n/2} \frac{1}{(M^2)^{2 - \frac{n}{2}}} \frac{\Gamma(2 - \frac{n}{2})}{\Gamma(2)} \quad (7.2)$$

The pole and its residue at $n = 4$ are:

$$- i\pi^2 \frac{g^3}{\kappa^2} \delta_{aa'} \frac{1}{n-4} \quad (7.3)$$

The Feynman rules include a factor $(2\pi)^4 i$ for every vertex; one must divide by this factor in going back to the transformation law. Moreover we provide a factor η as indicated in ref. 1, and a minus sign since we have to subtract this infinity. In this way we arrive at the renormalized transformation law for the Z field:

$$Z' = Z + \frac{1}{2} g \left(1 + \eta \frac{Z_0}{\kappa^2} \right) \Lambda^a \Psi^a$$

$$\text{with } Z_0 = \frac{g^2}{8 \pi^2 (n-4)} \quad .$$

In the following we will leave the factor Z_0 understood: every η is always accompanied by a Z_0 .

We have bluntly computed in this way the poles and their residues for the transformation laws of the W and Ψ fields as well as for all the two and three point vertices occurring in the Lagrangian.

$$W_{\mu}^{i,a} = W_{\mu}^a + g \left(1 + \eta \frac{1}{\kappa^2} \right) \epsilon_{abc} \Lambda^b W_{\mu}^c - \left(1 + \eta \left(\frac{1}{2\kappa^2} - \frac{3}{2} \right) \right) \partial_{\mu} \Lambda^a \quad (7.4)$$

$$Z' = Z + \frac{1}{2} g \left(1 + \eta \frac{1}{\kappa^2} \right) \Lambda^a \Psi^a \quad (7.5)$$

$$\begin{aligned} \Psi'^a = & \Psi^a + \frac{1}{2} g \left(1 + \eta \frac{1}{\kappa^2} \right) \epsilon_{abc} \Lambda^b \Psi^c - \frac{1}{2} g \left(1 + \eta \frac{1}{\kappa^2} \right) Z \Lambda^a \\ & - M \left(1 + \eta \left(\frac{1}{\kappa^2} - \frac{3}{4} \frac{\lambda}{\kappa} \right) \right) \Lambda^a \quad (7.6) \end{aligned}$$

The obtained Lagrangian can be written as

$$\hat{\mathcal{L}} = \mathcal{L}_{\text{inv}} - \frac{1}{2} (-\kappa \partial_{\mu} \bar{W}_{\mu}^a + \lambda M \psi^a)^2 + \mathcal{L}_{\varphi}. \quad (7.7)$$

Note that the second term in eq. (7.7) contains no poles.

$$\begin{aligned} \mathcal{L}_{\varphi} = & \left(1 + \eta \left(\frac{1}{2\kappa^2} - \frac{3}{2}\right)\right) \kappa (\varphi_a^* \square \varphi_a) - \left(1 + \eta \left(\frac{1}{\kappa^2} - \frac{3}{4} \frac{\lambda}{\kappa}\right)\right) \lambda M^2 \varphi_a^* \varphi_a \\ & + (1 + \eta \frac{1}{\kappa^2}) g \kappa \epsilon_{abc} \partial_{\mu} \varphi_a^* \varphi_b W_{\mu}^c \\ & + (1 + \eta \frac{1}{\kappa^2}) \frac{g}{2} \lambda M \epsilon_{abc} \varphi_a^* \varphi_b \psi^c \\ & - (1 + \eta \frac{1}{\kappa^2}) \frac{g}{2} \lambda M \varphi_a^* \varphi_a Z. \end{aligned} \quad (7.8)$$

\mathcal{L}_{φ} has been computed by considering self energy and triangle diagrams involving a through going ghost line, and turns out to be, as should, what corresponds to the renormalized gauge transformations applied to the original C_a .

The renormalized invariant Lagrangian is:

$$\begin{aligned} \mathcal{L}_{\text{inv}}^{\text{ren}} = & -\frac{1}{4} \left\{1 + \eta \left(-\frac{25}{6} + \frac{1}{\kappa^2}\right)\right\} (\partial_{\mu} W_{\nu}^a - \partial_{\nu} W_{\mu}^a)^2 \\ & - \frac{1}{2} M^2 \left\{1 + \eta \left(\frac{3}{4} - \frac{3}{2} \frac{\lambda}{\kappa} + \frac{7}{4} \frac{1}{\kappa^2}\right)\right\} W^2 - g \left\{1 + \eta \left(-\frac{8}{3} + \frac{3}{2} \frac{1}{\kappa^2}\right)\right\} \epsilon_{abc} \partial_{\mu} W_{\nu}^a W_{\mu}^b W_{\nu}^c \\ & - \frac{1}{2} \left\{1 + \eta \left(-\frac{9}{4} + \frac{3}{4} \frac{1}{\kappa^2}\right)\right\} (\partial_{\mu} Z)^2 - \frac{1}{2} \left\{4\alpha + \eta \left(-9\alpha \frac{\lambda}{\kappa} + \frac{15}{2} \alpha \frac{1}{\kappa^2} - 30\alpha^2 - \frac{27}{4}\right)\right\} M^2 Z^2 \\ & - \frac{1}{2} \left\{1 + \eta \left(-\frac{9}{4} + \frac{3}{4} \frac{1}{\kappa^2}\right)\right\} (\partial_{\mu} \psi)^2 - \frac{1}{2} \left\{\eta \left(-3\alpha \frac{\lambda}{\kappa} + \frac{3}{2} \alpha \frac{1}{\kappa^2} - 6\alpha^2 - \frac{9}{4}\right)\right\} M^2 \psi^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left\{ 1 + \eta \left(-\frac{3}{4} + \frac{5}{4} \frac{1}{\kappa^2} \right) \right\} g \epsilon_{abc} W_{\mu}^a \partial_{\mu} \psi^b \psi^c \\
& + \frac{1}{2} \left\{ 1 + \eta \left(-\frac{3}{4} + \frac{5}{4} \frac{1}{\kappa^2} \right) \right\} g W_{\mu}^a (Z \partial_{\mu} \psi^a - \psi^a \partial_{\mu} Z) \\
& - \frac{1}{2} \left\{ 1 + \eta \left(\frac{3}{4} - \frac{3}{4} \frac{\lambda}{\kappa} + \frac{7}{4} \frac{1}{\kappa^2} \right) \right\} g M W_{\mu}^2 Z \\
& - \left\{ \alpha + \eta \left(-\frac{3}{4} \alpha \frac{\lambda}{\kappa} + \frac{3}{2} \alpha \frac{1}{\kappa^2} - 6\alpha^2 - \frac{9}{8} \right) \right\} M g Z (\psi^2 + Z^2) \\
& - \left\{ \eta \left(-6\alpha \frac{\lambda}{\kappa} + 3\alpha \frac{1}{\kappa^2} - 12\alpha^2 - \frac{9}{2} \right) \right\} \frac{M^3}{g} Z \\
& + \left\{ 1 + \eta \left(-\frac{3}{4} - \frac{3}{4} \frac{\lambda}{\kappa} + \frac{5}{4} \frac{1}{\kappa^2} \right) \right\} M W_{\mu}^a \partial_{\mu} \psi^a \\
& + (4 \text{ point vertices}) \tag{7.9}
\end{aligned}$$

Indeed, one may verify that \mathcal{L}_{inv} is invariant under the transformations (7.4) - (7-6), up to terms of order η^2, Λ^2 . The Lagrangian \mathcal{L}_{inv} is nothing but the original one up to renormalization factors in the fields and the parameters, This may be seen by substituting into eq. (2.10):

$$\begin{aligned}
W_{\mu} &= (1 + \eta A) W'_{\mu} \\
Z &= (1 + \eta B) Z' + \eta C \\
\psi &= (1 + \eta B) \psi' \\
g &= (1 + D) g' \\
\alpha &= (1 + E) \alpha' \\
m &= (1 + F) m' \\
M &= (1 + F - \frac{1}{5}E) M' \tag{7.10}
\end{aligned}$$

(all up to terms of order η^2).

We then obtain eq. (7.9) for the primed variables if

$$\begin{aligned}
 A &= \eta \left(-\frac{25}{12} + \frac{1}{2\kappa^2} \right) \\
 B &= \eta \left(-\frac{9}{8} + \frac{3}{8\kappa^2} \right) \\
 C &= M g^{-1} \eta \left(-\frac{3\lambda}{2\kappa} + \frac{3}{4\kappa^2} - 3\alpha - \frac{9}{8\alpha} \right) \\
 D &= 43\eta/12 \\
 E &= \eta \left(-6\alpha - \frac{8}{3} - \frac{9}{8\alpha} \right) \\
 F &= \eta \left(-\frac{3\alpha}{2} + \frac{9}{8} \right). \tag{7.11}
 \end{aligned}$$

Of course, the same replacements(7.10) into the original gauge-transformation laws(2.4), (2.11) and (2.12), together with

$$\Lambda = \left(1 - 43\eta/12 + \eta/\kappa^2 \right) \Lambda', \tag{7.12}$$

lead to the rules (7.4) - (7.6).

It is important to observe that the physical parameters g , α and m renormalize in a gauge-independent way, i.e. D , E and F do not depend on the gauge parameters λ and κ .

Acknowledgements

Calculations necessary to obtain the renormalized Lagrangian (7.9) were performed with the help of the C.D.C. 6400 computer of the Zeeman Laboratory, Amsterdam. The authors gratefully acknowledge the hospitality and cooperation encountered.

References

1. G. 't Hooft and M. Veltman, Utrecht preprint, July 1972.
2. G. 't Hooft, Nucl.Phys. B35 (1971) 167.
3. See B.W. Lee, rapporteurs talk, XVI High Energy Conf., 1972, for an extensive list of references.