S-MATRIX THEORY FOR BLACK HOLES

G. 't Hooft

Institute for Theoretical Physics
Princetonplein 5, P.O. Box 80.006
3508 TA Utrecht, The Netherlands

ABSTRACT

We explain the principles of the laws of physics that we believe to be applicable for the quantum theory of black holes. In particular, black hole formation and evolution should be described in terms of a scattering matrix. This way black holes at the Planck scale become indistinguishable from other particles. This *S*-matrix can be derived from known laws of physics. Arguments are put forward in favor of a discrete algebra generating the Hilbert space of a black hole with its surrounding space-time including surrounding particles.

1. INTRODUCTION

There is quite a bit of controversy (and confusion) regarding the nature of physical law governing a black hole. Some of the difficulties have their origin in the deceptively clean picture given by the "classical" (here this means "non-quantum mechanical") solutions of Einstein's equations of gravity in the case of gravitational collapse. The metric tensor describing the fabric of space-time appears to be smooth and well-behaved in the vicinity of a region we call the "horizon", a surface beyond which there are space-time points from which no information can reach the outside world. It seems that one should be able to apply standard techniques from particle theory here to derive what a distant observer can perceive and, naturally, this exercise has been done[1].

There is one innocent-looking assumption that most practitioners then make. One observes that clouds of particles may venture into the "forbidden region", from which they can no longer escape or even emit any signal towards the outside world, and so one *assumes* that the corresponding states in Hilbert space may be treated the way one always does in quantum mechanics: the unseen modes are averaged over. Operators describing observations in the outside world are assumed to be diagonal in the sector of Hilbert space that is not seen, and hence in all computations one is obliged to sum over all unseen modes.

The immediate consequence of this practice is that the outside world alone is not anymore described by a single wave function but by a density matrix[2]. Even if one starts with a "pure" wave function, sooner or later one finds the system to be in a mixed state. It is as if part

New Symmetry Principles in Quantum Field Theory, Edited by
J. Fröhlich et al., Plenum Press, New York, 1992

of the wave functions "disappeared into the wormhole"; information escaped, as if the system were linked to a heat bath.

With large black holes one can perform Gedanken experiments, and consider observers who move semiclassically in the neighborhood of a black hole horizon. If we attach some sense of "reality" to these observers the correctness of the above assumptions seems to be an inescapable conclusion.

But what if the hole is small, so that classical observers are too bulky to enter? Or let us ask a question that is probably equivalent to this: suppose one keeps track of *all* possible states a black hole can be in, is it then still impossible to describe the hole in terms of pure quantum states alone? Will the very tiny black holes evolve according to conventional evolution equations in quantum physics or is the loss of information a fundamental new feature, even for them?

There is a big problem with any theory in which the loss of "quantum information" is accepted as a fundamental item. This is the fact that all effective laws become fuzzy. It is not difficult to construct an example of a theory in which pure states evolve into mixed states. Consider a system with a Hamiltonian that depends on a free physical parameter α (for instance the fine structure constant). A state $|\psi\rangle_0$ at $t=0$ evolves into the state

$$|\psi\rangle_t \quad = \quad e^{-iH(\alpha)t}|\psi\rangle_0 \tag{1.1}$$

at time t. The expectation value for an operator $\mathcal{O}$ evolves into

$$\langle\mathcal{O}\rangle_{t,\alpha} \quad = \quad {}_0\langle\psi|e^{iH(\alpha)t}\mathcal{O}\, e^{-iH(\alpha)t}|\psi\rangle_0 \quad , \tag{1.2}$$

and from the ψ dependence one can recover the information that the system remained in a pure quantum state. But now assume that there is an *uncertainty* in α. We only know the first few decimal places. There is a distribution of values for α, each with a probability $P(\alpha)$. Our theory now predicts for the "expectation value" of $\mathcal{O}$:

$$\langle\mathcal{O}\rangle_t \quad = \quad \int d\alpha\, \langle\mathcal{O}\rangle_{t,\alpha}\, P(\alpha) \quad \equiv \quad \mathrm{Tr}\,\rho(t)\mathcal{O} \quad ; \tag{1.3}$$

$$\rho(t) \quad = \quad \int d\alpha\, P(\alpha)\, e^{-iH(\alpha)t}|\psi\rangle_0\, {}_0\langle\psi|e^{iH(\alpha)t} \quad . \tag{1.4}$$

This is an impure density matrix, of the kind one obtains in doing calculations with black holes. The outcome of a by now standard calculation is a thermal distribution of outgoing particles. A thermal distribution is always a mixed state.

In our example we clearly see what the remedy is. the extra uncertainty had nothing to do with quantum mechanics; the Hamiltonian was not yet known because of our incomplete knowledge of the laws of physics (in this case the value of α). By doing extra experiments or by working harder on the theory we can establish a more precise value for α, and thus obtain a more precise prediction for $\langle\mathcal{O}\rangle_t$.

Returning with this wisdom to the black hole, what knowledge was incomplete? Here I think one has a situation that is common to all macroscopic systems: because of the large number of quantum mechanical states it was hopelessly difficult to follow the evolution of just one such state precisely. One was forced to apply thermodynamics. The outcome of our calculations with black holes got the form of thermodynamic expressions because of the impossibility, in practice, to follow in detail the evolution of any particular quantum state.

But this does mean that our basic understanding of black holes at present is incomplete. In a statistical system such as a vessel containing an ideal gas, we have *in principle* a quantum theory that is precise enough to study pure quantum states. In particular, if we dilute the gas so much that a single atom remains, the thermodynamic description will no longer be correct, and we must use the real quantum theory. Similarly, if we want to understand how a black hole behaves when it reaches the Planck mass, we expect the thermodynamic expressions to break down.

The importance of a good quantum mechanical description is that it would enable us to link black holes with ordinary particles. The Planck region may well be populated by a lot of different types of fundamental particles. their "high energy limit" will probably consist of particles small enough and heavy enough to possess a horizon and thus be indistinguishable from black holes. What we want is a consistent theory that covers all of this region. If we had a "conventional" Schrödinger equation in this region, it would be relatively straightforward (at least conceptually) to extrapolate to large distance scales using renormalization group techniques, and recover the "standard model" (or more!)

There have been many proposals concerning the nature of our physical world near the Planck length. we have seen "supergravity", "string theory", "heterotic strings", et cetera. My problem with these ideas is that they seem to be ad hoc. The models are "postulated" and then afterwards the authors try to argue why things have to be this way (basically the argument is that the new model is "more beautiful" than anything else known).

It would be a lot safer if we could *derive* the only possible correct setting of variables and forces, directly from the presently established laws of physics. In these lectures we will argue that it is possible to do this, or at least to make a good start, by doing Gedanken experiments with black holes. The reason why black holes should be used as a starting point in a theory of elementary particles is that *anything* that is tiny enough and heavy enough to be considered an entry in the spectrum of ultra heavy elementary particles (beyond the Planck mass), must be essentially a black hole.

Black holes are defined as solutions of the classical, i.e. unquantized, Einstein equations of General Relativity[3]. This implies that we only know how to describe them reliably when they are considerably bigger than the Planck length and heavier than the Planck mass. What was discovered by Hawking[1] in 1975 is that these objects radiate and therefore must decrease in size. It is obvious that they will sooner or later enter the domain that we presently do not understand.

Curiously, it is not easy to see why Hawking's derivation of the thermal black hole radiation would not be exactly correct. Even in a functional integral expression for this calculation one might still expect wormhole configurations through which quantum information leaks towards a mystical "other universe"[4]. We will now decide to be merciless: topologically non-trivial space-times are forbidden (until further notice) so that, at least at the microscopic level, pure quantum mechanics can be restored. More precisely, what we require is first of all some quantum mechanically pure evolution operator, and secondly that this operator be consistent with all we know of large scale physics, in particular general relativity.

At first sight these requirements are in conflict with each other. General relativity predicts unequivocally that gravitational collapse is possible, and this produces a horizon with all its difficulties. However, we claim that a pure quantum prediction that naturally blends into thermodynamic behavior in the large scale limit is not at all impossible, but it is true that the requirement for this to happen is extremely restrictive. Combining it with all we already know about large scale physics may well yield an unambiguous theory. Anyway, we know for sure that the amendments needed at the horizon all refer to Planck scale physics. As long as this physics is not completely understood it will also be impossible to refute our theories on the ground of inconsistencies with known physics.

So this is our program. We *assume* that, as for all spatially confined systems, there exists such a thing as a "scattering matrix". One then tries to reconcile this scattering matrix with the laws of physics already known. We will find that this scattering matrix, to some extent, can be derived. More precisely: *the exact quantum behavior at large distance scales* (the distance scales reached in present particle experiments) *can be derived uniquely*.

The problem is an apparent acausality. If we apply linearized quantum field theory in the black hole background it seems ununderstandable how information that is thrown into the black hole can reemerge as information in the outgoing states. This is because the outgoing radiation originates at $t = -\infty$ and the ingoing matter proceeds until $t = +\infty$, so the information had to go backwards in time. We simply claim that precisely for this reason linearized quantum field theory is inappropriate here. One *must* take gravitational (if not other) interactions between in- and outgoing matter into account. One way to interpret what happens then is to assume that there is a fundamental *symmetry principle*, because matter inside the horizon is unobservable. One can then perform a transformation that transforms away the singularity at $t=+\infty$, and produces one at $t=-\infty$.

It is of crucial importance to note that what we are deriving is not only the (quantum) behavior of the black hole itself. It is the entire system, black hole *plus* all surrounding particles, that we are talking about. Using our (assumed) knowledge of physics at large distance scales we derive the properties of the black hole *and all other forms of matter* at energies larger than the Planck energy.

In ordinary quantum field systems behavior at small distance, or equivalently, at high energies, determines the behavior at large distances and low energies. In the present case the interdependence goes both ways, or, in other words, the whole construction will be over-determined. We expect stringent constraints of consistency, which, one might hope, may lead to a single unique theory. The point is that the symmetry principle just mentioned affects matter in an essential way, and thus may perhaps continue to be of relevance at the low energy domain.

This is the motivation of this work. It may lead to "the unique theory". Even though our work is far from finished, we will be able to show that there will be a remarkable role for the old string theory[5]. The mathematical expressions we derive are so similar to those of string theory that perhaps some of its results will apply without any change. But both the physical interpretation and the derivations will be very different. As a consequence, the mathematics is not identical. One important difference is the string constant (determining the masses of the excitations), which in our case turns out to be imaginary[6].

In the usual string theory one uses the obvious requirements of unitarity and causality to derive that the string is governed by a local Lagrangean on the string world sheet. To derive similar requirements for the strings born from black holes is far from easy. This is presently what is holding us back from considerations such as tachyon elimination and anomaly cancellation that so successfully seem to have given us the superstring scenario. What we advertise is a careful though slow process establishing the correct demands for a full black hole/string theory. If successful, one will know exactly the rules of the game and the ways how to select good from false scenarios and models.

2. QUANTUM HAIR

Classical black holes are characterized by exactly three parameters[3]:

the *mass* M , the *angular momentum* L , and the *electric charge* Q . If magnetic monopoles exist in nature then there will be a fourth parameter, namely magnetic charge Q_m , and if besides electromagnetism there are other long range $U(1)$ gauge fields then also their charges correspond to parameters for the black hole.

However, the existence of long range $U(1)$ gauge fields other than electromagnetism seems to be rather unlikely. Then, since L , Q (and Q_m) are all quantized, the number of different values they can take is limited, and indeed one can argue convincingly (more about this in Ref[6]) that the black hole can be in much more different quantum states than the ones labeled by L and Q (and Q_m), or in other words, the mass M must be a function of much more variables than these quantum numbers alone.

An interesting attempt to formulate new quantum numbers for black holes was initiated by Preskill, Krauss, Wilczek and others[7]. They took as a model field theory a $U(1)$ gauge theory in which the local symmetry undergoes a Higgs mechanism *via* a Higgs field with charge Ne . In addition one postulates the presence of particles with charge e . In such a theory there exist vortices, much like the Abrikosov vortex in a super conductor. These vortices can be constructed as classical solutions with cylindrical symmetry, at which the Higgs field makes one full rotation if one follows it around the vortex.

The behavior near the vortex of particles whose charge is only e is more complicated. One finds that because of the magnetic flux in the Abrikosov vortex the fields of these particles undergo a phase rotation when they flow around the vortex, in such a way that an Aharonov-Bohm effect is seen. The Aharonov-Bohm phase is $2\pi/N$, or, if we take a particle with charge ne , this phase will be $2\pi n/N$.

The importance of this Aharonov-Bohm phase is that it will be detectable for any charged particle, at any distance from the vortex, in such a way that we will detect its charge *modulo* N . This is surprising because *there is no long range gauge field present*!

An observer who can only detect large scale phenomena may not be able to uncover the chemical composition of the particle, but he can determine its charge modulo N . All he needs is a vortex, which to him will look just like a Nambu-Goto string.

Even if a particle were absorbed by a black hole, its electric charge would still reveal itself. Thus, charge modulo N is a quantum number that will survive even for black holes. It must be a strictly conserved charge.

One can then formalize the argument using only strings and charges modulo N , without ever referring to the original gauge field. Then there may exist many kinds of strings/vortices, so that the black hole may have a rich spectrum of these pseudo-invisible but absolutely conserved charges.

Will this argument allow us to specify all quantum numbers for a black hole? There are several reasons to doubt this. One is that an extremely large number of different kinds of strings must be postulated, which seems to be a substantial departure from the Standard Model at large distance scales.

Secondly, it is not at all obvious that it will be possible to do Aharonov-Bohm experiments with black holes. One then has to assume *first* that black holes indeed occur in well-defined quantum states, just like atoms and molecules. So this argument that black holes have quantum hair is rather circular.

In my lectures there is no need for the mechanism advertised by Preskill et al. It is neither necessary nor likely that all quantum states can be distinguished by means of some conserved quantum number(s). In my other lectures I use just the assumption that quantum states exist, and nothing else. No large-scale strings are needed.

3. DECAY INTO SMALL BLACK HOLES

Due to Hawking radiation the black hole looses energy, hence also mass. The intensity of the radiation will be proportional to T^4 , where T is the temperature, and the total area of the horizon, which for the Schwarzschild black hole is $4\pi R^2$; $R = 2M$. Since one expects[#]

$$T = 1/8\pi M \quad , \tag{3.1}$$

the mass loss should obey

$$\frac{dM}{dt} = - C\, T^4\, R^2 = - C'/\, M^2 \quad . \tag{3.2}$$

The constants C , C' depend on the number of independent particle types at the corresponding mass scale, and this will vary slightly with temperature; the coefficients will however stay of order one (as long as M stays considerably larger than the Planck mass).

Ignoring this slight mass dependence of C' , one finds

$$M(t) = C''\, (t_o - t)^{\frac{1}{3}} \quad , \tag{3.3}$$

where t_o is a moment where the thing explodes violently. Conversely, the lifetime of any given Schwarzschild black hole with mass M can be estimated to be

$$t_1 = M^3/3C' \quad . \tag{3.4}$$

Now this is the time needed for the complete disappearance of the black hole. One may also ask for the average lifetime of a black hole in a given quantum mechanical state, i.e. the average time between two Hawking emissions.

A rough estimate reveals that the wavelength of the average Hawking particle is of the order of the black hole radius R , and that this is also the expected average spatial distance between two Hawking particles. Therefore the lifetime of a given quantum state is of order R , i.e. of order $1/M$ in Planck units.

In the language of particle physics this implies that the radiating black hole is a resonance state that in an S matrix would produce a pole at the complex energy value

$$E = M - C_3\, i/M \quad , \tag{3.4}$$

where C_3 is again a constant of order one. This corresponds to the value

$$E^2 = M^2 - 2C_3\, iM_{Pl}^{\ 2} \tag{3.5}$$

for the Mandelstam variable s . We see that all black hole poles are expected to be below the real axis of s at a universal average distance of order one in units of the Planck mass squared.

It is not altogether unreasonable to assume that a black hole is just a pole in the S matrix like any other tiny physical object.

[#]As was pointed out by this author[8], the derivation of this formula requires an assumption concerning the interpretation of quantum wave functions for particles disappearing into the black hole. Though plausible, one can imagine this assumption to be wrong, in which case the black hole temperature will be different from (3.1).

4. THE S-MATRIX ANSATZ AND THE SHIFTING HORIZON

The problem with linearised quantum field theory in the black hole background is that the ingoing particles then seem to be independent of the outgoing ones. Hilbert space is then a *product* space, $|\psi\rangle = |\psi\rangle_{in} \times |\psi\rangle_{out}$. If we were to describe a black hole that obeys an overall Schrödinger equation then these in- and out-spaces cannot be allowed to be independent of each other. In contrast, one would expect the existence of an S-matrix:

$$|\psi\rangle_{out} = S\ |\psi\rangle_{in} \ , \tag{4.1}$$

and with this mapping of in- to out-states the degrees of freedom pictured in Fig. 1*a* are replaced by the ones of Fig. 1*b* or Fig. 1*c*.

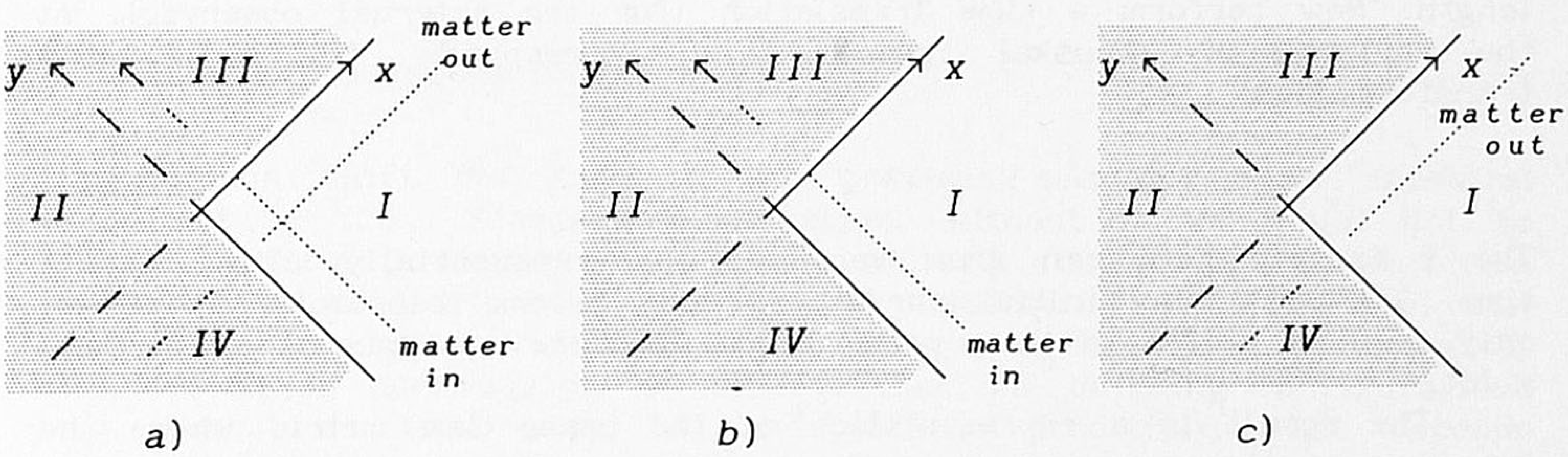

Fig. 1
a) Linearised quantum field theory produces a Hilbert space that is the product of two factors: $|\psi\rangle = |\psi\rangle_{in} \times |\psi\rangle_{out}$ Ingoing particles form the factor space $\{|\psi\rangle_{in}\}$ (*b*), and outgoing ones form $\{|\psi\rangle_{out}\}$ (*c*). x and y are the Kruskal coordinates for the Schwarzschild metric.

As stated earlier, the reason why superimposing in- and out-particles as in Fig. 1*a* is incorrect is the breakdown of linearised quantum field theory at distances closer than a Planck length from the horizon. Gravitational interactions there become super strong. We can obtain the black hole representations of Fig. 1*b* and Fig. 1*c* by adopting the following elementary procedure:

i) *Postulate* the existence of an S matrix, and

ii) take interactions between in- and out-states into account, in particular the gravitational ones.

We can look upon this procedure as a new and more precise formulation of the general coordinate transformation from Kruskal coordinates[3] to Schwarzschild coordinates, or from flat space-time to Rindler[9] space-time. There is no direct contradiction with anything we know about general relativity or quantum mechanics, but because of the crucial role attributed to the interactions the picture is only somewhat more complicated.

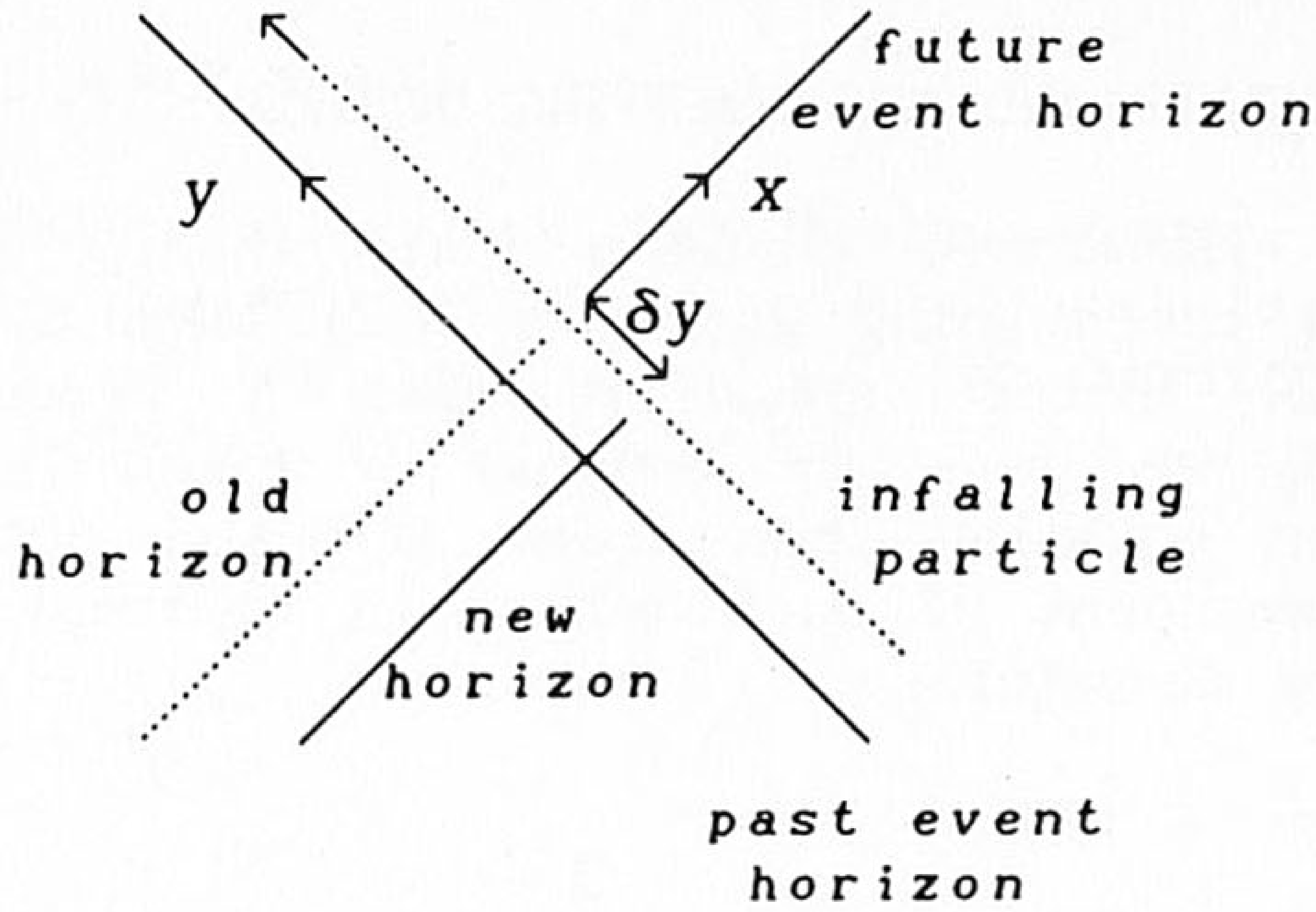

Fig. 2. The horizon displacement.

The most important ingredient of the gravitational interactions is the *horizon shift*[10]. Consider any particle falling into the black hole. Its gravitational field is assumed to be so weak that a linearised description of it, during the infall, is reasonable. The curvatures induced in the Kruskal frame are initially much *smaller* than the Planck length. Now perform a time translation (for the external observer). At the origin of Kruskal space this corresponds to a Lorentz transformation:

$$x \to \gamma^{-1} x \ ; \qquad y \to \gamma\, y \ . \tag{4.2}$$

The γ factors here can grow very quickly, exponentially with external time. The very tiny initial curvatures soon become substantial, but are only seen as shifts in the y coordinate, because y has expanded such a lot.

The result is a representation of the space-time metric where the incoming particle enters along the y axis, with a velocity that has been boosted to become very close to the speed of light. Its energy in terms of the boosted coordinates has become so huge that the curvature became sizable. It is described completely by saying that two halves of the conventional Kruskal space are glued together along the y axis with a *shift* δy, depending explicitly on the angular coordinates ϑ and φ.

The calculation of the function $\delta y(\vartheta,\varphi)$ is elementary. In Rindler space, $\delta y(\tilde{x})$ is simply[11]

$$\delta y(\tilde{x}) \;=\; 4G_N p \log(\tilde{x}^2) + C \ , \tag{4.3}$$

where G_N is Newton's constant, p is the ingoing particle momentum, and C is an arbitrary constant.

In Kruskal space the angular dependence of δy is a bit more complicated than the $\tilde{x}$ dependence in Rindler space. It is found by inserting Einstein's equation, which is $R_{\mu\nu} = 0$, everywhere except where the particle comes in. Starting with an arbitrary δy as an Ansatz, one finds Einstein's equation to correspond to

$$(1 - \Delta_{\vartheta,\varphi})\ \delta y(\vartheta,\varphi) \;=\; 0 \ , \tag{4.4}$$

where $\Delta_{\vartheta,\varphi}$ is the angular Laplacian. At the angles ϑ_o,φ_o where the particle enters we simply compare with the Rindler result (4.3) to obtain

$$(1 - \Delta_{\vartheta,\varphi})\ \delta y(\vartheta,\varphi) \;=\; \kappa\ p_{in}\ \delta^2(\vartheta,\varphi;\vartheta_o,\varphi_o) \ , \tag{4.5}$$

where κ is a numerical constant related to Newton's constant.

This equation can be solved:

$$\delta y(\vartheta,\varphi) \;=\; f(\vartheta,\varphi;\vartheta_o,\varphi_o)\; p_{in}(\vartheta_o,\varphi_o) \;;$$

$$f \;=\; \kappa_1 \int_0^\infty \frac{\cos\left(\frac{\sqrt{3}}{2}\, s\right)\, ds}{\left(\cosh s \;-\; \cos\vartheta_1\right)^{\frac{1}{2}}} \;, \tag{4.6}$$

where ϑ_1 is the angular separation between (ϑ,φ) and (ϑ_0,φ_o).

Other expressions for f are

$$f \;=\; \kappa_2 \int_{\vartheta_1}^{2\pi-\vartheta_1} dz\, \left(\cos\vartheta_1 - \cos z\right)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\sqrt{3}\, z} \;; \tag{4.7}$$

and[12]

$$f \;=\; \frac{\pi\kappa_1}{\sqrt{2}}\; \frac{P_{-\frac{1}{2}\,+\,\frac{1}{2}i\sqrt{3}}\left(-\cos\vartheta_1\right)}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)} \;, \tag{4.8}$$

where P is a Legendre function with complex index (conical function). From (4.7) one sees directly that for all angles ϑ_1 f is positive.

5. SPACE-TIME SURROUNDING THE BLACK HOLE

The horizon shift discussed in the previous section is an essential ingredient in the S-matrix construction. Without it we would not be able to perform this task. Now we are. We will discuss this construction in the next chapter. First one has to understand what the relevant degrees of freedom are and where in space-time they live. Here, partly anticipating on our results, we observe that the outgoing configurations will depend on what goes in, and with a sensitivity that depends exponentially with $\delta t/4M$, where δt is the time interval as seen by the distant observer. However, the particles coming out later than a

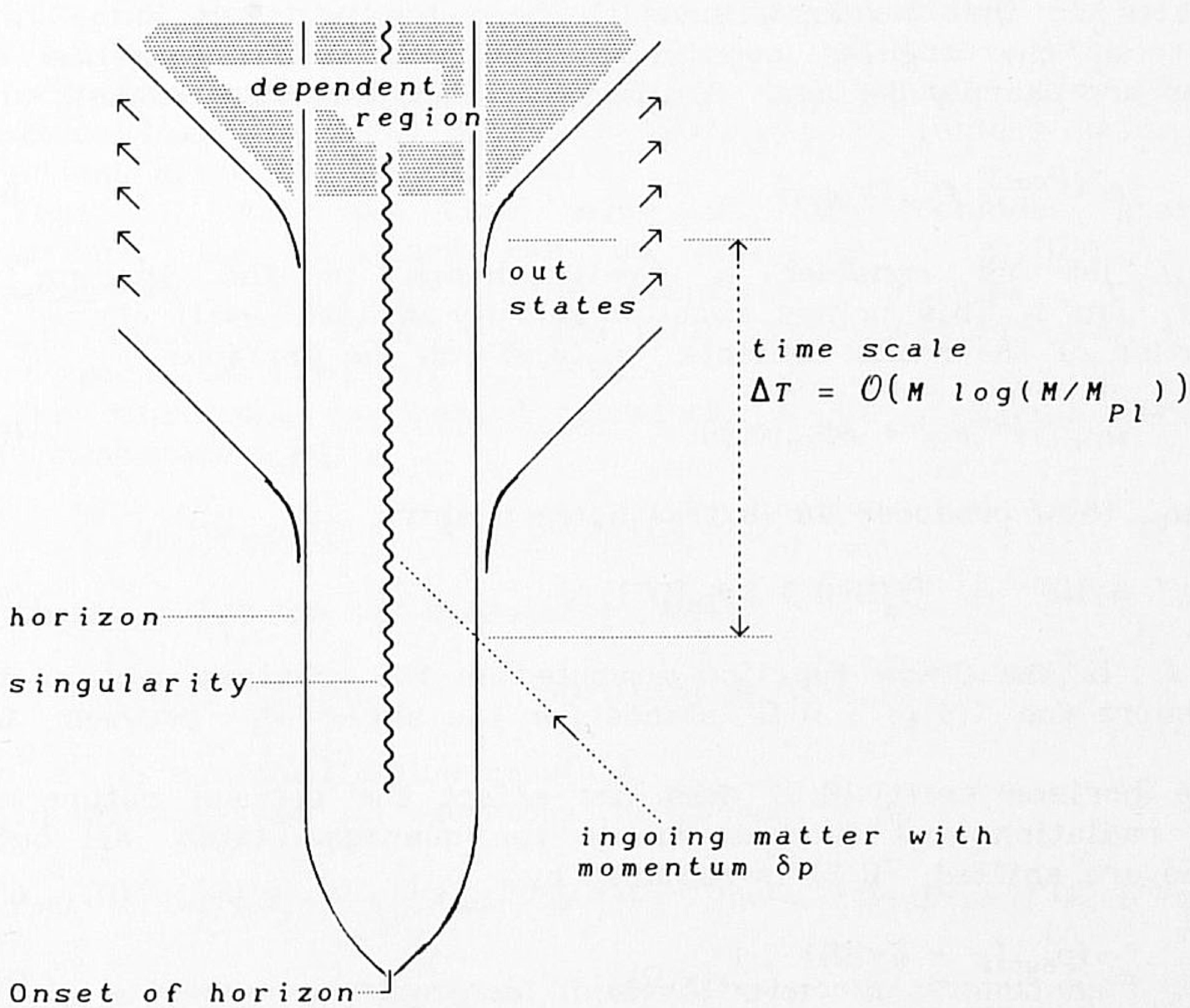

Fig. 3

lapse of time of the order of $\Delta t = 4M \log(M/M_{Pl})$, can no longer be described at all. We will assume, for simplicity, that these particles will be completely determined (in the quantum mechanical sense) by earlier events, or in other words, one is not allowed to choose these states any way one pleases. Simply counting states (as one can derive from the finite black hole entropy[14]) one notes that, given all incoming particles, the outgoing particles can be chosen freely only during an amount of time Δt , or *vise versa*.

Any incoming particle will not only affect the outgoing particles after a time lapse of order Δt , but also all that come out later than that. These limitations in choosing in- and outgoing states are just what they would be for any macroscopic system such as a finite size box containing a gas or liquid, connected to the outside world for instance via a tiny hole.

6. CONSTRUCTION OF THE S-MATRIX

The argument now goes as follows[6].

1. Consider one particular in-state and one particular out-state. Assume that someone gave us the amplitude defined by sandwiching the S-matrix between these two states:

$$\langle in | out \rangle \quad . \tag{6.1}$$

Both the in- and the out-state are described by giving all particles in some conveniently chosen wave packets. The ingoing wave packets look like

$$e^{-ip^i_{in}x} \; f^i_{in}(x,\vartheta,\varphi) \quad , \tag{6.2}$$

where i runs over all particles involved, and f^i_{in} are smooth functions. We assume them to be sharply peaked in the angular coordinates so that we know exactly where the particles enter into the horizon (so the *angular coordinates* and the *radial momenta* of all particles are sharply defined). Similarly the outgoing wave packets are

$$e^{-ip^i_{out}y} \; f^i_{out}(y,\vartheta,\varphi) \quad . \tag{6.3}$$

Now let us consider a small change in the ingoing state: $|in\rangle \rightarrow |in'\rangle$. This brings about a sharply defined small change in the distribution of the radial momenta $p_{in}(\vartheta,\varphi)$ on the horizon:

$$p_{in} \rightarrow p_{in} + \delta p_{in}(\vartheta,\varphi) \quad . \tag{6.4}$$

This δp_{in} now produces an (extra) horizon shift,

$$\delta y(\Omega) = \int f(\Omega-\Omega')\, \delta p_{in}(\Omega') \quad , \tag{6.5}$$

where f is the Green function computed in the previous section and Ω stands short for (ϑ,φ) ; $\Omega-\Omega'$ stands for the angle ϑ_1 between Ω and Ω' .

The horizon shift (6.5) does *not* affect the thermal nature of the Hawking radiation, but it *does* change the quantum states. All out-wave functions are shifted. (6.3) is replaced by

$$e^{-ip^i_{out}(y + \delta y(\Omega))} \; f^i_{out}(y,\Omega) \tag{6.6}$$

(the effect of the shift on f is of lesser importance). The shift δy

is assumed to be so small that the outgoing particle is not thrown over the horizon. This requires that we consider only those outgoing particles that are already sufficiently far separated from the horizon, or: they are not the ones that emerge later than the time interval Δt , as defined in the previous section, see Fig. 3. This is why the time interval Δt was necessary. Now such a restriction will also imply that we will have to reconsider the definition of inner products in our Hilbert space, and this will imply that the operator $\exp(-ip^i_{out}\delta y)$ might not be unitary. We will temporarily ignore this important observation.

We observe that the S-matrix element (6.1) is replaced:

$$\begin{aligned}\langle in'|out\rangle &= e^{-i\int p_{out}(\Omega)\ \delta y(\Omega)\mathrm{d}^2\Omega}\ \langle in|out\rangle \\ &= e^{-i\int\int p_{out}(\Omega)\ f(\Omega-\Omega')\ \delta p_{in}(\Omega')\ \mathrm{d}^2\Omega\mathrm{d}^2\Omega'}\ \langle in|out\rangle\ .\end{aligned} \tag{6.7}$$

Here, $p_{out}(\Omega)$ is the total outgoing momentum at the angular coordinates Ω . What we have achieved is that we have been able to compute *another* matrix element of S . Now simply repeat this procedure many times. We then find *all* matrix elements of S to be equal to

$$\langle in|out\rangle = \mathcal{N}\ e^{-i\int\int p_{out}(\Omega)\ f(\Omega-\Omega')\ p_{in}(\Omega')\mathrm{d}^2\Omega\mathrm{d}^2\Omega'}\ , \tag{6.8}$$

where $\mathcal{N}$ is one common unknown factor. Apart from an overall phase, $\mathcal{N}$ should follow from unitarity.

The derivation of (6.8) ignores all interactions other than the gravitational ones. We will be able to do better than that, but let us first analyze this expression.

What is unconventional in the S-matrix (6.8) is the fact that the in- and out-states must have been characterized *exclusively* by specifying the *total* radial momentum distribution over the angular coordinates on the horizon. If there are more parameters necessary to characterize these states, these extra parameters will not figure in the S-matrix. But this would mean that two different states $|A\rangle$ and $|B\rangle$ could evolve into the same state $|\psi_{out}\rangle$, so these extra parameters will not be consistent with unitarity. *We cannot allow for other parameters than the total momentum distributions* (unless more kinds of interactions are taken into account).

Thus, if for the time being we only consider gravitational interactions, the in-states can be given as $|p_{in}(\Omega)\rangle$ and the out-states as $|p_{out}(\Omega)\rangle$. The operators $p_{in}(\vartheta,\varphi)$ commute for different values of ϑ and φ , and their representations span the entire Hilbert space; the same for $p_{out}(\vartheta,\varphi)$.

The canonically conjugated operators $u_{in}(\Omega)$, $u_{out}(\Omega)$ are defined by the commutation rules

$$[p_{in}(\Omega),u_{in}(\Omega')] = -i\delta^2(\Omega-\Omega') \tag{6.9}$$

(and similarly for the *out* operators), or,

$$\langle u_{in}(\Omega)|p_{in}(\Omega)\rangle = C\ \exp\ i\int\mathrm{d}^2\Omega\ p_{in}(\Omega)u_{in}(\Omega)\ , \tag{6.10}$$

where C is a normalization constant.

In terms of the u operators the S-matrix is

$$\langle u_{out}(\Omega)|u_{in}(\Omega)\rangle = \int\mathcal{D}p_{out}\mathcal{D}p_{in}\ \exp\left(-ip_{in}u_{in} + ip_{out}u_{out} - ip_{out}fp_{in}\right)\ , \tag{6.11}$$

which is a Gaussian functional integral over the functions p_{out} and p_{in} . Since the inverse f^{-1} of f is $\kappa^{-1}(1-\Delta_\Omega)$, the outcome of this functional integral is

$$\langle u_{out}(\Omega)|u_{in}(\Omega)\rangle = C\exp\left[-i\kappa^{-1}\int d^2\Omega\; u_{in}(\Omega)\,(1-\Delta_\Omega)\,u_{out}(\Omega)\right] =$$
$$C\exp\left[-i\kappa^{-1}\int d^2\Omega\;(\partial_\Omega u_{in}(\Omega)\,\partial_\Omega u_{out}(\Omega) + u_{in}(\Omega)\,u_{out}(\Omega))\right] \quad . \tag{6.12}$$

The last term in the brackets is something like a mass term and becomes subdominant if we concentrate on small subsections of the horizon. Therefore it will be ignored from now on. Eq. (6.12) seems to be more fundamental than (6.8) because it is local in Ω .

Fourier transforming back we get

$$\langle p_{out}(\Omega)|p_{in}(\Omega)\rangle =$$
$$\mathcal{N}\int\mathcal{D}u_{in}\mathcal{D}u_{out}\;\exp\int d^2\Omega\left[ip_{in}u_{in} - ip_{out}u_{out} - i\kappa^{-1}\partial_\Omega u_{in}(\Omega)\,\partial_\Omega u_{out}(\Omega)\right] \quad . \tag{6.13}$$

We reobtain (6.8) written as a functional integral.

It is illuminating to redefine

$$p_{out} = p_- \quad ; \quad p_{in} = p_+ \quad ; \quad u_{out} = x^- \quad ; \quad u_{in} = -x^+ \quad , \tag{6.14}$$

and to replace the angular coordinate Ω by a transverse coordinate $\tilde{x}$, so that if we define the transverse momentum components $\tilde{p} \cong 0$ one can write

$$\langle p_{out}(\tilde{x})|p_{in}(\tilde{x})\rangle = \mathcal{N}\int\mathcal{D}x^\mu(\tilde{x})\;\exp\int d^2\tilde{x}\left[ip_\mu(\tilde{x})x^\mu(\tilde{x}) - i\kappa^{-1}(\partial_\Omega x^\mu(\tilde{x}))^2\right] \quad . \tag{6.15}$$

Suppose now that both the in-state and the out-state are written as sets containing a finite number of particles, having not only fixed longitudinal momenta but also transverse momenta $\tilde{p}^i$. We then have to convolute the amplitude (6.15) with transverse wave functions $\exp(i\tilde{p}^i\tilde{x}^i)$ for each particle i . It becomes

$$\mathcal{N}\prod_i\int d\tilde{x}^i\int\mathcal{D}x^\mu(\tilde{x})\;\exp\left[i\sum_i p_\mu{}^i x^\mu(\tilde{x}^i) + \int d^2\tilde{x}\left(-i\kappa^{-1}(\partial_\Omega x^\mu(\tilde{x}))^2\right)\right] \quad . \tag{6.16}$$

This functional integral is very similar to the functional integral for a string amplitude, including the integration over Koba-Nielsen variables[13], except for the unusual imaginary value for the string constant:

$$T = 8\pi G_N i \quad . \tag{6.17}$$

7. ELECTROMAGNETISM

What happens if more interactions are included? The simplest to handle turn out to be the electromagnetic forces. Suppose that the particles that collapsed to form the black hole carried electric charges. The angular charge distribution was

$$\rho_{in}(\Omega) \quad . \tag{7.1}$$

As in the previous section, we consider a small change in this setting, so

$$\rho_{in}(\Omega) \rightarrow \rho_{in}(\Omega) + \delta\rho_{in}(\Omega) \quad . \tag{7.2}$$

The $\delta\rho_{in}(\Omega)$ produces an extra contribution to the vector potential at the horizon which is not difficult to compute[♩].

$$\delta A_\mu = \frac{1}{r_o^2} \delta_{\mu x} \delta(x) A(\Omega) , \tag{7.3}$$

where r_o is the radius of the horizon, and $A(\Omega)$ must satisfy

$$\Delta_\Omega A(\Omega) = \delta\rho_{in}(\Omega) . \tag{7.4}$$

The field (7.3) is only non-vanishing on the plane $x=0$, where it causes a sudden phase rotation for all wave packets that go through. An outgoing wave undergoes a phase rotation

$$e^{iQ\Lambda(\Omega)} \quad ; \quad \Lambda(\Omega) = r_o^{-1} A(\Omega) . \tag{7.5}$$

This rotation must be performed for all outgoing particles with charge Q. All together the outgoing wave is rotated as follows:

$$|p_{out}(\Omega),\rho_{out}(\Omega)\rangle \rightarrow$$
$$\exp -i \int d^2\Omega \int d^2\Omega' \; f_1(\Omega-\Omega') \; \rho_{out}(\Omega)\delta\rho_{in}(\Omega') \times |p_{out}(\Omega),\rho_{out}(\Omega)\rangle , \tag{7.6}$$

where $f_1(\Omega-\Omega')$ is a Green function that satisfies

$$\Delta_\Omega f_1(\Omega-\Omega') = -\kappa_e \delta^2(\Omega-\Omega') . \tag{7.7}$$

κ_e is a numerical constant.

And using arguments identical to the ones of the previous section we repeat the infinitesimal changes to obtain the S-matrix dependence on $\rho_{in}(\Omega)$ and $\rho_{out}(\Omega)$:

$$\langle p_{out}(\Omega),\rho_{out}(\Omega)|p_{in}(\Omega),\rho_{in}(\Omega)\rangle =$$
$$\mathcal{N} \, e^{-i\int\int p_{out}(\Omega) \; f(\Omega-\Omega') \; p_{in}(\Omega')d^2\Omega d^2\Omega'} \times \tag{7.8}$$
$$e^{-i\int\int \rho_{out}(\Omega) \; f_1(\Omega-\Omega') \; \rho_{in}(\Omega')d^2\Omega d^2\Omega'} .$$

Now let us replace $\rho_{out}(\Omega)\rho_{in}(\Omega')$ by

$$-\tfrac{1}{2}\left(\rho_{out}(\Omega)-\rho_{in}(\Omega)\right)\left(\rho_{out}(\Omega')-\rho_{in}(\Omega')\right) . \tag{7.9}$$

This differs from the previous expression by two extra terms in the exponent, one depending on $\rho_{out}(\Omega)$ only and the other depending on $\rho_{in}(\Omega)$ only. These would correspond to external "wave function renormalization factors" that do not describe interaction between the in- and the out-state. So we ignore them.

The electromagnetic contribution in (7.8) can then be written as a functional integral of the form

$$\int \mathcal{D}\Phi(\Omega) \; \exp \int d^2\Omega \left(\frac{-i}{2\kappa_e}(\partial_\Omega\Phi)^2 + i\Phi(\rho_{out}-\rho_{in}) \right) . \tag{7.10}$$

Now it may also be observed that the charge distribution ρ is actually a combination of Dirac delta distributions,

---

[♩]The unit e of electric charge of the ingoing particle is here included in ρ_{in}.

$$\rho(\Omega) \;=\; \sum Q_i \delta^2(\Omega-\Omega_i) \quad ; \quad Q_i = n_i e \quad . \tag{7.11}$$

Therefore, if we add an integer multiple of $2\pi/e$ to the field Φ the integrand does not change. In other words: Φ is a periodic variable.

Adding (7.10) to (6.15) we notice that the field Φ acts exactly as a fifth, periodic dimension. Hence, electromagnetism emerges naturally as a Kaluza-Klein theory.

8. HILBERT SPACE

In this section we briefly recapitulate the nature of the Hilbert space in which these S matrix elements are defined. As explained in Section 4, the states whose momentum and charge distribution over the horizon were given by $p(\Omega)$ and $\rho(\Omega)$ *include* all particles in the black hole's vicinity. But (if for simplicity we ignore electromagnetism) we can also form a complete basis in terms of states for which the canonical operators $x(\Omega)$ are given. These $x(\Omega)$ (eq. 6.12) may be interpreted as the coordinates of the horizon. Apparently, *the precise shape of the horizon determines the state of the surrounding particles!*

Furthermore, the *in*-horizon and the *out*-horizon do not commute. Therefore, the positions of the future event horizon and the past event horizon do not commute with each other. If we define a "black hole" as an object for which the location in space-time of the future event horizon is precisely determined, we can define a "white hole" as a state for which the past event horizon is precisely determined. *The white hole is a linear superposition of black holes* (and vice versa); operators for white holes do not commute with the ones for black holes. In our opinion this resolves the issue of white holes in general relativity.

Obviously, it is important that the horizon of the quantized black hole is not taken to be simply spherically symmetric. In a black hole with a history that is not spherically symmetric, the onset of the horizon, i.e. the point(s) in space-time (at the bottom of Fig. 3) where for the first time a region of space-time emerges from which no timelike geodesic can escape to $\mathcal{I}^+$, has a complicated geometrical structure. Its mathematical construction has the characteristics of a caustic. One might conjecture that the topological details of this caustic specify the quantum state a black hole may be in.

The fact that the geometry of the (future or past) horizon should determine the quantum state of the surrounding particles gives rise to interesting questions and problems. In ordinary quantum field theory the Hilbert space describing particles in a region of space-time is Fock space; an arbitrary, finite, number of particles with specified positions or momenta together define a state. But now, close to the horizon, a state must be defined by specifying the *total* momentum entering (or leaving) the horizon at a given solid angle Ω . Apparently we are not allowed to specify further how many particles there were, and what their other quantum numbers were. Together all these possibilities form just one state. So, our Hilbert space is set up differently from Fock space. The difference comes about of course because we have strong gravitational interactions that we are not allowed to ignore.

The best way to formulate the specifications of our basis elements here is to assume a lattice cut-off in the space of solid angles (one "lattice point" for each unit of horizon surface area somewhat bigger than $\delta\Sigma$ (the Planck distance squared), and then to specify that there should be *exactly one ingoing and one outgoing particle* at each $\delta\Sigma$. The momenta are given by the operators $p_{in}(\Omega)$ and $p_{out}(\Omega)$ (and the charges by $\rho_{in}(\Omega)$ and $\rho_{out}(\Omega)$). The in- and out-operators of course do not commute.

One may speculate that since $\delta\Sigma$ is extremely small, the totality of all these particles may be indistinguishable from an ordinary Dirac sea for the large-scale observers.

Also one may notice that the way conventional string theory deals with in- and outgoing particles is remarkably similar. Before integrating over the Koba-Nielsen variables the string amplitudes also depend exclusively on the distribution of total in- and outgoing momenta (see concluding remarks in Sect. 6).

If Hilbert space is constructed entirely from the operators $p(\tilde{x})$ and $x(\tilde{x})$ then these operators are hermitean by construction. But we have also seen that in terms of ordinary Fock space $p(\tilde{x})$, and hence also $x(\tilde{x})$ are probably not hermitean. There are different ways to approach this hermiticity problem, but we shall not elaborate here on this point.

9. RELATION BETWEEN TERMS IN THE HORIZON FUNCTIONAL INTEGRAL AND BASIC INTERACTIONS IN 4 DIMENSIONS

In principle one can pursue our doctrine to obtain more precise expressions for our black hole S matrix by including more and more interactions that we actually know to exist from ordinary particle theory. We should be certain to obtain a result that is accurate apart from a limitation in the angular resolution, because particle interactions are known only up to a certain energy. In this section we indicate some qualitative results.

The details of our "presently favored Standard Model" may well change in due time. We will denote anything used as an input regarding the fundamental interactions among in- and outgoing particles near the horizon, at whatever scale, by the words "standard model".

Suppose the standard model contains a scalar field. The effects of this field will be felt by slowly moving particles at some distance from the horizon. But at the horizon itself these effects are negligible. Consider namely a particle such as a nucleon, surrounded by a scalar field such as a pion field. Close to the horizon this particle will be Lorentz boosted to tremendous energies. The scalar field configuration will become more and more flattened. But unlike vector or tensor fields, its intensity will not be enhanced (it is Lorentz invariant). So the cumulated effect on particles traversing it will tend to zero.

However, one effect due to the scalar field will not go away. Suppose our standard model contains a Higgs field, rendering a $U(1)$ gauge boson massive. This means that the electromagnetic field surrounding a fast electrically charged particle will be of short range only. One can derive that the field equation (7.4) will change into

$$(\Delta_\Omega - M_A{}^2)A(\Omega) \;=\; \delta\rho_{in}(\Omega) \quad . \tag{9.1}$$

One may say that the incoming charge density $\rho_{in}(\Omega)$ is screened by charges coming from the Higgs particles.

This implies that the equations for the Φ field in Sect. 7 will obtain a mass term:

$$\int \mathcal{D}\Phi(\Omega)\ \exp \int d^2\Omega\Big(\frac{-i}{2\kappa_e}[(\partial_\Omega\Phi)^2 + M_A{}^2\Phi^2] \;+\; i\Phi\ \rho \Big) \quad . \tag{9.2}$$

Note that this mass term breaks explicitly the symmetry $\Phi \to \Phi + \Lambda$. This explicit symmetry breaking may be seen as a result of the finite and constant value of the Higgs field at the origin of Kruskal space-time.

Next, we may ask what happens if our standard model exhibits confinement. This means that at long distance scales no effect of the gauge field is seen and all allowed particles are neutral.

Confinement is usually considered to be the *dually opposite* of the Higgs mechanism: Bose condensation of magnetic monopoles. A magnetic monopole is an object to which the end point of a Dirac string is attached. A Dirac string is a singularity in a gauge transformation such that the gauge transformation makes one full rotation if we follow a loop around the string.

We must know how to describe the operator field of a monopole at the horizon. Suppose a monopole entered at the solid angle Ω_1. This means that a Dirac string connects to the black hole at that point. The outgoing charged particles undergo a gauge rotation that rotates a full cycle if we follow a closed curve around Ω_1 (an anti-monopole may neutralize this elsewhere on the horizon).

The gauge jump for the vector potential field A can be identified with the periodic field Φ of Sect 7. So adding an entering monopole to the in-state implies that this field Φ is shifted by an amount $\Lambda(\Omega)$ where Λ makes a full cycle when followed over a loop around Ω_1. This is an operation that is called *disorder operator* in statistical physics and field theory. This operator, Φ_D, is dual to the original field Φ. We find that the dual transformation electricity $\leftrightarrow$ magnetism corresponds to the duality between Φ and Φ_D.

Thus, if we have confinement, a mass term will result in the equations for Φ_D. It explicitly breaks the symmetry $\Phi_D \rightarrow \Phi_D + C$. And this bars the transformation back to Φ. Therefore, *if confinement occurs, the field* Φ *is no longer well-defined, we have only* Φ_D. Its mass will be the glueball mass.

In Table 1 we list peculiarities of the mapping from 4 to 2 dimensions. The *generators* of local symmetry transformations in 4 dimensions correspond to the dynamic variables in 2 dimensions. Thus one expects that if the standard model includes a gravitino (requiring a supersymmetry generator of spin $\frac{1}{2}$) then a fermionic field variable will emerge in 2 dimensions.

But the above are merely qualitative features. They should be turned into precise quantitative rules and principles, for which further work is needed.

10. OPERATOR ALGEBRA ON THE HORIZON

A fundamental shortcoming of the procedure described above is that the dimensionality of Hilbert space is infinite from the start. The functions $p(\tilde{x})$ and $x(\tilde{x})$ generate an infinite set of basis elements. Yet the black hole entropy, as calculated from Hawking radiation, is finite. Indeed, we have not yet been able to reproduce Hawking radiation from our S-matrix. This is because we have ignored the *transverse* components of the gravitational shifts, and the string functionals we produced thus far only allow for infinitesimal string excitations. We shall now try to improve our description of the basis elements of Hilbert space. This we do by setting up an operator algebra. First we consider the algebra generated by the amplitudes we have.

Our starting point here is that states in Hilbert space are uniquely determined by specifying any one of the following four functions: the distribution of ingoing momenta $p_+(\tilde{x})$, the outgoing momenta $p_-(\tilde{x})$, the conjugated operators $x^+(\tilde{x})$, or $x^-(\tilde{x})$. They obey the algebra

$$[p_+(\tilde{x}),\ p_+(\tilde{x}')] = 0 \quad ; \quad [p_+(\tilde{x}),\ x^+(\tilde{x}')] = -i\delta^2(\tilde{x},\tilde{x}') \quad ; \tag{10.1}$$

$$[p_-(\tilde{x}),\ p_-(\tilde{x}')] = 0 \quad ; \quad [p_-(\tilde{x}),\ x^-(\tilde{x}')] = -i\delta^2(\tilde{x},\tilde{x}') \quad , \tag{10.2}$$

and we have the relation

$$x^-(\tilde{x}) \;=\; 4\pi G \int d^2\tilde{x}' \; f(\tilde{x},\tilde{x}')\; p_+(\tilde{x}') \quad . \tag{10.3}$$

This implies

$$[x^-(\tilde{x}),\; x^+(\tilde{x}')] \;=\; -4\pi i G\; f(\tilde{x},\tilde{x}') \quad , \tag{10.4}$$

so that we have also

$$x^+(\tilde{x}) \;=\; -4\pi G \int d^2\tilde{x}' \; f(\tilde{x},\tilde{x}')\; p_-(\tilde{x}') \quad . \tag{10.5}$$

Table 1

STANDARD MODEL IN 3+1 DIMENSIONS	INDUCED 2 DIMENSIONAL FIELD THEORY ON BLACK HOLE HORIZON
• *Spin 2:* $g_{\mu\nu}(\mathbf{x},t)$ local gauge generator: $u^\mu(\mathbf{x},t)$	String variables (*spin 1*): $x^\mu(\Omega)$
• *Spin 1:* $A_\mu(\mathbf{x},t)$ local gauge generator: $\Lambda(\mathbf{x},t)$ *mod* $2\pi/e$	Scalar variable (*spin 0*): $\Phi(\Omega)$ *mod* $2\pi/e$
• *Spin 0:* $\phi(\mathbf{x},t)$	No field at all
• Higgs mechanism: "spontaneous" mass M_A for vector field	explicit symmetry breaking; $\Phi(\Omega)$ gets same mass M_A .
• Confinement in vector field A_μ	Φ must be replaced by disorder op. Φ_D ; its symmetry broken.
• Non-Abelian gauge theory	only scalars Φ_i corresponding to Cartan subalgebra
• *Spin* $\frac{1}{2}$: fermions	no field at all
• *Spin* $\frac{3}{2}$: gravitino local gauge generator spin $\frac{1}{2}$	Spin $\frac{1}{2}$ fermion (?)

The algebraic relations among $\rho(\tilde{x})$ and $\phi(\tilde{x}')$ are slightly more subtle because of the quantization of electric charge and the ensuing periodic boundary conditions on ϕ . We will disregard these from here on.

The relations (10.1-5) are not infinitely accurate. This is because we neglected any gravitational curvature in the sideways directions. This is fine as long as transverse distance scales are kept considerably larger than the Planck scale. One may convince oneself that this implies neglecting higher orders in the derivatives $\partial x^{\pm}/\partial\tilde{x}$. Is there any way to obtain a more precise algebra? It is natural to search for an algebra that is invariant under Lorentz transformations. One might hope that such an algebra could generate the correct degrees of freedom at the Planck scale (in particular *quantized* degrees of freedom).

It was proposed in Ref[6] that Hilbert space on the horizon may be generated by the operator algebra of fundamental surface elements,

$$W^{\mu\nu}(\tilde{\sigma}) \quad = \quad \varepsilon^{ab}\,\frac{\partial x^\mu}{\partial\sigma^a}\,\frac{\partial x^\nu}{\partial\sigma^b} \quad . \tag{10.6}$$

where the transverse coordinates $\tilde{x}$ were replaced by more arbitrary surface coordinates σ^1, σ^2 . The relations (10.1-5) may be used in the case $\tilde{\sigma} = \tilde{x}$, when the derivatives are small. This means

$$W^{12} = 1 \quad ; \quad W^{1\mu} = \frac{\partial x^\mu}{\partial\sigma^2} \quad ; \quad W^{2\mu} = -\frac{\partial x^\mu}{\partial\sigma^1} \quad ; \quad W^{34} = \mathcal{O}(\partial x^\mu)^2 \ . \tag{10.7}$$

The commutation rules can then be rewritten in the form

$$\sum_\lambda\,[W^{\lambda\mu}(\tilde{\sigma}),W^{\lambda\nu}(\tilde{\sigma}')] \quad = \quad \tfrac{1}{2}T\ \varepsilon^{\mu\nu\kappa\lambda}W^{\kappa\lambda}(\tilde{\sigma})\delta^2(\tilde{\sigma}-\tilde{\sigma}') \quad , \tag{10.8}$$

which is written in such a way that it remains true in all coordinate frames. T is a constant ('string constant') equal to 8π in Planck units. In stead of (10.1-5) we can take this to be the equation that generalizes to arbitrary surfaces. It has the advantage of being linear in W .

Now (10.8) is not a closed algebra, because the left hand side still contains a summation. A complete algebra is obtained as follows.

Let K be i times the self dual part of W :

$$K^{\mu\nu} \quad = \quad i(W^{\mu\nu} + \tfrac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}W^{\kappa\lambda}) \quad . \tag{10.9}$$

It has three independent components:

$$K_1 = i(W^{23} + W^{14}) \quad ; \quad K_2 = i(W^{31} + W^{24}) \quad ; \quad K_3 = i(W^{12} + W^{34}) \quad . \tag{10.10}$$

Now from (10.8) we derive that these obey a complete commutator algebra,

$$[K_a(\tilde{\sigma}),K_b(\tilde{\sigma}')] \quad = \quad iT\varepsilon_{abc}K_c(\tilde{\sigma})\delta^2(\tilde{\sigma}-\tilde{\sigma}') \quad . \tag{10.11}$$

Apart from a complication to be mentioned shortly, this is a local and complete algebra of the kind we were looking for. At first sight it seems to generate an infinite dimensional Hilbert space because the operators K , like the W , are distributions . But let us introduce test functions $f(\sigma)$, $g(\sigma)$ and define operators

$$L_a^{(f)} \quad = \quad T^{-1}\!\int K_a(\tilde{\sigma})f(\tilde{\sigma})\mathrm{d}^2\tilde{\sigma} \quad , \tag{10.12}$$

then these satisfy commutation rules:

$$[L_a^{(f)},L_b^{(g)}] \quad = \quad i\varepsilon_{abc}L_c^{(fg)} \quad . \tag{10.13}$$

Let us now restrict to test functions $f(\tilde{\sigma})$ that can only take the values 0 or 1 . Then $L_a^{(f)}$ satisfy the commutation rules of ordinary angular momentum operators. Note that for such an f the integral (10.12) is nothing but a boundary integral:

$$L_1^{(f)} \quad = \quad iT^{-1}\oint_{\delta f}(x^2\mathrm{d}x^3 + x^1\mathrm{d}x^4) \quad , \quad \text{etc.,} \tag{10.14}$$

where δf stands for the boundary of the support of f . We conclude that for every closed curve δf on $\tilde{\sigma}$ space we have three 'angular momentum' operators $L_a^{(f)}$ that satisfy the usual commutation rules and addition rules for angular momenta. Given such a bunch of closed curves f_i we can characterize the contribution of that part of the horizon to Hilbert space by the usual quantum numbers l_i and m_i. These are discrete and so, in some sense, we seem to come close to our aim of realizing a discrete Hilbert space for black holes. We note an important resemblance with the loop variable approach to quantum gravity[15].

Unfortunately, there is a snag. The operators L_a are not hermitean. If we take x^i to be hermitean and x^4 anti-hermitean then in the definition (10.6) W^{ij} are hermitean and W^{i4} anti-hermitean. Therefore, $L_a^\dagger$ correspond to the *anti*-self dual parts of $W^{\mu\nu}$. The commutation rules between L_a and $L_a^\dagger$ are non-local (they follow from (10.1-5)). The operators L^2 are hermitean, but not necessarily positive (they are only nonnegative for *time-like* surface elements). If we may assume the smallest surface elements to be timelike we can still build our surface using quantum numbers l_i and m_i but the states we get are *not properly normalized* (it is for finding the norms of the states that we need hermitean conjugation). If

$$\psi\{l_i, m_i\}$$

are the basis elements constructed using the self dual operators L_i, and

$$\phi\{l_i, m_i\}$$

the basis elements generated by the anti-self dual $L_i^\dagger$, then we have

$$\langle\phi\{l_i', m_i'\}|\psi\{l_i, m_i\}\rangle = \prod_i \delta_{l_i, l_i'}\delta_{m_i, m_i'} , \qquad (10.15)$$

but the ψ themselves, or the ϕ themselves, are not orthonormal.

Now remember our realization earlier that actually the operators $x^+(\tilde{x})$ and $x^-(\tilde{x})$ are *not* hermitean, when we pass from the "horizon Hilbert space" to ordinary Fock space, because the shift operators may move particles behind the horizon. It is conceivable that this will lead to hermiticity conditions altogether different from (10.15).

But it is far from clear whether or not we actually obtained a complete representation of our Hilbert space.

REFERENCES

1. S.W. Hawking, Commun. Math. Phys. **43** (1975) 199; J.B. Hartle and S.W. Hawking, Phys.Rev. **D13** (1976) 2188; W.G. Unruh, Phys. Rev. **D14** (1976) 870; R.M. Wald, Commun. Math. Phys. **45** (1975) 9
2. S.W. Hawking, Phys. Rev. **D14** (1976) 2460; Commun. Math. Phys. **87** (1982) 395; S.W. Hawking and R. Laflamme, Phys. Lett. **B209** (1988) 39; D.N. Page, Phys. Rev. Lett. **44** (1980) 301, Gen. Rel. Grav. **14** (1987) 299; D.J. Gross, Nucl. Phys. **B236** (1984) 349
3. C.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation", Freeman, San Francisco, 1973; S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-time", Cambridge: Cambridge Univ. Press, 1973; E.T. Newman et al, J. Math. Phys. **6** (1965) 918; B. Carter, Phys. Rev. **174** (1968) 1559; K.S. Thorne, "Black Holes: the Membrane Paradigm", Yale Univ. press, New Haven, 1986; S. Chandrasekhar, "The Mathematical Theory of Black Holes", Clarendon Press, Oxford University Press
4. S. Coleman, Nucl. Phys. **B310** (1988) 643; S.B. Giddings and A. Stroninger, Nucl. Phys. **B321** (1989) 481; *ibid.* **B306** (1988) 890
5. P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn, Nucl. Phys. **B56** (1973) 109; M.B. Green, J.H. Schwarz and E. Witten, "Superstring

Theory", Cambridge Univ. Press; D.J. Gross, et al, Nucl. Phys. **B 256** (1985) 253
6. G. 't Hooft, Phys. Scripta **T15** (1987) 143; *ibid.* **T36** (1991) 247; Nucl. Phys. **B335** (1990) 138; G. 't Hooft, "Black Hole Quantization and a Connection to String Theory" 1989 Lectures, Banff NATO ASI, Part 1, "Physics, Geometry and Topology, Series B: Physics Vol. 238. Ed. H.C. Lee, Plenum Press, New York (1990) 105-128; G. 't Hooft, "Quantum gravity and black holes", in: Proceedings of a NATO Advanced Study Institute on Nonperturbative Quantum Field Theory, Cargèse, July 1987, Eds. G. 't Hooft et al, Plenum Press, New York. 201-226
7. L. Kraus and F. Wilczek, Phys. Rev. Lett. **62** (1989) 1221; J. Preskill, L.M. Krauss, Nucl. Phys. **B341** (1990) 50; L.M. Krauss, Gen. Rel. Grav. **22** (1990); S. Coleman, J.Preskill and F. Wilczek, preprint IASSNS-91/17 CALT-68-1717/ HUTP-91-A016
8. G. 't Hooft, J. Geom. and Phys. **1** (1984) 45
9. W. Rindler, Am.J. Phys. **34** (1966) 1174
10. T. Dray and G. 't Hooft, Nucl Phys. **B253** (1985) 173
11. W.B. Bonner, Commun. Math. Phys. **13** (1969) 163; P.C. Aichelburg and R.U. Sexl, Gen.Rel. and Gravitation **2** (1971) 303
12. C. Lousto, private communication
13 Z. Koba and H.B. Nielsen, Nucl. Phys. **B10** (1969) 633 , *ibid.* **B12** (1969) 517; **B17** (1970) 206; Z. Phys. **229** (1969) 243
14. J.D. Bekenstein, Phys. Rev. **D7** (1973) 2333; R.M. Wald, Phys. Rev. **D20** (1979) 1271; G. 't Hooft, Nucl. Phys. **B256** (1985) 727; V.F. Mukhanov, "The Entropy of Black Holes", in "Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity, vol IX, Ed. W. Zurek, Addison-Wesley, 1990. *See also* M. Schiffer, "Black Hole Spectroscopy", São Paolo preprint IFT/P - 38/89 (1989)
15. A. Ashtekar, *Phys. Rev.* **D36** (1987) 1587; A. Ashtekar *et al*, *Class. Quantum Grav.* **6** (1989) **L**185; C. Rovelli, Class. Quant. Grav. **8** (1991) 297, *ibid.* **8** (1991) 317