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COMBINATORICS OF GAUGE FIELDS

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Abstract: Gauge field theories can be described by many different sets of Feynman rules, depending on the particular gauge chosen. In this paper a prescription for obtaining the Feynman rules in different gauges is given. A rigorous combinatorial proof of the independence of the S-matrix of the chosen gauge is presented. The proof is general and applies to Yang-Mills type theories as well as to gravitation. For renormalizable Yang-Mills type theories it is shown that the renormalized theory is invariant with respect to renormalized gauge transformations.

1. INTRODUCTION

A gauge field theory is defined to be any quantum field theory which has a local gauge invariance [1]. The local gauge group may be any (compact or non compact) Lie group. Examples are:

(i) Massless Yang-Mills fields [2];

(ii) Massive Yang-Mills fields if the mass of the vector bosons is due to the Higgs-Kibble mechanism [3, 4, 5];

(iii) Quantum theory of gravitation.

It appears that a gauge field theory can be described by many different sets of Feynman rules corresponding to different gauges. One of these sets will only contain internal lines that correspond to physical particles (that is, no ghosts). In such a set, unitarity of the S-matrix is relatively simple to prove. It will be called the physical gauge from now on. On the other hand, some gauge field theories turn out to have also a manifestly renormalizable set of Feynman rules [4]. If indeed these different sets of Feynman rules describe the same S-matrix, and if this remains true after renormalization then we have unitary and renormalizable vector field theories.

The fact that the S-matrix is gauge independent has only been shown in a very formal way, based on path integrals (apart from some special cases [2, 5]). This "proof" has the shortcoming that it ignores all infinities, and it is precisely those infinities which

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can cause the wellknown anomalies. These anomalies can invalidate the whole theory [8, 9]. Moreover we have no insight in the renormalization procedure and the properties of the renormalized theory.

In this paper a rigorous combinatorial proof is given of the equivalence of many different gauges, for a large variety of gauge field theories. We limit ourselves to those cases for which a gauge invariant regularization procedure is known which makes all diagrams occurring in the proof finite. Indeed, such a procedure exists for a large class of gauge field theories [10, 11]. The condition which the theory then should fulfill is the following: parity changing transformations in the gauge-group are only admitted if the Bell-Jackiw-Adler anomalies that then arise in diagrams with one fermion loop, cancel [11]. If now the theory posesses a renormalizable gauge, then renormalization may be carried through in the usual way. It is essential that the renormalized theory still possesses gauge invariance since this is needed to show that the theory is unitary. Due to the occurence of ghost particles the renormalization procedure is not trivial, and it will be shown that in fact the renormalized theory is invariant with respect to gauge transformations that are different from those of the unrenormalized theory. One would expect that the group associated with the renormalized gauge transformations is the same as the group associated with the unrenormalized gauge transformation. If then the original Lagrangian is the most general for the given fields involving only renormalizable type vertices, the renormalized Lagrangian will contain just as many free parameters as the original Lagrangian.

2. PRELIMINARIES

Let the fields in our model be denoted by $A_i(x)$ and an infinitesimal gauge transformation be described by $\Lambda_a(x)$ (indices $i, j, k \ldots$ count all possible fields, indices $a, b, c \ldots$ can have as many values as there are generators in the group, and μ, ν are Lorentz indices).

Infinitesimal gauge transformations are given by

$$A_i' = A_i + g\, \hat{s}_{ia}(A)\, \Lambda_a + \hat{t}_{ia}\, \Lambda_a \quad . \tag{2.1}$$

Here g is the coupling constant with respect to which we expand. The $\hat{s}_{ia}$ may depend on the fields. The hat on both $\hat{s}$ and $\hat{t}$ symbolizes the fact that derivatives may occur. Some explicit examples are:

(i) Quantum electrodynamics of photons and electrons. See appendix C.

(ii) The model of ref. [4], sect. 6. This model resembles most closely the pure massive Yang-Mills theory. There is one more physical particle, called the Z-particle, and a triplet of Higgs-Kibble ghosts, called the ψ^a. The Lagrangian ref. [4], eq. (6.6) is invariant for the transformations

$$W_\mu^{a\prime} = W_\mu^a - g\, \epsilon_{abc}\, W_\mu^b\, \Lambda_c - \partial_\mu \Lambda_a \quad ,$$

$$Z' = Z + \tfrac{1}{2} g\, \psi^a \Lambda_a \;,$$

$$\psi^{a\prime} = \psi^a - \tfrac{1}{2} g\, \epsilon_{abc}\, \psi^b\, \Lambda_c - \tfrac{1}{2} g\, Z\, \Lambda_a - M\, \Lambda_a \;.$$

In general fields that do have a non-zero $\hat{t}$ in their transformation law are unphysical: the longitudinal W-mesons and the ψ field in this example.

Let the Lagrangian of the model, L_{inv}, be invariant under the transformation (2.1). We first give a prescription how to obtain Feynman rules for such a model in a permissible gauge, that is a non-singular gauge in the sense defined below. Next we will prove that the S-matrix so defined is the same in all continuously connected non-singular gauges as defined at the end of this section. In practice this means all non-singular gauges.

The gauge is fixed by choosing a function $C_a(x)$ of the fields $A_i(x)$. The index a has as many values as there are generators in the group. Let $C_a(x)$ transform under (2.1) as

$$C_a'(x) = C_a(x) + g\, \hat{l}_{ab}(A)\, \Lambda_b(x) + \hat{m}_{ab}\, \Lambda_b(x) \tag{2.2}$$

Again, the hat indicates the possible presence of derivatives. C_a specifies a permissible or non-singular gauge if the operator $\hat{m}_{ab}$ has an inverse $\hat{m}_{ab}^{-1}$ (in k-space non-singular for Euclidean k).

The prescription is to remove the gauge invariance of the Lagrangian by subtracting $\frac{1}{2} C_a^2$:

$$L = L_{\mathrm{inv}} - \tfrac{1}{2} C_a^2 \tag{2.3}$$

and adding a Faddeev-Popov ghost Lagrangian L_φ

$$L_\varphi = \varphi_a^* (\hat{m}_{ab} + g\, \hat{l}_{ab}(A))\, \varphi_b \;. \tag{2.4}$$

The Faddeev-Popov ghost occurs only in closed loops and obeys Fermi statistics, i.e. there is a -1 for each closed φ-loop. Furthermore the S-matrix is defined in terms of these rules with the additional prescription that all external lines are provided with a factor called Z_{e}^{-1} in the following. This factor is $(1 + F)^{-\frac{1}{2}}$ where F is the wave renormalization factor due to self energy insertions. It is needed both to make the S-matrix gauge invariant and unitary, even for finite F †.

Examples of possible C_a are:

$C = \partial_\mu A_\mu$	(Feynman gauge for q.e.d.) ,
$C_a = \partial_\mu W_\mu^a$	(Feynman gauge for the massless Y–M theory),
$C_a = \alpha\, \partial_\mu W_\mu^a$, $\alpha \to \infty$	(Landau gauge) ,
$C_a = \alpha\, \psi^a$, $\alpha \to \infty$	(Physical gauge for the model of ref. [4], sect. 6) ,

† This point, which is also pertinent to quantum electrodynamics was emphasized by Bialynicki-Birula [12]. We are indebted to J.S. Bell for bringing this paper to our attention.

$$C_a = -\partial_\mu W^a_\mu + M\,\psi^a \qquad \text{(Renormalizable gauge for the model of ref. [4], sect. 6)}\,.$$

The particles are divided into physical and unphysical ones. The Faddeev-Popov ghost is always unphysical; for the other fields this division is defined in terms of sources. A source contribution is added to the Lagrangian

$$L = L_{\text{inv}} - \tfrac{1}{2} C_a^2 + L_\varphi + J_i A_i \tag{2.5}$$

and a source $J_i(x)$ is said to be a physical source † if

$$J_i \langle 0 | g\,\hat{s}_{ia}\,\varphi^a + \hat{t}_{ia}\,\varphi^a | \varphi \rangle = 0 \tag{2.6}$$

and $|\varphi\rangle$ denotes an incoming Faddeev-Popov ghost on or off-mass-shell. In zeroth order of perturbation theory this means that $J_i\,\hat{t}_{ia}$ must be zero for all a if J_i is physical; in most cases occurring in practice the matrix element in (2.6) is simply proportional to $\hat{t}_{ia}$ or a linear combination of the $\hat{t}_{ia}$, and the zeroth order definition remains valid to all orders.

From the Lagrangian (2.5) the Feynman rules can be obtained. The bilinear terms in this Lagrangian now have an inverse; minus this inverse is by definition the A-field propagator. The remaining terms describe the vertices.

Suppose now that we choose another gauge function C'_a, with $C'_a \equiv C_a + \epsilon R_a$, ϵ infinitesimal. Let R_a transform as

$$R'_a = R_a + \hat{r}_{ab}\,\Lambda_b + g\,\hat{\rho}_{ab}(A)\,\Lambda_b \quad . \tag{2.7}$$

Here also $\hat{r}_{ab}$ is independent of the fields A_i, and the field dependent terms are of higher order in g.

We see that the Lagrangian then changes by an amount

$$\Delta L = -\epsilon\, C_a R_a + \epsilon\, \varphi_a^* (\hat{r}_{ab} + g\,\hat{\rho}_{ab}(A))\,\varphi_b \quad . \tag{2.8}$$

Equivalence of the gauges C_a and C'_a follows if in the gauge C_a a change in the Lagrangian proportional to ΔL does not change S-matrix elements between physical states. We can formulate this condition in terms of a Ward identity, which we have written in terms of diagrams in fig. 2.

The vertices have been defined in fig. 1. The + signs in the blobs denote the fact that an arbitrary number of Faddeev-Popov ghost loops may occur. All additional external lines must be connected to a physical source on mass shell, as defined below. We have written explicitly the minus sign for the ghost closed loop (Fermi statistics).

Instead of fig. 2 we will prove another Ward identity which differs from that of fig. 2 in that one of the loop momenta, k, is not integrated over and furthermore all external lines have been replaced by sources J^R_{1i}, J^R_{2i}, etc, to which field combinations R_{1i}, R_{2i}, etc. are coupled. These field combinations are taken to transform

† Sources that emit or absorb only physical particles satisfy (2.6), but the converse is not necessarily true.

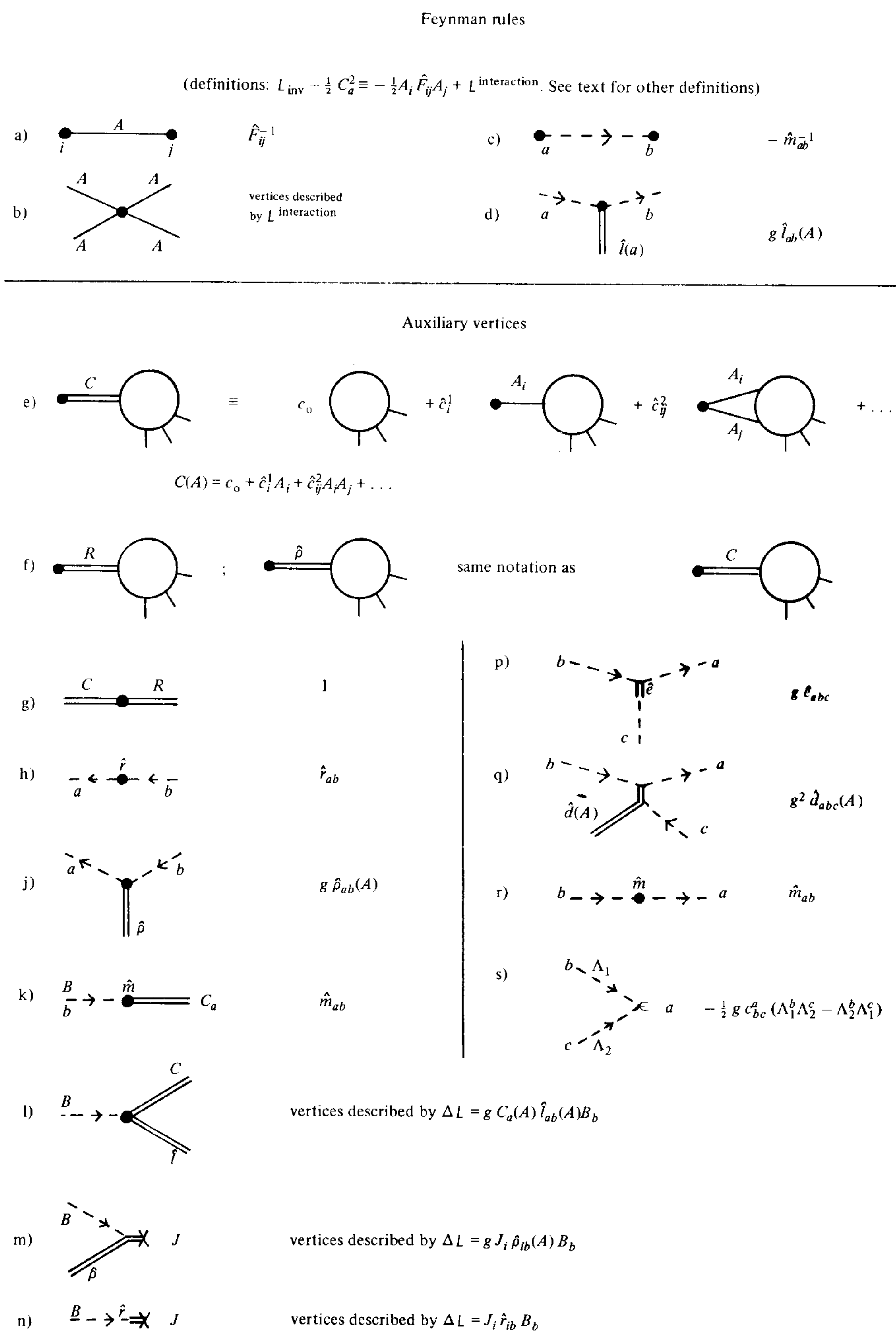

Feynman rules
(definitions: $L_{\text{inv}} - \frac{1}{2} C_a^2 \equiv - \frac{1}{2} A_i \hat{F}_{ij} A_j + L^{\text{interaction}}$. See text for other definitions)
a) $\hat{F}_{ij}^{-1}$
b) vertices described by $L^{\text{interaction}}$
c) $-\hat{m}_{ab}^{-1}$
d) $g\,\hat{l}_{ab}(A)$
Auxiliary vertices
e) $C(A) = c_0 + \hat{c}_i^1 A_i + \hat{c}_{ij}^2 A_i A_j + \ldots$
f) same notation as
g) 1
h) $\hat{r}_{ab}$
j) $g\,\hat{\rho}_{ab}(A)$
k) $\hat{m}_{ab}$
l) vertices described by $\Delta L = g\, C_a(A)\, \hat{l}_{ab}(A) B_b$
m) vertices described by $\Delta L = g\, J_i\, \hat{\rho}_{ib}(A)\, B_b$
n) vertices described by $\Delta L = J_i\, \hat{r}_{ib}\, B_b$
p) $g\, \hat{e}_{abc}$
q) $g^2\, \hat{d}_{abc}(A)$
r) $\hat{m}_{ab}$
s) $-\frac{1}{2} g\, c^a_{bc} (\Lambda_1^b \Lambda_2^c - \Lambda_2^b \Lambda_1^c)$

Fig. 1.

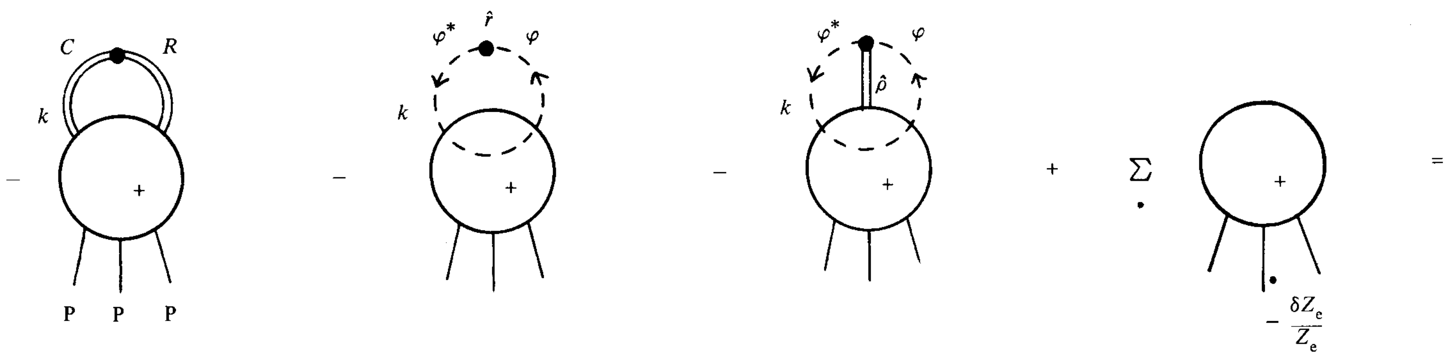


Fig. 2. The lines with a P are physical lines on mass-shell including the external line factors Z_e. The quantities δZ_e are the changes in the Z_e due to the change of gauge.

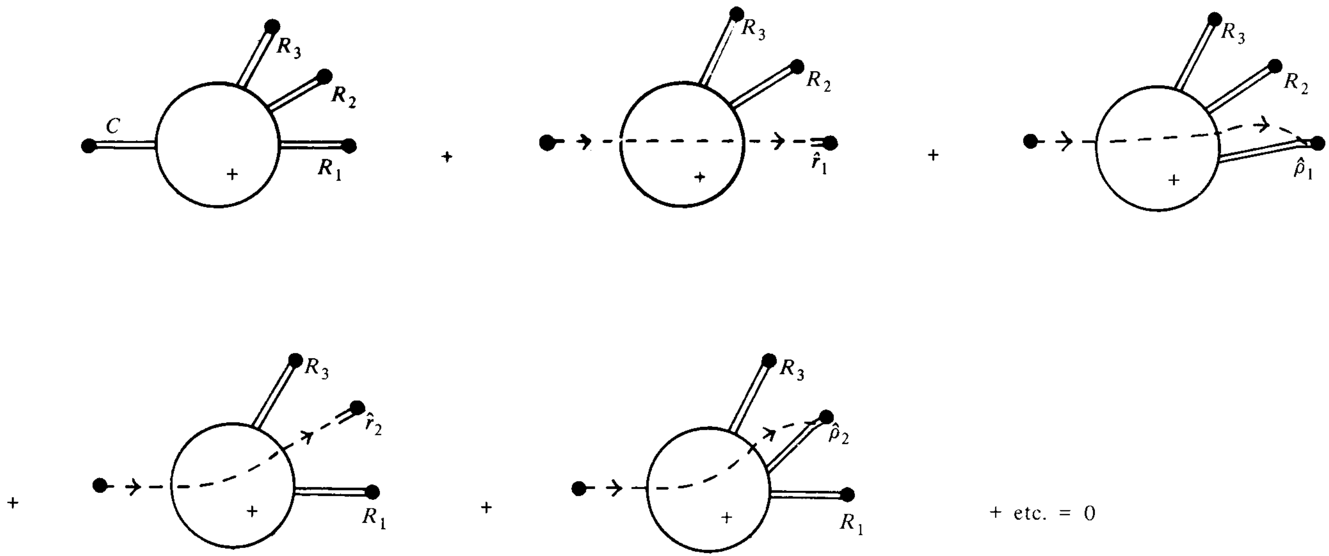


Fig. 3. Generalized Ward identity for gauge fields. C specifies the gauge. The Feynman rules inside the blobs are those that correspond to this gauge choice, and include ghost propagators and vertices. The R_i are arbitrary functions of the fields A_i. The $\hat{\rho}$ and $\hat{r}$ are given by the behaviour of the R under gauge transformation.

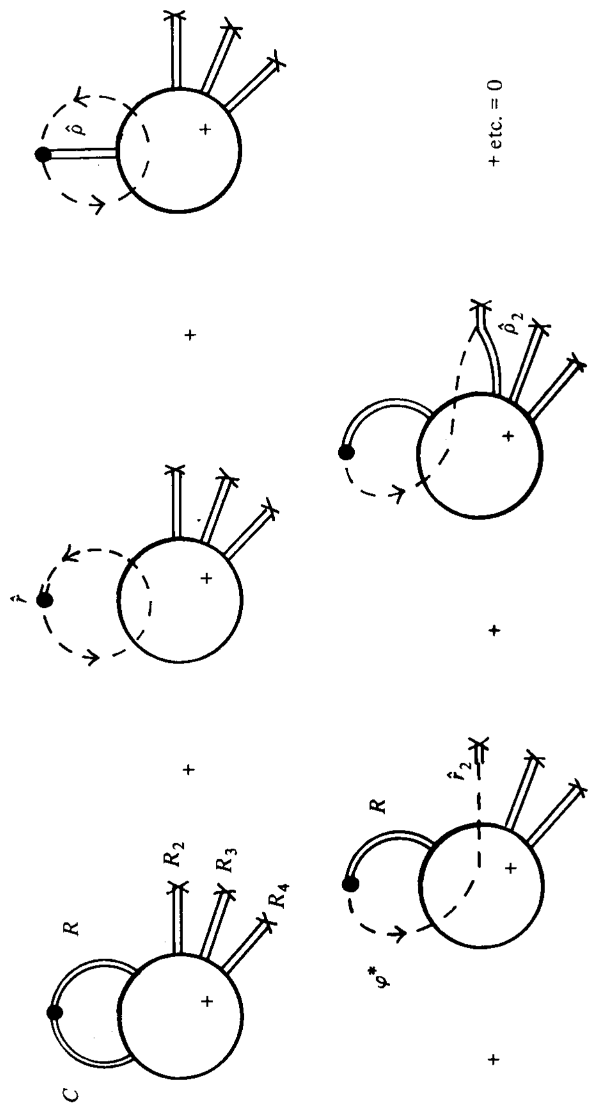


Fig. 4.

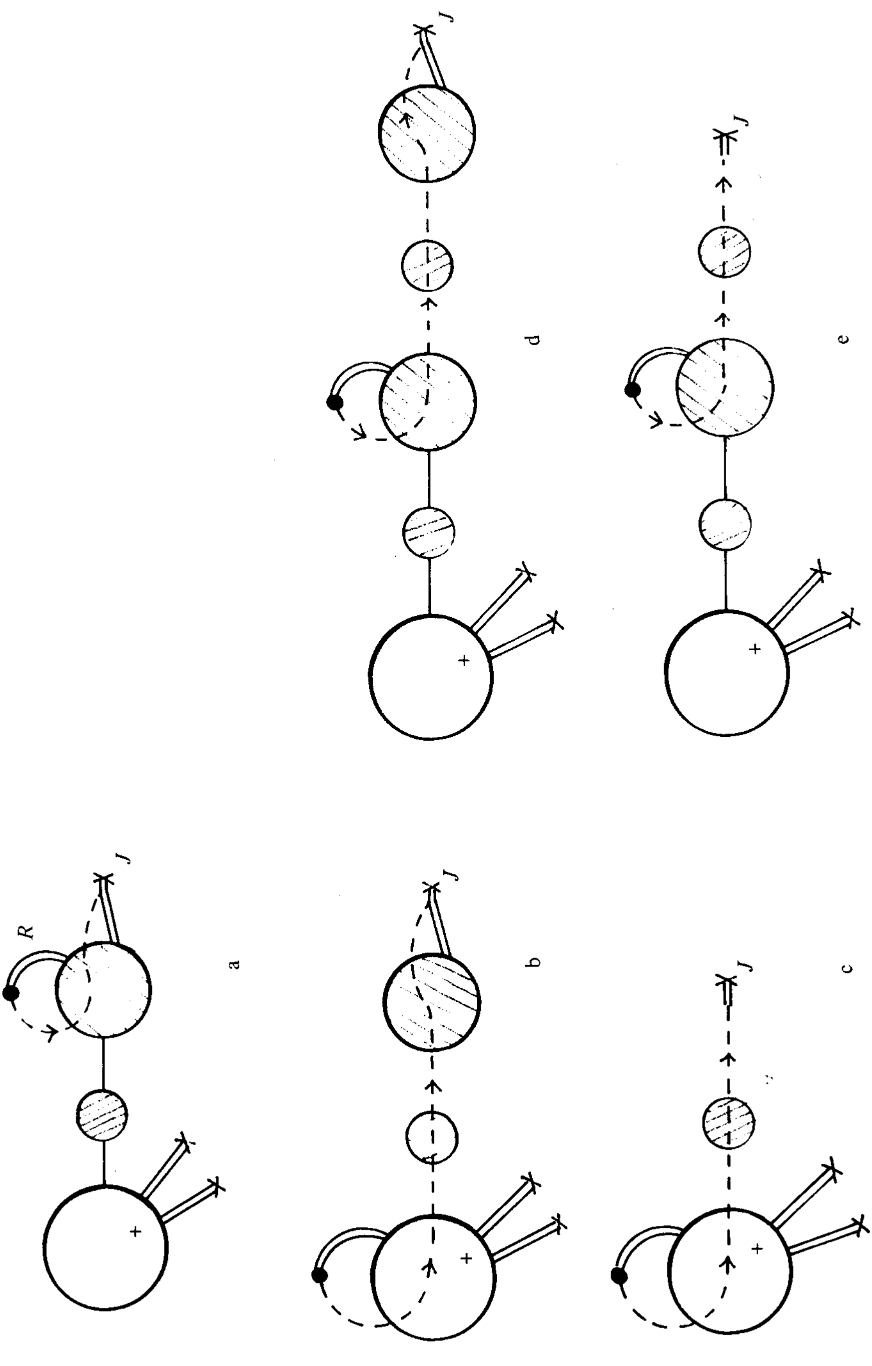


Fig. 5.

under a gauge transformation as

$$R'_{ij} = R_{ij} + g\,\hat{\rho}_{ija}(A)\,\Lambda_a + \hat{r}_{ija}\,\Lambda_a \quad .$$

The R_{ij} may be linear , quadratic etc. in the fields. We will then prove the generalized Ward identity of fig. 3.

Suppose that fig. 3 is true. Then we may fold the C-source and the R_1 source together to get fig. 4.

Next we take for $R_2, R_3, \ldots$ sources that emit physical particles and, moreover, put them on the mass shell. For simplicity we only consider stable particles at the external lines, but this is no limitation. Thus we multiply by $k^2 + m^2$ (where m is the mass of the emitted particle) and take the limit $k^2 + m^2 = 0$. We get contributions from those diagrams in fig. 4 that can be decoupled from R_i by cutting one (dressed) propagator. They are represented in more detail in fig. 5a-e. Fig. 5b and c only contribute if the ghost mass happens to be m also.

However, the set of diagrams contained in figs. 5b-e cancel, because of the condition that we have a physical source, see fig. 6. If now we assume that fig. 7 corresponds to the change of the factor Z_e associated with the external line, then we get precisely fig. 2 (the blob in fig. 7 is taken to be irreducible). This assumption can easily seen to be correct: in fig. 2 the sources are connected through a dressed propagator to the rest of the diagram. Therefore the correct definition of the S-matrix amounts to a factor

$$\sqrt{1+F}\;\left.\frac{k^2+m^2}{k^2+m_0^2+E(k^2)}\right|_{k^2+m^2=0} \quad ,$$

where

$$m^2 = m_0^2 + E(-m^2) \quad ,$$

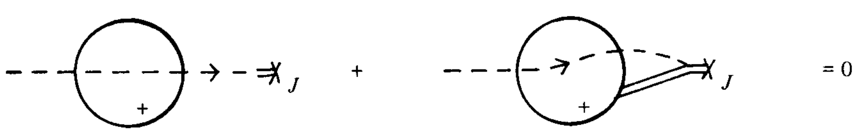


Fig. 6. Condition for a source J to be physical.

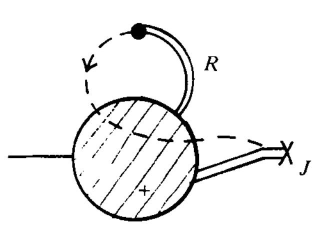


Fig. 7.

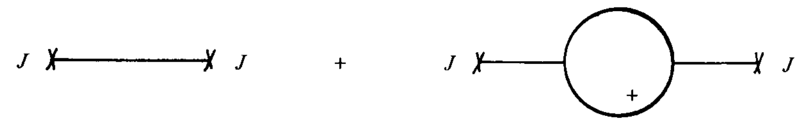


Fig. 8a.

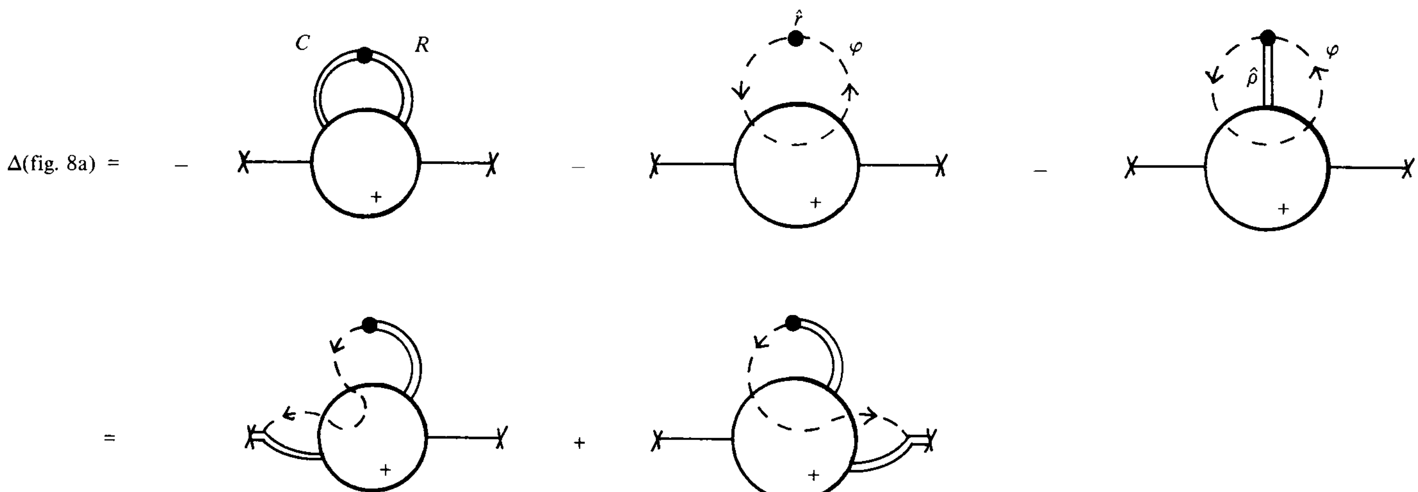


Fig. 8b.

$$F = \left. \frac{E(k^2) - E(-m^2)}{k^2 + m^2} \right|_{k^2 + m^2 = 0}$$

The factor Z_e in fig. 2 is defined to be $\sqrt{1 + F}$. The residue of the pole of the dressed propagator (fig. 8a) is $(1 + F)^{-1} = (Z_e)^{-2}$. The blob in fig. 8a contains all self energy insertions including the repeated insertions of irreducible self-energy graphs, i.e. fig. 8a is the completely dressed propagator. The effects of a change in gauge, taking into account that both sources are physical, are given in fig. 8b.

Thus the change in the residue of the propagator at the pole $k^2 = -m^2$ is $2\alpha/(1 + F)$, where α is by definition the graph of fig. 7. (In general, α is a matrix that needs not be diagonal; we shall ignore this complication). Hence, $\delta(1 + F)^{-1} = 2\alpha(1 + F)^{-1}$, or

$$\alpha = -\frac{\delta Z_e}{Z_e} \quad .$$

So now we have shown that if we prove the generalized Ward identity of fig. 8b then we have proven that infinitesimally different gauges give the same S-matrix between physical states. It is clear then that two non-singular gauges C_{a1} and C_{a2} will give the same S-matrix if there exists a continuous set of non-singular gauges $C_{a\lambda}$ that connects these two gauges.

The Ward identity of fig. 3 will be proven by induction. Our proof differs slightly from ref. [2]; we follow more closely the method of ref. [13].

3. WARD IDENTITIES FOR DIAGRAMS WITHOUT THE INCLUSION OF FADDEEV-POPOV GHOSTS

First we consider the "field theory" defined by the Lagrangian (2.3) without the Faddeev-Popov correction (2.4), but with sources that emit particles:

$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \tfrac{1}{2} C_a^2 + J_i^R R_i(A) \tag{3.1}$$

The J_i^R may be anything, and $R_i(A)$ may be any linear or nonlinear combination of fields A.

The trick of ref. [13] is to add a free particle field B_a, with arbitrary mass m_a:

$$\mathcal{L} = \mathcal{L}_{\text{inv}} - \tfrac{1}{2} C_a^2 + J_i R_i(A) - \tfrac{1}{2} (\partial_\mu B_a)^2 - \tfrac{1}{2} m_a^2 B_a^2 \tag{3.2}$$

Now we make a redefinition of our fields A_i by performing an infinitesimal gauge transformation:

$$A_i \to A_i + \epsilon\, g\, \hat{s}_{ia}(A)\, B_a + \epsilon\, \hat{t}_{ia} B_a \tag{3.3}$$

with ϵ infinitesimal, and $\hat{s}$ and $\hat{t}$ are the matrices as defined in eq. (2.1). The Lagrangian (3.2) now becomes

$$L = L_{\text{inv}} - \tfrac{1}{2} C_a^2 + J_i R_i(A) - \tfrac{1}{2} (\partial_\mu B_a)^2 - \tfrac{1}{2} m^2 B_a^2$$
$$- \epsilon C_a (g \hat{l}_{ab}(A) + \hat{m}_{ab}) B_b + \epsilon J_i (\hat{r}_{ib} + g \hat{\rho}_{ib}(A)) B_b \tag{3.4}$$

with $\hat{l}$, $\hat{m}$, $\hat{r}$ and $\hat{\rho}$ as defined in eqs. (2.2) and (2.7).

Now this redefinition (3.3) of the fields A_i does not alter the fact that the B-fields are free fields, whether on or off the mass shell, even in the presence of the sources J_i; hence the contribution of the "interaction" terms in (3.4) must cancel in the amplitudes. Precisely: according to the L.S.Z. formalism we have

$$_{\text{out}}\langle A \ldots A|B, A \ldots A\rangle_{\text{in}} = {}_{\text{out}}\langle A \ldots A|B_{\text{out}}|A \ldots A\rangle_{\text{in}}$$
$$+ {}_{\text{out}}\langle A \ldots A|B, A \ldots A\rangle_{\text{in}}^{B\text{-connected}} ,$$

with

$$_{\text{out}}\langle A \ldots A|B, A \ldots A\rangle_{\text{in}}^{B\text{-connected}} = -\int \mathrm{d}^4 x \, \frac{\mathrm{e}^{ipx}}{2p_0 V} (\Box - m_a^2)$$
$$_{\text{out}}\langle A \ldots A|B(x)|A \ldots A\rangle_{\text{in}} . \tag{3.5}$$

Thus, because B obeys the free field equation of motion, we have

$$_{\text{out}}\langle A \ldots A|B, A \ldots A\rangle_{\text{in}}^{B\text{-connected}} = 0 \tag{3.6}$$

as a functional of J_i. "B-connected" means that we select the diagrams where the B-line is connected. Note that $_{\text{out}}\langle A \ldots A|B_{\text{out}}|A \ldots A\rangle_{\text{in}}$ need not be zero because the A fields contain a small admixture of B-fields. Fig. 9 is a graphical notation for eq. (3.6), where all B-vertices are written down.

4. IDENTITIES FOR TREE DIAGRAMS

In sect. 5 we shall prove fig. 3 up to all orders, but it is illustrative to consider first tree diagrams. Of course, as fig. 9 is correct for all orders in g, it holds in particular for tree diagrams alone (fig. 10).

Now let us define a B propagator and vertices as in fig. 1 c, d. The vertex fig. 1k then satisfies the identity of fig. 11 †. Inserting fig. 11 into the second graph of fig. 10 we find that we can iterate. The result of the iteration procedure is given in fig. 12. After multiplication with the ghost propagator $-\hat{m}^{-1}$ this is precisely fig. 3 for tree diagrams.

† Note that fig. 11 only holds if the sign of $i\epsilon$ in the ghost propagator is the same as that of the fields that have the same mass as the ghost and occur in C.

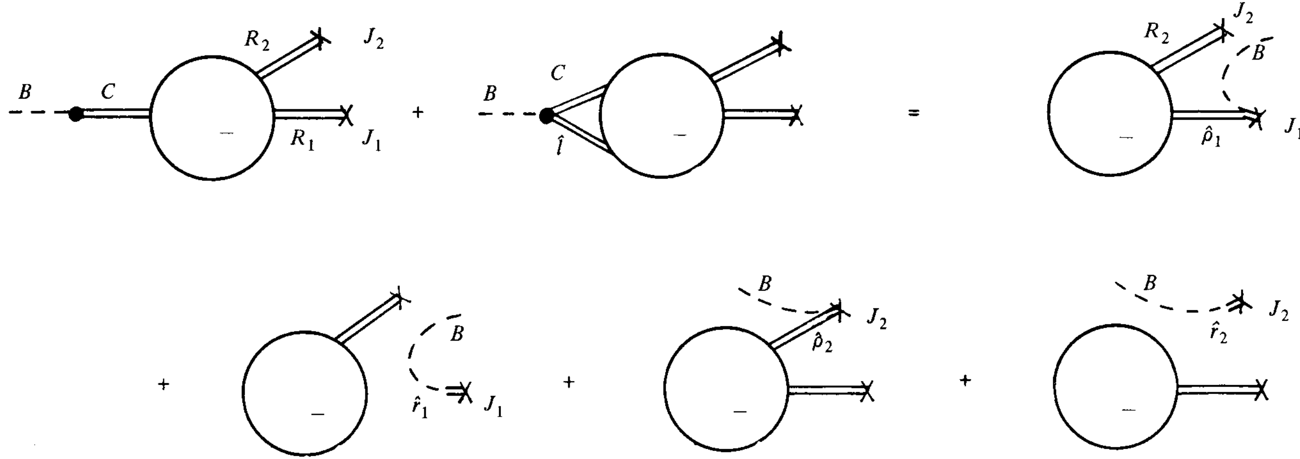


Fig. 9. Ward identities for B-connected diagrams containing any number of closed loops, but no Faddeev-Popov ghosts (indicated by the minus sign).

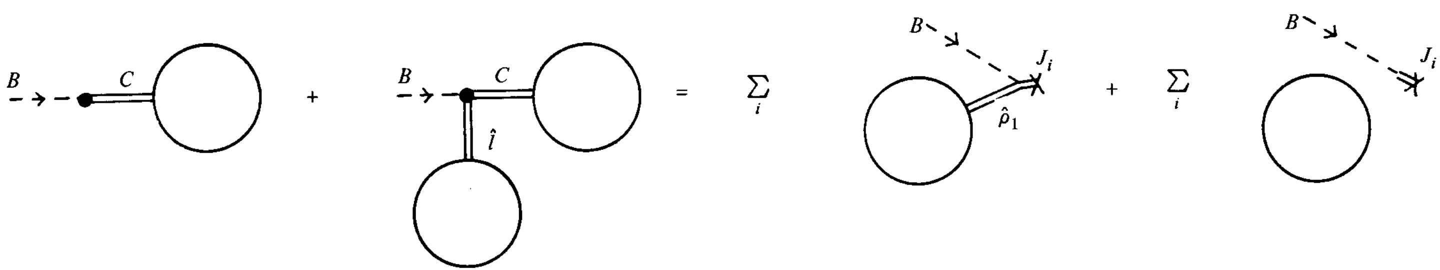


Fig. 10. The equations of fig. 5 for tree diagrams. The blobs contain sources J_i. The summation sign indicates that the B-line must be connected in the way shown to each of the sources consecutively.

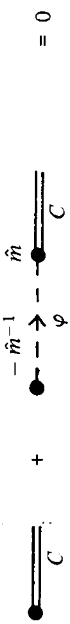


Fig. 11.

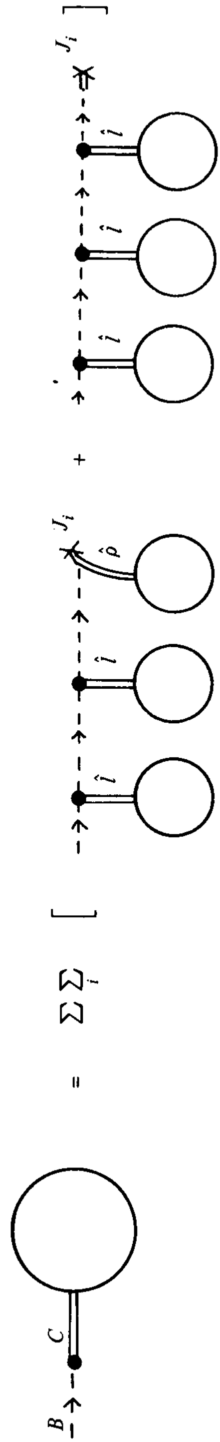


Fig. 12. Ward identity of fig. 3 for tree diagrams. The symbol Σ denotes the sum over all the numbers of blobs. The blobs contain (other) sources.

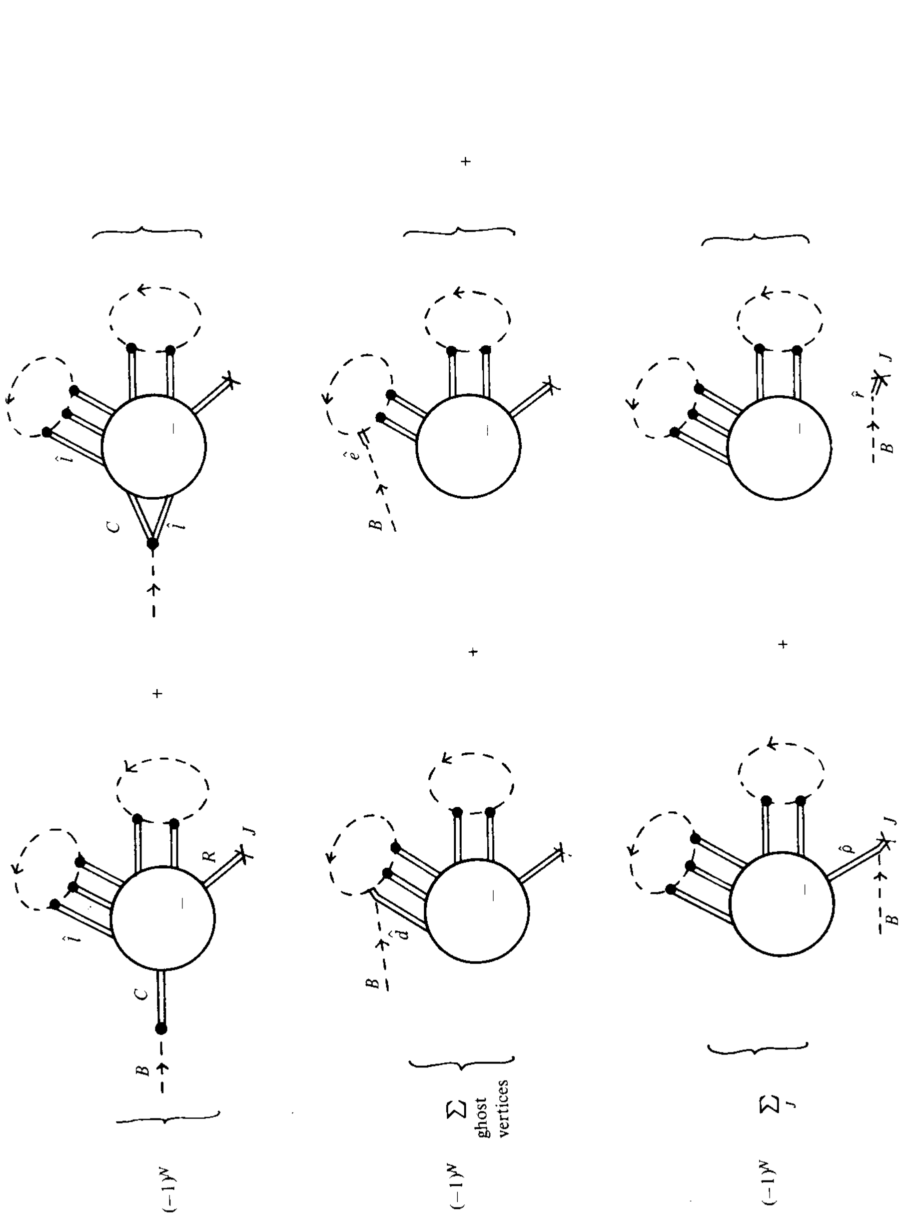


Fig. 13. Ward identity for complete gauge field theories derived from fig. 9. All ghost vertices have been denoted explicitly. See the definition of vertices in fig. 1. N is the number of ghost loops.

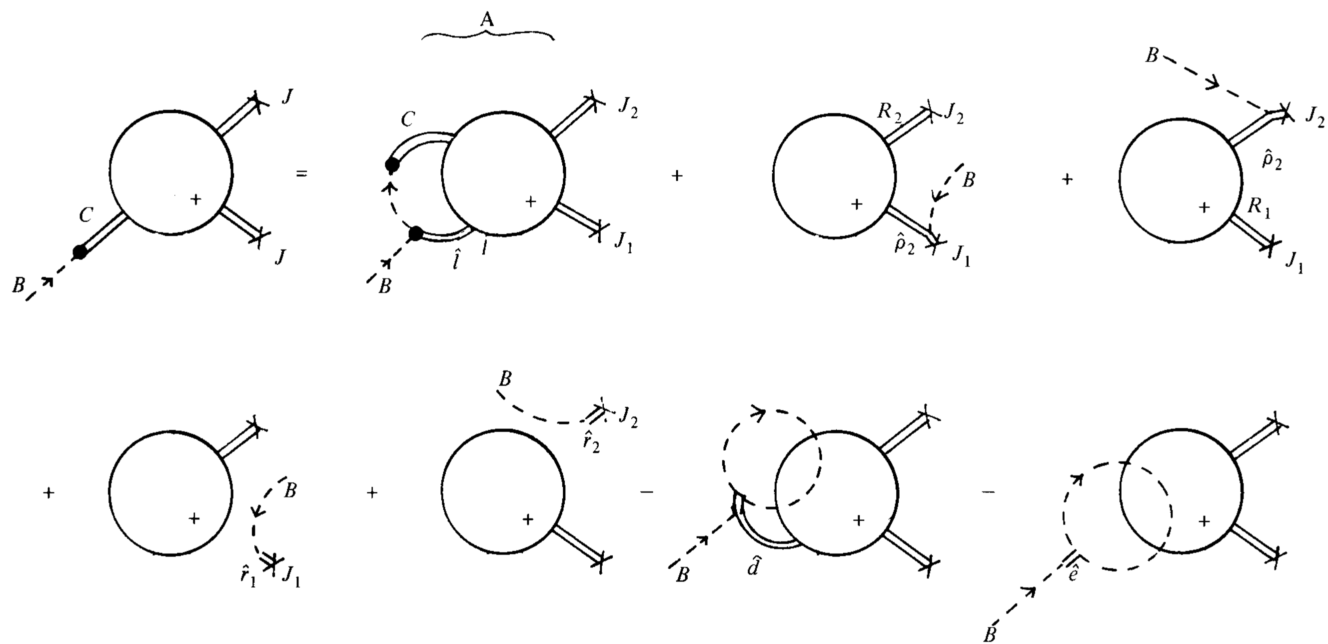


Fig. 14. Same equation as fig. 13. This is the analogue of fig. 9, for real gauge field theories, where the Faddeev-Popov ghosts are included. Fig. 11 has also been substituted.

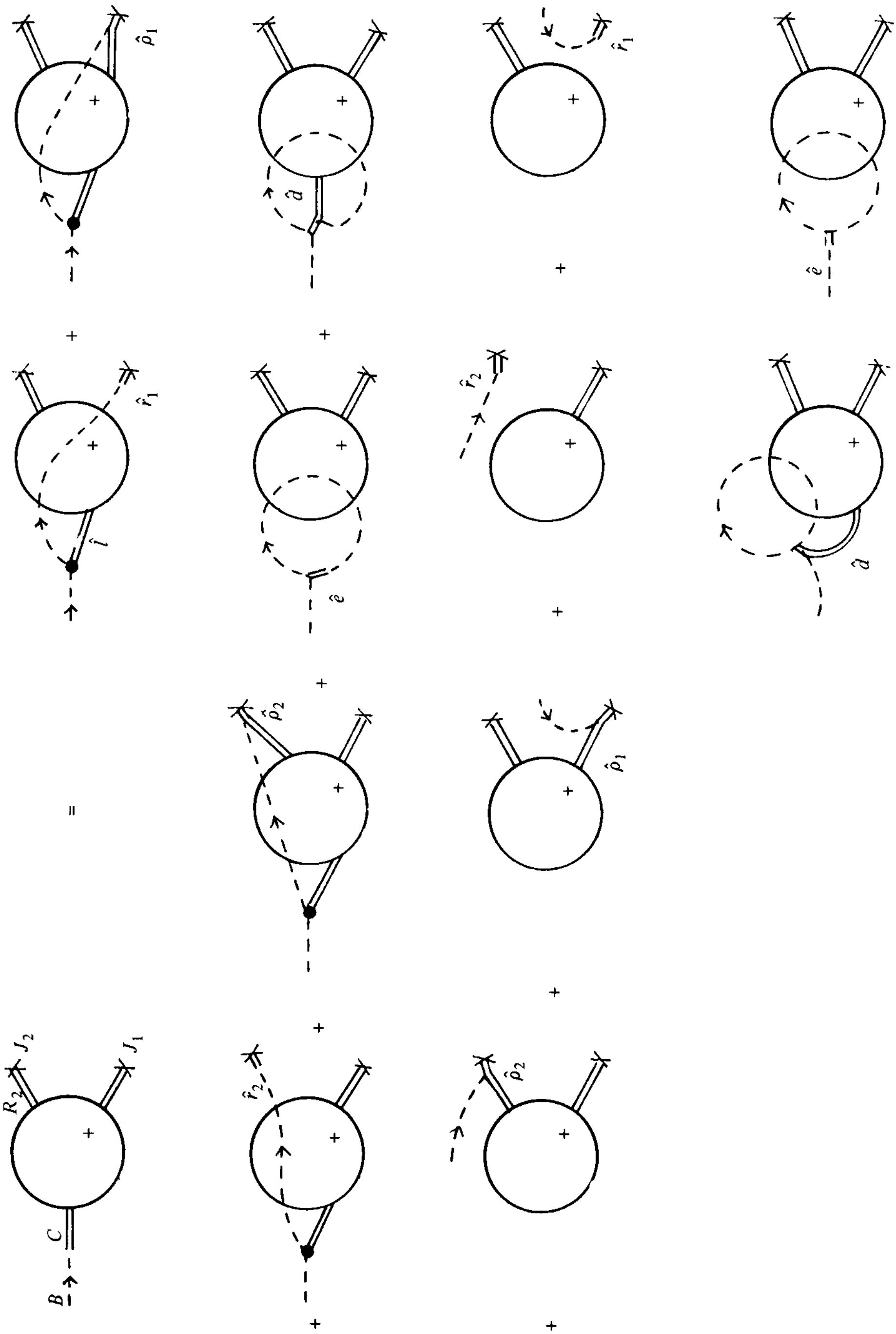


Fig. 15.

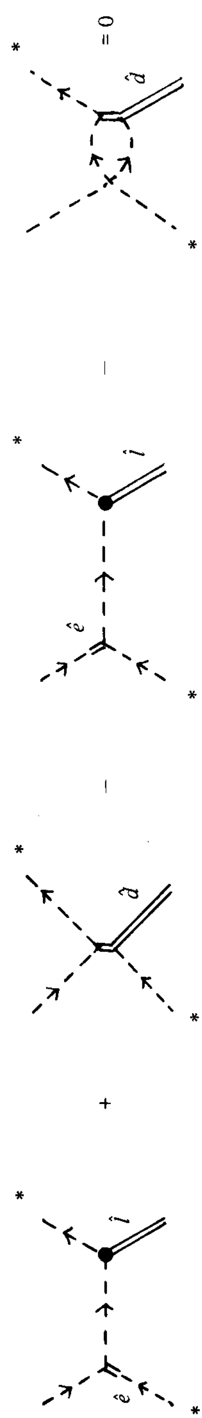


Fig. 16. Identity related to second order gauge transformations of C. $\hat{l}$ results after a gauge transformation of C; $\hat{e}$ and $\hat{d}$ after a gauge transformation of $\hat{l}$. For the * see text.

5. THE CASE OF CLOSED LOOPS

If we want to prove fig. 3 up to higher orders in g, we must take the Faddeev-Popov ghosts that circulate in the blob into consideration. The "field theory" for which we proved fig. 9 is now changed into a real gauge field theory by adding the ghosts with the propagators and vertices of fig. 1c, d, and Fermi-statistics. The ghosts are coupled to the quantity $\hat{l}_{ab}(A)$. These coupling are not gauge invariant and therefore cannot be included into L_{inv}. Nevertheless, we can use fig. 9 if we treat all couplings of A-fields with ghosts as additional source contributions. This may be done by replacing $J_i R_i(A)$ in eq. (3.1) by $J^*_a \hat{l}_{ab} J_b$, and treating $J^*_a J_b$, $J^*_a \partial_\mu J_b$, etc. as sources.

We get new vertices by considering the transformation properties of $\hat{l}_{ab}(A)$:

$$\hat{l}_{ab}(A') = \hat{l}_{ab}(A) + g\,\hat{d}_{abc}(A)\,\Lambda_c + \hat{e}_{abc}\,\Lambda_c \quad . \tag{5.1}$$

Note that the new auxiliary vertices fig. 1p and q are not symmetric under interchange of the B and φ lines. Clearly, by introducing blobs "+" which differ from the blobs "−" by the inclusion of all possible ghost loops and their associated minus signs, fig. 13 may be rewritten as fig. 14. (also we substituted fig. 11). So now we acquired the analogue of fig. 9 for complete gauge field theories.

From this equation we prove fig. 3 by induction with respect to powers of the coupling constant g. Suppose fig. 3 holds up to g^N. Consider the blob in the graph denoted by A in fig. 14. We may apply fig. 3 by the induction assumption, because one power of g has been extracted by the explicitly written vertex. One of the arbitrary R_i in fig. 3 must now be taken to be $\hat{l}$. Thus we get fig. 15.

Now let us call the vertices fig. 1d and q "principal" vertices because they contain A fields, and let us sort those terms in fig. 15 which contain an explicitly drawn ghost loop into groups according to the number of principal vertices they contain. Then we can apply an amusing identity (fig. 16) (c.f. eq. (4.11), ref. [2]).

This identity is proven in appendix A. It may be derived from the group properties of the gauge transformations or from the tree-graph Ward identities of fig. 12. If we take the lines with an * to be part of the closed loops shown in fig. 15, we see that each of the above mentioned groups of diagrams contains the full combination of fig. 16 and therefore vanishes, except for the group with no principal vertex. This group contains the ghost in a tadpole and is absent in most models, but anyway this group vanishes automatically when it is regularized properly. This is also shown in appendix A. And thus the equation of fig. 15 reduces to fig. 3 now up to order g^{N+1}, which completes our ptoof by induction.

6. THE TREE-LOOP THEOREM

In the previous sections the procedure was to start from a Lagrangian with known invariance properties and then to derive identities among diagrams. Now we will do roughly the opposite. Suppose there is a Lagrangian of which no symmetry properties are known, but where on the other hand the following properties hold for the diagrams associated with that Lagrangian:

(i) there exists a function $C_a(A)$ of the fields $A_i(x)$, and there exist matrices $\hat{s}_{ia}(A)$, $\hat{t}_{ia}$, $\hat{l}_{ab}(A)$ and $\hat{m}_{ab}$, such that

$$C_a(A_i + g\,\hat{s}_{ia}(A)\,\Lambda_a + \hat{t}_{ia}\,\Lambda_a)$$

$$= C_a + g\,\hat{l}_{ab}(A)\,\Lambda_b + \hat{m}_{ab}\,\Lambda_b + \mathrm{O}(\Lambda^2) \tag{6.1}$$

for all Λ (compare eqs. (2.1) and (2.2)).

(ii) The Ward identity fig. 12 holds for tree diagrams, if we use the transformation law

$$A_i' = A_i + g\,\hat{s}_{ia}\,\Lambda_a + \hat{t}_{ia}\,\Lambda^a \tag{6.2}$$

for the external sources, and furthermore use the $\hat{m}$ and $\hat{l}$ of (6.1) as ghost propagator and vertex respectively. (Note that here it is not a given fact that the infinitesimal transformation law of eq. (6.2) corresponds to a gauge transformation group).

From (i) and (ii) we shall deduce that †

(a) The Lagrangian can be written in the form

$$\mathcal{L} = \mathcal{L}_1 - \tfrac{1}{2}\,C_a^2 \ , \tag{6.3}$$

where $\mathcal{L}_1$ is invariant under the infinitesimal transformation (6.2).

(b) Fig. 3 holds for diagrams with closed loops, provided that Faddeev-Popov ghosts with propagators $-\,\hat{m}^{-1}$ and vertices $\hat{l}$ are included in the Feynman rules.

This theorem will be called the tree-loop theorem, and it is of importance in studying the renormalization counterterms (sect. 7).

Statement (a) is proven by going backwards through sects. 4 and 3. From fig. 12 fig. 10 follows, which is then fig. 9 for tree graphs.

As we only work with tree graphs, the fields may be considered as unquantized, and there is a one to one correspondence between the Lagrangian and the vertices and propagators of the tree diagrams.

Fig. 9 implies that to the Lagrangian

$$\mathcal{L} + J_i R_i(A) - \tfrac{1}{2}\,(\partial_\mu B_a)^2 - \tfrac{1}{2}\,m^2 B_a^2 \tag{6.4}$$

one may add

$$-\,\epsilon\,C_a(g\,\hat{l}_{ab}(A) + \hat{m}_{ab})\,B_b + \epsilon\,J_i(\hat{r}_{ib} + g\,\hat{\rho}_{ib}(A))\,B_b \tag{6.5}$$

without changing the vacuum-vacuum transition amplitude in the presence of sources (cf. eq. (3.4)).

Redefining the fields A_i in eq. (6.4) just as in sect. 3

† Our proof does not cover all cases, but it applies to renormalizable theories. See appendix B.

$$A_i \to A_i - \epsilon g \hat{s}_{ia} B_a - \epsilon \hat{t}_{ia} B_a \tag{6.6}$$

does not change the fact that B is a free field. Next substituting (6.3) and (6.1) and using (6.5), we find that if the replacement is done in L_1 alone:

$$L_1(A) \to L_1(A - \epsilon g \hat{s}_{ia} B_a - \epsilon \hat{t}_{ia} B_a) \tag{6.7}$$

the B-field remains a free field, even in the presense of the sources J_i. This implies that none of the off-mass shell amplitudes are altered by the replacement (6.7), which is only true if L_1 is invariant.

Statement (b) follows from (a) except for the fact that the group property of the transformation law (6.2) is not known. Here we have an argument that works for many models, see appendix B. Otherwise this must be verified separately.

7. RENORMALIZATION

In the previous sections we have proved equivalence of the S-matrices for different gauges. Essential in this context are the generalized Ward identities of fig. 3. However, the graphs are still divergent, or, in the language of ref. [10], they still contain poles for $n = 4$. If for a given theory a so-called renormalizable gauge exists we may try to subtract the infinities, that is, renormalize the theory. This requires the introduction of counter terms in the Lagrangian, and it is not clear that those counter terms will not spoil the symmetry. For instance, there will be counter terms associated with the Faddeev-Popov self energy diagrams, and a priori such terms could certainly spoil the symmetry of the theory.

Thus we must investigate the structure of the counter terms, and establish what the Ward identities are in the presence of these terms. It will become clear that we obtain new Ward identities, the renormalized Ward identities, and the theory will be invariant for certain gauge transformations that are in general different from the gauge transformations of the unrenormalized theory. Thus not only mass, charge etc. but also Ward identities and gauge transformations are renormalized.

We will work within the framework of the continuous dimension method of ref. [10]. Suppose for a given L_{inv} we have found a renormalizable gauge. Order by order one may find counter terms that have to be introduced in the Lagrangian so that the S-matrix becomes finite. For instance, one will have counter terms associated with ghost self-energy diagrams. Having introduced those counter terms we must investigate if there still exists some gauge invariance; even if this gauge invariance need not be the same as the original gauge invariance one nevertheless needs some kind of gauge invariance in order to be able to show equivalence to a physical gauge. That is, we must have Ward identities as in fig. 3 but now for the subtracted theory. As input for our considerations we have that fig. 3 holds for the unrenormalized theory, in particular fig. 3 holds separately for poles of any given order in $(n-4)^{-1}$. That is not enough, because as is well known the subtraction procedure corresponding to the introduction of local counterterms in the Lagrangian is not equivalent simply to disregarding poles and their residues in the S-matrix. As is emphasized in ref. [10], the correct subtraction procedure is to consider amplitudes order by order in the perturbation expansion. Subdivergencies in a diagram are then cancelled by lower order counter terms, and the remaining overall divergencies in a diagram can then always be subtracted by local counter terms.

In other words, according to ref. [10], the Lagrangian can be written in the form

$$L(\eta) = L_{(0)} + \eta\, L_{(1)} + \eta^2\, L_{(2)} \cdots \,, \tag{7.1}$$

where $L_{(0)}$ is the complete Lagrangian that describes the propagators and vertices of the unrenormalized theory, including those of the ghosts; η is an expansion parameter. $L_{(k)}$ are counter terms that contain poles for $n = 4$†, which are at most of degree k. The $L_{(k)}$ are obtained from the requirement that all amplitudes with k loops should remain finite as $n \to 4$ and $\eta \equiv 1$.

In order to keep the discussion transparant we will make the simplifying restriction that C_a is a simple field, i.e. C_a contains linear but not quadratic or higher terms in the fields A_i. Also we will consider only sources that are coupled to simple fields, so that also $\hat{t}$ should be simple as is generally true for renormalizable theories. This restriction avoids the necessity of renormalizing C and R, for which one should have to introduce counter terms involving the sources R and the source that emits C in fig. 3. After renormalization one may go over from this simple C_a to any other C_a, without changing the S-matrix between physical states.

Let us now consider the Ward identity of fig. 3, and apply the procedure sketched above to make all diagrams occurring there finite. Then we find that apart from the counter terms described by the $L_{(k)}$ from eq. (7.1) we also need counter terms to cancel those infinities where source vertices are involved. See fig. 17. These counter terms correspond to a change in the quantities $\hat{r}$ and $\hat{\rho}$ in fig. 3, or $\hat{s}$ and $\hat{t}$ in eq. (2.1):

$$\hat{s}(\eta) = \hat{s}_{(0)} + \eta\, \hat{s}_{(1)} + \eta^2\, \hat{s}_{(2)} \cdots \,, \tag{7.2}$$

$$\hat{t}(\eta) = \hat{t}_{(0)} + \eta\, \hat{t}_{(1)} + \eta^2\, \hat{t}_{(2)} \cdots \,.$$

In this section we now prove the following. Suppose the counter terms $L_{(1)}$, $\hat{s}_{(1)}$, $\hat{t}_{(1)}$, $L_{(2)}$, $\hat{s}_{(2)}$, ... are constructed so as to make all diagrams occurring in the Ward identities finite, order by order in the perturbation expansion. To be precise, $L_{(k)}$, $\hat{s}_{(k)}$, $\hat{t}_{(k)}$ are the counter terms associated with the overall divergencies of diagrams with k closed loops. Then the Lagrangian $L(\eta)$ of eq. (7.1) can be written in the form

$$L(\eta) = L_{\text{inv}}(\eta) - \tfrac{1}{2} C_a^2 + L_{\varphi}(\eta) \,, \tag{7.3}$$

with $L_{\text{inv}}(\eta)$ invariant under the infinitesimal transformation

$$A_i' = A_i + g\, \hat{s}_{ia}(\eta, A)\, \Lambda_a + \hat{t}_{ia}(\eta) \Lambda_a \tag{7.4}$$

† Finite renormalizations can always be considered afterwards.

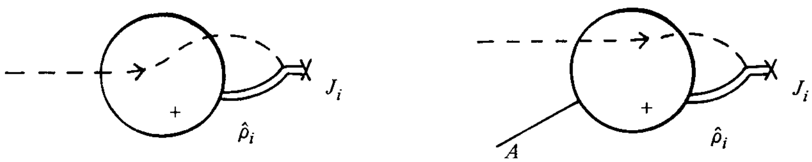


Fig. 17. Possibly divergent diagrams which necessitate renormalization of $\hat{s}$ and $\hat{t}$ in eq. (2.1).

for arbitrary η, while C_a is the same C_a as in the unrenormalized Lagrangian. The quantity $L_\varphi(\eta)$ is derived in the usual way by considering the behaviour of C_a under the transformation (7.4) According to the work of the previous sections this implies that the renormalized S-matrix is the same in different gauges.

This assertion is proved by treating η in eqs. (7.1) and (7.2) as a separate expansion parameter, to be distinguished from the coupling constant. The proof goes as usual by induction. Suppose the assertion is true up to order η^k, that is, all necessary Ward identities hold up to order η^k. Now consider

(i) diagrams described by $L_{(0)}$ alone, with $k + 1$ closed loops. These diagrams exhibit poles up to degree $k + 1$, and satisfy Ward identities as in fig. 3;

(ii) diagrams of the same order in g containing the counter terms $L_{(1)}$, $\hat{s}_{(1)}$, $\hat{t}_{(1)}$, etc. but with at least one closed loop. They are of order $\eta, \eta^2, \ldots, \eta^k$. From our induction assumption we have that also these diagrams satisfy Ward identities as in fig. 3 (note that Ward identities hold between diagrams of a fixed order in η).

(iii) The diagrams of order η^{k+1}. These diagrams contain no closed loops.

Now from ref. [10] we know that (i) + (ii) + (iii) are finite $n \to 4$ and $\eta \to 1$. On the other hand (i) and (ii) satisfy the Ward identities. We conclude that the residues of the poles in (iii) must satisfy the Ward identities. The diagrams (iii) are tree diagrams and therefore contain the dimension n only in terms of poles. Therefore also (iii) satisfies the Ward identities. Next we wish to apply the tree-loop theorem, using the same C_a however with $\hat{l}(\eta)$ and $\hat{m}(\eta)$ as given by the renormalized ghost vertices and propagators. Of course we must then verify that the requirements of the tree loop theorem are met. First, taking for the R_i the fields A_i themselves we may determine the quantities $s(\eta)$ and $t(\eta)$ by considering the poles in $n = 4$ of the diagrams of the unrenormalized theory. Next we must allow also for linear combinations of the fields, and assure that the overall divergencies as in fig. 17 associated with linear combinations of the fields are the same linear combination of the overall divergencies of the single fields A_i. This is trivially true if we observe that any linear combination of the A_i coupled to one source can be written as a sum of terms each containing one field coupled to one source with a certain strength. Finally we must convince ourselves of the property (6.1) with our choice of C, $\hat{l}$ and $\hat{m}$. This is most easily seen to be correct by substituting C for one of the R in the Ward identities and considering the divergencies of the unrenormalized theory.

Since the Ward identities hold for tree diagrams up to order η^{k+1} we have, according to the tree-loop theorem that the Lagrangian $L_{\text{inv}}(\eta)$ occuring in $L(\eta)$ of eq. (7.3) is invariant under the transformation (7.4) up to order η^{k+1}, while L_φ describes up to order η^{k+1} the correct Faddeev-Popov ghost Lagrangian. This completes our proof by induction.

The result is: the renormalized theory satisfies the Ward identities of fig. 3 with the same C and R as for the unrenormalized theory id the C and R are simple fields, but the quantities $\hat{s}$ and $\hat{t}$ in the transformation law (2.1) require higher order counter terms, infinite for $n \to 4$.

In the renormalized theory one may go over from simple to other C thereby introducing infinities in the non-physical sector of the S-matrix. The S-matrix between physical states is however not changed.

8. CONCLUSIONS

If our prescription for obtaining the Feynman rules is followed one can prove for all types of gauge field theories, by pure combinatorics, that the unrenormalized S-matrix is independent of the gauge chosen.

As outlined in sect. 7, the renormalization procedure of ref. [10] leads to a unitary renormalized S-matrix if

(i) a gauge, specified by a $C_a = C_a^{\text{ren}}$, exists such that the resulting rules are renormalizable by power counting;

(ii) a gauge, specified by a $C_a = C_a^{\text{phys}}$, exists such that the resulting rules contain no longer ghosts, and the propagators of massive vector particles are of the form $(\delta_{\mu\nu} + k_\mu k_\nu/M^2)/(k^2 + M^2 - i\epsilon)$;

(iii) a continuous set of gauge functions as for instance

$$C_a = \lambda\, C_a^{\text{ren}} + (1-\lambda)\, C_a^{\text{phys}}$$

exists in such a way that the operator $\hat{m}$ defined in sect. 2 has an inverse for all λ between 0 and 1;

(iv) no anomalies of the Bell-Jackiw-Adler type occur.

All these requirements are met with in many models [4, 5, 9].

It is of importance to note that the renormalization counterterms given by $L(\eta) - L(0)$ need not be gauge invariant. The reason is that $L(\eta)$ and $L(0)$ are both invariant, but under different gauge transformations.

The assumptions of appendix B in particular do not hold in the case of quantum gravity. It may well be that the invariance properties, if any, of an eventual renormalized theory of gravity are very different from those of the unrenormalized theory.

The authors are indebted to the participants of the Marseille Colloquium, June 1972, for their criticism on a preliminary version of this article. In particular they are indebted to J.S. Bell and R. Stora for their remarks concerning the treatment of external lines as well as the renormalization problem.

NOTE ADDED IN PROOF

Several preprints that deal with the subject of this paper have recently been circulated. See ref. [14].

APPENDIX A.

Group property

Fig. 16 is an identity containing the vertices of fig. 1p and q, which are obtained by performing two gauge transformations (eqs. (2.2) and (5.1)). Let us consider small gauge transformations Ω_t, described by small functions $\Lambda_t^a(x)$. The Ω_t form a group of transformations on the fields A_i, and the action of Ω_t on A_i can be expressed in terms of a power series in Λ_t^a, of which eq. (2.1) contains the first terms. The group property implies that the product of two gauge transformations is again a gauge transformation. If

$$\Omega_3 = \Omega_2 \Omega_1 \ , \tag{A.1}$$

then Ω_3 is described by

$$\Lambda_3^a = \Lambda_1^a + \Lambda_2^a - \tfrac{1}{2} g\, \hat{c}_{bc}^a \Lambda_1^b \Lambda_2^c + O(\Lambda_{1,2}^3) \ , \tag{A.2}$$

where $\hat{c}_{bc}^a$ are the structure constants of the group (in most cases simply numbers; in the theory of gravitation they contain the differentiation operator).

Let us apply eq. (A.2) on some function R_i:

$$\Omega_1 R_i = R_i + (\hat{r}_{ia} + g\,\hat{\rho}_{ia}(A))\,\Lambda_1^a + O(\Lambda_1^2) \ ,$$

$$\Omega_2\Omega_1 R_i = R_i + (\hat{r}_{ia} + g\,\hat{\rho}_{ia}(A))\,\Lambda_2^a + O(\Lambda_2^2)$$

$$+ (\hat{r}_{ia} + g\,\hat{\rho}_{ia}(A))\,\Lambda_i^a + g(\hat{u}_{iab} + g\,\hat{v}_{iab}(A))\,\Lambda_1^a\Lambda_2^b + O(\Lambda_1^2) \ , \tag{A.3}$$

where $\hat{u}$ and $\hat{v}$ are defined by

$$\hat{\rho}_{ia}(A') = \hat{\rho}_{ia}(A) + \hat{u}_{iab}\Lambda_b + g\,\hat{v}_{iab}(A)\,\Lambda_b \ . \tag{A.4}$$

Note that we disregard terms of order Λ_1^2 and Λ_2^2, but we do take into account terms of order $\Lambda_1\Lambda_2$.

$$\Omega_3 R_i = R_i + (r_{ia} + g\,\hat{\rho}_{ia}(A))\,(\Lambda_1^a + \Lambda_2^a - \tfrac{1}{2} g\,\hat{c}_{bc}^a \Lambda_1^b\Lambda_2^c) + O((\Lambda_1 + \Lambda_2)^2) \ . \tag{A.5}$$

Now let us consider only the antisymmetric bilinear terms in Λ_1 and Λ_2 (note that the omitted terms in (A.5) only contain symmetric bilinear terms). We then have, according to (A.1):

$$g(\hat{u}_{iab} + g\,\hat{v}_{iab}(A))\,(\Lambda_1^a\Lambda_2^b - \Lambda_2^a\Lambda_1^b)$$

$$= -\tfrac{1}{2} g(\hat{r}_{ia} + g\,\hat{\rho}_{ia}(A))\,\hat{c}_{bc}^a(\Lambda_1^b\Lambda_2^c - \Lambda_2^b\Lambda_1^c) \ . \tag{A.6}$$

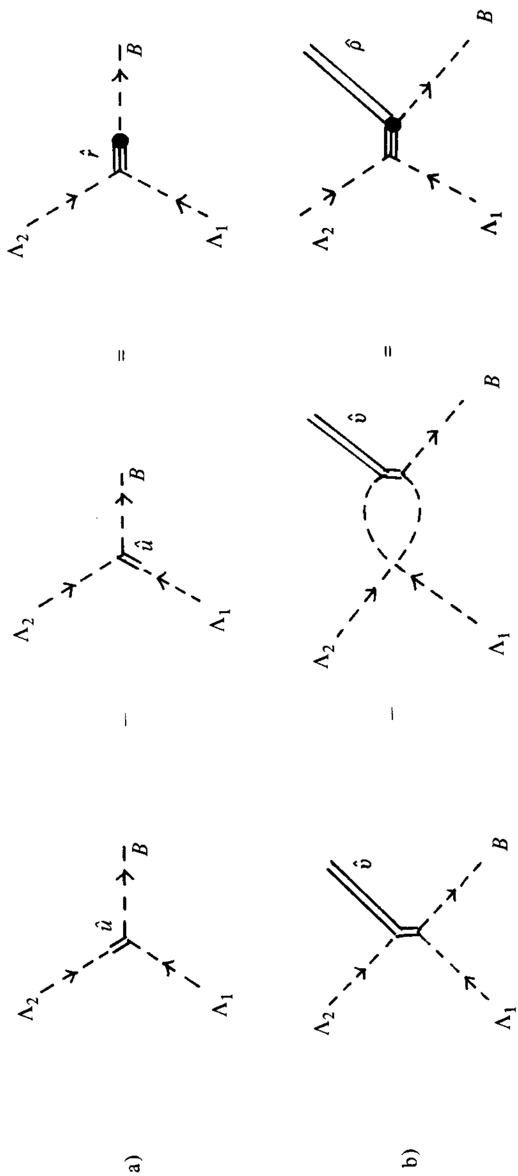


Fig. A1.

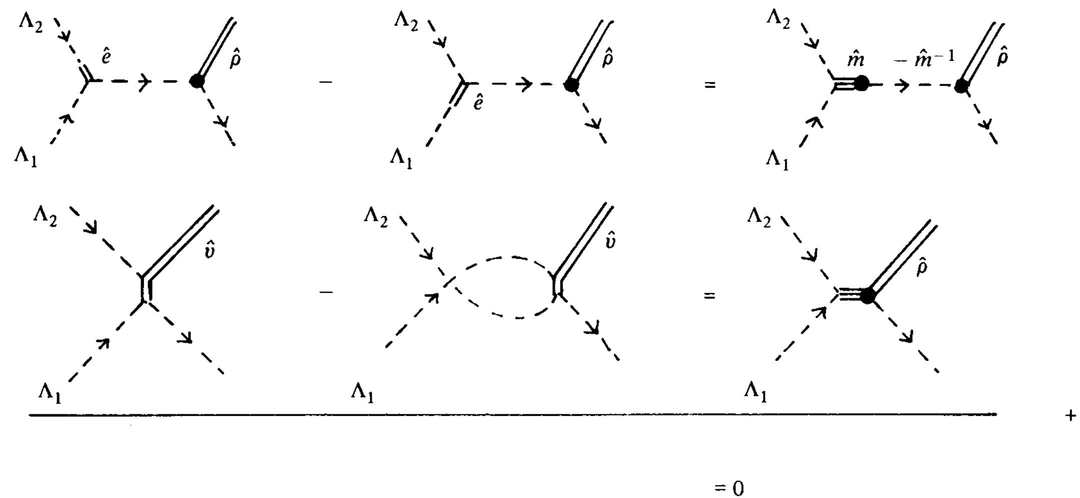


Fig. A2. Group property for an arbitrary function R_i of the fields. $\hat{\rho}_{ia}$ is the field dependent part obtained after a gauge transformation is applied to R_i. Next $\hat{v}_{iab}$ is the field dependent part obtained after transforming $\hat{\rho}_{ia}$. Further $\hat{e}_{abc}$ is the field independent part resulting from $\hat{l}_{ab}$; the latter is the field dependent part obtained after subjecting the gauge function C_a to a gauge transformation. The triple line stands for the structure constants of the gauge group.

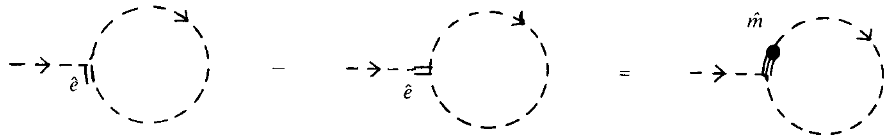


Fig. A3.

This is an equation between vertices as for example fig. 1d, p, q and r and holds for all functions Λ_1 and Λ_2.

Defining finally the auxiliary vertex part of fig. 1s, which is antisymmetric between Λ_1 and Λ_2 and multiplying eq. (A.6) on the left with B_i^* we can write the vertex identities of fig. A.1. We may use fig. A.1 a for the gauge function C_a. In that case we must take $\hat{e}$ for $\hat{u}$ and $\hat{m}$ for $\hat{r}$. Next we add on a ghost propagator $-\hat{m}^{-1}$, and a ρ-ghost vertex. We so obtain fig. A.2.

Finally we show the cancellation of the two tadpole graphs (fig. A.3). The ghost propagator is cancelled by the vertex $\hat{m}$ in fig. A.3; hence, the loop integration at the right hand side is an integral over some polynomial. Indeed, in the regularization scheme of ref. [10] integrals over polynomials are identically zero.

APPENDIX B

Group property and tree diagrams

The aim of this appendix is to show that the Ward identities of fig. 3 for tree diagrams imply in many cases sufficient information to deduce the group property (fig. 16, or fig. A.2).

Suppose we have a Lagrangian $L_{\text{inv}}(A)$ that is invariant for infinitesimal transformations of the type (2.1), however it is not known if the $\hat{s}$ and $\hat{t}$ satisfy the group property (i.e. eq. (A.6), with $\hat{r}$ and $\hat{\rho}$ replaced by $\hat{t}$ and $\hat{s}$, and with $\hat{u}$ and $\hat{v}$ being the transformed of $\hat{s}$). We fix a gauge by chosing a C_a and add source terms of the form J_iA_i. Thus there are as many sources as fields, and the sources emit simple fields only.

Next we may redefine our field A_i as done in sect. 3, eq. (3.3) involving a field B. Subsequently we may again redefine the fields but now involving a field B'. Next with $-B$, and finally with $-B'$. Neglecting terms of order B^2, B'^2 we obtain

$$L_{\text{inv}} - \tfrac{1}{2} C_a^2 + J_iA_i + J_i(\hat{u}_{iab} + g\,\hat{v}_{iab}(A))(B_aB'_b - B'_aB_b)$$
$$- \; C_a(\hat{e}_{abc} + g\,\hat{d}_{abc}(A))(B_bB'_c - B'_bB_c)\,. \qquad \text{(B.1)}$$

The infinitesimal transformations relate the various objects

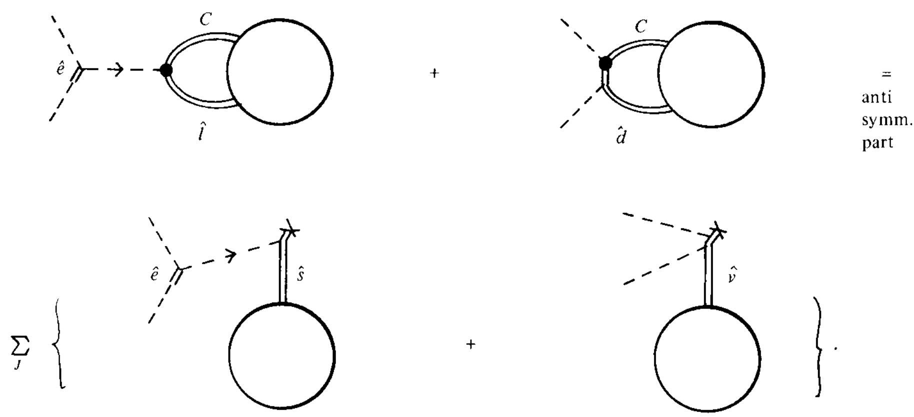


Fig. B.1.

$$C \to \hat{m},\ \hat{l}(A)\quad,\qquad \hat{l} \to \hat{e},\ \hat{d}(A)\quad,$$

$$A \to \hat{t},\ \hat{s}(A)\quad,\qquad \hat{s} \to \hat{u},\ \hat{v}(A)\quad.$$

Since $\hat{m}$ has an inverse we may finally do a field redefinition involving the quantitiy $-\hat{m}^{-1}_{ad}\,\hat{e}_{dbc}(B_b B'_c - B'_b B_c)$. The resulting Lagrangian is:

$$L_{\text{inv}} - \tfrac{1}{2} C_a^2 + J_i A_i$$

$$+ J_i(\hat{u}_{iab} + g\,\hat{v}_{iab}(A) - \hat{t}_{ic}\,\hat{m}^{-1}_{cd}\,\hat{e}_{dab} - g\,\hat{s}_{ic}(A)\,\hat{m}^{-1}_{cd}\,\hat{e}_{dab})(B_b B'_c - B'_b B_c)$$

$$- C_a(g\,\hat{d}_{abc}(A) - g\,\hat{l}_{ad}(A)\,m^{-1}_{df}\,\hat{e}_{fbc})(B_b B'_c - B'_b B_c)\quad. \tag{B.2}$$

The sum of diagrams involving one BB' pair must be zero. Considering diagrams involving one J and one BB' pair one deduces straightaway that

$$\hat{u}_{iab} - \hat{t}_{ic}\,\hat{m}^{-1}_{cd}\,\hat{e}_{dab} = 0\quad. \tag{B.3}$$

This is an identity, not an equation of motion, and may be substituted in the Lagrangian (B.2). The requirement that diagrams involving one BB' pair sum up to zero is then depicted in fig. B.1. The desired group property follows if either side of the equation of fig. B.1 is zero separately. Now the singularity structure (as a function of the momenta emitted by the sources) of the left and right hand side is in general very different. This may be seen as follows. Limiting ourselves to tree diagrams we may work out the ingoing C-line to obtain fig. B.2.

We cannot go further unless we make some assumptions on the momentum dependence of the various propagators and vertices. Suppose that

(i) the ghost vertices (except the $\hat{e}$-ghost vertex, see eq. (B.3)) have no momen-

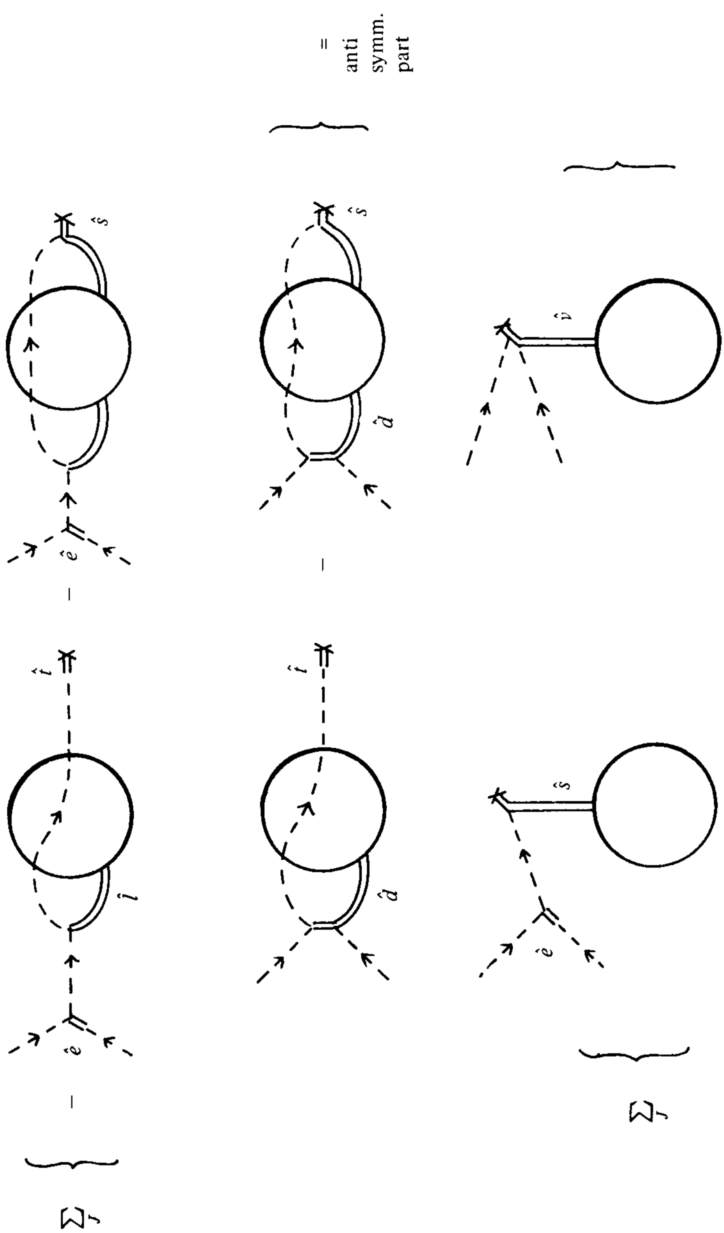


Fig. B.2.

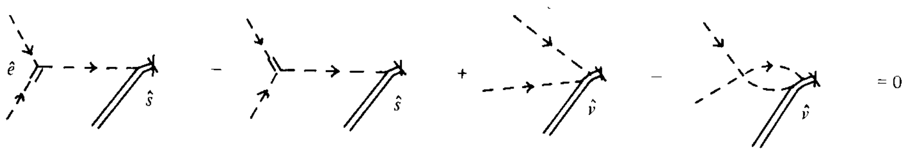


Fig. B.3.

tum dependence such that the pole of the attached ghost propagator cancels (in practice this means that $\hat{l}$, $\hat{s}$, $\hat{t}$, $\hat{d}$ may contain at most one derivative);

(ii) the vertices that are separated from the sources by one propagator only, have no momentum dependence such that the pole of that propagator cancels.

Then we may separate out in a unique way those terms where a given source is connected directly to the ghost combination (without any intervening vertex where momentum can flow away to other sources). These terms can be found only in the right hand side of the equal sign in fig. B.2. The result is the group property, see fig. B.3.

It may be noted here that if the group property holds for the transformation law of the fields it also holds for a large class of functions of the fields.

The above argument works for many theories, in particular those that possess a renormalizable gauge (then the vertices have at most one derivative) and that have at most one derivative in $\hat{l}$, $\hat{d}$, $\hat{s}$ and $\hat{t}$.

APPENDIX C

Gauge invariance in quantum-electrodynamics

Consider photons interacting with electrons. The Lagrangian is invariant for the gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \Lambda \ , \qquad \psi' = \psi + ig\,\Lambda\psi \ . \tag{C.1}$$

These are the eqs. (2.1). The invariant Lagrangian is

$$L_{\text{inv}} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}(\gamma D + m)\,\psi \ , \tag{C.2}$$

$$D_\mu = \partial_\mu - igA_\mu \ .$$

This must be supplemented by an L_{C}. If we take $L_{\text{c}} = -\frac{1}{2}(\partial_\mu A_\mu)^2$ we get the usual Feynman gauge, and the ghost Lagrangian involves no photon field. An illustrative different choice is:

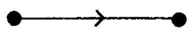
$$C = \partial_\mu A_\mu - \tfrac{1}{2}\alpha g A_\mu^2 \tag{C.3}$$

with α arbitrary.

Under a gauge transformation (C.1):

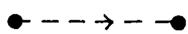
$$\frac{\delta_{\mu\nu}}{k^2 - i\epsilon}$$

photon propagator



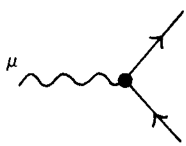
$$-\frac{i\gamma k - m}{k^2 + m^2 - i\epsilon}$$

electron propagator



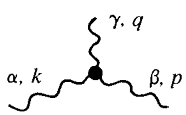
$$\frac{1}{k^2 - i\epsilon}\ ,$$ -1 for every closed loop.

ghost propagator



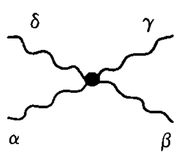
$$i\,g\,\gamma^{\mu}$$

$i\,g\,A_\mu\,\bar{\psi}\,\gamma^\mu\,\psi$



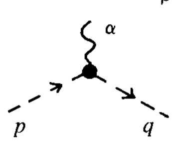
$$i\,\alpha\,g\,(\delta_{\alpha\beta}q_\gamma + \delta_{\alpha\gamma}p_\beta + \delta_{\beta\gamma}\,k_\alpha)$$

$\frac{1}{2}\,\alpha\,g\,A^2\,\partial_\mu A_\mu$



$$-\alpha^2\,g^2(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma})$$

$-\frac{1}{8}\,\alpha^2\,g^2\,A_\mu^2\,A_\nu^2$



$$-\,i\,\alpha\,g\,p_\alpha$$

$-\,\alpha\,g\,A_\mu\,\varphi^*\partial_\mu\varphi$

Fig. C.1.

$$C' = C - \alpha g A_\mu\,\partial_\mu\Lambda + \partial^2\Lambda\ . \qquad \text{(C.4)}$$

Thus, in the notation of (2.2):

$$\hat{l} = -\,\alpha A_\mu\partial_\mu\ , \qquad \hat{m} = \partial^2\ . \qquad \text{(C.5)}$$

This is a permissible gauge because $\hat{m}$ has an inverse. The corresponding ghost Lagrangian is (see 2.4):

$$L_\varphi = \varphi^*\,\partial^2\varphi - \alpha g A_\mu\,\varphi^*\,\partial_\mu\varphi\ . \qquad \text{(C.6)}$$

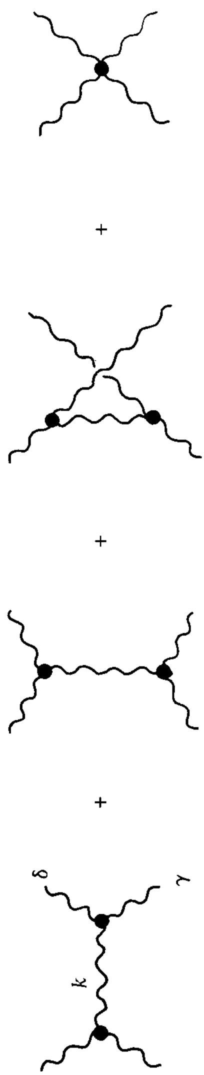


Fig. C.2.

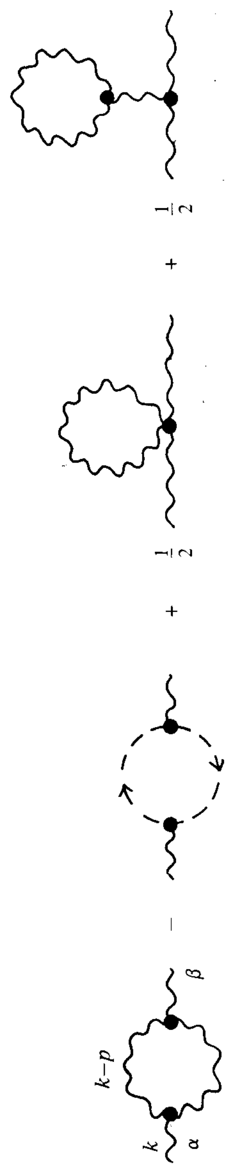


Fig. C.3.

The propagators corresponding to $L_{\rm inv} - \frac{1}{2} C^2 + L_\varphi$ are given in fig. C.1. The diagrams composed out of these elements reproduce ordinary Q.E.D., which is the theory without the last three vertices of fig. C.1. First we note that physical photons are those that are according to (2.6) emitted by a source J_μ that satisfies $\partial_\mu J_\mu = 0$ (from C_2 one has $\hat{s} = 0$, $\hat{t}_\mu = \partial_\mu$ in the notation of (2.6). These are indeed the ordinary transverse photons.

If the vertices of fig. C.1 reproduce ordinary Q.E.D. then the contributions of the last three vertices must cancel in the physical matrix elements. We will consider some examples; Photon-photon scattering, fig. C.2.

The first diagram of fig. C.2 gives, between physical states

$$i \alpha g\, \delta_{\alpha\beta} k_\lambda \frac{1}{k^2 - i\epsilon} (- i\alpha g\, \delta_{\delta\gamma} k_\lambda) = \alpha^2 g^2\, \delta_{\alpha\beta} \delta_{\delta\gamma}$$

Similarly for the second and third diagram. Indeed, this cancels against the fourth.

Vacuum polarization in electron-electron scattering, fig. C.3. The ghost tadpole diagram gives zero after symmetrical integration. The first diagram gives

$$-\tfrac{1}{2} \alpha^2 g^2 \int \mathrm{d}^4 p\, \frac{\{\delta_{\alpha\delta}(k-p)_\gamma - \delta_{\alpha\gamma} p_\delta\}\{-\delta_{\beta\delta}(k-p)_\gamma + \delta_{\beta\gamma} p_\delta\}}{(p^2 - i\epsilon)((k-p)^2 - i\epsilon)}$$

$$= \alpha^2 g^2\, \delta_{\alpha\beta} \int \mathrm{d}^4 p\, \frac{1}{p^2} - \alpha^2 g^2 \int \mathrm{d}^4 p\, \frac{p_\alpha (k-p)_\beta}{(k-p)^2} \quad .$$

The ghost diagram gives

$$-(-\alpha^2 g^2) \int \mathrm{d}^4 p\, \frac{(-p_\alpha)\{-(k-p)_\beta\}}{p^2 (k-p)^2} = \alpha^2 g^2 \int \mathrm{d}^4 p\, \frac{p_\alpha (k-p)_\beta}{p^2 (k-p)^2} \quad .$$

The last two diagrams give

Fig. C.4.

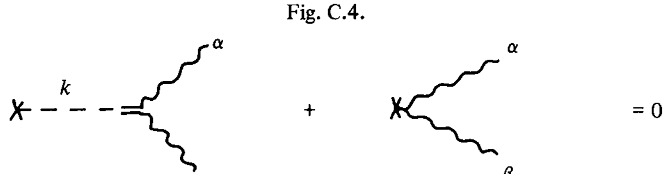


Fig. C.5.

$$2\,\alpha^2 g^2\,\delta_{\alpha\beta}\int \mathrm{d}^4p\,\frac{1}{p^2}\ ,\qquad -3\,\alpha^2 g^2\,\delta_{\alpha\beta}\int \mathrm{d}^4p\,\frac{1}{p^2}$$

respectively. We see that the whole lot adds up to zero.

Ward identities in this model are obtained by multiplying a source J by C. This gives the extra terms

$$J(\partial_\mu A_\mu - \tfrac{1}{2}\,\alpha g A_\mu^2)\ ,$$

which implies the vertices of fig. C.4.

In lowest order of g the identity of fig. 3 in the absence of further sources R reads as in fig. C.5. Indeed, remembering that the external photons are physical

$$\alpha g\,\frac{1}{k^2}\,k^2\,\delta_{\alpha\beta} - \alpha g\,\delta_{\alpha\beta} = 0\ .$$

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