SOME OBSERVATIONS ON QUANTUM CHROMODYNAMICS

G. 't Hooft*

Stanford Linear Accelerator Center

Stanford, California

and

Institute for Theoretical Physics

University of Utrecht, Netherlands

1. INTRODUCTION

It need not be repeated here why the theory referred to as "quantum chromodynamics" is considered to be the best candidate for a successful theory for the strong interactions. The Lagrangean is written as

$$L(A,\bar{\psi},\psi) = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \sum_f \bar{\psi}_f(\gamma D+m_f)\psi_f, \quad (1.1)$$

where the gauge group is SU(3):

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu ,$$

$$a,b,c = 1,\ldots,8, \quad (1.2)$$

*Present address: Institute for Theoretical Physics, University of Utrecht, Princetonplein 5, Utrecht, The Netherlands.

and the quark fields ψ_f come in triplets, for which

$$D_\mu \psi_f = \partial_\mu \psi_f - \frac{1}{2} ig \lambda^a A^a_\mu \psi_f ,$$

$$[\lambda^a, \lambda^b] = 2i f_{abc} \lambda^c . \tag{1.3}$$

The free parameters of the theory are (seem to be) the masses m_f and a dimensionless coupling constant g.

In spite of the simplicity of the basic Lagrangean the proposed system is the most complicated field theory particle physicists ever dealt with. The reason is that there does not exist a simple free field theory that even remotely describes the physical particle spectrum and therefore could be used as the first of a successive series of approximation. In other quantum field theories such as quantum electrodynamics and the weak interactions the perturbation series in terms of the coupling constant is successful because the coupling is small enough to guarantee rapid convergence.

In quantum chromodynamics the coupling is large, and in a certain sense even infinite (the quarks cannot come free). Only in the far Euclidean region for the momenta involved is the effective coupling small and the perturbation expansion with respect to it can be used as an asymptotic expansion (the radius of convergence is zero). It is therefore understandable that most particle theorists seek other ways to approximate the infinity of field variables by some simpler model. Those approximations, however useful they may be, do not answer the question whether the given Lagrangean really defines a unique theory. It is easy to argue (as we will do in the following chapters) that the formal series in the coupling constant g diverges badly for all values of g. Due to the renormalization group the series has a direct physical interpretation as an asymptotic expansion for very large (Euclidean) momenta. Although the expansion diverges, does it perhaps in combination with physical requirements such as unitarity and causality define a theory uniquely? Can we in principle replace the divergent series by a convergent one, no matter how complicated? It is this question that we shall investigate.

In section 2 we show that the most essential renormalization factors in our theory are determined by the one- and two loop Feynman graphs only, due to asymptotic freedom, and are therefore known. This implies that we can give a simple definition of the theory and its physical parameters in such a way that all physical quantities remain finite at all orders of perturbation theory. It is tempting, then, to assume that this defines also the nonperturbative theory, but that is not true (except possibly when one of the quark masses is put equal to zero). There is known to be yet another free parameter in the form of an angle θ that has to be defined as well. Its effects (arising from the so called instantons) simply do not show up in any finite order of the perturbation expansion. Are there no other such parameters θ', θ'', ...? Perhaps an infinity of them? As yet we were unable to settle that question.

In the next chapters we assume that a theory with physically acceptable properties is defined this way and consider it as a function of the renormalized coupling constant g^R and fixed value(s) for θ (θ', θ'',...). In the massless case ($m_f = o$) we find the complete analytic structure in the complex $(g^R)^2$ plane (fig. 2).

In view of the analytic structure that we find we consider it unlikely that any attempts to solve QCD using Padé approximants will be successful; they will certainly diverge sooner or later. Much more interesting is the claim that perhaps Borel resummation can be applied[1]. We show how the instantons enter in the Borel resummation expression. They are an interesting complication but do not seem to be the most essential obstable in formulating convergent expressions. Much harder we think is the problem of justifying the interchange of this resummation procedure with the limit where renormalization and infrared cutoffs Λ go to infinity. And there is another complication that might make the Borel approach essentially worthless here: we show that the Borel integral diverges for all values of the coupling constant. This is again a consequence of the bad analytic structure for complex g^R. We conclude that there is still no convergent resummation procedure for QCD, the same conclusion as in ref. 2) but on different grounds.

Finally we observe some infrared divergencies of QCD in less than four dimensions (independent of the masses of the quarks). In four dimensions the relevant diagrams (fig. 4) are not infinite but may become so large that the Borel summability properties of the theory are endangered.

2. DEFINITION OF THE THEORY

The Lagrangean (1.1) defines the unrenormalized perturbation expansion in the coupling constant g. A convenient way to define a (perturbatively) finite expansion parameter g_D is through dimensional renormalization. The subscript D stands for dimensionally renormalized. At 4-ε dimensions, with ε nonrational, the integrations in momentum space can be defined unambiguously. The S-matrix in the limit $\varepsilon \to o$ is finite if g (the bare coupling constant occurring in (1.1)) is taken to be a function of g_D and ε (see ref. 3)

$$g^2 = \mu^\varepsilon \left[g_D^2 + \frac{g_D^4\, b_1(g_D^2)}{\varepsilon} + \frac{g_D^6\, b_2(g_D^2)}{\varepsilon^2} + \ldots\right], \quad (2.1)$$

and similarly the bare mass m:

$$m_f = \mu m_{fD} \left[1 + \frac{g_D^2\, a_1(g_D^2)}{\varepsilon} + \frac{g_D^4\, a_2(g_D^2)}{\varepsilon^2} + \ldots\right]. \quad (2.2)$$

Here, the quantities μ^ε and μ are there to show that in 4-ε dimensions g^2 has dimension ε and m_f has dimension one, as can easily be read off from the Lagrangean. The functions $a_i(g_D^2)$ and $b_i(g_D^2)$ are uniquely determined[3] if we require only poles in 1/ε in (2.1) and (2.2), but the definition of g_D^2 depends on the choice of the parameter μ.

At this point we emphasize that field theory in a noninteger number of dimension has never been defined beyond the perturbation expansion. Therefore the above defined parameters g_D^2 and m_D may not exist beyond perturbation theory. We prefer to construct other parameters whose existence follows undisputably from the assumption that the theory is approached by the asymptotic expansion in g in the deep Euclidean region.

First we go back to the perturbation series for g_D^2 and m_D. The S-matrix should not depend on the choice of μ (invariance under the renormalization group), but the above definitions of g_D and m_D do, so we write $g_D(\mu)$ and $m_D(\mu)$. Invariance of the S-matrix under changes of μ is guaranteed if $g_D(\mu)$ and $m_D(\mu)$ satisfy the equations[3]

$$\frac{\mu d g_D^2}{d\mu} \equiv \beta^D(g_D^2) = \left(1 - g_D^2 \frac{d}{dg_D^2}\right) g_D^4\, b_1(g_D^2) \quad , \tag{2.3}$$

and

$$\frac{\mu d m_D}{d\mu} \equiv m_D(-1+\alpha^D(g_D^2)) =$$

$$= -m_D - m_D\, g_D^2 \frac{d}{dg_D^2}\left(g_D^2\, a_1(g_D^2)\right) \; . \tag{2.4}$$

The lowest order terms of α and β for our theory are known[4]:

$$\beta^D(g_D^2) = \beta_1\, g_D^4 + \beta_2\, g_D^6 + \beta_3^D\, g_D^8 + \ldots \quad ,$$

$$\alpha^D(g_D^2) = \alpha_1\, g_D^2 + \alpha_2^D(g_D^4) + \ldots \quad ,$$

with

$$\beta_1 = -\frac{1}{8\pi^2}\left(11 - \frac{2}{3} N_f\right) \; ,$$

$$\beta_2 = \frac{1}{(8\pi^2)^2}\left(\frac{19}{3} N_f - 51\right) \; ,$$

$$\alpha_1 = -1/2\pi^2 \; . \tag{2.5}$$

We limit ourselves to $N_f \leqslant 16$, so that $\beta_1 < o$.

The importance of eqs. (2.3) and (2.4) is that we can let our choice of μ depend on the problem considered. For any process that is free of infrared divergencies in the massless limit the essential expansion parameter is

$$g_D^2(\mu) \log \frac{k^2}{\mu^2} \quad , \tag{2.6}$$

where k^2 stands for the typical external momenta. This is smallest if μ^2 is chosen to be of the same order as k^2. If β_1 is negative (as is the case here) then $g_D(\mu) \to o$ if $\mu \to \infty$. Thus, if the external momenta k^2 go to infinity then the expansion parameter (2.6) goes to zero and we have a rapidly converging series. This phenomenon is called asymptotic freedom.

Numerous authors discuss the consequences of a non-trivial zero of the function β. We stress that with our definition the presence or absence of zero's is irrelevant. We say this because g_D is not directly connected with a physically measurable quantity such as a gauge-invariant Green's function. Suppose there were a zero at g_D^o. Any redefinition of the form

$$\frac{1}{g_x^2} = \frac{1}{g_D^2} - \frac{1}{(g_D^o)^2} \qquad (2.7)$$

would remove the zero from the β function for the new g_x^2. The $\theta(g^4)$ corrections to the renormalized coupling constants are usually defined in a rather arbitrary fashion and the "correction" (2.7) to g_D^2 is indeed of order g_D^4. Secondly, we repeat that g_D may not have a finite meaning at all.

Let us now consider a substitution of the form

$$g_D^2 = g_R^2 + p_1\, g_R^4 + p_2\, g_R^6 + \dots \quad ,$$

$$m_D = m_R\,(1 + q_1\, g_R^2 + q_2\, g_R^4 + \dots). \qquad (2.8)$$

We will choose p_i and q_i and obtain a preferred set of variables g_R and m_R. We have

$$\frac{\mu d}{d\mu} g_R^2 = \beta^R(g_R^2) = \beta_1 g_R^4 + \beta_2 g_R^6 + \beta_3^R g_R^8 + \dots \quad ,$$

$$\frac{\mu d}{d\mu} m_R(-1 + \alpha_R(g_R^2)) ;$$

$$\alpha_R = \alpha_1 g_R^2 + \alpha_2^R g_R^4 + \dots \quad . \tag{2.9}$$

We find that β_1, β_2 and α_1 are unaffected by the change. These parameters are universal. But

$$\beta_3^R = \beta_3^D + \beta_1(2p_2+p_1^2) + \beta_2 p_1 \quad ,$$

$$\alpha_2^R = \alpha_2^D - \beta_1 q_1 - q_1 p_1 \quad . \tag{2.10}$$

We now choose $p_1 = o$ and $p_2, p_3, \dots$ and $q_1, q_2, \dots$ in such a way that all coefficients β_3^R, $\beta_4^R, \dots$ and α_2^R, $\alpha_3^R, \dots$ are equal to zero. Thus we have obtained parameters g_R and m_R that are equal to g_D and m_D up to computable higher order corrections, and they have known Callan-Symanzik functions α and β. Clearly, if this β function has a zero one should not attach any physical relevance to that.

We can now solve* for $g_R(\mu)$ and $m_R(\mu)$:

$$\frac{1}{|\beta_1| g_R^2(\mu)} + \frac{\beta_2}{\beta_1^2} \log \left(\frac{|\beta_1|}{g_R^2(\mu)} - \beta_2\right) = \log (\mu/\mu_o) ; \tag{2.11}$$

$$\frac{\mu m_{fR}(\mu)}{m_{fo}} = \left(\frac{|\beta_1|}{g_R^2(\mu)} - \beta_2\right)^{\alpha_1/\beta_1} \quad . \tag{2.12}$$

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*By further redefinition of g_R with counter terms of order g_R^3 one can simplify this solution, but with no particular merit.

Here μ_o and m_{fo} are integration constants. They all have dimension of a mass and they are the true renormalization group invariant physical parameters of the theory. The advantage of the above definition is that it is finite to all orders of the perturbation expansion and even for the summed theory, if it approaches the asymptotic expansion in the deep Euclidean region. A similar definition of quark masses has been given in ref. 5.

As already noted in the Introduction we should not be tempted to assume that μ_o and m_{fo} are therefore necessarily the only parameters of the theory. Due to the "instantons" (classical field configurations in Euclidean space-time) there are effects that give rise to amplitudes[6] proportional to

$$g_R^{-C} \exp\,[-8\pi^2/g_R^2 \pm i\theta][1+\theta(g_R^2)]. \tag{2.13}$$

For the gauge group SU(3) we have

$$C = 12\ . \tag{2.14}$$

(C counts the number of zero eigenmodes of the gauge fields in the presence of an instanton). θ is a free parameter, observable if all parameters $m_{fo} \neq o$. Clearly, the effects that depend on θ do not show up in the usual perturbation expansion in g_R. The mere existence of phenomena that do not show up in the usual perturbation expansion shows the importance of the questions we will consider.

3. THE COMPLEX COUPLING CONSTANT PLANE FOR THE MASSLESS THEORY

The massless theory is defined by

$$m_{fo} = o \ , \ f = 1,\ldots, N_f \leqslant 16 \ . \qquad (3.1)$$

In this limit we have a global chiral $SU(N_f) \times SU(N_f) \times U(1)$ symmetry. There is only one known parameter left, which is μ_o (the parameter θ in this limit is meaningless because it only fixes the preference coordinate in the broken chiral $U(1)$ group).

Instead of taking the variable μ_o we consider the theory as a function of $g_R(\mu)$ at some fixed μ. The relation between $g_R(\mu)$ and μ_o is given by the solution (2.11) of the renormalization group equation. This we do because the theory is known as a series of integer powers of $g_R(\mu)$.

First we must give a definition of the Green's functions which is as accurate as our definition of the physical variables (that is, the result must be finite to any order of the perturbation expansion and remain finite after integration of the renormalization group equation). One restriction will always be made. We only consider gauge-invariant amplitudes. A very good example to work with is the time ordered product of two (or more) bilinear quark operators: one adds to the Lagrangean a source term

$$\bar{\psi}(x)\ J(x)\ \psi(x) \ ,$$

where J may contain Dirac and flavor indices but no color indices, and consider that part of the vacuum-vacuum amplitude that is linear both in $J(o)$ and $J(x)$.

Let us first renormalize dimensionally,

$$J(x) = J_D(x) + \frac{g_D^2}{\varepsilon} J_{D1}(x,g_D^2) + \frac{g_D^4}{\varepsilon^2} J_{D2}(x,g_D^2) + \dots \quad . \tag{3.2}$$

As before, J_{D1}, J_{D2},... are chosen such that we obtain a perturbatively finite Green's function. In momentum space

$$\Gamma^D_{pert}(k^2,\mu,g_D^2) = \Gamma_o(k^2,\mu) + g_D^2\,\Gamma_1(k^2,\mu) + \dots \quad . \tag{3.3}$$

The renormalization group equation is

$$[\frac{\mu d}{d\mu} + \beta^D(g_D^2)\,\frac{\partial}{\partial g_D^2} + \gamma^D_{pert}(k^2,\mu,g_D^2) = o \quad . \tag{3.4}$$

Here γ also contains the canonical part of the dimension of Γ(depending on the number and type of external lines). The general solution is

$$\Gamma_{pert}(k^2,\mu,g_D^2) = Z(g_D^2)\,\Gamma(g_D^2(\mu),\,k^2/\mu^2) \quad , \tag{3.5}$$

with

$$\frac{d}{dg_D^2}\log Z(g_D^2) = -\,\frac{\gamma^D(g_D^2)}{\beta^D(g_D^2)} \quad , \tag{3.6}$$

and

$$\frac{d}{d\mu}\,\Gamma_D\,(g_D^2(\mu),\,\frac{k^2}{\mu^2}) = o \quad . \tag{3.7}$$

In the simplest example, where we look at the σ-channel ($\bar{\psi}J\psi = J(\bar{\psi}\psi)$), we have

$$\gamma^D = \gamma_o + \gamma_1\,g_D^2 + \gamma_2^D\,g_D^4 \dots \quad ,$$

with
$$\gamma_o = 2 ,$$
$$\gamma_1 = - 1/\pi^2 . \qquad (3.8)$$

The right hand side of eq. (3.6) is then

$$\frac{z_o}{g_D^4} + \frac{z_1}{g_D^2} + z_2 + \ldots , \qquad (3.9)$$

with
$$z_o = - \gamma_o/\beta_1 ,$$
$$z_1 = - \gamma_1/\beta_1 + \gamma_o\beta_2\beta_1^2 . \qquad (3.10)$$

Thus

$$\log Z(g_D^2) = - \frac{z_o}{g_D^2} + z_1 \log g_D^2 + z_2 g_D^2 + \ldots . \qquad (3.11)$$

Not only will we replace g_D here by g_R, but also we will multiply Z with finite corrections of order g_R^2 such that the coefficients z_2, z_3,... all vanish. Again we make use of the fact that the subtraction procedure for Γ^D_{pert} was arbitrary anyhow and thus we obtain Γ^R_{pert}, equal to Γ^D_{pert} up to higher order computable corrections, and

$$Z^R(g_R^2) = g_R^{2z_1} \exp(-z_o/g_R^2) \qquad (3.12)$$

exactly.

According to eq. (3.7) we can write

$$\Gamma_D(g_D^2(\mu), k^2/\mu^2) \equiv \Gamma_R(g_R^2(\mu), k^2/\mu^2) =$$
$$= \Gamma(k^2/\mu_o^2) , \qquad (3.13)$$

where μ_o is given by (2.11). And we have

$$\Gamma^R_{pert}(k^2,\mu,g_R^2) = Z^R(g_R^2)\ \Gamma(k^2/\mu_o^2)\ . \tag{3.14}$$

Notice that we now have one unknown function Γ of one variable $z = k^2/\mu_o^2$. The analytic structure of Γ_{pert} for complex k^2 follows from general physical requirements (fig. 1). In mesonic channels we expect a cut starting at the origin, because massless pions can be produced. In baryonic channels the cut will start at some finite negative value for z. The discontinuity across the cut will show peaks where the resonances are. It is important to consider the second Riemann sheet across the cut. There we expect series of poles due to the resonances. In a simplified version of the theory SU(3) is replaced by SU(N) and the limit $N \to \infty$, g^2N fixed is taken. Then the poles move to the real axis and the cut disappears[7].

Substituting (2.11) into (3.14) we find the behavior for complex g_R^2 as well. The singularities described above occur at

$$\log(k^2/\mu^2) + \frac{2}{(-\beta_1)g_R^2(\mu)} + \frac{2\beta_2}{\beta_1^2}\log\left(\frac{-\beta_1}{g_R^2(\mu)} - \beta_2\right) = a + (2n+1)\pi i\ , \tag{3.15}$$

where a is real for the cut and may have a small imaginary part for the poles. If k^2 and μ are kept fixed and positive then (3.15) gives the singularities in complex g_R^2 plane (see fig. 2). We assume that β_1 is negative. The origin of g_R^2 plane is a density point of cuts (or poles if $N \to \infty$) both for the meson and for the baryon channels. In the case of a finite gauge group

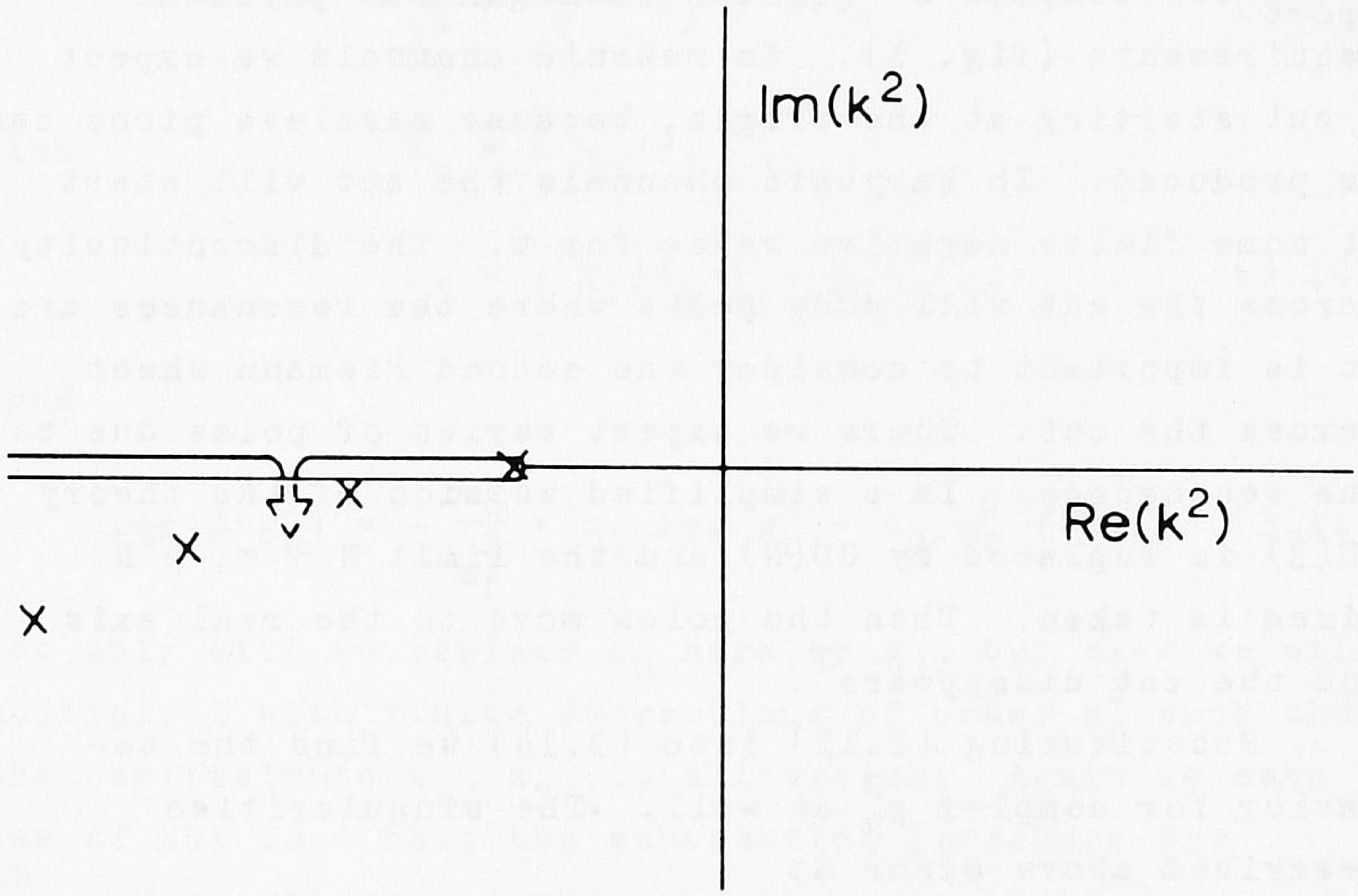

Fig. 1. Our basic assumption is that Green's functions are analytic except for a cut along the negative real axis in k^2 plane. If we try to continue analytically beyond the cut (arrow) then we will encounter poles due to the resonances (crosses).

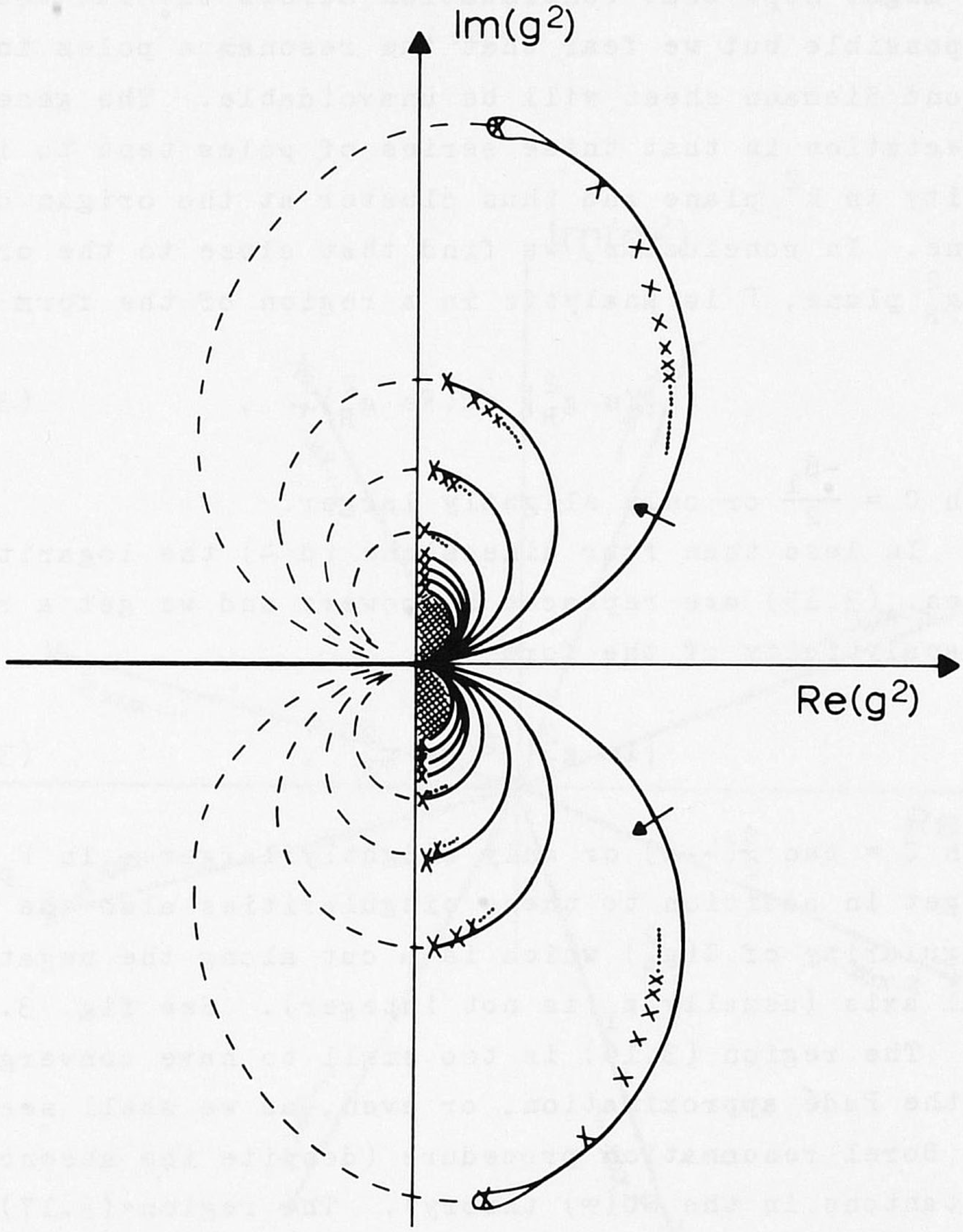

Fig. 2. Singularities in complex $(g^R)^2$ plane. The solid lines are cuts that will always be present. In the baryonic channels the cuts stop halfway; in the mesonic channels they continue to form full circles (dotted lines) because we have massless mesons. (The circles are slightly deformed due to the β_2 term). If we try to continue across the cut (arrows) we encounter poles (crosses).

one might hope that continuation across the cut would be possible but we fear that the resonance poles in the second Riemann sheet will be unavoidable. The general expectation is that these series of poles tend to infinity in k^2 plane and thus cluster at the origin of g_R^2 plane. In conclusion, we find that close to the origin of g_R^2 plane, Γ is analytic in a region of the form

$$|\mathrm{Im}\ g_R^2| < C(\mathcal{R}e\ g_R^2)^2 \quad , \tag{3.16}$$

with $C = \frac{-\beta_1}{2}$ or only slightly larger.

In less than four dimensions ($d<4$) the logarithms of eq. (3.15) are replaced by powers and we get a region of analyticity of the form

$$|\mathrm{Im}\ g_R^2| < C\mathcal{R}e\ g_R^2 \quad , \tag{3.17}$$

with $C = \tan \frac{\pi}{2}(4-d)$ or only slightly larger. In Γ_{pert} we get in addition to these singularities also the singularity of $Z(g_R^2)$ which is a cut along the negative real axis (usually z_1 is not integer). See fig. 3.

The region (3.16) is too small to have convergence of the Padé approximation, or even, as we shall see, the Borel resummation procedure (despite the absence of instantons in the SU(∞) theory). The region (3.17) is large enough for Borel resummation to converge only if $d \leqslant 3$, were it not that there may be other troubles with that procedure in less than four dimensions (see chapter 6).

4. BOREL RESUMMATION

The Borel resummation procedure for a function $\Gamma_{pert}(g^2)$ consists of writing the Laplace transformation

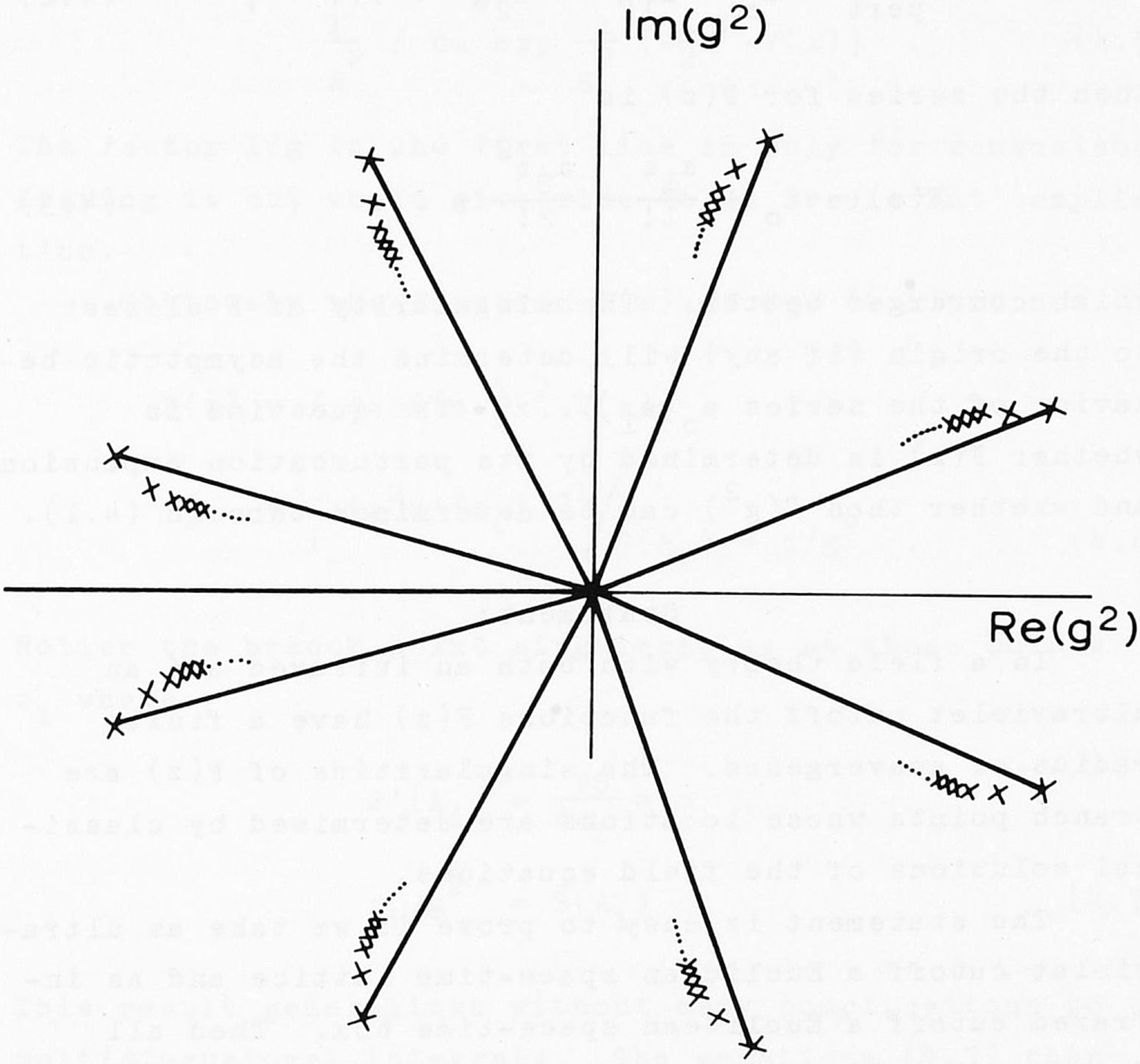

Fig. 3. Singularities in complex g^2 plane if $\varepsilon = 0.26$. Thick lines are cuts.

$$\Gamma^{\cdot}_{pert}(g^2) = \int_0^\infty F(z)\, e^{-z/g^2}\, dz/g^2 . \tag{4.1}$$

If Γ_{pert} has the perturbation expansion

$$\Gamma_{pert} = a_0 + a_1 g^2 + a_2 g^4 + \ldots , \tag{4.2}$$

then the series for F(z) is

$$F(z) = a_0 + \frac{a_1 z}{1!} + \frac{a_2 z^2}{2!} + \ldots , \tag{4.3}$$

which converges better. The singularity of F closest to the origin (if any) will determine the asymptotic behavior of the series $a_0, a_1, \ldots$. The question is whether F(z) is determined by its perturbation expansion and whether then $\Gamma(g^2)$ can be determined through (4.1).

Statement:

In a field theory with both an infrared and an ultraviolet cutoff the functions F(z) have a finite radius of convergence. The singularities of F(z) are branch points whose locations are determined by classical solutions of the field equations.

The statement is easy to prove if we take as ultraviolat cutoff a Euclidean space-time lattice and as infrared cutoff a Euclidean space-time box. Then all functional integrals are just finite dimensional integrals.

Let us illustrate the proof for a "field theory" with just one field variable A at just one space-time point x. The action is

$$S(A) = -\frac{1}{2} A^2 - \frac{1}{g^2} V(gA) , \tag{4.4}$$

with $V(x) = V_3x^3 + V_4x^4 + \dots$. Here g^2 is the expansion parameter.

$$W(g^2) = \frac{1}{g} \int dA \exp S(A) =$$

$$\frac{1}{g^2} \int dx \exp \frac{1}{g^2} [-\tfrac{1}{2}x^2 - V(x)] \quad . \qquad (4.5)$$

The factor 1/g in the first line is only for convenience. Leaving it out would give rise to an irrelevant complication.

Obviously the function $F(z)$ of eq. (4.1) is now

$$F(z) = \int dx \; \delta[z - \tfrac{1}{2}x^2 - V(x)]$$

$$= \sum_i (g^2 S'[A_i])^{-1} \Big/ S(A_i) = z/g^2 \quad . \qquad (4.6)$$

Notice the branch point singularities at those points z_i where

$$S'[A_i] = \frac{\partial S}{\partial A_i} = 0 \quad ,$$

$$z_i/g^2 = S(A_i) \quad . \qquad (4.7)$$

This result generalizes without many complications to multidimensional integrals. The equations (4.7) then correspond to the classical Lagrange field equations.

Still, however, we are far away from a real field theory. Rather than complete functional integrals as a function of a <u>bare</u> expansion parameter g^2 we are interested in renormalized connected Green's functions as a function of a renormalized parameter g_R^2. Now an interesting feature of the Laplace transformation (4.1) is that the positions of the branch point singularities of $F(z)$ are the same as those of the singularities of the Laplace transforms of

$$\frac{\partial\Gamma}{\partial g^2} \; , \; f(g^2)\Gamma \; , \; \exp \Gamma \; , \; \ln(1+g^2\Gamma) \; , \; \text{etc.},$$

as one can easily verify. Thus, by exponentiation and differentiation it will be possible to obtain the analytic structure of the Laplace transforms of connected Green's functions. The location of their branch points is the same as those of $F(z)$. Also, finite renormalizations,

$$g^2 \rightarrow g^2 + \theta(g^4)$$

leave the positions of the singularities unaffected. This is how refs. 1,2 obtained the result that the singular points z_i of $F(z)$ arise from classical solutions of $\partial S/\partial A_i = o$ (the Lagrange field equations) where $S(A_i) = z_i/g^2$, even if Γ_{pert} is a connected Green's function and if g^2 is renormalized.

Real solutions of the field equations in Euclidean space-time, with finite total action (instantons) give singularities in $F(z)$ on the positive real axis. These singularities correspond to the opening up of new types of field configurations with increasing action (those with a nonvanishing total Pontryagin winding number). Comparison of (4.1) and (4.5) suggests how to integrate over these singular points, without difficulties. By doing the perturbation expansion around the instanton field configurations we find the behavior of $F(z)$ close to and beyond these singular points. All in all we expect that when all classical solutions of the field equations with finite total action are known then F(z) can be determined completely by convergent perturbative methods. The instantons give rise to a complication that is not insurmountable[2].

5. PROBLEMS ASSOCIATED WITH THE BOREL RESUMMATION

The first problem we encounter if we try to pursue this program is that the analytic structure of F(z) is determined by all solutions of the classical equations that have a finite total action, including the complex solutions. The solutions where the fields are real usually have some physical interpretation and can be categorized. But to determine all nonreal solutions may be very hard if not impossible. There could be very many, possibly infinitely many of them with small or even vanishing total action. There seems to be no simple positivity principle to rule out such an infinite class. Not all of these solutions will be equally important though. Each of them gives rise to a branch point and thus to additional Riemann sheets in the z-variable. Many of the branch points will be in those other Riemann sheets and thus not affect the asymptotic behavior of the perturbation expansion in z. The criterion will be provided by connecting the given solutions continuously to the vacuum configuration through a series of field configurations $A(x,\lambda)$, $o < \lambda < 1$. For $\lambda = o$ we have the vacuum, say, and for $\lambda = 1$ the given solution. The action $S[A(\lambda)]$ will follow a certain path in the complex plane. If that path has to go around another solution point S' in the complex plane then we are in a different Riemann sheet than the origin. Not $S[A(1)]$ but S' will determine the radius of convergence of F(z). Even so, we do not know at present how to find all relevant complex solutions.

We note that the single BPST instanton is an example of a solution that cannot be connected continuously to the vacuum because of its nonvanishing winding number. The instanton anti-instanton pair is the first known solution that will limit the radius of convergence of $F(z)$ in QCD to $16\pi^2$ (because the total action of that pair is $16\pi^2/g^2$). Instantons also give rise to effects in theories with Higgs mechanism although no finite classical solution exists. Therefore we expect a singularity of F at $z = 16\pi^2$ also in these theories.

The second problem we wish to mention may be equally troublesome. It has been noted that the formal result can be proven in an alternative way in finite theories such as the anharmonic oscillator (quantum field theory in 0+1 dimensions) by counting and estimating the Feynman diagrams[8]. In renormalized theories however the diagrams must undergo subtractions and a consequence of these subtractions is a more complicated dependence of external momenta through powers of logarithms. At higher orders these logarithms make it much harder to estimate bounds for these diagrams. Indeed, naive guesses for these diagrams suggest that the perturbation coefficients diverge so badly that even the series (4.3) will diverge for all z. This is a new feature due to renormalization and one therefore wonders whether perhaps the interchange of the limit $n \to \infty$ (high terms in the perturbation expansion) and the limit $\Lambda \to \infty$ (renormalization cutoff) do not commute. One can merely conjecture that interchange is allowed because the resulting expressions seem to make sense even after renormalization, but there seems to be no way of proving that.

And here is the third problem. It concerns the integral in eq. (4.1). Does it converge at infinity? Let us consider the one theory where we know the analytic behavior as a function of g_R^2, which is the massless gauge theory, considered in chapter 3. There are cuts and/or poles at

$$- 2/\beta_1 g_R^2 \simeq a_i + (2n+1)\pi i \quad , \tag{5.1}$$

which is a simplification of eq. (3.15). Here a_i is real or nearly real and can become arbitrarely large. Consequently the integral

$$\int_0^\infty F(z) \left[\exp(-\beta_1/2)(-az)\right] \left[\exp(-\beta_1/2)(2n+1)\pi i z\right] dz$$

will be infinite, i.e. diverges regardless of how large a is (remember that $-\beta_1$ is positive). Therefore F(z) diverges for z real and large, faster than any exponential of z, and oscillates with periods of the order of $\left|\frac{4}{\beta_1}\right|$ and fractions thereof. So, for this theory we answered the question: no, the integral diverges very badly. Attempts to redefine it in terms of some cutoff procedure will be as ambiguous as the original divergent series.

This divergence does not occur in 3-dimensional massless QCD but, as we shall see, there are other problems there.

We think there is no reason to assume that adding masses to the quarks would improve the situation, since we are not able to apply our analysis there.

6. AN INFRARED DIVERGENCE IN EUCLIDEAN SPACE

In four dimensions we have no infrared divergences

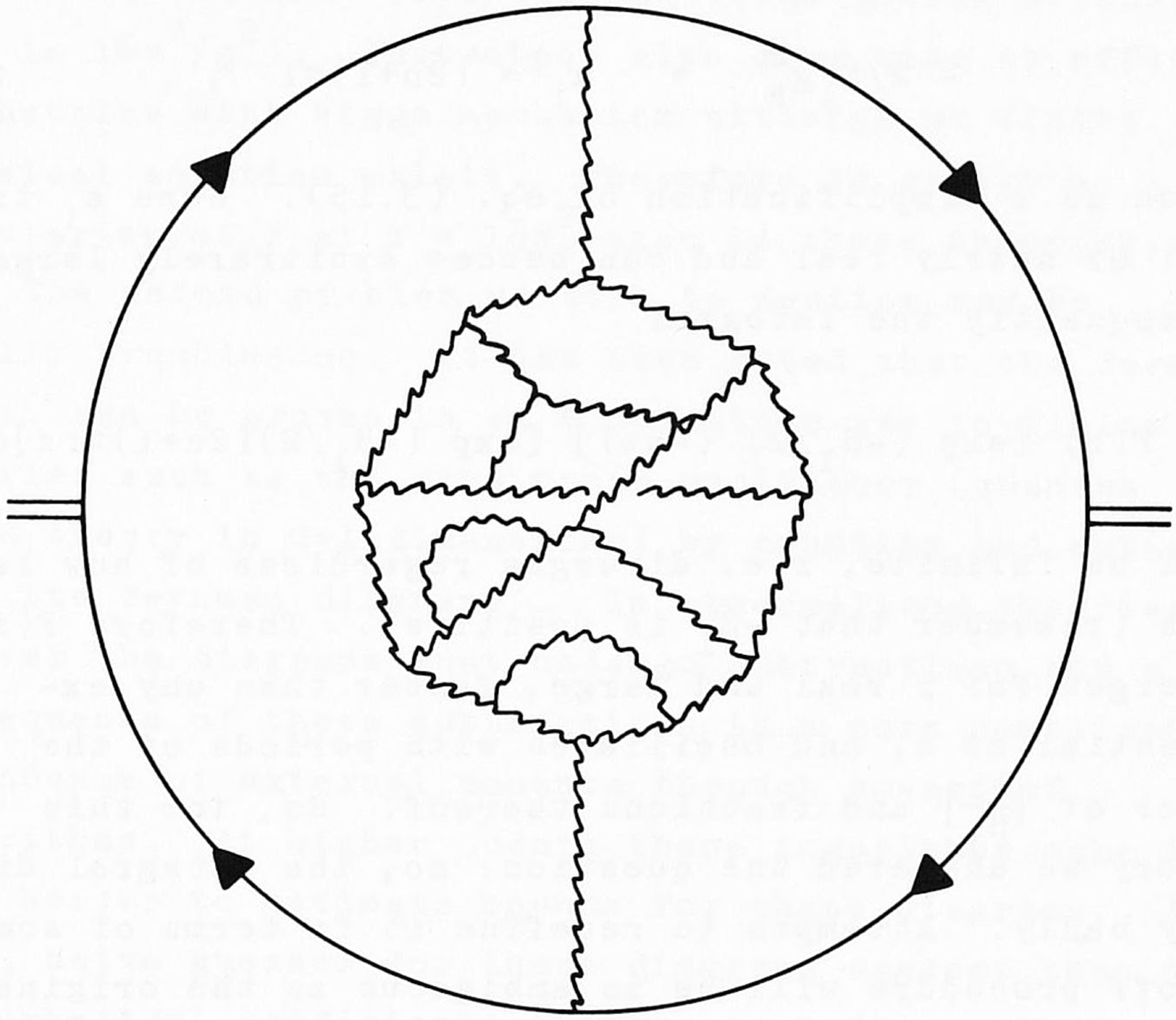

Fig. 4. Diagrams of this type give the first infrared divergences in less than four dimensions in Euclidean space. In 4 dimensions they might spoil Borel summability.

in Euclidean space. But for renormalization it is convenient to consider the theory in $4-\varepsilon$ dimensions, ε small and positive. Then, if we look at fixed order in the perturbation expansion we expect an infrared divergence, from the following arguments. If we consider the L-loop corrections to the gluon propagator, then, by power counting, we find the following momentum dependence:

$$\Delta P_{\mu\nu}(k) \propto (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})(k^2 - i\varepsilon)^{-1-\frac{\varepsilon L}{2}} \quad . \tag{6.1}$$

If we insert this into a larger diagram (fig. 4), we encounter integrals of the form

$$\int d^{4-\varepsilon} k \; \Delta P_{\mu\nu}(k) \; f_{\mu\nu}(k) \quad . \tag{6.2}$$

Since we consider only gauge invariant amplitudes, $f_{\mu\nu}$ will satisfy a Ward identity:

$$\begin{aligned} k_\mu f_{\mu\nu}(k) &= 0 \quad , \\ k_\nu f_{\mu\nu}(k) &= 0 \quad . \end{aligned} \tag{6.3}$$

From this we derive that $f_{\mu\nu}$ must have a second order zero at $k = 0$:

$$f_{\mu\nu}(k) \propto (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \text{ if } k \to o \quad . \tag{6.4}$$

So if $k \to o$, the integrand in (6.2) behaves as

$$d^{4-\varepsilon} k \; (k^2)^{-\frac{\varepsilon L}{2}} \quad . \tag{6.5}$$

This starts to diverge if

$$L > \frac{4}{\varepsilon} - 1 \ . \qquad (6.6)$$

The total diagram is at least of order

$$(g^2)^{4/\varepsilon} \ . \qquad (6.7)$$

So at finite ε we have an infrared divergence at large but finite order in the perturbation expansion. It is not clear how to give a direct physical interpretation of this divergence. In more than two dimensions there seems to be no reason for expecting cancellations between these divergences for different diagrams at the same order. But, if we sum the gluon selfenergy diagrams to obtain a new gluon propagator, then the "divergent" ineducible parts behave as a mass insertion and at high orders we expect logarithmic dependence of this mass insertion, so then the infinite coefficient for the diagrams of order $(g^2)^{4/\varepsilon}$ is replaced by a term of order

$$(g^2)^{4/\varepsilon} \log g \ . \qquad (6.8)$$

The Laplace transform will consequently contain terms of order

$$z^{4/\varepsilon} \log z \ .$$

Notice that then F(z) must have a cut starting at the origin of z-plane. This is the complication for theories in less than 4 dimensions as we alluded to in the beginning.

7. CONCLUSION

We do think perturbation theory up to two loops is essential to obtain an accurate definition of the theory.

But we were not able to obtain sufficient information on the theory to formulate a mathematically self-consistent procedure for accurate computations.

Our results made us skeptical against approaches that rely on Padé or Borel resummation procedures.

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