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Three-Dimensional Einstein Gravity: Dynamics of Flat Space

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In three spacetime dimensions, the Einstein equations imply that source-free regions are flat. Localized sources can therefore only affect geometry globally rather than locally. Some of these effects, especially those generated by mass and angular momentum are discussed.

I. INTRODUCTION

Because Einstein and curvature tensors are equivalent in three spacetime dimensions, general relativity is dynamically trivial there. Outside sources, spacetime is flat. All effects of localized sources are on the global geometry, which is fixed by singularities of the worldlines of the particles; these are arbitrary flat space geodesics. This means in particular that the conserved quantities, total energy-momentum and angular momentum, are related to topological invariants. As we shall see, there is a static $\mathcal{N}$-body solution with conical spatial geometry, whose total energy is additive and determines the Euler invariant of the spatial surface. Moving particles will also be treated. When angular momentum is present, novel phenomena involving time appear.

We shall derive the global $\mathcal{N}$-body geometry both analytically, in terms of an explicit metric solution, and geometrically. Angular momentum will be similarly discussed both in terms of an explicit "Kerr" solution corresponding to a localized spinning source and through its orbital effects. We shall also comment on the linearized approximation, and on the absence of a Newtonian limit.

When a cosmological term is present, the curvature is a (non-vanishing) constant, and the situation changes considerably; that analysis will be given elsewhere [1]. We emphasize that we are discussing ordinary Einstein gravity, rather than the quite

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different "topologically massive" gravity model [2], where additional terms of topological origin provide both graviton dynamics and matter couplings.

Flat but globally nontrivial external solutions were actually first obtained in four-dimensional gravity by Marder[1] [3] (following an observation by Fierz, as quoted in [4]). He gave cylindrically and axially symmetric solutions which had similar structure to our static case. The three-dimensional theory is the natural setting for those results, since here gravity has no life of its own. Staruskiewicz [5] first discussed this model, and obtained the one- and two-body static solutions.

II. Static $\mathcal{N}$-Body Solutions

Because of the identity

$$R^{\mu\nu}_{\alpha\beta} = \varepsilon^{\mu\nu\sigma} \varepsilon_{\alpha\beta\lambda} G^{\lambda}_{\sigma} \tag{2.1}$$

linking curvature and Einstein tensors, empty regions where $G^{\sigma}_{\lambda} = 0$ are flat, although interior ones, with a non-vanishing stress tensor $T_{\mu\nu}$ and

$$G^{\lambda}_{\sigma} = 8\pi G T^{\lambda}_{\sigma} \tag{2.2}$$

are not. Here G is the gravitational constant, with dimensions of inverse mass in $c = 1$ units. Since we are mainly interested in the "soluble model" aspects of this theory, with a view towards quantization, we shall deal only with point sources which concentrate curvature on worldlines and so affect the exterior geometry purely in a global way.

Consider the static case, where the metric decomposes into

$$-g_{00} = N^2(\mathbf{r}), \qquad g_{0i} = 0, \qquad g_{ij} = \gamma_{ij}(\mathbf{r}), \qquad \sqrt{-g} = N\sqrt{\gamma}, \tag{2.3}$$

where g and γ are determinants of $g_{\mu\nu}$ and γ_{ij}.

The Einstein tensor depends on the intrinsic spatial geometry and on the 2-scalar N as

$$-\sqrt{\gamma}\, G^0_0 = \frac{1}{2}\sqrt{\gamma}\, R, \qquad G^0_i = 0, \qquad G_{ij} = -\frac{1}{2N}(D_i D_j - \gamma_{ij} D^2)\, N. \tag{2.4}$$

Here D_i is the covariant derivative with respect to the spatial metric γ_{ij}. The spatial components of the Einstein tensor simplify because of the identical vanishing of the two-dimensional Einstein tensor. The form (2.4) of the Einstein tensor follows from the static part of the Einstein–Hilbert action, $I_{ES} = \int d^2x \sqrt{\gamma}\, NR$. (Throughout R denotes the intrinsic scalar curvature of the 2-surface; we never refer to the three-curvature scalar to avoid confusion.) Note that $\sqrt{\gamma}\, G^0_0$ is the Euler invariant density, a total 2-divergence. Outside sources, R vanishes, so the spatial 2-surface is flat.

[1] We thank J. S. Dowker for this reference.

We now consider the source to be a set of point particles at rest, with masses m_n located at $\mathbf{r}_n$, and stress tensor density

$$T^{00} = \sum_n m_n \delta^2(\mathbf{r} - \mathbf{r}_n), \qquad T^{0i} = 0 = T^{ij}. \tag{2.5}$$

Recall that the covariant conservation law $D_\mu T^{\mu\nu} = 0$ is equivalent, for a point particle, to the geodesic equation, $\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$. For initially static bodies, this reduces to $\ddot{x}^i = -\frac{1}{2}g^{ij}\partial_j g_{00}$, or equivalently $D_\mu T^{\mu\nu} = 0$ reduces to $\Gamma^i_{00} T^{00} = 0$. The acceleration will consistently vanish for our solution since, as we shall see, $g_{00} = -1$. These results are of course in accord with the Bianchi identity on G_{ij}, since $\sqrt{\gamma}[D_i, D_j]\, D^j N = \sqrt{\gamma} R^j_i \partial_j N = \frac{1}{2}\sqrt{\gamma} R \partial_i N = T^{00}\partial_i N$. We mention that the choice of sign of the gravitational constant G is not physically fixed a priori here, in contrast to four dimensions: First there is no static interaction to be made attractive and, more fundamentally, the gravitational field itself has no energy whose sign must be positive (*i.e.*, the same as that of a particle).

Irrespective of the spatial gauge choice at our disposal, the $G_{ij} = 0$ equations clearly imply that $D^2 N = 0$, $D_i D_j N = 0$, so that N is indeed constant; the convention $N = 1$ is just a calibration of time. We shall solve the G_{00} equation in isotropic coordinates $\gamma_{ij} = \phi\delta_{ij}$ (which are always permitted in two dimensions) and then transform to curvature ("Schwarzschild") coordinates to exhibit the global aspects. In this frame, $\frac{1}{2}\sqrt{\gamma}\, R$ reduces to $-\frac{1}{2}\nabla^2 \ln\phi$, where ∇^2 is the flat Laplacian. Then, since its Green's function is $\ln r$, with $\nabla^2 \ln r = 2\pi\delta^2(\mathbf{r})$, our solution to the time-time component of (2.2) is

$$\ln\varphi = -8\pi G \sum_n m_n \ln|r - r_n| + \ln C \tag{2.6}$$

and the metric becomes

$$g_{00} = -1, \qquad g_{0i} = 0, \qquad g_{ij} = C\delta_{ij} \prod_n |r - r_n|^{-8Gm_n}. \tag{2.7}$$

The constant C (which is neessarily positive to preserve signature) can be removed by a rescaling of r, except in the singular case $\sum m_n = 1/4G$, which will be discussed separately. The metric (2.7) represents the general static $\mathcal{N}$-body solution; the two-body form was first obtained in [5] by a different argument. A general geometrical treatment is given in Section V. There are neither particle interactions nor any hidden "rods" holding the particles fixed, since there is no curvature between them. Reassurance that this metric is indeed locally flat is provided by transforming to curvature coordinates. For the one-body case, this is easily carried out by explicitly transforming our line element

$$dl^2 = r^{-8Gm}[dr^2 + r^2\, d\theta^2] \tag{2.8a}$$

according to

$$\mathfrak{r} = \alpha^{-1} r^\alpha, \qquad \theta' = \alpha\theta, \qquad \alpha \equiv 1 - 4Gm \tag{2.8b}$$

to the flat element

$$dl^2 = d\rho^2 + \rho^2(d\theta')^2. \tag{2.8c}$$

However, the range of θ' is now $0 \leqslant \theta' \leqslant 2\pi\alpha$, and the space is a cone, the unique 2-space which is metrically flat except at one point—its vertex.[2]

In the above we have taken α to be positive, or $m < 1/4G$. When m exceeds this limit ($\alpha < 0$), the metric near the particle becomes singular, e.g., the distance from the particle to any other point diverges as r^{α}. This limitation on the mass may also be understood in terms of the formal equivalence between $\alpha \leftrightarrow -\alpha$ and the coordinate inversion $r \leftrightarrow 1/r$ implied by (2.8). In other words, a high-mass particle at the origin with $m > 1/4G$ ($\alpha < 0$) is actually a particle at infinity with acceptable mass $1/2G - m$ ($-\alpha > 0$). The geometrical counterpart of this argument is given in Section V, where it is also shown that composite (many-body) sources can have a higher (up to $1/2G$) total mass. There is no upper limit on α, hence arbitrary negative Gm are permitted.

At $4Gm = 1$, ($\alpha = 0$), the transformation (2.8c) becomes singular. In conformal coordinates, (2.8b) becomes

$$dl^2 = d(\ln r)^2 + d\theta^2, \qquad -\infty < \ln r < \infty, \qquad 0 \leqslant \theta \leqslant 2\pi, \tag{2.9}$$

which is a periodic strip or cylinder in the "Cartesian" coordinates $(\ln r, \theta)$; the integration constant C in (2.6) is absorbed into the strip height $\sqrt{C}\,\theta$ and into the choice of "origin" for $\ln r$. The Schwarzschild metric (2.8a) degenerates correspondingly.[3]

In the $\mathcal{N}$-body case, the asymptotic form of the metric (for $|\mathbf{r}| \gg |\mathbf{r}_n|$) reduces to the conical one-body form (2.8a) with $M = \sum m_n$. Values of $M > 1/4G$ actually represent a closed space, as we shall see in Section V. The mapping to a flat-space metric is now considerably more intricate. The analysis is aided by the observation that the n-body metric has simple form in complex notation

$$dl^2 = dx^i\, dx^i \prod_n |r - r_n|^{-8Gm_n} = dz\, dz^* \prod_n [(z - z_n)(z^* - z_n^*)]^{-4Gm_n}, \tag{2.10}$$

where $z \equiv x + iy$. Hence the transformation via the generalized incomplete Beta function,

$$Z = \int^z dz' \prod_n (z' - z_n)^{-4Gm_n} \tag{2.11}$$

[2] The angular measure $2\pi\alpha$ is clearly given by the ratio of circumference to radius of the cone. The singularity at the origin is manifest from the Cartesian form of the metric, $g_{ij} = \delta_{ij} + (\alpha^{-2} - 1)\, x^i x^j / r^2$, whose value is not well defined there.

[3] In the frame $dl^2 = dr^2 + \rho^2(r)\, d\theta^2$, the field equation yields the one-body solution $\rho = Br + C$, and $B = 0$ is the degenerate case.

reduces the above to manifestly flat form, $dl^2 = dZ\,d\bar{Z}$,[4] but with restrictions on the range of the new variables dictated by the external source masses and locations.

Finally we note that, in this classical background space, a quantized system will have its angular momentum altered by a factor α, i.e., the eigenvalues will be α^{-1} times an integer or half-integer value.[4] A way of measuring the mass would be through an Aharonov–Bohm effect in which the local curvatures would be reflected in the phase of a quantum system moving through the otherwise flat space. Light-bending would provide another way of locating the sources: A test beam impinging on a source would be split and reunited on the other side of the singularity (but light still follows a free null worldline in the process).

These results are quite special to gravity. Two-dimensional electrodynamics (or Yang–Mills) is also a model without excitations, but the field strength does not vanish outside the charges, hence generic static many-body solutions of the coupled Maxwell-charged particle system are excluded. The Gauss equation for the electric field $\mathscr{E}(r)$ reads $\mathscr{E}' = \sum e_n \delta(x - x_n)$. Its general solution is $\mathscr{E} = \frac{1}{2} \sum e_n \varepsilon(x - x_n) + \mathscr{E}_0$, where ε is the sign function and $\mathscr{E}_0$ is a constant "background field." For two charges with $e_1 = -e_2$, one can always arrange $\mathscr{E}_0$ such that there is no force on either of the two initially static particles, but this clearly cannot be accomplished for three or more (except for special configurations such as chains of arbitrarily located alternating charges of equal magnitude or infinite arrays of equal charges; these constitute a rather trivial set compared to the gravitational case). We have taken the Lorentz force law on initially static particles, $\ddot{x}_n = -e_n \mathscr{E}(x_n)$, to exclude the self-force.

III. Energy-Momentum

We now turn to the energy. Since this model is a gauge theory, there should be a flux integral expression for its total "charge," i.e., energy. But since there is also no asymptotic curvature, the only possible energy measure must be topological, and the Euler invariant is the only candidate. Indeed, $\sqrt{\gamma}\, G_0^0$ is (by (2.4)) both the sum of source energy densities and the Euler density,

$$-\sqrt{\gamma}\, G_0^0 = \tfrac{1}{2} \sqrt{\gamma}\, R = \tfrac{1}{4}\, \varepsilon^{ij} \varepsilon_k^l R^k_{lij} = \text{total divergence}. \tag{3.1}$$

This is another way of understanding why there is no gravitational field contribution to the energy: the full nonlinear $\sqrt{\gamma}\, G_0^0$ is already a divergence, whereas in four dimensions this is only true for its linearized part about an asymptotic background, so the total source there is the sum of matter and (nonlinear) gravitational contributions. From (2.4), (3.1), we see that the energy is given by

$$E = 1/16\pi G \int d^2x \sqrt{\gamma}\, R = -1/16\pi G \oint dS \cdot \nabla \ln \varphi = \sum_n m_n. \tag{3.2}$$

As expected, it is just the sum of source masses. Note that a closed space with S_2

[4] We thank F. Wilczek for discussions on this point.

topology has Euler invariant equal to 8π; consequently the total mass equals $1/2G$ there.

One may also calculate invariant (geodesic) distances between points in the usual way; these will be rather complicated in general, to take into account the "matching" of cones from the various sources, and the identification of points along the seams.

IV. Angular Momentum: Rotating Sources

Whenever there are two or more moving (non-collinear) particles, the system will possess (orbital) angular momentum as well as energy. We shall analyze the geometry for moving particles in Section V; here we obtain the "Kerr" solution corresponding to a time independent spatially localized spinning source with no energy density but only angular momentum density. The metric will of course be a transform of the Minkowski $\eta_{\mu\nu}$, since exterior spacetime is still flat. However, *which* transform it is can only be understood by analysis of the time-space component of the Einstein equations (2.2) whose sources, the momentum densities T_{0i}, determine angular momentum, here the single number J,

$$J = \tfrac{1}{2}\varepsilon_{ij} J^{ij} = \tfrac{1}{2}\varepsilon_{ij} \int d^2x (x^i T^{oj} - x^j T^{0i}). \tag{4.1}$$

Because there are no gravitational field contributions, our solution will be much simpler than the 4-dimensional Kerr metric.

We consider the stationary (time-independent but not static) axially symmetric interval in circular coordinates

$$ds^2 = dr^2 + f^2(r)\, d\theta^2 + 2g_{02}(r)\, dt\, d\theta - g_{00}(r)\, dt^2. \tag{4.2}$$

For later convenience we have chosen the spatial gauge $g_{11} = 1$. (A change of radial coordinates would yield any other desired form, e.g., curvature coordinates: $dl^2 = dr^2 + \rho^2(r)\, d\theta^2 = (dr/d\rho)^2\, d\rho^2 + \rho^2\, d\theta^2$.) Note that the coefficient of g_{02}, written in Cartesian coordinates ($x \equiv r\cos\theta$, $y \equiv r \sin\theta$) is

$$dt\, d\theta = dt r^{-2}(x\, dy - y\, dx) = dt(dy\, \partial_x - dx\, \partial_y) \ln r. \tag{4.3}$$

Since $\ln r$ is the Green's function, this part of the metric may be expected to to give rise, in the field equations, to a singularity appropriate to a localized spin source. This is a first indication that angular momentum is present. Note also that g_{02} has dimensions of length, like angular momentum itself: $GJ \propto (Gm)\, vr \propto r$ in gravitational units.

The component $\sqrt{-g}\, G^{0i}$ is identically $-2D_j \pi^{ij}$, where in the stationary case,

$$\pi_{ij} = -\sqrt{\gamma}/2N[D_i N_j + D_j N_i - 2\gamma_{ij} D_l N^l] \tag{4.4}$$

with $N_i \equiv g_{0i}$, $N^2 \equiv \gamma^{ij} N_i N_j - g_{00}$; all operations are with respect to the spatial metric, γ_{ij}. The exterior G_i^0 equation is

$$D_j \pi^{ij} = 0. \tag{4.5}$$

Our metric (4.2) implies that

$$\begin{aligned} \pi_{11} &= \pi_{22} = \pi^i_i = 0, \\ \pi_{12} &= [2(\rho'/\rho)\, N_2 - N_2']\, \rho/2N = -(\rho^{-2} N_2)'\, \rho^3/2N, \end{aligned} \tag{4.6}$$

where a prime denotes r-differentiation. Consequently, in this frame, $D_j \pi^j_i = \partial_j \pi^j_i$ and (4.5) reduces to an ordinary divergence equation

$$\partial_j \pi^j_1 \equiv 0, \qquad \partial_j \pi^j_2 = \pi'_{12} = 0 \Rightarrow \pi_{12} = A. \tag{4.7}$$

The absence of covariant differentiation in (4.7) is characteristic of vanishing gravitational contribution to angular momentum. To determine the spatial metric, we next calculate $\rho(r)$ from the G_0^0 equation, which is

$$\sqrt{\gamma}\, R = \pi_{ij} \pi^{ij} / \sqrt{\gamma} = 2(\pi_{12})^2 \rho^{-3}. \tag{4.8}$$

Although here there is an apparent "gravitational energy density" arising from π_{ij}^2, we shall see that the angular defect (i.e., the energy) actually vanishes for our spinning model with $T^{00} = 0$. The spatial curvature does not vanish, of course; this is due to our choice of spacetime slicing in terms of curved 2-surfaces in flat spacetime. Equation (4.8) is an equation for ρ,

$$\sqrt{\gamma}\, R \equiv -2\rho'' = 2A^3 \rho^{-3} \tag{4.9}$$

whose solution is

$$\rho^2 = (Br + C)^2 - (A^2/B^2). \tag{4.10}$$

The integration constant C is just a shift of origin and may be set to zero. We set the scale B to unity, or else there will be a conical singularity, unrelated to our source. This is exemplified by going back to the case $A = 0$, where $\rho'' = 0$ has the solution $\rho = Br + C$ and $B \neq 1$ produces the "mass defect" in θ as we saw in the static case.[3] We thus have

$$\rho^2 = r^2 - A^2. \tag{4.11}$$

Finally, we must determine N from the $G_{ij} = 0$ equations. The latter are[5]

[5] The full set of equations (4.5), (4.8), (4.12), is conveniently obtained from the time-independent part of the Einstein action as expressed in terms of our variables

$$I_{\mathrm{ES}} = \int d^2x \{ N[\sqrt{\gamma}\, R - 1/\sqrt{\gamma}(\pi_{ij}\pi^{ij} - \pi^i_i \pi^j_j)] + 2N_i D_j \pi^{ij} \}.$$

$$(N/2\gamma)\,\gamma_{ij}(\pi_{lm}\pi^{lm}-\pi^l_l\pi^m_m)-(2N/\gamma)(\pi^m_i\pi_{mj}-\pi^m_m\pi_{ij})$$
$$+(D_iD_j-\gamma_{ij}D^2)\,N+1/\sqrt{\gamma}\{D_m(\pi_{ij}N^m)-\pi^m_iD_mN_j-\pi^m_jD_mN_i\}=0. \quad (4.12)$$

The content of (4.12) is fully determined by its 1,1-component,

$$\rho' N'/\rho + A^2N/\rho^4 = 0. \quad (4.13)$$

Equation (4.13) fixes N as

$$N^2=\tau^2(\rho^2+A^2)/\rho^2=\tau^2r^2(r^2-A^2)^{-1} \quad (4.14)$$

and the constant τ can be absorbed in a time rescaling. Finally, we determine N_2 from (4.4) or (4.6) to be

$$N_2 = A + F\rho^2. \quad (4.15)$$

Choosing $F=0$, we therefore find that N_2 is a constant, which, as we shall see, is proportional[6] to the angular momentum J. Furthermore,

$$-g_{00}\equiv N^2-N_iN^i=\rho^{-2}(A^2+\rho^2)-\rho^{-2}A^2=1. \quad (4.16)$$

Our solution can now be summarized in the line element

$$ds^2=(dr^2+r^2\,d\theta^2)-(dt^2-2A\,dt\,d\theta+A^2\,d\theta^2). \quad (4.17)$$

We may now check the singularity structure of π^{ij} and calculate the angular momentum (4.1). The latter is given in terms of gravitational variables by the standard formula, in Cartesian coordinates

$$J^{ik}=-(8\pi G)^{-1}\int d^2x[x^i\partial_j\pi^{kj}-x^k\partial_j\pi^{ij}]$$
$$=-(8\pi G)^{-1}\oint dS_j(x^i\pi^{kj}-x^k\pi^{ij}), \quad (4.18)$$

where we have used $8\pi G\sqrt{-g}\,T^{0i}=\sqrt{-g}\,G^{0i}=-2\partial_j\pi^{ij}$. From $\pi^{r\theta}=g^{rr}g^{00}\pi_{r\theta}=A(r^2-A^2)^{-1}$, we obtain the Cartesian components π^{ij} by the usual circular to Cartesian transformation and find

$$\pi^{ij}=-A(r^2-A^2)^{-1}\,r^{-2}(\varepsilon^{im}x^mx^j+\varepsilon^{jm}x^mx^i)$$
$$\xrightarrow[r\to\infty]{}-A/r^4(\varepsilon^{im}x^mx^j+\varepsilon^{jm}x^mx^i)$$
$$=\tfrac{1}{2}A(\varepsilon^{im}\partial_m\partial_j+\varepsilon^{jm}\partial_m\partial_i)\ln r. \quad (4.19)$$

The second line of (4.19) is the asymptotic value of π^{ij} to be inserted in the surface

[6] The $D=4$ Kerr solution, evaluated in the equatorial plane, has $g_{0\varphi}\equiv N_3\sim J/r$ at large distances. The extra r factor, compared to our $g_{0\theta}\equiv N_2\sim J$, is kinematical, being needed to offset the additional dimension in the Gaussian surface integral expression.

integral of (4.18). Asymptotically, $\partial_j \pi^{kj}$ (or $\partial_j \pi^j_k$) is $\frac{1}{2} A \varepsilon^{km} \partial_m (\nabla^2 \ln r)$, which indeed represents an effective localized spin source. The value of J^{ik} is

$$J^{ik} = -(4G)^{-1} A \varepsilon^{ik} = J \varepsilon^{ik}. \tag{4.20}$$

This determines the constant A to be $-4GJ$.

The interval (4.17) can be recast into Minkowski form

$$ds^2 = dr^2 + r^2 \, d\theta^2 - dT^2 \tag{4.21}$$

by a change of the time coordinate according to

$$T = t + 4GJ\theta. \tag{4.22}$$

But there is a singularity, analogous to the mass defect in the static case. Namely, at constant t, as θ reaches 2π, which is identified with $\theta = 0$, T jumps by $8\pi GJ$ and we must identify times which differ by $8\pi GJ$ to preserve single-valuedness. This "time-helical" structure may have interesting consequences in the quantized theory.

We conclude that angular momentum yields a flat spacetime, but the coordinate time T has the jump property. Alternatively, in the original form (4.17) the metric is singular at the spatial origin and at $r = A$ (as is also clear from the Cartesian forms of g_{0i} and g_{ij}) and this singularity structure has been replaced by the jump in the new time coordinate.[7] Thus the whole effect of these calculations has again been to uncover anomalies in the range of the Minkowski coordinates, and to relate them to the source strengths. In the next Section, we shall rederive our results in a totally geometrical fashion.

V. Geometrical Approach

Since spacetime outside sources is locally flat, it is tempting to describe our various analytic solutions purely in terms of spacetime patches with Minkowski metric, but connected by somewhat twisted matching conditions. Henceforth we consider the case in which all $Gm_i > 0$. Figure 1 shows the static spinless one-particle solution (2.8) in 2-space with an excised wedge (as we saw, there is no effect in the time direction). The matching condition is expressed by identifying points (x', x) along the edges which are related by the rotation matrix Ω,

$$\mathbf{x}' = \Omega(\beta)\, \mathbf{x},$$

$$t' = t,$$

$$\Omega(\beta) \equiv \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta \end{pmatrix}, \qquad \beta \equiv 4\pi G m \equiv \pi(1 - \alpha). \tag{5.1}$$

[7] If a nonvanishing localized T^{00} were also present, there would be in addition a mass defect in θ, whose altered range would have to be included and would modify the eigenvalues of the quantized angular momentum, precisely as discussed earlier for conical geometries.

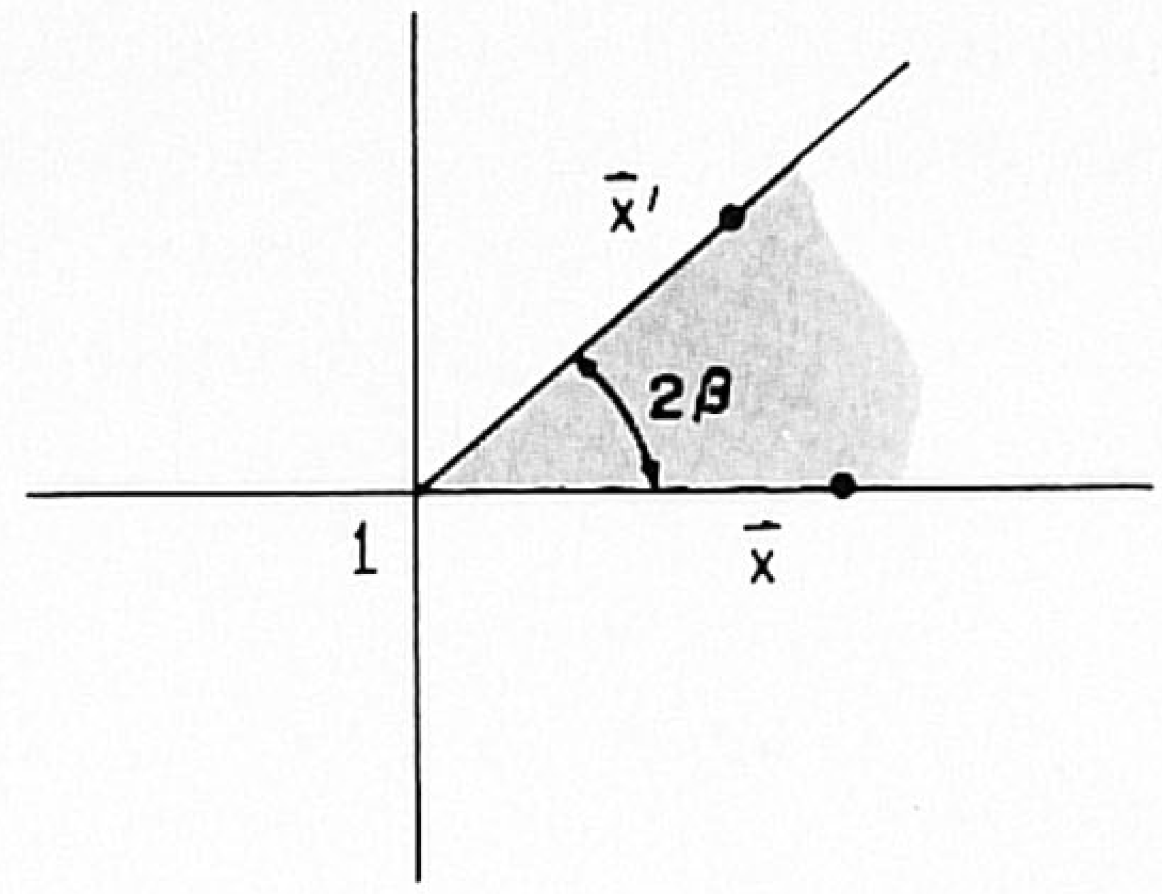

FIG. 1. Two-space around a static spinless particle at the origin. The shaded region is to be excluded. The dots (●) indicate parts of space that are to be identified with each other. The angle 2β is proportional to the mass: $\beta = 4\pi GM = \pi(1-\alpha)$.

The two-particle static solution can easily be obtained by combining two of these spaces, as was done in [5]. Alternatively this solution can be pictured as in Fig. 2. Mathematically the matching conditions for particles at the origin and at **a** consist in the identifications

$$\mathbf{x}' = \Omega_1 \mathbf{x}, \tag{5.2a}$$

$$\begin{aligned} \mathbf{x}'' &= \Omega_1\{\mathbf{a} + \Omega_2(\mathbf{x} - \mathbf{a})\} \\ &= \mathbf{b} + \Omega_1 \Omega_2(\mathbf{x} - \mathbf{b}), \end{aligned} \tag{5.2b}$$

with

$$\mathbf{b} = \frac{\sin\beta_2}{\sin(\beta_1 + \beta_2)}\,\Omega_1^{1/2}\mathbf{a}, \qquad \Omega_{1,2} \equiv \Omega(\beta_{1,2}). \tag{5.3}$$

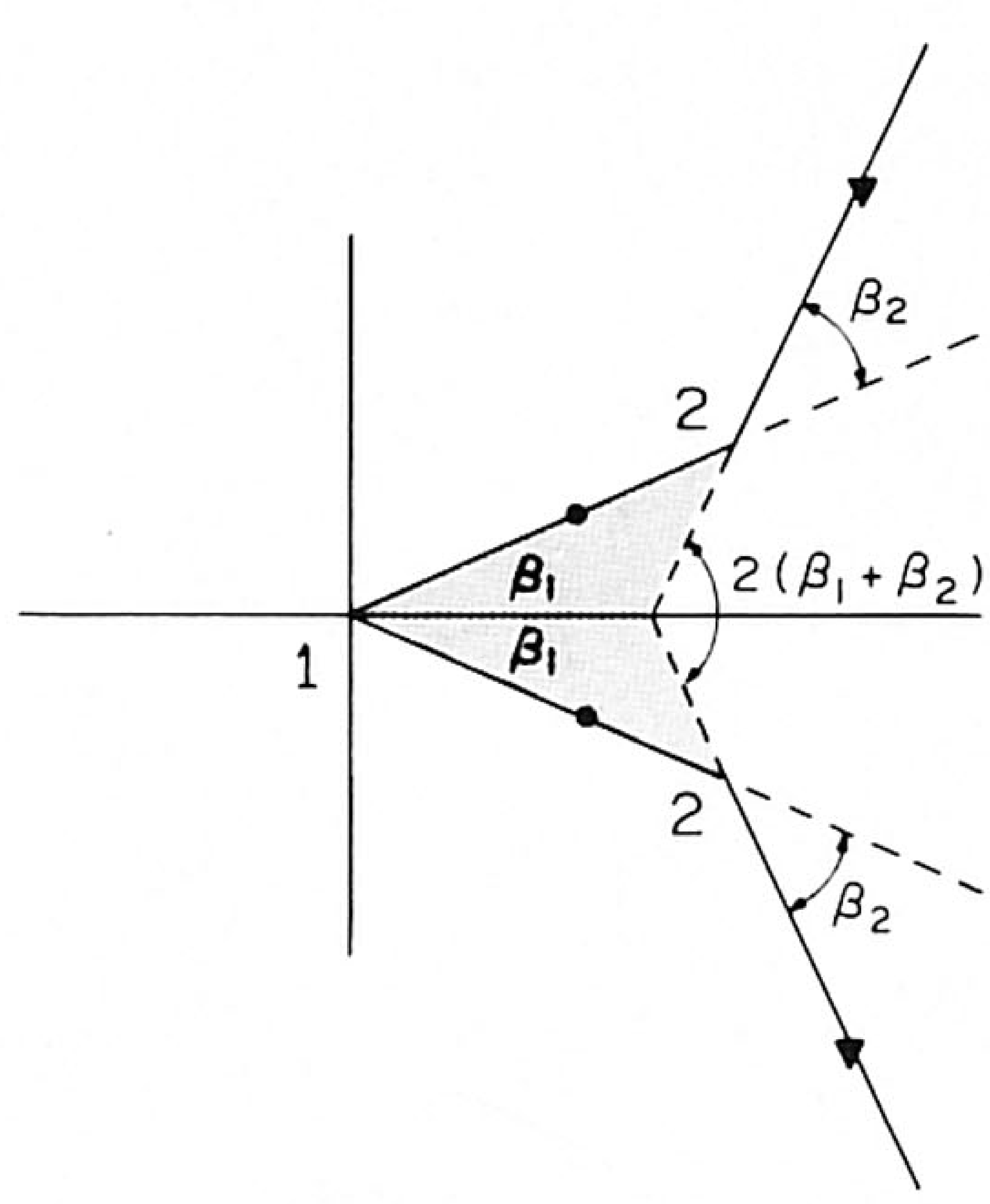

FIG. 2. Two-particle static solution. Dots (●) must be identified with each other, as must triangles (▲). The shaded region shows the *extra* missing part of space as compared with the one-particle solution with mass parameter $\beta_1 + \beta_2$.

Here (5.2a), (5.2b) refer to the loci of the dots and triangles, respectively, in Fig. (2). We see that this solution coincides with a one-particle solution with mass $m' = m_1 + m_2$ located at $\mathbf{x} = \mathbf{b}$, except that a further (shaded) patch of space is excised. The mass of a single point source must be less than $1/4G$, i.e., $\alpha > 0$ in (2.8b). In the geometric picture (Figs. 1 and 2) this is evident. In (2.8b) positive α ensures that

$$d\rho/dr > 0, \tag{5.4}$$

which is a locality requirement: the location of the source at $r = 0$ should correspond to $\rho = 0$. Formally, as we saw earlier, if $\alpha < 0$ the source is at $\rho = \infty$, which is hardly acceptable in the geometric picture. The above procedure can readily be extended to describe more particles. However, two (or more) particles may have a total mass equal or less than $1/2G$. This is illustrated in Fig. 3, where $m_1 + m_2 > 1/4G$. The "effective" particle with mass $m_1 + m_2$ is now a virtual one, and necessarily at least one other particle has appeared with total mass $m_3 = (1/2G) - m_1 - m_2$. The sum of the three masses is $1/2G$ and space becomes compact, since

$$1/8\pi \int d^2x \sqrt{\gamma}\, R = 1 \tag{5.5}$$

which is the value of the Euler characteristic for spaces with S_2 topology.

Now consider one particle at rest and one, located at $\mathbf{a}$, moving past the first with velocity v in the x direction. The matching condition for the second particle is then given by the spacetime vector relation

$$x' = a + L\Omega_2 L^{-1}(x - a), \tag{5.6}$$

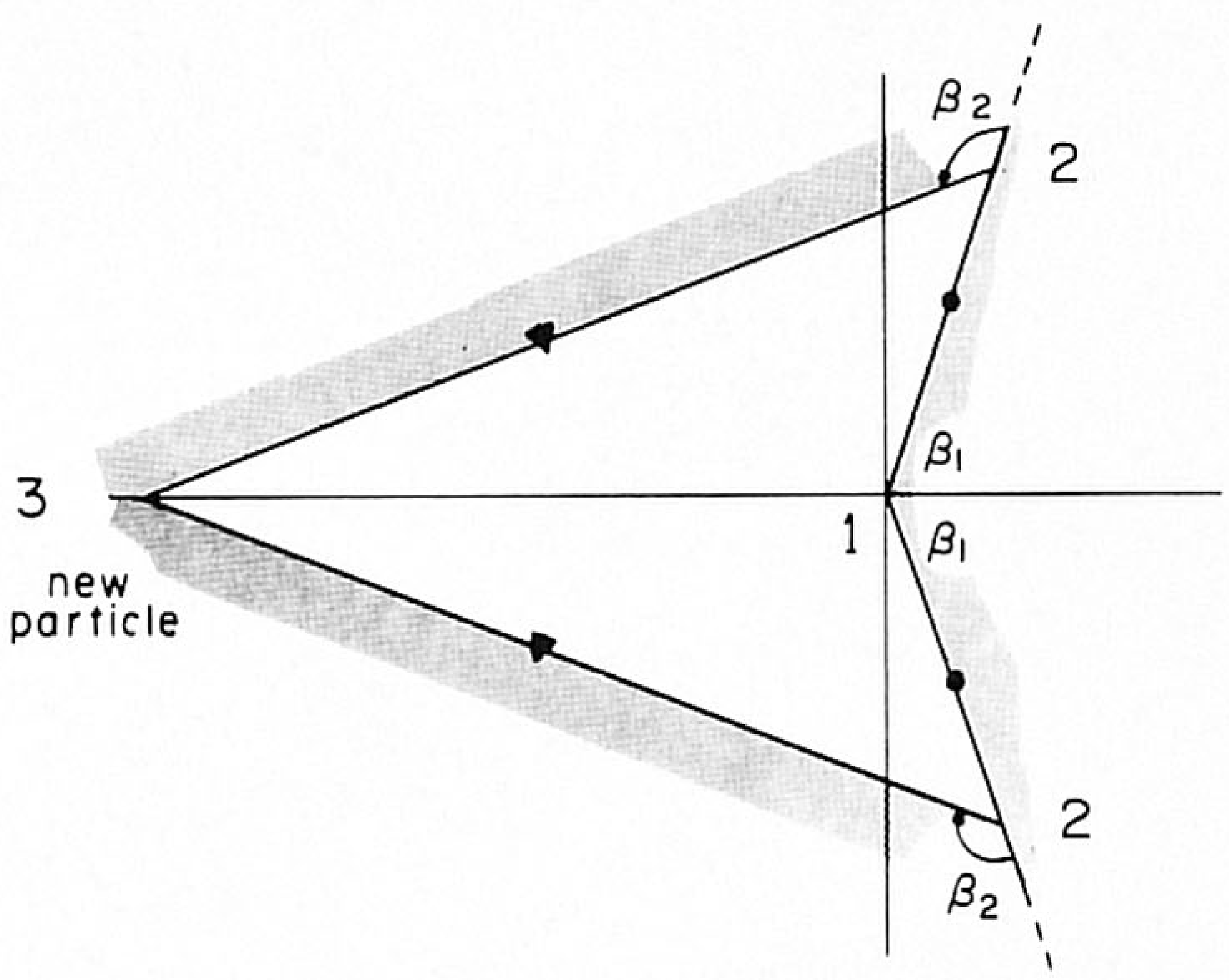

FIG. 3. Particles 1 and 2 have masses such that $\beta_1 + \beta_2 > \pi$. Space now necessarily closes (unless negative masses are allowed) with at least one new particle forming at the left. The sum of all β's is 2π.

where Ω_2 is the rotation appropriate to a particle at rest (determined by the angle deficit corresponding to the rest mass of the particle as in (5.2b)), and L is a Lorentz boost,

$$\Omega_2 = \begin{pmatrix} \cos 2\beta_2 & \sin 2\beta_2 & 0 \\ -\sin 2\beta_2 & \cos 2\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} \cosh \gamma & 0 & \sinh \gamma \\ 0 & 1 & 0 \\ \sinh \gamma & 0 & \cosh \end{pmatrix}, \tag{5.7}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}, \qquad \tanh \gamma = v.$$

The total system behaves like a single particle with the matching condition

$$x'' = \Omega_1(a + L\Omega_2 L^{-1}(x - a)). \tag{5.8}$$

Let us try to write this as

$$x'' = L_3[b + \Omega_3(L_3^{-1}x - b) + c], \tag{5.9}$$

where Ω^3 and b are again spacelike, so that we have an "effective" particle moving with a Lorentz boost L_3. The need for the timelike vector c will become apparent shortly. To determine the unknowns in (5.9) we first equate the coefficients of x in (5.8) and (5.9):

$$\Omega_1 L\Omega_2 L^{-1} \equiv M = L_3\Omega_3 L_3^{-1}. \tag{5.10}$$

Clearly,

$$\mathrm{tr}\, M = \mathrm{tr}\, \Omega_3 = 1 + 2\cos 2\beta_3, \tag{5.11}$$

from which we find the solutions

$$\pm\cos\beta_3 = \cos\beta_1 \cos\beta_2 - \sin\beta_1 \sin\beta_2 \cosh\gamma. \tag{5.12}$$

By continuity we must choose the + sign. Knowing Ω_3 gives us enough equations to find L_3 from (5.10),

$$|v_3| \equiv \tanh\gamma_3; \qquad M_{00} = \cosh^2\gamma_3 - \sinh^2\gamma_3 \cos 2\beta_3. \tag{5.13}$$

We shall not compute L_3 further because (5.12) is what we are after: it is our analog of the flat space relativistic mass addition formula,

$$m_3^2 = m_1^2 + m_2^2 + 2m_1 m_2 \cosh\gamma. \tag{5.14}$$

(Equation (5.14) follows from (5.12) for small values of β.) To fix b and c it is convenient to rewrite (5.8) and (5.9) in the center of mass system. In that system particles 1 and 2 are Lorentz boosted by L_1 and L_2, respectively, with distances a_1 and a_2 from the origin. Let us take a_1 and a_2 to be spacelike; (5.8) then reads

$$x'' = a_1 + L_1 \Omega_1 L_1^{-1}\{-a_1 + a_2 + L_2 \Omega_2 L_2^{-1}(x - a_2)\}, \tag{5.15}$$

which we want to write as

$$x'' = \Omega_3 x + c. \tag{5.16}$$

We have arrived at the center of mass system if Ω_3 is purely spacelike and c purely timelike. We find

$$\Omega_3 = L_1 \Omega_1 L_1^{-1} L_2 \Omega_2 L_2^{-1}, \tag{5.17}$$

and

$$\begin{aligned} c &= L_1 \Omega_1 L_1^{-1}(a_2 - a_1) \qquad +\text{spacelike} \\ &= L_2 \Omega_2^{-1} L_2^{-1}(a_2 - a_1) \qquad +\text{spacelike}. \end{aligned} \tag{5.18}$$

If L_1 is a boost in the x direction with magnitude γ_1, as in (5.7), then the timelike component of c is

$$c_0 = 2 \sinh \gamma_1 \sin \beta_1 \{\cos \beta_1 (a_2 - a_1)_y - \sin \beta_1 \cosh \gamma_1 (a_2 - a_1)_x\}. \tag{5.19}$$

We recognize that for small v and β_1, the first term in the bracket is the leading one, and it is proportional to the angular momentum

$$c_0 \to 2\beta_1 (\mathbf{a}_1 - \mathbf{a}_2) \times \mathbf{v}_1 = 8\pi G J. \tag{5.20}$$

This time component in the matching condition (5.16) implies that there is a jump in time as one travels around the system, which is exactly what we found in Section IV. A possible spacelike component in c would imply that the effective spinning particle is not at the origin; it can be absorbed by a displacement, yielding the vector $\mathbf{b}$ in (5.9). We remark that a localized, pointlike, spinning particle would give rise to difficulties with causality: it would become possible to travel along a closed timelike contour. One can show that such closed timelike contours are not possible in a space with n moving spinless particles, where angular momentum is purely orbital.

The most general boundary condition for spacetime surrounding any possible composite source can be written as

$$x' = \mathscr{L} x + z, \tag{5.21}$$

where $\mathscr{L}$ is any Lorentz transformation and z any vector. Causality requires $\mathscr{L}_{00} > 0$, and if we restrict ourselves to orientable spaces, then $\det \mathscr{L} = +1$. The eigenvalues of

$\mathscr{L}$ are $e^{\pm 2i\beta}$ and 1, corresponding to eigenvectors e_1, e_1^*, and e_0. The angular momentum is then evidently

$$J = 1/8\pi G(e_0 z) \tag{5.22}$$

and the trajectory of the "effective" particle is given by

$$x' = x + (e_0 z)\, e_0. \tag{5.23}$$

We have seen in the static case (Fig. 3) that this trajectory sometimes creates a new physical singularity outside the original system, and it must then necessarily represent a physical particle that closes the universe. The causality argument tells us that this can only happen if $J = 0$. Finally, $\beta = 4\pi G M_{\text{tot}} > 0$.

Clearly the geometric approach gives a powerful way of understanding the global and topological features of this (2 + 1)-dimensional world. We find it suggestive that quantization of angular momentum would correspond to quantization of the jumps in the time coordinate, but we do not yet understand how to make use of such an observation in a quantized version of this theory.

VI. Linearized and Newtonian Limits

In the full theory we have seen that particles at rest do not interact, whereas moving particles do interact in some sense, if only because they move in a globally "curled" space, although they follow free (locally straight) worldlines. The linearized approximation, while still dynamically trivial (vanishing of the linearized Einstein tensor implies that the potentials $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ are pure gauges) also leads to coupling among its conserved (prescribed) sources. In any spacetime of dimension D ($\neq 2$), the linearized Einstein equations in harmonic gauge are

$$\square(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h^\alpha_\alpha) = -16\pi G T_{\mu\nu}, \qquad \partial^\nu(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h^\alpha_\alpha) = 0. \tag{6.1}$$

The resulting source-source interaction is D-dependent:

$$I_{\text{INT}} \propto \int d^D x T^{\mu\nu} \square^{-1} (t_{\mu\nu} - (D-2)^{-1} \eta_{\mu\nu} t^\alpha_\alpha). \tag{6.2}$$

The factor $(D-2)^{-1}$ has the consequence for $D = 3$ that (as already noted in [2]) there is no static coupling, i.e., no $T_{00} - t_{00}$ interaction, in agreement with our rigorous static results. However, there are residual, $O(v^2)$ forces. These are actually instantaneous because the $\square^{-1}$ factor in (6.2) is cancelled when the $T_{\mu\nu}$ are expressed in terms of their three unconstrained components. This absence of retardation is in accord with the absence of propagating graviton modes (and occurs also in $D = 2$

electrodynamics). Conservation enables one to write all six components of $T^{\mu\nu}$ in terms of the three quantities T_{00}, T_T^{0i} and T, defined by the decomposition

$$\begin{aligned} T^{0i} &= T_T^{0i} + \lambda \partial_i \dot{T}_{00}, \qquad \partial_i T_T^{0i} \equiv 0, \\ T_{ij} &= (\delta_{ij} + \lambda \partial_i \partial_j)\, T + \lambda^{3/2} \partial_i \partial_j \dot{T}_{00} + \lambda(\partial_i \dot{T}_T^{0j} + \partial_j \dot{T}_T^{0i}), \\ (-\nabla^{-2}) &\equiv \lambda \end{aligned} \tag{6.3}$$

The interaction then reduces to

$$I_{\text{INT}} \propto \int d^3x [2T_T^{0i} \nabla^{-2} t_T^{0i} - T\nabla^{-2} t_{00} - T_{00} \nabla^{-2} t] \tag{6.4}$$

and clearly is of order (v^2) or $(v_1 v_2)$. Even if one source is static there is still an interaction $\propto T\nabla^{-2} t_{00}$ (e.g., bending of light). This result is compatible with our full static metric, since the geodesic equation, $\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$ will yield an acceleration for a moving test particle from the $\Gamma^i_{jk} \dot{x}^j \dot{x}^k$ term. Conversely, an initially static particle will feel a force from the Γ^i_{00} term in the field of a moving source, since the latter's spatial stresses will generate a g_{00} field. Of course, this "force" is not real (there is no geodesic deviation between nearby test particles); nevertheless as we have seen, it has a counterpart in the linearized approximation where all motion is projected into a Cartesian coordinate description. This description corresponds to a (harmonic) gauge where the metric is not Minkowski. We conclude that there is no conflict between exterior flatness and coordinate-induced apparent forces (light bending is another example) in the linearized approximation. The same apparent non-geodesic motion is found in the full theory if one projects the worldlines on the cone onto a "flat" map.

Finally, we comment briefly on the discontinuity in the full theory between the Newtonian limit and Newtonian gravity, implicit in the fact that $g_{00} = -1$ in the static solution, so that $\ddot{x}^i = -\Gamma^i_{00} = 0$ for slow test particles. This discontinuity was noted long ago [6], essentially on the basis of the absence of linearized $T_{00} - t_{00}$ interaction at $D = 3$. There is no paradox here, for Newtonian correspondence is *not* guaranteed a priori for Einstein theory. While a Newtonian theory exists in any dimension, being defined by $\nabla^2 V = -4\pi G\rho$, $\ddot{x}^i = -\partial^i V$, only the first of these elements has relativistic antecedent for $D = 3$: The familiar $D = 4$ connection between that spatial metric component which is determined by T_{00} from the time-time Einstein equation (and hence identified with V in the Newtonian limit) and the component g_{00} relevant to the geodesic equation breaks down here. Indeed, for $D = 2$ neither element follows from a relativistic action principle, since there is no Einstein action at all: $\sqrt{-g}\, R$ is a total divergence, and $G_{\mu\nu} \equiv 0$. The $R = 0$ equation can only be obtained from a covariant action involving non-geometrical variables, e.g., $\int d^2x\, N \sqrt{-g}\, R$, where the scalar field N is a Lagrange multiplier.

While this paper was in preparation we received two preprints [7, 8] also dealing with three-dimensional gravity.

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