TRANSPLANCKIAN PARTICLES
AND THE QUANTIZATION OF TIME

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Abstract

Trans-Planckian particles are elementary particles accelerated such that their energies surpass the Planck value $\sqrt{\hbar c^5/8\pi G}$. There are several reasons to believe that trans-Planckian particles do not represent independent degrees of freedom in Hilbert space, but they are controlled by the cis-Planckian particles. A way to learn more about the mechanisms at work here, is to study black hole horizons, starting from the scattering matrix Ansatz.

By compactifying one of the three physical spacial dimensions, the scattering matrix Ansatz can be exploited more efficiently than before. The algebra of operators on a black hole horizon allows for a few distinct representations. It is found that this horizon can be seen as being built up from string bits with unit lengths, each of which being described by a representation of the $SO(2,1)$ Lorentz group. We then demonstrate how the holographic principle works for this case, by constructing the operators corresponding to a field $\phi(\tilde{x}, t)$. The parameter t turns out to be quantized in units t_{Planck}/R, where R is the period of the compactified dimension.

1. INTRODUCTION

A satisfactory theory that unifies Quantum Mechanics with General Relativity must start with a description of the fundamental degrees of freedom that are relevant at the Planck scale. Standard quantum field theories, including perturbative Quantum Gravity as well as supergravity theories, use as a starting point that Hilbert space can be thought of as spanned by the set of states of N approximately freely moving physical particles:

$$\mathcal{H} \overset{?}{=} \left\{ |\vec{p}_{(1)}, \sigma_{(1)}, \ \vec{p}_{(2)}, \sigma_{(2)}, \ \ldots, \ \vec{p}_{(N)}, \sigma_{(N)} \rangle \ , \qquad N = 0, \ 1, \ 2, \ \ldots \right\} , \qquad (1.1)$$

where $\vec{p}_{(i)}$ are the momenta, and $\sigma_{(i)}$ discrete degrees of freedom such as spin and other quantum numbers. Even though the particles may interact, one still describes a state of Hilbert space at any given instant of time using this language*. After constructing wave packets using these states as a basis, one can define the scattering matrix as the transformation matrix between ingoing and outgoing wave packets.

For non-perturbative Quantum Gravity, this picture cannot be maintained. Not only do trans-Planckian particles (elementary particles boosted towards energies beyond the Planck energy) cause non-renormalizable divergences in perturbation theory, but they also form mini-black holes, as soon as one would allow for the presence of several trans-Planckian particles moving in different directions. Therefore, the true Quantum Gravity Hilbert space must be very different from (1.1).

Given a box with linear dimensions of the order of some length R. The highest energy state that can be described as being confined to this box must be a black hole with mass $M \approx R/Gc^2$. It is also the state with the largest entropy allowed inside the box. This observation dictates that the dimensionality of Hilbert space grows exponentially with the surface area of the box, as if we had a conventional field theory entirely restricted to the surface, with only fermionic basic fields and a cut-off at the Planck length. We refer to this doctrine as the 'Holographic Principle' [1].

On the one hand, the dimensionality of this Hilbert space is hardly big enough to describe all cis-Planckian states (states containing only particles with energies less than the Planck energy). On the other hand, we still would like to maintain Lorentz-invariance, which allows a single particle to be boosted to arbitrary energies. The way out of this dilemma must be that, although single particle states allow to be Lorentz boosted to trans-Planckian energies, it will not be allowed to boost one particle in one direction and an other particle in an other direction to beyond the Planckian regime, without creating states that can also be described in a different way. If two trans-Planckian particles approach one another too closely, they should actually be described as being a black hole, and their combined state cannot be distinguished from black holes formed in some different way.

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* In non-Abelian gauge theories, there are further refinements: ghost particles have to be introduced, which play a role in the renormalization procedure, but must be removed from the spectrum of physical states.

Thus we see that any attempt at a description of Hilbert space that is as sharply defined as possible, forces us to consider the presence of black holes. According to the equivalence principle in General Relativity, the behavior of a single, large black hole can be derived from field theories in flat space-time by means of a Rindler space transformation. If this would not be true at all for states in the immediate vicinity of a horizon, then it would be difficult to imagine why this equivalence principle appears to work flawlessly in ordinary General Relativity. We consider it to be much more likely that this principle can be replaced by some more complicated symmetry transformation even in the full theory of Quantum Gravity. The author has advertised before, that one should be able to use this symmetry transformation in the reverse direction, *i.e.*, it should be possible to derive features of the Hilbert space of flat space-time by studying the horizon of one black hole, preferrably of infinite size. Surely, the Hilbert space (1.1) should be replaced by something else, but whatever it is, one should still be able to define asymptotic wave packets of in- and out-states – it is alright if these contain trans-Planckian particles, as long as each of these are very far separated from all the others. Therefore, we do assume that the notion of a scattering matrix still makes sense in Quantum Gravity [2].

Although this idea was originally taken with much skepticism, it is now generally being confirmed that also in terms of D-brane theories black holes can be represented as pure quantum states [3]. However, insights in the dynamics of a black hole horizon can be obtained without assuming the validity of string theories or their relatives. What one has to assume is that not only quantum fields in the vicinity of a horizon, but also the black hole as an entire unit should obey the laws of quantum mechanics, or, more precisely, the processes of particle emission and absorption should be described by a scattering matrix. This is referred to as the S-matrix Ansatz [2].

We insist that the S-matrix Ansatz should be appliccable first and foremost for the prototype of all black holes, the Schwarzschild solution in 3+1 dimensions. It is not difficult to argue that the S-matrix Ansatz may well be a local property of any horizon, since the entropy of a black hole is proportional to the horizon area. Thus, the properties of Reissner-Nordstrøm and Kerr-Newmann black holes should follow directly from the pure Schwarzschild case. In contrast, extreme black holes will be much more dubious objects. Only string theorists appear to be able to imagine extreme black holes as regular objects, but the physical black holes, among which the astronomical ones, will never come close to the extreme case through either accretion or decay. No further attention will be given to extreme black holes in this paper.

The mutual interactions between in- and outgoing particles are known, as soon as[†]

$$|g_{\mu\nu}(x)\, p_{\text{in}}^\mu p_{\text{out}}^\nu| \lesssim M_{\text{Pl}}^2 \ , \qquad M_{\text{Pl}} = \sqrt{\hbar c/8\pi G} \ , \tag{1.2}$$

where p_{in}^μ and p_{out}^ν are the momenta of an ingoing and an outgoing particle in a locally regular coordinate frame with metric[‡] $g_{\mu\nu}(x)$, and we assume ordinary quantum field theory

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[†] The reason for the factor 8π will become evident soon. As has been proposed earlier, we could use the symbol $\overline{G}$ for $8\pi G$, and the associated units the $\overline{\text{Planck}}$-units.

[‡] Our signature convention for the metric $g_{\mu\nu}$ is $(-,+,+,+)$.

and perturbative quantum gravity to apply below Planckian energies. However, since one is forced to limit oneself to the case that the in- and the outgoing particles stay separated by distances in the transverse direction that are large compared to the Planck length, we have not yet been able to cover all possible elements of Hilbert space this way, which may explain why as yet this approach did not naturally yield the Hawking entropy.

The next step in the right direction could be made by replacing the algebra of known operators in Hilbert space by an obviously Lorentz covariant algebra [4, 5, 6], but then it was found that one is forced to choose either to have non-local commutators or incomplete commutators. Choosing the latter option, we found that some of the horizon surface elements can be arranged according to a representation of the self-dual part of the Lorentz group $SO(3,1)$, but this representation is not unitary, and the commutators with the anti-self-dual parts are non-local. This made the physical interpretation difficult, although one clearly finds hints of a space-time discreteness [7].

In our approach, it is essential that the dimensionality of space-time is limited to the physical value 3+1. Excursions to other dimensionalities may be useful exercises, but eventually we will have to deal with the physical case anyhow. It was observed, however, that adding the electromagnetic force to the S-matrix Ansatz, essentially corresponds to the addition of a fifth, compactified Kaluza-Klein dimension, so it will certainly be of interest to study the S-matrix Ansatz also in 4+1 dimensions.

Now this suggests that we can also venture towards the opposite direction. The proposal followed in this paper is, temporarily, to compactify also the third, physical, dimension. This may amount simply to restricting oneself to 'solutions' which are strictly periodic with respect to the z coordinate. At a later stage, one could decide to lift this restriction, for instance by taking the limit where the period R of the compactified coordinate tends to infinity. Concretely, what we propose to do is to study the S-matrix Ansatz in 2+1 dimensions, adding as a refinement two conserved $U(1)$ charges, one for momentum conservation in the compactified dimension, and one for ordinary electric charge, and of course at a later stage we can accommodate for the other known interactions as well.

The paper is organised as follows. In Sect. 2, we summarise the derivation, by now standard, of the horizon algebra, first in 3+1 dimensions. Then we show how to derive a similar algebra for the case that one dimension is compactified. If, for the time being, we ignore the $U(1)$ charges altogether, the result is the simple commutator algebra of Eq. (2.13), which is used as a starting point. Its representations are briefly described in Sect. 3. Then, in Sect. 4, we describe the notion of a field operator in a space-time point $(\tilde{x}, t)$, in terms of the states in the horizon representation, which illustrates the holographic principle. [1]

In Sect. 5, we return to the question of decompactifying the compactified dimension. This, unfortunately, is not quite so straightforward, as the results appears not to be Lorentz invariant, but the problem is seen to be related to the fact that in more than 3 dimensions, the horizon has more than one dimension, and this complicates the algebra.

2. THE HORIZON ALGEBRA

An extensive discussion of the horizon algebra can be found in Ref [5]. We start by characterising ingoing states by their distribution of the ingoing momentum $p_{\text{in}}^- = \frac{1}{\sqrt{2}}(p^1 - p^0)$ as a function of the transverse coordinates $\tilde{x} = (y, z)$ of the impact point (as yet two-dimensional):

$$|\text{in}\rangle \overset{\text{def}}{=} |\{p_{\text{in}}^-(\tilde{x})\}\rangle . \tag{2.1}$$

Any change in the in-state of the form $\delta p_{\text{in}}^-(\tilde{x})$, produces a shift δx_{out}^- among the outgoing particles, of the form

$$\delta x_{\text{out}}^-(\tilde{x}) = \int \mathrm{d}\tilde{x}' \, f(\tilde{x} - \tilde{x}')\delta p_{\text{in}}^-(\tilde{x}') , \tag{2.2}$$

where $f(\tilde{x})$ obeys the Poisson equation (if the background is flat)

$$\tilde{\partial}^2 f(\tilde{x}) = -\delta^2(\tilde{x}) , \tag{2.3}$$

in units where $8\pi G = 1$. Representing the outgoing state by the distribution $p_{\text{out}}^+(\tilde{x})$, one obtains the amplitude

$$\langle \text{out}|\text{in}\rangle = \mathcal{N} \exp\left[-i \int \mathrm{d}^2\tilde{x} \, \mathrm{d}^2\tilde{x}' p_{\text{in}}^-(\tilde{x}')f(\tilde{x} - \tilde{x}')p_{\text{out}}^+(\tilde{x})\right] . \tag{2.4}$$

This allows us to identify the Hilbert space of outgoing states with that of the ingoing particles. Introducing operators

$$\begin{aligned}
p^-(\tilde{x}) = p_{\text{in}}^-(\tilde{x}) , && x^+(\tilde{x}) , && [p^-(\tilde{x}), x^+(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') , \\
p^+(\tilde{x}) = p_{\text{out}}^+(\tilde{x}) , && x^-(\tilde{x}) , && [p^+(\tilde{x}), x^-(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') ,
\end{aligned} \tag{2.5}$$

allows us to write

$$p^-(\tilde{x}) = -\tilde{\partial}^2 x^-(\tilde{x}) , \qquad p^+(\tilde{x}) = \tilde{\partial}^2 x^+(\tilde{x}) . \tag{2.6}$$

In Ref [5], from Eq. (15.11) onwards, it is attempted to rewrite these equations in a completely Lorentz-invariant way, leading to partly local commutation rules for the surface elements

$$W^{\mu\nu}(\tilde{\sigma}) = \varepsilon^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} . \tag{2.7}$$

The route we will now take instead, is to choose one of the transverse coordinates, $z = x_3$, to be compactified, having a period R. Here, the horizon is taken to be running in the $(y, z) = (x_2, x_3)$ direction, the longitudinal coordinates being x^1, x^0). We only insist that the equations (2.5) and (2.6) hold at transverse distances $|\delta y| \gg R$, in which case the explicit z-dependence can be ignored. Taking in mind that $p^\pm(\tilde{x})$ should be treated as distributions, and $x^\pm(\tilde{x})$ should be averaged over in the z direction, we define

$$p^\pm(y) = \int \mathrm{d}z \, p^\pm(y, z) , \qquad x^\pm(y) = \frac{1}{R} \int \mathrm{d}z \, x^\pm(y, z) . \tag{2.7}$$

This turns Eqs (2.5) and (2.6) into

$$[p^{\pm}(y),\, x^{\mp}(y')] = -i\delta(y-y')\,, \qquad p^{\pm}(y) = \pm R\frac{\partial^2}{\partial y^2}x^{\pm}(y)\,. \tag{2.8}$$

For the time being, we remove the factor R here by replacing our units such a way that

$$8\pi G/R = 1\,. \tag{2.9}$$

These are now the equations that hold for $|\delta y| \gg 1$. From now on, in Eq. (2.8), $R = 1$.

Invariance under the $SO(1,1)$ subgroup of the Lorentz group that corresponds to $(x,\,t)$ transformations, is obvious:

$$\begin{aligned}
p^+ &\to A\,p^+\,; & x^+ &\to A\,x^+\,; \\
p^- &\to A^{-1}p^-\,; & x^- &\to A^{-1}x^-\,.
\end{aligned} \tag{2.10}$$

Before restoring $SO(2,1)$ invariance, we rewrite our algebraic equations as follows:

$$[\partial_y^2\, x^+(y),\, x^-(y')] = -i\delta(y-y') \quad \to \quad \begin{cases} [\partial_y x^+(y),\, \partial_y x^-(y')] &= i\delta(y-y')\,, \\ [\partial_y x^0(y),\, \partial_y x^1(y')] &= i\delta(y-y')\,. \end{cases} \tag{2.11}$$

Replacing the y coordinate by an arbitrary coordinate $\sigma(y)$, turns this equation into

$$[\partial_\sigma x^0(\sigma),\, \partial_\sigma x^1(\sigma')] = i\partial_\sigma y(\sigma)\,\delta(\sigma-\sigma')\,, \tag{2.12}$$

where $\partial_\sigma y(\sigma)$ is required to reproduce the necessary Jacobian factor. Now we see that, writing $y = x_2(\sigma)$, allows us to rewrite Eq. (2.12) as

$$[\partial_\sigma x^\mu(\sigma),\, \partial_\sigma x^\nu(\sigma')] = i\varepsilon^{\mu\nu\lambda}\,g_{\lambda\kappa}\delta(\sigma-\sigma')\partial_\sigma x^\kappa(\sigma)\,, \tag{2.13}$$

if $(\mu,\,\nu) = (0,\,1)$ or $(+,\,-)$. The sign of the $2+1$ dimensional ε symbol is set by

$$\varepsilon^{012} = \varepsilon^{+-2} = +1\,. \tag{2.14}$$

For $g_{\lambda\kappa}$ we take the flat metric diag(-1,1,1). It is important to note that in the transformation from y to σ, the Jacobian $\partial_\sigma y$ was assumed to be positive. This implies that the *orientation* of the horizon, with respect to the observer, is of relevance to our considerations. Note furthermore that the contribution of $\partial y/\partial\sigma$ helps to make the expression look Lorentz-covariant. Indeed, requiring invariance under the Lorentz group $SO(2,1)$, leads us to expect Eq. (2.13) to remain valid also when μ or ν reprsents the y coordinate. This extremely non-trivial step hinges upon the assumption that the operators $x^{\pm}(\tilde{x})$ in Eqs. (2.5) indeed represent the position of the horizon in the longitudinal coordinates, thus allowing us to view $(x^{\pm},\,\tilde{x})$ as a Lorentz vector. The result of this assumption is that we obtained what we believe to be the correct commutation rules for all coordinates, longitudinal and transverse alike (though as yet reduced to 2+1 dimensions, see Sect. 5).

It is the new commutator (2.13) that we will use as our starting point. In contrast to the general case in 3+1 dimensions, we are still dealing with entirely local commutators (that is, containing only Dirac delta functions in σ and σ'), and in the next section, we exploit this observation.

3. REPRESENTATIONS

The commutator algebra (2.11) was derived directly from the S-matrix Ansatz, and is valid when the transverse distance scales Δy are large, so that the transverse momenta p_y are negligible. This is important to remember; in (2.11) the sideways shifts had been ignored. The representations of (2.11) are simply described by single functions,

$$|\{p^-(y)\}\rangle \qquad \text{or} \qquad |\{p^-(y)\}\rangle \qquad \text{or} \qquad |\{x^+(y)\}\rangle \qquad \text{or} \qquad |\{x^-(y)\}\rangle\,, \qquad (3.1)$$

which can be transformed one into the other using (2.8).

The algebra (2.13) is assumed to have a wider validity; being Lorentz invariant it handles the sideways shifts equally well as the longitudinal shifts. It replaces (2.11) when the transverse momenta also become large.

As Eq. (2.13) contains derivatives with respect to the (arbitrary) coordinate σ, we are invited to integrate over σ. Let us divide the horizon into segments $A = [\sigma_1, \sigma_2]$, $B = [\sigma_2, \sigma_3]$, etc.. Define

$$a_{(1)}^\mu = x^\mu(\sigma_2) - x^\mu(\sigma_1)\,; \qquad a_{(2)}^\mu = x^\mu(\sigma_3) - a^\mu(\sigma_2)\,; \qquad \text{etc.} \qquad (3.2)$$

Integrating Eq. (2.13) over σ yields the new commutation rules for the line segments $a_{(i)}^\mu$ (see Fig. 1):

$$[a_{(i)}^\mu, a_{(j)}^\nu] = i\delta_{ij}\, \varepsilon^{\mu\nu\lambda}\, g_{\lambda\kappa} a_{(i)}^\kappa\,. \qquad (3.3)$$

Since the different segments commute, we can now concentrate on one segment A only. As Eq. (3.3) corresponds to the algebra of the Lorentz group $SO(2,1)$ (for each segment), its non-trivial representations are infinite dimensional. The Casimir operator is

$$\left(\int_A d\sigma\, \partial_\sigma x^\mu\right)^2 = \mathbf{a}^2 = g_{\mu\nu}a^\mu a^\nu = a^{1\,2} + a^{2\,2} - a^{0\,2}\,. \qquad (3.4)$$

This operator gives us some information about the separation between the end points of the segment A. If we fix it to some definite value, this corresponds to a gauge fixing condition for the coordinate σ, which, after all, had been chosen arbitrarily. Therefore, it is legitimate to give a constraint on the allowed value(s) for $\mathbf{a}^2$. We write

$$\mathbf{a}^2 = -\ell(\ell+1)\,, \qquad (3.5)$$

although ℓ does not have to be an integer or half-integer. Special cases of importance will be the choices $\ell = {}^1\!/_2$, $\ell = 0$, and $\ell = -{}^1\!/_2$.
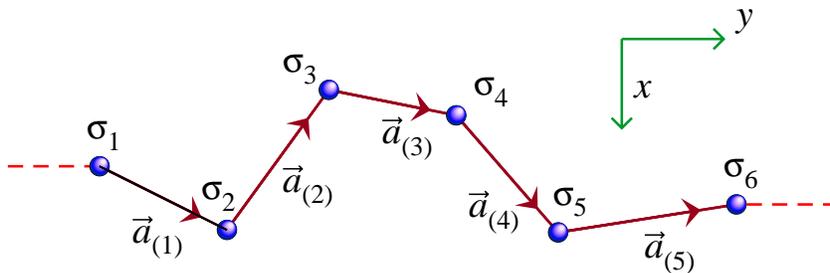

Fig. 1. Segmentation of the horizon.

The representations are constructed by diagonalising $a^0 = m$, and using the raising and lowering operators $a^\pm$:

$$a^\pm = a^1 \pm ia^2 \,, \qquad [a^0,\, a^\pm] = \pm a^\pm \,, \tag{3.6}$$

$$a^\mp a^\pm = \mathbf{a}^2 + a^0(a^0 \pm 1)\,. \tag{3.7}$$

From this:

$$a^\pm|m,\, \ell\rangle = \sqrt{m(m \pm 1) - \ell(\ell + 1)}\,|m \pm 1,\, \ell\rangle \,. \tag{3.8}$$

If ℓ is real and positive, the values of m may be taken to be

$$m = \ell + 1,\, \ell + 2, \ldots,\, \infty\,, \qquad \text{or} \qquad m = -\ell - 1,\, -\ell - 2, \ldots,\, -\infty\,. \tag{3.9}$$

In addition, for all real values of $\mathbf{a}^2$, with ℓ either real or complex, one may have series of m values that are unbounded above and below:

$$m = m_0,\, m_0 \pm 1,\, m_0 \pm 2,\, \ldots \pm \infty\,. \tag{3.10}$$

The representations (3.9) will be called timelike. Even though $\mathbf{a}^2$ may be positive ($\mathbf{a}^2 < \frac{1}{4}$), the vector $\mathbf{a}$ may be seen to lie either on the positive or on the negative wing of a "timelike" hyperbola. The series (3.10) will be called spacelike, although one could also admit negative values for $\mathbf{a}^2$ here.

A special case is $\ell = -\frac{1}{2}$, which we will call lightlike. In this case, the square root in Eq. (3.8) can be drawn:

$$a^\pm|m,\, -\tfrac{1}{2}\rangle = (m \pm \tfrac{1}{2})\,|m \pm 1,\, -\tfrac{1}{2}\rangle \,. \tag{3.11}$$

For negative m values, this implies a minor revision of the sign conventions for the states $|m,\, -\frac{1}{2}\rangle$.

We conclude that the representations of our horizon algebra are characterised by a number ℓ that is either real or of the form $\ell = -\frac{1}{2} + i\,\mathrm{Im}(\ell)$, and that there is a special case $\ell = -\frac{1}{2}$, where the explicit expressions for the operators simplify. The full Hilbert space $\mathcal{H}$ for the representations of the algebra (2.13) is the product of the hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$, ... of the (connected) line segments A, B, etc. The vectors $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, ..., can be added to describe larger horizon segments, in a way similar to the procedure of adding angular momenta, although, of course, there are fewer selection rules, due to the fact that the vectors $\mathbf{a}_{(i)}$ are Lorentz vectors.

The number of states is always infinite, but if $\left(a^{1^2} + a^{2^2}\right)^{1/2}$ can be taken to be a measure for the "length" of a section of the horizon, and if we look at all states where the total length of the horizon is limited to some value, we find that the number of allowed values for $\{a^0_{(i)}\}$ is bounded after all, and it will naturally diverge as the exponential of the horizon area (= length), as one should expect from the entropy formula.

4. HOLOGRAPHIC FIELDS

As it was emphasized in the Introduction, and in earlier papers, this Hilbert space not only describes the horizon of a black hole, but it should also describe the Hilbert space of the entire universe, since the Rindler space transformation allows us to relate horizon dynamics with the dynamics of flat space-time. Since we have chosen the metric $g_{\mu\nu}$ to be that of flat space-time, the black hole under considerartion is of infinite size, and therefore all points of space-time that are at a finite metric distance from our horizon, belong to the black hole's environment. All field operators $\phi(\tilde{x}, t)$, for finite values of $\tilde{x}$ and t, should operate within the same Hilbert space.

How to define a field operator in our Hilbert space, be it (3.1) or the representation of (2.13) as derived in Sect. 3, is not at all evident. A scalar field (in 2+1 dimensions) would normally be defined as[§]

$$\phi(\tilde{x}, t) = \int \frac{\mathrm{d}^2 \tilde{k}}{\sqrt{2k^0(\tilde{k})(2\pi)^2}} \Big(a(\tilde{k}) e^{-i\tilde{k}\cdot\tilde{x}+ik^0 t} + a^\dagger(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}-ik^0 t} \Big) , \tag{4.1}$$

where a and $a^\dagger$ annihilate and create exactly one particle. Concentrating first on the algebra (2.11) with representation (3.1), we write

$$\phi(\tilde{x}, t) = \sqrt{(2\pi)^{-2}} \int \mathrm{d}^2 \tilde{k} \hat{\phi}(\tilde{k}, t) \, e^{i\tilde{k}\cdot\tilde{x}} , \tag{4.2}$$

and we may impose that the sideways momenta p_y and the mass μ of the particles involved are negligible[||]. In that case, the equation of motion for the field reads

$$(k^+ k^- + k_y^2 + \mu^2)\hat{\phi}(\tilde{k}, t) \approx k^+ k^- \hat{\phi}(\tilde{k}, t) \approx 0 , \tag{4.3}$$

which implies that either k^+ or k^- vanish; our field then consists of an ingoing component ϕ_{in} and an outgoing component ϕ_{out}. Since, in this case, k^0 only depends on k_x and not on k_y, we may introduce the mixed Fourier transform,

$$\phi_{\mathrm{in}}(\tilde{x}, t) = \sqrt{(2\pi)^{-1}} \int \mathrm{d}k_x \, e^{ik_x(x+t)} \check{\phi}_{\mathrm{in}}(k_x, y) , \tag{4.4}$$

to obtain

$$\check{\phi}_{\mathrm{in}}(k_x, y') \Big| p^-(y) \Big\rangle = \sqrt{\frac{N}{2|k_x|}} \Big| p^-(y) + \delta(y-y')k_x\sqrt{2} \Big\rangle , \tag{4.5}$$

where N refers to the number of identical particles already present in the ket-state in case of annihilation, or in the bra state in case of creation. We used $k^0 = |k_x|$. The factor $\sqrt{2}$ is a normalization factor.

[§] Sign conventions in the exponent are here chosen such that $\partial\phi/\partial t = -i[H, \phi]$.

[||] In this section, we use the notation $\tilde{x} = (x, y)$ and $\tilde{k} = (k_x, k_y)$.

Temporarily omitting the square root in Eq. (4.5), we find that the field operator can be written as

$$\check{\phi}_{\text{in}}(k_x, y') = C \exp\left(-ix^+(y')\, k_x\sqrt{2}\right),\qquad(4.6)$$

since $-x^+(y')$ is precisely the operator that produces the required shift in the function $p^-(y)$, according to (2.5). Using (4.4),

$$\begin{aligned}\phi_{\text{in}}(x^+, x^-, y') &= \frac{C}{\sqrt{2\pi}}\int \mathrm{d}k_x e^{i\sqrt{2}\left(k_x x^+ - k_x x^+(y')\right)}\\[4pt]&= C'\delta\left(x^+ - x^+(y)\right),\end{aligned}\qquad(4.7)$$

which is independent of x^- just because it is an ingoing field. C' is just a new constant.

In order to express the action of this operator in terms of operators of the algebra (2.13) or (3.2), we convert to the σ coordinates:

$$\phi_{\text{in}}(x^+, x^-, y') \approx C'\int \mathrm{d}\sigma\left|\frac{\partial y}{\partial\sigma}\right|\delta\bigl(y(\sigma) - y'\bigr)\delta\bigl(x^+(\sigma) - x^+\bigr).\qquad(4.8)$$

Similarly,

$$\phi_{\text{out}}(x^+, x^-, y') \approx C'\int \mathrm{d}\sigma\left|\frac{\partial y}{\partial\sigma}\right|\delta\bigl((y(\sigma) - y'\bigr)\delta\bigl(x^-(\sigma) - x^-\bigr),\qquad(4.9)$$

and the original field $\phi(\tilde{x}, t)$ could be defined as

$$\phi(\tilde{x}, t) = \phi_{\text{in}}(x^+, y) + \phi_{\text{out}}(x^-, y).\qquad(4.10)$$

In these expressions, we used the 'approximate' symbol ($\approx$) because the expressions were derived for the case that they act on states with smooth y dependences (small values for p_y). The exact expressions are not very useful for fields since the operators $x^+(\sigma)$ and $y(\sigma)$ do not commute, so that the order of the Dirac deltas may not be altered, and hence the transformation properties of these fields under rotations an Lorentz transformations would be problematic.

Yet, it is of importance to try to define as precisely as possible field operators with local commutation rules. Eq. (4.10) may be required to hold only in the limit of vanishing $p_y(\sigma)$. It is not clear to the author what the best possible procedure for this is. Our approximate expressions (4.8)–(4.10) suggest for instance to define

$$\phi(x, y, t) = C''\sum_{\text{orderings}}\int \mathrm{d}\sigma\, \delta\bigl(x^1(\sigma) - x\bigr)\delta\bigl(x^2(\sigma) - y\bigr)\delta\bigl(x^0(\sigma) - t\bigr),\qquad(4.11)$$

where the delta functions must be in all possible orders, and C'' is again a new numerical constant. The equations of motion for such fields, and from there the particle content of the theory, should follow from these expressions. One thing appears to be obvious [5]: the time coordinate t tends to be quantized in units $8\pi G\hbar/c^5 R$. The fact that the time quantum is inversely proportional to R should not come as a surprise. The maximal energy admitted in a compactified world is linearly proportional to R.

5. A NOTE ON DECOMPACITFICATION

In our compactified world, the momenta in the compactified dimension z in all respects act as new electric charges. Electric charge has been considered in Refs. [2, 5], and by using these results, incorporation of these compactified dimensions should be straightforward. It was found that electric charge dnsity operators $\varrho_{\text{in}}(y)$ and $\varrho_{\text{out}}(y)$ must be defined separately for the in-Hilbet space and for the out-Hilbert space. Similarly, we have the phase operators $\phi_{\text{in}}(y)$ and $\phi_{\text{out}}(y)$, living in the in- and the out-Hilbert space, respectively (strictly speaking, these objects are only defined *modulo* their period 2π, so that they are not good as operators, but we can give unambiguous meaning to their derivatives, $\partial_y \varrho_{\text{in, out}}(y)$).

Correspondingly, this leads to introducing (de-)compactified coordinate operators $a_{\text{in}}^i(y)$ and $a_{\text{out}}^i(y)$ ($i = 3$ or more), obeying the commutation rule

$$[\partial_y a_{\text{in}}^i(y), \, \partial_y a_{\text{out}}^j(y')] = ig^{ij}\delta(y - y'), \tag{5.1}$$

which, in σ coordinates would read

$$[\partial_\sigma a_{\text{in}}^i(\sigma), \, \partial_\sigma a_{\text{out}}^j(\sigma')] = ig^{ij}(\sigma)\partial_\sigma y \, \delta(y - y'), \tag{5.2}$$

and a^i, $i = 3, \ldots$ is expected to commute with a^μ, $\mu = 0, 1, 2..$

It is difficult to see how to turn these equations into Lorentz covariant ones. The reason for this is, presumably, that the horizon should not be taken to depend on a single coordinate σ but on two (or more, if space-time has more than 4 dimensions) coordinates $\tilde\sigma = (\sigma^1, \sigma^2)$. For the correct, Lorentz covariant algebra, we have to return to the original treatment of Ref [4, 5].

Alternatively, there is the following possibility to be considered. We could rewrite the algebra (3.3) in terms of the operators (temporarily suppressing the indices i),

$$T_{\mu\nu} = \varepsilon_{\mu\nu\lambda}a^\lambda, \tag{5.3}$$

which then obey

$$[T_{\mu\nu}, \, T_{\kappa\lambda}] = g_{\mu\kappa}T_{\nu\lambda} - g_{\mu\lambda}T_{\nu\kappa} - g_{\nu\kappa}T_{\mu\lambda} + g_{\nu\lambda}T_{\mu\kappa}. \tag{5.4}$$

It is tempting to generalize this commutator to larger numbers of dimensions. This would slightly deviate from what one can derive directly in 4-dimensional spacetime [5], but it is conceivable that a consistent theory can be obtained along these lines.

Our theory did result in a scheme where Hilbert space is represented by integers, and if the length of the horizon, to be defined as $\sum_i \left(a_i^{1\,2} + a_i^{2\,2}\right)^{1/2}$, is kept finite, there are only a finite number of states. Yet there is Lorentz-invariance. This must imply that the transPlanckian particle contents os a state are not freely adjustable, but follow from the configurations of the cis-Planckian particles present.

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