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# The gravitational effect of colliding planar shells of matter

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**Abstract.** Using methods similar to those in a previous paper we construct exact solutions to Einstein's equations containing two colliding planar shells of matter which divide spacetime into four regions, three of which are flat. In the appendices we consider some more general cases.

## 1. Introduction

In two previous papers (Dray and 't Hooft 1985a, b, hereafter referred to as I and II) we first showed the existence of a $C^0$ solution to the Einstein field equations representing a single massless particle sitting at the horizon of a Schwarzschild black hole (I), and then generalised this solution to spherical shells of null matter, both on the horizon and elsewhere (II). The latter solutions are much simpler mathematically, and we were also able to construct exact models for the collision of two such shells.

The analogous situation in flat space would start with a model for a single massless particle in flat space. Such a solution was given by Aichelburg and Sexl (1971). Indeed, this solution formed the basis for discovering its Schwarzschild analogue in I. In the present paper we will first perform the 'smearing out' of this single-particle solution to a model for a planar shell of matter in flat space and then consider the collision of two such shells. A summary of what has been accomplished appears in table 1. We have been unable to find solutions for the remaining cases in column 1 and can only say that they do not appear to be simply related to the known cases. We do not expect exact solutions for column 2 to exist; even approximate solutions for the remaining cases will be at best difficult to construct.

As already pointed out in II, the scattering results for spherical shells of matter in the Schwarzschild metric bear a close resemblance to earlier results on the scattering of *sourceless* planar shock waves in flat space‡. For the planar shells considered in this paper the analogy is even stronger. However, whereas in the Schwarzschild case we found (II) that the 'out' region could also be described by the Schwarzschild metric we obtain here a somewhat different result: the 'out' region turns out to be described

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‡ See D'Eath (1978), Curtis (1978), Khan and Penrose (1971), Szekeres (1970, 1972), Nutku and Halil (1977) and Chandrasekhar and Xanthopoulos (1985). For related results see also Chandrasekhar and Ferrari (1984), Carr and Verdaguer (1983), Ibañez and Verdaguer (1983, 1985), Centrella (1980), Centrella and Matzner (1979, 1982), Tipler (1980) and Ipser and Sikivie (1984). We thank one referee for bringing some of these references to our attention.

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**Table 1.** A summary of known shock wave solutions with null source, giving the reference where each case was first considered. The type of shock wave is shown across the top and the 'background' spacetime is given at the left; the numbers in parentheses refer to the Schwarzschild masses on either side of the shock wave. I and II refer to Dray and 't Hooft (1985a, b), III refers to the present paper. The arrows in column 3 refer to the fact that these three cases can all be considered to be limiting forms of the Vaidya (1951) metric.

| | Particle | | Shell | |
|---|---|---|---|---|
| | Single particle | Scattering of two particles | Single shell | Scattering of two shells |
| flat (o/o) | Aichelburg and Sexl (1971) | D'Eath (1978) (approximate) | III | III |
| Schwarzschild (m/m) | I | | Eardley (1975) | II |
| Schwarzschild (m/M) | | | Vaidya (1951) | II |
| flat/Schwarzschild (o/m) | | | Synge (1957) | II |

by the Robinson and Trautman (1962) 'nullicle' solution which can be thought of as an $m=\infty$ Schwarzschild solution.

We give the solution corresponding to a single planar shell of null matter in § 2 and consider two ways of colliding such shells in § 3. In § 4 we analyse the global structure of these solutions and construct Penrose diagrams for them. A more general argument, which includes both of these models as special cases, is relegated to appendix 1 and in appendix 2 we then give an exhaustive treatment of the possible cases. Finally, in § 5 we discuss our results.

## 2. One planar shell in flat space

The metric associated with a single null particle in flat space was first given by Aichelburg and Sexl (1971) (see also Pirani 1959). In I we rederived their results using a general method (appendix B of I) for cutting and pasting a spacetime along a null hypersurface. The resulting metric was shown to be ((B4) with (6*b*) of I)

$$\mathrm{d}s^2 = -\mathrm{d}U(\mathrm{d}V - F(X, Y)\delta(U)\,\mathrm{d}U) + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{1a}$$

Using (B5) of I we can transform this to coordinates which make explicit the $C^0$ nature of the metric, obtaining

$$\begin{aligned}\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 + u\theta[f_{,xx}\,\mathrm{d}x^2 + 2f_{,xy}\,\mathrm{d}x\,\mathrm{d}y + f_{,yy}\,\mathrm{d}y^2]\\ + \tfrac{1}{4}u^2\theta[(f_{,xy}\,\mathrm{d}x + f_{,yy}\,\mathrm{d}y)^2 + (f_{,xx}\,\mathrm{d}x + f_{,xy}\,\mathrm{d}y)^2]\end{aligned} \tag{1b}$$

where $\theta = \theta(u)$ is the step function and where $f(x, y) = F|_{X=x, Y=y}$ satisfies†

$$\Delta f := f_{,xx} + f_{,yy} = -2\kappa\delta(x)\delta(y) \tag{2}$$

† $\delta$ is the usual Dirac delta function. Note that in I the two-dimensional distributions '$\delta(\rho)$' and '$\delta(\theta)$' were never properly defined. For example, (7) of I means *by definition* $\int_{R^2} f\,\Delta g\,\mathrm{d}x\,\mathrm{d}y = g(0, 0)$ for all suitable $g$, so that one can identify '$\delta(\rho)$' with $\delta(x)\delta(y)$. A similar statement can be made for '$\delta(\theta)$'.

with $\kappa=$ constant. The only non-zero component of the Ricci tensor is†

$$R_{uu}=\kappa\delta(x)\delta(y)\delta(u) \tag{3}$$

as desired.

To smear this out to a plane shell of null matter we must simply remove the $\delta(x)\delta(y)$ term from (2) and (3), obtaining (cf II)

$$\Delta f=-2\kappa=\text{constant} \tag{2'}$$

$$R_{uu}=\kappa\delta(u). \tag{3'}$$

Up to solutions of the homogeneous equation (for which see Penrose (1972) or I) the general solution to (2') is

$$f=-\tfrac{1}{2}\kappa(x^2+y^2) \tag{4}$$

and the metric is now‡

$$\mathrm{d}s^2=-\mathrm{d}u\,\mathrm{d}v+[1-\kappa u(1-\tfrac{1}{4}\kappa u)\theta](\mathrm{d}x^2+\mathrm{d}y^2). \tag{5}$$

As shown in II, the metric (5) (and also (1)) is in fact flat everywhere except at $u=0$. This is obvious for $u<0$. For $u>0$ the metric is

$$\mathrm{d}s^2=-\mathrm{d}u\,\mathrm{d}v+(1-\tfrac{1}{2}\kappa u)^2(\mathrm{d}x^2+\mathrm{d}y^2) \qquad (u>0). \tag{6a}$$

Making the following substitutions (of which the first is only for later convenience)

$$\begin{aligned} U &:= u-2/\kappa \\ V &:= v-\tfrac{1}{2}\kappa(1-\tfrac{1}{2}\kappa u)(x^2+y^2) \\ X &:= (1-\tfrac{1}{2}\kappa u)x \\ Y &:= (1-\tfrac{1}{2}\kappa u)y \end{aligned} \tag{6b}$$

yields§

$$\mathrm{d}s^2=-\mathrm{d}U\,\mathrm{d}V+\mathrm{d}X^2+\mathrm{d}Y^2 \qquad (u>0). \tag{6c}$$

## 3. Colliding plane shells

We can now try to collide two such planar shells. We will consider two inequivalent ways of doing this, illustrated schematically in figure 1. The first of these (figure 1($a$)) describes the scattering of two incoming shells along the null surfaces $u=0$ and $v=0$.

† Throughout this paper we make use of the fact that $\theta(1-\theta)\equiv 0$ in the sense of distributions.

‡ A metric similar to (5) can be found, e.g., in Misner *et al* (1973, p 958) for the more general case of a sandwich wave, where the support of $R_{uu}$ is larger than (3'). Sandwich waves were first discussed by Bondi *et al* (1959); see also Penrose (1965).

§ The metrics (5) and (6) are special cases of

$$\mathrm{d}s^2=-\mathrm{d}u\,\mathrm{d}v+h^2(u)(\mathrm{d}x^2+\mathrm{d}y^2)$$

in which all Ricci tensor elements vanish except $R_{uu}$ (cf (A1.6))

$$h''(u)+\tfrac{1}{2}R_{uu}h(u)=0.$$

For any *given* function $R_{uu}(u)$ all solutions $h(u)$ to this equation can be shown to be gauge equivalent, i.e. the resulting metrics are related by coordinate transformations. Here we choose $R_{uu}$ as in (3'); (6$b$) is now the gauge transformation which transforms the solution with $h(u)=1$ for $u<0$ into the solution with $h(U)=1$ for $u>0$. This gauge equivalence is the underlying reason why spacetime is flat both before and after the shock wave.

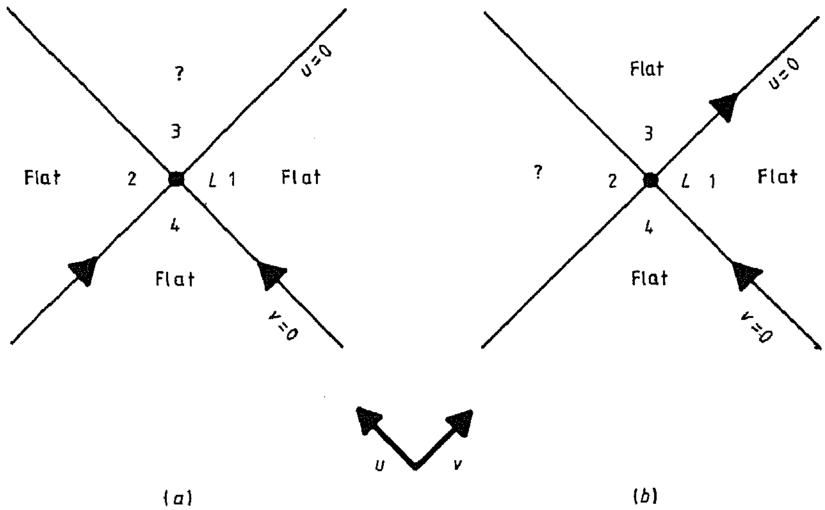


**Figure 1.** Two possible collisions involving two planar shells of the type described in § 2 in an otherwise flat space. ($a$) represents the case where the two incoming shells are planar, while ($b$) shows the case where one incoming and one outgoing shell are planar; the location of the planar shells is indicated by arrows. Spacetime is divided into four regions, three of which are flat. As shown in §§ 3 and 4 the fourth region, indicated by '?', *and its boundary* are highly curved. The hypersurface where the shells collide is labelled $L$.

This is the most natural case because the initial data consist only of flat spacetime regions connected by planar shells of matter. However, we will also be interested in the second possibility (figure 1($b$)) in which the regular 'initial' data are imposed 'at the right'. This describes the interaction of an ingoing planar shell with matter such that an outgoing planar shell is produced.

Turning first to figure 1($a$) we see from § 2 that the (flat) metrics in regions 1, 2 and 4 can be given by

$$\begin{aligned} &\text{region 4:} \quad ds^2 = -du\, dv + dx^2 + dy^2 \\ &\text{region 2:} \quad ds^2 = -du\, dv + (1 - \tfrac{1}{2}\kappa u)^2 (dx^2 + dy^2) \\ &\text{region 1:} \quad ds^2 = -du\, dv + (1 - \tfrac{1}{2}\lambda v)^2 (dx^2 + dy^2) \end{aligned} \tag{7}$$

where $\kappa$, $\lambda$ are constants and we assume $1 - \frac{1}{2}\kappa u > 0$, $1 - \frac{1}{2}\lambda v > 0$. We wish to find a $C^0$ (in fact piecewise $C^\infty$) metric which agrees with the above metrics in the appropriate regions and which satisfies

$$R_{uu} \sim \delta(u) \qquad R_{vv} \sim \delta(v) \tag{8}$$

all other components being zero; this is the Ricci tensor produced by two pressureless shells of matter moving at the speed of light. Proceeding as in appendix 1 we make the ansatz (A1.1) which allows us to write the metric in the form (A1.9), namely

$$ds^2 = -\frac{\alpha(u)\beta(v)\, du\, dv}{[1 + A(u) + B(v)]^{1/2}} + [1 + A(u) + B(v)](dx^2 + dy^2) \tag{9}$$

with $\alpha$, $\beta$, $A$, $B$ $C^0$. But requiring that this agree with (7) forces

$$\begin{aligned} A(u) &= \kappa u (\tfrac{1}{4}\kappa u - 1)\theta(u) \\ B(v) &= \lambda v (\tfrac{1}{4}\lambda v - 1)\theta(v) \end{aligned} \tag{10}$$

$$\begin{aligned}\alpha(u) &= -1+\tfrac{1}{2}\kappa u\theta(u)\\ \beta(v) &= -1+\tfrac{1}{2}\lambda v\theta(v)\end{aligned} \tag{11}$$

where the overall sign in (11) has been chosen to agree with the notation in the appendix. The full metric is thus

$$-\left(1-\theta(u)\theta(v)+\frac{(1-\frac{1}{2}\kappa u)(1-\frac{1}{2}\lambda v)}{(1+\rho+\sigma)^{1/2}}\theta(u)\theta(v)\right)\mathrm{d}u\,\mathrm{d}v + (1+\rho\theta(u)+\sigma\theta(v))(\mathrm{d}x^2+\mathrm{d}y^2) \tag{12}$$

where we have introduced the notation

$$\begin{aligned}\rho(u) &= \kappa u(\tfrac{1}{4}\kappa u-1)\\ \sigma(v) &= \lambda v(\tfrac{1}{4}\lambda v-1).\end{aligned} \tag{13}$$

The only non-vanishing components of the Ricci tensor are

$$\begin{aligned}R_{uu} &= \kappa\delta(u)(1-\sigma\theta(v))\\ R_{vv} &= \lambda\delta(v)(1-\rho\theta(u)).\end{aligned} \tag{14}$$

Requiring that the energy density of the shells be positive forces $\kappa, \lambda > 0$ so that by making the substitutions (A1.15) in region 3 we can bring the metric there to the form

$$\text{region 3:}\quad \mathrm{d}s^2 = -R\,\mathrm{d}R^2+\mathrm{d}T^2/R+R^2(\mathrm{d}p^2+\mathrm{d}q^2). \tag{15}$$

Before discussing the properties of this metric in more detail we first consider the case depicted in figure 1($b$). Instead of (7) we now impose

$$\begin{aligned}&\text{region 1:}\quad \mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v+\mathrm{d}x^2+\mathrm{d}y^2\\ &\text{region 3:}\quad \mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v+(1-\tfrac{1}{2}\kappa u)^2(\mathrm{d}x^2+\mathrm{d}y^2)\\ &\text{region 4:}\quad \mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v+(1-\tfrac{1}{2}\lambda v)^2(\mathrm{d}x^2+\mathrm{d}y^2)\end{aligned} \tag{16}$$

and again assume $1-\frac{1}{2}\kappa u > 0$ and $1-\frac{1}{2}\lambda v > 0$. Proceeding as above and requiring that (9) agree with (16) forces

$$\begin{aligned}A(u) &= \kappa u(\tfrac{1}{4}\kappa u-1)\theta(u)\\ B(v) &= \lambda v(\tfrac{1}{4}\lambda v-1)\theta(-v)\\ \alpha(u) &= -1+\tfrac{1}{2}\kappa u\theta(u)\\ \beta(v) &= -1+\tfrac{1}{2}\lambda v\theta(-v)\end{aligned} \tag{17}$$

and the full matrix is now

$$-\left(1-\theta(u)\theta(v)+\frac{(1-\frac{1}{2}\kappa u)(1-\frac{1}{2}\lambda v)\theta(u)\theta(-v)}{1+\rho+\sigma}\right)\mathrm{d}u\,\mathrm{d}v + (1+\rho\theta(u)+\sigma\theta(-v))(\mathrm{d}x^2+\mathrm{d}y^2) \tag{18}$$

with $\rho$ and $\sigma$ as in (13). The only non-vanishing components of the Ricci tensor are

$$\begin{aligned}R_{uu} &= \kappa\delta(u)(1-\sigma\theta(-v))\\ R_{vv} &= -\lambda\delta(u)(1-\rho\theta(u)).\end{aligned} \tag{19}$$

Requiring that the energy density of the shells be positive forces $\kappa > 0$ as before but $\lambda < 0$ so that making the substitutions (A1.15) in region 2 now yields

$$\text{region 2:} \quad ds^2 = -dT^2/R + R\,dR^2 + R^2(dp^2 + dq^2). \tag{20}$$

We note that the metrics (15) and (20) are the same except for the sign of $R$. This metric was first discovered by Levi-Civita (1917, 1918, 1919) and was classified by Ehlers and Kundt (1962) as being of type A3 (see also Jordan *et al* 1960). It was subsequently rediscovered by Kasner (1921) and more recently by Robinson and Trautman (1962); this solution is sometimes called Robinson's nullicle. It also appears as one of the examples of 'maximally symmetric' spacetimes discussed by Aichelburg (1970). As was first pointed out by Robinson and Trautman (1962) (see also Geroch 1969) this solution can be obtained as an infinite mass limit of the Schwarzschild metric. To see this, set

$$\begin{aligned} T &= (2m)^{1/3} t \\ R &= r/(2m)^{1/3} \\ p &= \left(\theta - \frac{\pi}{2}\right)(2m)^{1/3} \\ q &= \varphi (2m)^{1/3} \end{aligned} \tag{21}$$

in the usual Schwarzschild metric $d\sigma^2$ to obtain

$$d\sigma^2 = -\left(\frac{1}{(2m)^{2/3}} - \frac{1}{R}\right) dT^2 + \frac{dR^2}{(2m)^{-2/3} - R^{-1}} + R^2\left(dp^2 + \cos^2\frac{p}{(2m)^{1/3}}\, dq^2\right). \tag{22}$$

We can thus think of (15) as being the $m \to +\infty$ limit of the Schwarzschild metric. Similarly, *first* replacing $m$ by $-m$ in (21) and *then* taking the limit we can think of (20) as being the $m \to -\infty$ limit of the Schwarzschild metric.

We summarise the results of this section in figure 2 and give the metrics in each region below. For the scattering of two incoming shells (figure 2($a$)) we have

$$\text{region 4:} \quad ds^2 = -du\, dv + dx^2 + dy^2 \tag{23$a$}$$

$$\text{region 2:} \quad ds^2 = -dU\, dV + dX^2 + dY^2 \tag{23$b$}$$

with

$$\begin{aligned} U &= u - 2/\kappa \\ V &= v - \tfrac{1}{2}\kappa(1 - \tfrac{1}{2}\kappa u)(x^2 + y^2) \\ X &= (1 - \tfrac{1}{2}\kappa u)x \\ Y &= (1 - \tfrac{1}{2}\kappa u)y \end{aligned} \tag{23$c$}$$

$$\text{region 1:} \quad ds^2 = -d\mathfrak{U}\, d\mathfrak{V} + d\mathfrak{X}^2 + d\mathfrak{Y}^2 \tag{23$d$}$$

with

$$\begin{aligned} \mathfrak{U} &= u - \tfrac{1}{2}\lambda(1 - \tfrac{1}{2}\lambda v)(x^2 + y^2) \\ \mathfrak{V} &= v - 2/\lambda \\ \mathfrak{X} &= (1 - \tfrac{1}{2}\lambda v)x \\ \mathfrak{Y} &= (1 - \tfrac{1}{2}\lambda v)y \end{aligned} \tag{23$e$}$$

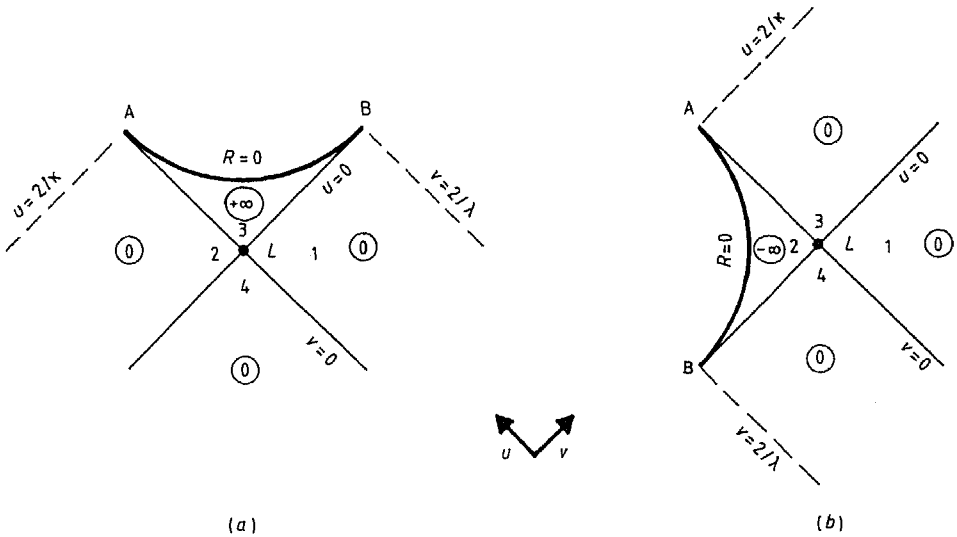


**Figure 2.** The results of § 3. (*a*) shows an exact model for the collision of two incoming planar shells with metric given by (23). (*b*) shows an exact model for the interaction of an ingoing with an outgoing shell with metric given by (24). In $(u, v)$ coordinates the $R=0$ singularity takes the form of an arc of a circle. There are also 'focusing singularities' at $u=2/\kappa$, $v=\pm 2/\lambda$, which are indicated by the broken lines. The circled numbers give the 'Schwarzschild mass' of each region. The point labelled $L$ represents the $u=0=v$ hypersurface where the two shells collide, while the points labelled A and B show the intersection of the shells with the $R=0$ singularity.

and

$$\text{region 3:} \quad ds^2 = -R\, dR^2 + dT^2/R + R^2(dp^2 + dq^2) \tag{23$f$}$$

with $(R, T, p, q)$ as in (A1.15). For the interaction of an ingoing with an outgoing shell (figure 2(*b*)) we have

$$\text{region 1:} \quad ds^2 = -du\, dv + dx^2 + dy^2 \tag{24$a$}$$

$$\text{region 3:} \quad ds^2 = -dU\, dV + dX^2 + dY^2 \tag{24$b$}$$

with

$$\left.\begin{aligned} U &= u - 2/\kappa \\ V &= v - \tfrac{1}{2}\kappa(1 - \tfrac{1}{2}\kappa u)(x^2 + y^2) \\ X &= (1 - \tfrac{1}{2}\kappa u)x \\ Y &= (1 - \tfrac{1}{2}\kappa u)y \end{aligned}\right\} \tag{24$c$}$$

$$\text{region 4:} \quad ds^2 = -d\mathfrak{U}\, d\mathfrak{V} + d\mathfrak{X}^2 + d\mathfrak{Y}^2 \tag{24$d$}$$

with

$$\left.\begin{aligned} \mathfrak{U} &= u + \tfrac{1}{2}|\lambda|(1 + \tfrac{1}{2}|\lambda|v)(x^2 + y^2) \\ \mathfrak{V} &= v + 2/|\lambda| \\ \mathfrak{X} &= (1 + \tfrac{1}{2}|\lambda|v)x \\ \mathfrak{Y} &= (1 + \tfrac{1}{2}|\lambda|v)y \end{aligned}\right\} \tag{24$e$}$$

and

$$\text{region 2:} \quad \mathrm{d}s^2 = -\mathrm{d}T^2/R + R\,\mathrm{d}R^2 + R^2(\mathrm{d}p^2 + \mathrm{d}q^2) \tag{24f}$$

with $(T, R, p, q)$ again given by (A1.15), and where we have written $|\lambda|$ to emphasise that, although $\kappa$ is positive, $\lambda$ here is negative.

## 4. Global structure

We now turn to the global structure of these two models. The problem is that not only are the infinite-mass Schwarzschild regions highly curved, but the boundaries of these regions are *also* curved and in no sense represent 'plane shells'. Thus, figure 2 does *not* accurately represent the global structure of these solutions. We note that Matzner and Tipler (1984) have discussed similar problems for colliding sourceless waves.

Let us for instance consider region 1 of Figure 2($a$). The natural coordinates to use here are of course the flat coordinates (23$e$). Note that, although the $v=0$ boundary is given by $\{\mathfrak{V}=0\}$ and is thus planar, as expected, the $u=0$ boundary is not. We have

$$\{u=0\} \leftrightarrow \{\mathfrak{U} = (\mathfrak{X}^2 + \mathfrak{U}^2)/\mathfrak{V}\}$$

which is the equation of the light cone of the origin in coordinates (23$e$)! Region 1 (and similarly also region 2) is thus the interior of a null cone, bounded by a null plane. This is shown schematically in figure 3.

Identifying the two parabolas (but *not* their interiors!) as indicated in figure 3 and adding the singularity at $R=0$ results in figure 4 (see also figure caption) and the full Penrose diagram of figure 5. Note that in order to be able to draw these figures some artistic freedom has been taken. In particular, the coordinates are *not* flat, so that the incoming plane shells do not *look* planar. The lines $u=2/\kappa$ and $v=2/\lambda$ representx focusing singularities: the incoming plane waves focus each other. An equivalent statement is that the incoming plane shells have the structure of collapsing *spherical* shells after colliding which then converge to the vertices (A and B) of the cones.

Applying similar techniques to the model shown in figure 2($b$) leads successively to figures 6 and 7, for which similar statements concerning artistic freedom and the lines $u=2/\kappa$ and $v=2/\lambda$ hold. Note that these two models (figures 2($a$) and ($b$)) have vastly different global structures!

Finally, we wish to emphasise that, although we have drawn Penrose diagrams for these models, the spacetimes which we have constructed are not, in any standard usage of the term, asymptotically flat. Furthermore, it is not possible to draw these diagrams in coordinates such that all local light cones are at 45°; Penrose diagrams are usually only drawn for spherically symmetric spacetimes.

## 5. Discussion

Matzner and Tipler (1984) considered the global structure of the case analogous to figure 2($a$) for *sourceless* plane waves, i.e. for a spacetime which is (everywhere) a vacuum solution. The global structure they find is similar to ours. In particular, their figure 7 (which was drawn by Penrose) resembles our figures 3 and 4.

† Note that these lines are *not* actually covered by the coordinates (23). Each line represents the intersection of one shell with the $R=0$ singularity. More precisely, any neighbourhood of a point on one of these lines intersects *both* the $R=0$ singularity *and* one of the shells. Points on these lines therefore cannot be regular points of spacetime.

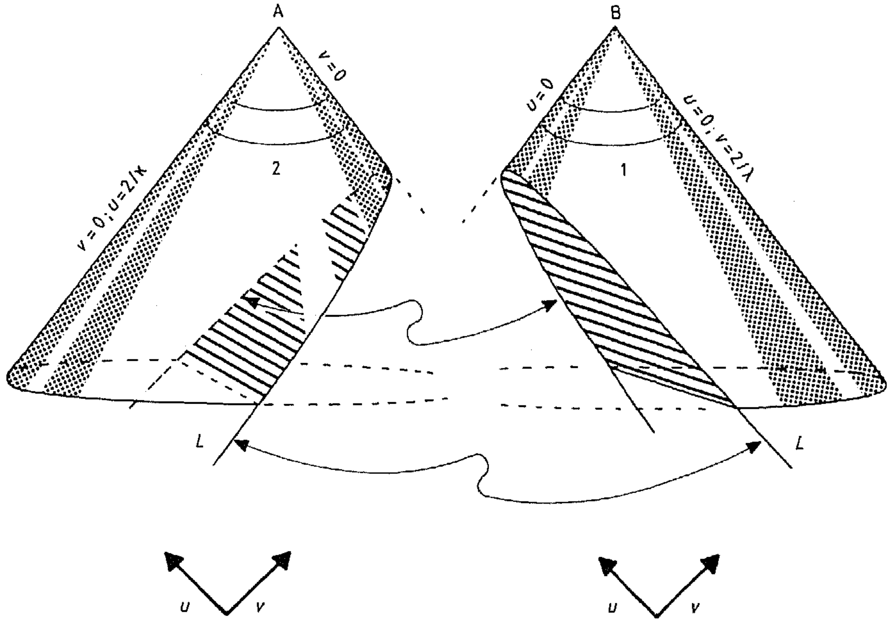


**Figure 3.** The first step in analysing the global structure of the model depicted schematically in figure 2($a$). The global structure of regions 1 and 2 (the interior of the cones) is shown in the flat coordinates ((23$e$) and (23$c$), respectively) appropriate to each region. The parabolas labelled $L$ (but not their interiors!) both represent the $u = 0 = v$ hypersurface where the two shells collide and must be identified, as indicated by the arrows. The vertices A and B of the two cones correspond to the identically labelled points in figure 2($a$). One spatial dimension has been suppressed.

The physical motivation for studying the case shown in figure 2($b$) may be less obvious: our spacetime consists of flat regions glued together by shells of matter (or 'gravitons') at the right, requiring singularities both in the past and in the future. Our special interest in this case stems from a study of Rindler space (region 1). If one attempts to construct a Boulware vacuum in region 1 (and possibly also in region 2) then a transformation to Minkowski coordinates would produce large amounts of matter concentrated near the $u$ and $v$ axes. If we assume that nature might provide us with a natural cut-off so that these amounts of matter are actually finite, then the quantum mechanical Hilbert space in Rindler space might be constructed from a Minkowski spacetime with shells of matter on the $u$ and $v$ axis. If we require region 1 to be flat, and put shells of matter on the $u = 0$ and $v = 0$ planes, we get the spacetimes pictured in Figures 6 and 7.

In some sense then what we have done is to compute the gravitational back-reaction of matter in Rindler space. We believe that such calculations are crucial for a complete understanding of the Rindler Hamiltonian, although there are still many mysteries here to be resolved.

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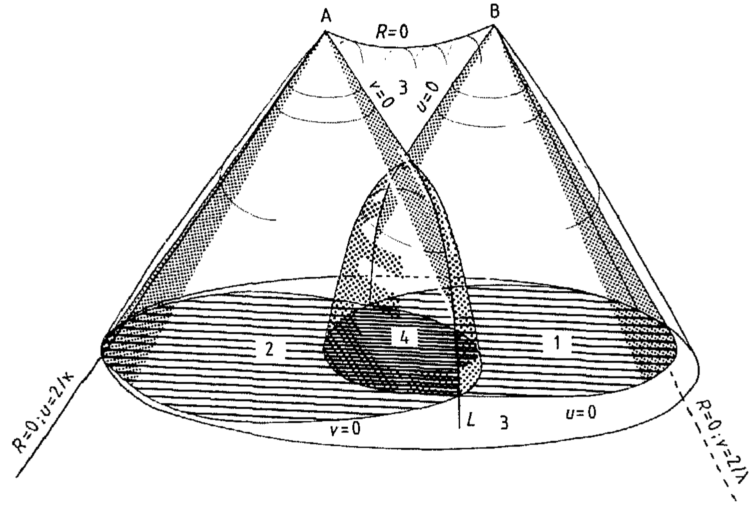


**Figure 4.** The result of performing the identification indicated in figure 3. The cones have been distorted and the planar nature of the incoming shells is no longer readily discernible. An essentially conical surface representing the $R=0$ singularity has been added and sits atop the two cones. The four regions are indicated: 1 and 2 inside the cones, 4 inside the intersection of the two cones and 3 between the cones and the $R=0$ surface. All four regions meet at $u=0=v$, shown in the figure as the curve $L$. One spatial dimension has been suppressed.

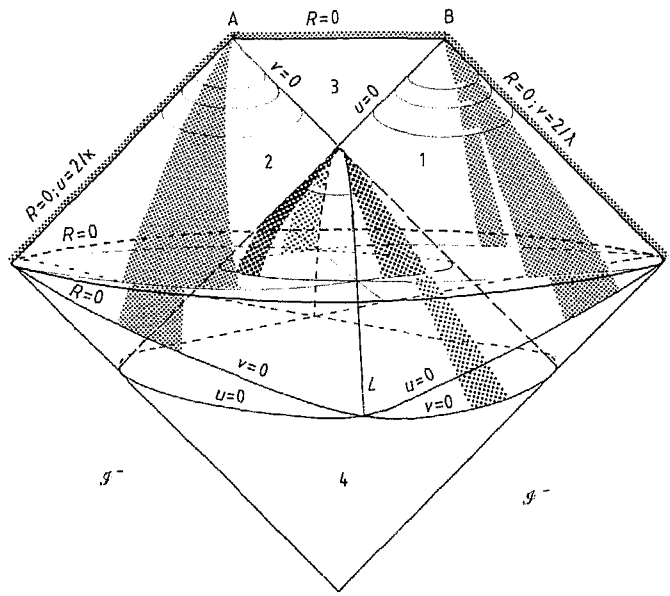


**Figure 5.** The Penrose diagram for the model depicted schematically in figure 2($a$) showing the relationship of the structure in figure 4 to null infinity ($\mathscr{I}$). The lines where the $u=0$ and $v=0$ planes meet $\mathscr{I}$ are indicated. One spatial dimension has been suppressed. Compare with figure 4.

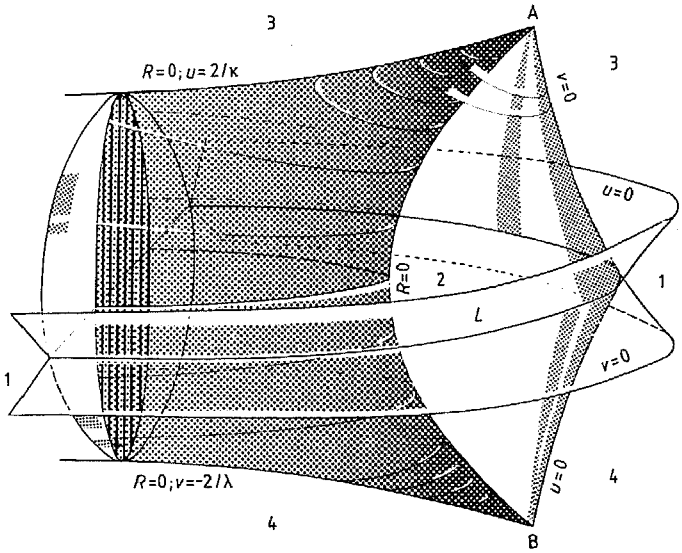


**Figure 6.** The global structure of the model depicted schematically in figure 2(*b*). The shells as seen from regions 3 and 4 now appear to be cones with vertices at A and B; these cones have been distorted and joined to produce this figure. The four regions are indicated: regions 3 and 4 each now sit *outside* one cone, region 1 is outside *both* cones and region 2 is inside both cones. All four regions meet at $u=0=v$, shown in the figure as the line *L*. The $R=0$ singularity bounds region 2 from the inside and here appears as a sideways saddle-shaped surface, indicated by dots. The 'inside' of the $R=0$ surface as drawn is *not* part of spacetime. One spatial direction has been suppressed.

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## Appendix 1. General argument

We now give a more general derivation of a $C^0$ metricx representing colliding shells of matter which includes the metrics (12) and (18) as special cases. We wish to consider *planar* shells and will thus build an $R^2$ invariance to our metric. We therefore make the ansatzx

$$ds^2 = -g(u, v)\, du\, dv + h^2(u, v)(dx^2 + dy^2) \tag{A1.1}$$

where $g$ and $h$ are $C^0$, which immediately implies that

$$R_{ux} = R_{uy} = 0 = R_{vx} = R_{vy}. \tag{A1.2}$$

Assuming a pressureless source moving at the speed of light implies that

$$R_{xx} = R_{xy} = R_{yy} = 0 = R_{uv} \tag{A1.3}$$

† We will in fact only need the special case where the metric is piecewise $C^\infty$.
‡ These metrics have a long history; see, e.g., footnote 9 of Penrose (1965). All vacuum solutions were given, e.g., by Taub (1951). Ipser and Sikivie (1984) considered the similar case of vacuum regions separated by domain walls.

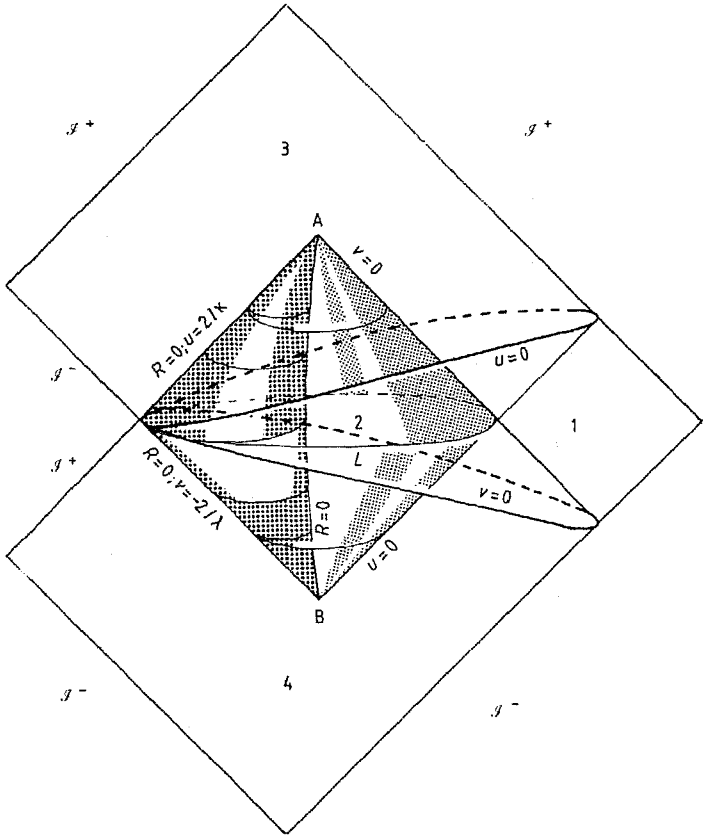


**Figure 7.** The Penrose diagram for the model depicted schematically in figure 2(*b*) showing the relationship of the structure of figure 6 to null infinity ($\mathscr{I}$). The lines where the $u=0$ and $v=0$ planes meet $\mathscr{I}$ are indicated. One spatial dimension has been suppressed. Compare with figure 6.

which enables us to write

$$\begin{aligned} h^2 &= 1+A(u)+B(v) \\ g &= \alpha(u)\beta(v)h^{-1} \end{aligned} \tag{A1.4}$$

with $A$, $B$, $\alpha$, $\beta$, $C^0$ so that the metric becomes

$$ds^2 = -\frac{\alpha(u)\beta(v)\,du\,dv}{[1+A(u)+B(v)]^{1/2}} + [1+A(u)+B(v)](dx^2+dy^2). \tag{A1.5}$$

The remaining components of the Ricci tensor are now

$$\begin{aligned} R_{uu} &= \frac{A'}{1+A+B}\left(\frac{\alpha'}{\alpha}-\frac{A''}{A'}\right) \\ R_{vv} &= \frac{B'}{1+A+B}\left(\frac{\beta'}{\beta}-\frac{B''}{B'}\right) \end{aligned} \tag{A1.6}$$

where the primes denote derivatives in the sense of distributions with respect to $u$ or

$v$ as appropriate. Assuming that the source is localised at $u=0$ and $v=0$, i.e.

$$\mathrm{supp}(R_{ab}) \subset \{uv=0\} \tag{A1.7}$$

we can divide spacetime into four regions, namely

$$\begin{aligned} &\text{region } 1 := \{u<0;\ v>0\} \\ &\text{region } 2 := \{u>0;\ v<0\} \\ &\text{region } 3 := \{u>0;\ v>0\} \\ &\text{region } 4 := \{u<0;\ v<0\} \end{aligned} \tag{A1.8}$$

as shown, e.g., in figure 2. In each of these regions we must have $R_{uu}=0=R_{vv}$; there are four possibilities.

*Case 1.* $A'=0=B'$.
We have $n^2 := 1+A+B=\text{constant}$, so that making the substitutions

$$\begin{aligned} X &:= nx \\ Y &:= ny \\ U &:= \frac{1}{n^{1/2}}\int \alpha(u)\,\mathrm{d}u \\ V &:= \frac{1}{n^{1/2}}\int \beta(v)\,\mathrm{d}v \end{aligned} \tag{A1.9}$$

brings the metric to the form

$$\mathrm{d}s^2 = -\mathrm{d}U\,\mathrm{d}V + \mathrm{d}X^2 + \mathrm{d}Y^2 \tag{A1.10}$$

which is flat.

*Case 2.* $A'=\kappa\alpha$; $B'=0$ $(0\neq\kappa=\text{constant})$.
We have $C := 1+B=\text{constant}$, so that making the substitutions

$$\begin{aligned} U-\frac{2}{\kappa} &:= \int \frac{\alpha(u)\,\mathrm{d}u}{[C+A(u)]^{1/2}} \equiv \frac{2}{\kappa}(C+A)^{1/2} \\ V &:= \int \beta(v)\,\mathrm{d}v \end{aligned} \tag{A1.11}$$

brings the metric to the form

$$\mathrm{d}s^2 = -\mathrm{d}U\,\mathrm{d}V + (1-\tfrac{1}{2}\kappa U)^2(\mathrm{d}x^2+\mathrm{d}y^2). \tag{A1.12}$$

But this is just the metric (6*a*) and is thus flat!

*Case 3.* $A'=0$, $B'=\lambda\beta$ $(0\neq\lambda=\text{constant})$.
Proceeding as in case 2 we make the substitutions

$$\begin{aligned} V-\frac{2}{\lambda} &:= \int \frac{\beta}{h}\,\mathrm{d}u \\ U &:= \int \alpha\,\mathrm{d}u \end{aligned} \tag{A1.13}$$

to obtain

$$ds^2 = -dU\,dV + (1 - \tfrac{1}{2}\lambda V)^2(dx^2 + dy^2) \tag{A1.14}$$

which is also flat.

*Case 4.* $A' = \kappa\alpha$, $B' = \lambda\beta$ ($\kappa \neq 0 \neq \lambda$ constant).
Making the substitutions

$$\begin{aligned} R &:= (1 + A + B)^{1/2}\omega^{-1/3} \\ T &:= \tfrac{1}{2}(A - B)\omega^{-2/3} \\ p &:= x\omega^{+1/3} \\ q &:= y\omega^{+1/3} \end{aligned} \tag{A1.15}$$

with $\omega := |\kappa\lambda|$ leads to

$$ds^2 = \mp dT^2/R \pm R\,dR^2 + R^2(dp^2 + dq^2) \tag{A1.16}$$

depending on the sign of $\kappa\lambda$. The metrics (A1.16) are sometimes referred to as Robinson's nullicle and are discussed in more detail at the end of § 3.

## Appendix 2. Classification

In this appendix we give a complete list of the possible ways of colliding two planar shells of matter within the ansatz (A1.1), (A1.3) and (8). We first note that the coordinate transformations

$$\begin{aligned} \mathfrak{U} &:= A(u)/\kappa \\ \mathfrak{V} &:= B(v)/\lambda \end{aligned} \tag{A2.1}$$

bring the metric for case 4 of appendix 1 to the form

$$ds^2 = -\frac{d\mathfrak{U}\,d\mathfrak{V}}{(1 + \kappa\mathfrak{U} + \lambda\mathfrak{V})^{1/2}} + (1 + \kappa\mathfrak{U} + \lambda\mathfrak{V})(dx^2 + dy^2) \tag{A2.2}$$

and that furthermore similar transformations can be used to bring the metrics in cases 1, 2 and 3 ((A1.10), (A1.12) and (A1.14)) *to the same form*; the metric (A2.2) corresponds to

$$\begin{aligned} &\text{case 1} \Leftrightarrow \kappa = 0 = \lambda \\ &\text{case 2} \Leftrightarrow \kappa \neq 0;\ \lambda = 0 \\ &\text{case 3} \Leftrightarrow \kappa = 0;\ \lambda \neq 0 \\ &\text{case 4} \Leftrightarrow \kappa \neq 0 \neq \lambda. \end{aligned} \tag{A2.3}$$

We can now unite the general $C^0$ (piecewise smooth) model for the collision of two planar shells of matter satisfying (A1.1), (A1.3) and (8) as

$$ds^2 = -\frac{d\mathfrak{U}\,d\mathfrak{V}}{[1 + \mathcal{A}(\mathfrak{U}) + \mathcal{B}(\mathfrak{V})]^{1/2}} + (1 + \mathcal{A}(\mathfrak{U}) + \mathcal{B}(\mathfrak{V}))(dx^2 + dy^2) \tag{A2.4}$$

with

$$\mathscr{A}(\mathfrak{U}) = \kappa_1 \mathfrak{U}\, \theta(-\mathfrak{U}) + \kappa_2 \mathfrak{U}\, \theta(+\mathfrak{U})$$
$$\mathscr{B}(\mathfrak{V}) = \lambda_1 \mathfrak{V}\, \theta(-\mathfrak{V}) + \lambda_2 \mathfrak{V}\, \theta(+\mathfrak{V}) \tag{A2.5}$$

where $\kappa_i$, $\lambda_j$ are constants. The only non-zero components of the Ricci tensor are now

$$R_{\mathfrak{U}\mathfrak{U}} = \frac{\kappa_1 - \kappa_2}{1 + \mathscr{A} + \mathscr{B}}\, \delta(\mathfrak{U})$$
$$R_{\mathfrak{V}\mathfrak{V}} = \frac{\lambda_1 - \lambda_2}{1 + \mathscr{A} + \mathscr{B}}\, \delta(\mathfrak{V}). \tag{A2.6}$$

Requiring that the energy density of the shells be positive forces

$$\kappa_1 > \kappa_2$$
$$\lambda_1 > \lambda_2. \tag{A2.7}$$

A complete list of the possible ways of colliding two non-trivial planar shells compatible with (A2.7) is shown in figure 8. We have not attempted to analyse the global structure of these spacetimes except for the two cases treated in § 4.

Finally we note that if we allow one of the shells to be trivial (e.g. $\lambda_2 = \lambda_1$) then *all* possible combinations of two regions, each of which is either flat or a ($\pm\infty$) Robinson nullicle, can be achieved within the restriction of positive energy density of the shell of matter joining them ($\kappa_2 > \kappa_1$).

| | $\lambda_2<\lambda_1<0$ | $\lambda_2<0;\ \lambda_1=0$ | $\lambda_2<0;\ \lambda_1>0$ | $\lambda_2=0;\ \lambda_1>0$ | $0<\lambda_2<\lambda_1$ |
|---|---|---|---|---|---|
| $\kappa_2<\kappa_1<0$ | $+\infty$<br>$+\infty$ $+\infty$<br>$+\infty$ | $+\infty$<br>0 $+\infty$<br>0 | $+\infty$<br>$-\infty$ $+\infty$<br>$-\infty$ | 0<br>$-\infty$ 0<br>$-\infty$ | $-\infty$<br>$-\infty$ $-\infty$<br>$-\infty$ |
| $\kappa_2<0;\ \kappa_1=0$ | $+\infty$<br>$+\infty$ 0<br>0 | $+\infty$<br>0 0<br>0 | $+\infty$<br>$-\infty$ 0<br>0 | 0<br>$-\infty$ 0<br>0 | $-\infty$<br>$-\infty$ 0<br>0 |
| $\kappa_2<0;\ \kappa_1>0$ | $+\infty$<br>$+\infty$ $-\infty$<br>$-\infty$ | $+\infty$<br>0 $-\infty$<br>0 | $+\infty$<br>$-\infty$ $-\infty$<br>$+\infty$ | 0<br>$-\infty$ 0<br>$+\infty$ | $-\infty$<br>$-\infty$ $+\infty$<br>$+\infty$ |
| $\kappa_2=0;\ \kappa_1>0$ | 0<br>0 $-\infty$<br>$-\infty$ | 0<br>0 $-\infty$<br>0 | 0<br>0 $-\infty$<br>$+\infty$ | 0<br>0 0<br>$+\infty$ | 0<br>0 $+\infty$<br>$+\infty$ |
| $0<\kappa_2<\kappa_1$ | $-\infty$<br>$-\infty$ $-\infty$<br>$-\infty$ | $-\infty$<br>0 $-\infty$<br>0 | $-\infty$<br>$+\infty$ $-\infty$<br>$+\infty$ | 0<br>$+\infty$ 0<br>$+\infty$ | $+\infty$<br>$+\infty$ $+\infty$<br>$+\infty$ |

**Figure 8.** A summary of the results of appendix 2. All possible ways of colliding two non-trivial planar shells of matter with positive energy density within the ansatz (A1.1) are shown. $\kappa_i$, $\lambda_j$ are constants and refer to the metric (A2.4) and (A2.5) describing the collision. In the figure a zero refers to flat space, while $\pm\infty$ refer to the Robinson nullicle, described by the metrics (15) and (20) respectively.

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