

QUANTUM GRAVITY AS A DISSIPATIVE DETERMINISTIC SYSTEM

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Abstract

It is argued that the so-called holographic principle will obstruct attempts to produce physically realistic models for the unification of general relativity with quantum mechanics, unless determinism in the latter is restored. The notion of time in GR is so different from the usual one in elementary particle physics that we believe that certain versions of hidden variable theories can – and must – be revived. A completely natural procedure is proposed, in which the dissipation of information plays an essential role. Unlike earlier attempts, it allows us to use strictly continuous and differentiable classical field theories as a starting point (although discrete variables, leading to fermionic degrees of freedom, are also welcome), and we show how an effective Hilbert space of quantum states naturally emerges when one attempts to describe the solutions statistically. Our theory removes some of the mysteries of the holographic principle; apparently non-local features are to be expected when the quantum degrees of freedom of the world are projected onto a lower-dimensional black hole horizon. Various examples and models illustrate the points we wish to make, notably a model showing that massless, non interacting neutrinos are deterministic.

1. Introduction.

At present, many elementary particle physicists appear to agree that superstring theory¹ and its descendants such as “M-theory”² are the only candidates for a completely unified theory that incorporates the gravitational force into elementary particle physics. This consensus is based on the very rich mathematical structure of these theories that shows some resemblance to the observed mathematical structure of the Standard Model as well as that of General Relativity.

It also appears to be a satisfactory feature³ of these theories, that they manage to reproduce the so-called ‘holographic principle’⁴. This principle states that any complete theory combining quantum mechanics with gravity should exhibit an upper limit to the total number of independent quantum states that is quite different from what might be expected in a quantum field theory: it should increase exponentially with the surface area of a system, rather than its volume.

Yet this only adds to the suspicion that these theories are far removed from a description of what one might call ‘reality’. One would have expected that the quantum degrees of freedom can be localised, as in a quantum field theory, but this cannot really be the case if theories with different dimensionalities are being mapped one onto the other. How can notions such as causality, unitarity, and local Lorentz invariance make sense if there is no trace of ‘locality’ left? In this paper, a theory is developed that will *not* postulate the quantum states as being its central starting point, but rather classical, deterministic degrees of freedom. Quantum states, being mere mathematical devices enabling physicists to make statistical predictions, will turn out to be derived concepts, with a not strictly locally formulated definition.

At the time this is written, the quantum mechanical doctrine, according to which all physical states form a Hilbert space and are controlled by non-commuting operators, is fully taken for granted in string theory. No return to a more deterministic description of “reality” is being foreseen; to the contrary, string theorists often give air to their suspicion that the real world is even crazier than quantum mechanics. Consequently, the description of what really constitutes concepts such as space, time, matter, causality, and the like, is becoming increasingly and uncomfortably obscure. By many, this is regarded as an inescapable course of events, with which we shall have to learn to live.

But there are also other difficulties associated to such starting points, for instance when space-time curvature is being used to close an entire universe. We get “quantum cosmology”. An extremely important example of a quantum cosmological model, is a model of gravitating particles in 1 time, 2 space dimensions⁵. Here, a complete formalism for the quantum version at first sight seems to be straightforward⁶, but when it comes to specifying exact details, one discovers that we cannot rigorously define what quantum mechanical amplitudes are, what it means when it is claimed that “the universe will collapse with such-and-such probability”, what and where the observers are, what they are made of, and so on. Yet such questions are of extreme importance if one wants to check a theory for its self-consistency, by studying unitarity, causality, etc.

Since the entire hamiltonian of the universe is exactly conserved, the “wave function

of the universe” is in an exact eigenstate of the hamiltonian, and therefore, the usual Schrödinger equation is less appropriate than the description of the evolution in the so-called Heisenberg representation. Quantum states are space-time independent, but operators may depend on space-time points – although only if the location of these space-time points can be defined in a coordinate-free manner!*

We have learned to live with the curious phenomenon that our wave functions can be eigenstates of operators which at different space-time points usually do not commute. A “physical state” can be an eigenstate of an arbitrary set of mutually commuting operators, but then other operators are not diagonalized, and so, these observables tend to be smeared, becoming “uncertain”. The idea that such uncertainties may be due to nothing other than our limited understanding of what really is going on, has become unpopular, for very good reasons. Attempts at lifting these uncertainties by constructing theories with ‘hidden variables’, have failed. It is the author’s suspicion, however, that these hidden variable theories failed because they were based far too much upon notions from everyday life and ‘ordinary’ physics, and in particular because general relativistic effects have not been taken into account properly. The interpretation adhered to by most investigators at present is still not quite correct, and a correct interpretation is crucial for making further progress at very technical levels in quantum gravity.

Earlier attempts by this author to obtain further insights led to the idea that space, time, and matter all had to be discrete⁷. If this were the case, it would seem to be easy to set up a deterministic model of the universe, and a mathematically rigorous procedure to handle probabilities by introducing an *auxiliary* Hilbert space, spanned by all possible states, whose evolution is accurately described by an evolution operator, leading to Schrödinger’s equation in the continuum limit. Indeed, some models constructed along these lines look very much like genuine quantum field theories.

They showed, however, one very important shortcoming. This is the fact the the hamiltonian, emerging naturally from the basic equations, invariably fails to have a lower bound, and so it appeared to be impossible to construct the vacuum state. One possible exception is a model of (second quantized) non-interacting massless fermions. They can be viewed *exactly* as a continuum limit of a discrete, deterministic theory, see Appendix A. Here it is shown that massless non-interacting neutrinos are deterministic. Unfortunately, however, we have been unable to generalize this system into something more interesting.

Since quantum mechanics is described by a *unitary* evolution operator, it was natural to expect that, in a cellular automaton model, only time-reversible evolution laws would be acceptable. However, a little bit of thought suffices to realise that this is not the case. If an evolution law is not time reversible, it just means that some states will be absolutely forbidden (their amplitudes will vanish), and others will evolve into states that, after a while, become indistinguishable from states with a different past. If a pair of states evolve in such a way that their futures are identical, then these states should be called physically identical from the very start. To be precise, we must introduce *equivalence classes* of

* Note that, besides energy, also total momentum and angular momentum of the universe must be conserved (and they too must be zero).

states, defined by collecting all states which some time in the future, after a given lapse of time, will become identical to one another. The evolution from one equivalence class to a different equivalence class is then again unitary, by construction.

An early attempt to construct a deterministic model with built-in information loss, appeared to bring improvement: in a certain approximation, the hamiltonian did appear to develop a lower bound⁸. Nevertheless, there were shortcomings, as in more precise calculations the lower bound disappears again, and anyway, the model was unattractive. On the other hand, once it is realized that, *at the classical level*, information loss is permitted, we can return to strictly continuous underlying deterministic equations. All that is needed is that the equivalence classes are discrete. At later stages of the theory, one might reconsider the option to regard the continuous theories as the continuum limit of some discrete system.

The advantages of returning to continuum theories are numerous. One is, that it becomes much easier to account for the many observed continuous symmetries such as rotational and Lorentz invariance. Even more important is the fact that a strictly continuous time coordinate implies that the hamiltonian is unbounded, so that realistic models may be easier to achieve. But making information dissipate is not easy in continuum theories. It may well be that discrete degrees of freedom must be added. This would be no real problem. Discrete degrees of freedom often manifest themselves as fermions in the quantum formalism. It is also conceivable that the continuum theories at the basis of our considerations will have to include string- and D -brane degrees of freedom, and it would be beautiful if we could make more than casual contact with the mathematics of string- and M -theories.

In Section 2, we expand on the definition of physical states as being equivalence classes of deterministic states, first illustrated for the discrete case, but it has a sensible continuum limit, so that a continuous time parameter can be employed. It is shown how dissipation of information leads to a reduction in the number of quantum levels, but in terms of these reduced states, unitarity is restored.

In Section 3, we treat one of the continuum versions of a model with information loss, and show how they lead to discrete quantization even if the original degrees of freedom form a continuous multi- (or infinite-) dimensional space.

In Section 4, we show how to couple different degrees of freedom gravitationally. Gravity theory naturally exhibits information loss when black holes are considered, and thus we argue that incorporating the gravitational force will actually help us to understand quantum mechanics.

An example of a non-quantum theory that could be considered for use as an input is a liquid obeying the Navier Stokes equation, and developing turbulent behaviour. Viscosity induces information loss. The ultraviolet structure of Navier Stokes fluids however does not quite meet the requirements of our theories. These matters are discussed in Section 5.

The reader will criticize our arguments on the basis of the well-known Einstein-Rosen-Podolsky paradox⁹. We elucidate our viewpoints on this matter in Section 6. Here also we discuss an other fundamental quantum feature of our world that may appear

to be irreconcilable with a non-quantum or pre-quantum interpretation: the ‘quantum computer’. Indeed we formulate a conjecture concerning the practical limits of a quantum computer.

Will the Copenhagen interpretation survive the 21st century? This is discussed in Section 7. Here, we also define the notions of *beables* and *changeables*.

Dropping the requirement that information is preserved at the deterministic level, settles the problem how to treat quantum mechanical black holes. We explain how to handle them in our theory, and what now to think of the ‘holographic principle’, in Sect. 8.

Conclusions are formulated in Sect. 9. In Appendix A, we discuss the massless neutrino model, and explain why massless neutrinos may be called quantum-deterministic objects.

2. Quantum States

Consider a discrete system that can be in any one of the states e_1 , e_2 , e_3 or e_4 . We shall call these states *primordial* states. Let there be an evolution law such that after every time step,

$$e_1 \rightarrow e_2, \quad e_2 \rightarrow e_1, \quad e_3 \rightarrow e_3, \quad e_4 \rightarrow e_1. \quad (2.1)$$

This evolution is entirely deterministic, but it will still be useful to introduce the Hilbert space spanned by all four states, in order to handle the evolution statistically. Now, in this space, the one-time-step evolution operator would be

$$U = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.2)$$

and this would not be a unitary operator. Of course, the reason why the operator is not unitary is that the evolution rule (2.1) is not time reversible. After a short lapse of time, only states e_1 , e_2 and e_3 can be reached. In this simple example, it is clear that one should simply erase state e_4 , and treat the upper 3×3 part of Eq. (2.2) as the unitary evolution matrix. Thus, the quantum system corresponding to the evolution law (2.1) is three-dimensional, not four-dimensional.

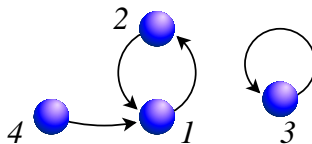


Fig. 1. The transitions of Eq. (2.1).

In more complicated non-time-reversible evolving systems, however, the ‘genuine’ quantum states and the false ones (the ones that cannot be reached from the far past) are actually quite difficult to distinguish, so it is more fruitful to talk of *equivalence classes*.

Two states are called equivalent if, after some finite time interval, they evolve into the same state. The system described above has three equivalence classes,

$$E_1 = \{e_1\} , \quad E_2 = \{e_2, e_4\} , \quad E_3 = \{e_3\} . \quad (2.3)$$

and the evolution operator in terms of the states E_1, E_2, E_3 is

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (2.4)$$

One may consider constructing a hamiltonian operator H such that $U = e^{-iH}$. Our model universe (2.1) would be in an eigenstate of this hamiltonian. Since the phases of the states $|e_i\rangle$ are arbitrary anyway, our universe can be assumed to be either in the state $|E_3\rangle$, or in

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle) . \quad (2.5)$$

Global time is not a directly observable quantity (time translations can be regarded as being gauge transformations), which is why only $|\psi^+\rangle$ is a physical state, together with $|E_3\rangle$. So, the physical Hilbert space is only two dimensional: $\{|\psi^+\rangle, |E_3\rangle\}$.

In a system with discrete time coordinates, the hamiltonian has a periodic energy spectrum, and it is impossible to identify any of the energy states as the true ‘ground state’, or vacuum. This was found to be a major obstacle impeding the construction of physically more interesting models. On the other hand, discretization of the states is imperative for any statistical analysis. The above model now shows that, if information is allowed to dissipate, we have to treat the equivalence classes of states as the basis of a quantum Hilbert space, and we observe that these equivalence classes can form a smaller set than the complete set of primordial states that one starts off with.

In the Heisenberg picture, the dimensionality of a limit cycle does not change if we replace the time variable by one with smaller time steps, or even a continuous time. Working with a continuous time variable has the advantage that the associated operator, the hamiltonian, is unambiguous in that case. The hamiltonian will play a very important role in what follows.

3. A continuum model with information loss.

In this section, it will be shown that even in theories containing many continuous degrees of freedom, the equivalence classes will tend to form discrete, ‘quantum’ sets, much like the situation in the real world, only if one allows information to dissipate. The simplest model goes as follows.

If there is a single limit cycle, we have one periodic degree of freedom $q(t) \in [0, 1)$, evolving according to

$$\dot{q}(t) = v , \quad (3.1)$$

so that the period is $T = 1/v$. In the Schrödinger picture, the dimensionality of Hilbert space is infinite, but if this model represents an entire universe, then only the state $E = 0$ is physically acceptable. Also the fact that time is not a gauge-invariant notion implies that only the single state $\langle q|\psi\rangle = 1$ is physical. Therefore, in the Heisenberg picture, the dimensionality of Hilbert space is just one.

Now imagine two such degrees of freedom:

$$q_1, q_2 \in [0, 1); \quad \dot{q}_1(t) = v_1, \quad \dot{q}_2(t) = v_2. \quad (3.2)$$

First, take v_1 and v_2 to be constants. The Schrödinger Hilbert space is spanned by the states $|q_1, q_2\rangle$, and our formal hamiltonian is

$$H = v_1 p_1 + v_2 p_2; \quad p_j = -i\partial/\partial q_j. \quad (3.3)$$

In this case, even the zero-energy states span an infinite Hilbert space, so, in the Heisenberg picture, there is an infinity of possible states.

Information loss is now introduced by adding a tiny perturbation that turns the flow equations into a non-Jacobian one:

$$v_1 \rightarrow v_1^0 + \varepsilon f(q_1, q_2); \quad v_2 \rightarrow v_2^0 + \varepsilon g(q_1, q_2). \quad (3.5)$$

The effect of these extra terms can vary a lot, but in the generic case, one expects the following (assuming ε to be a sufficiently tiny number):

Let the ratio v_1^0/v_2^0 be sufficiently close to a rational number N_1/N_2 . Then, at specially chosen initial conditions there may be periodic orbits, with period

$$P = v_1^0/N_1 = v_2^0/N_2, \quad (3.6)$$

where now v_1^0 and v_2^0 have been tuned to exactly match the rational ratio – possible deviations are absorbed into the perturbation terms. Nearby these stable orbits, there are non-periodic orbits, which in general will converge into any one of the stable ones, see Fig. 2. After a sufficiently large lapse of time, we will always be in one of the stable orbits, and all information concerning the extent to which the initial state did depart from the stable orbit, is washed out. Of course, this only happens if the Jacobian of the evolution, the quantity $\sum_i (\partial/\partial q_i) \dot{q}_i$, departs from unity. Information loss of this sort normally does not occur in ordinary particle physics, although of course it is commonplace in macroscopic physics, such as the flow of liquids with viscosity (see Sect. 5).

The stable orbits now represent our equivalence classes (note that, under time reversal, there are new stable orbits in between the previous ones). Most importantly, we find that the equivalence classes will form a discrete set, in a model of this sort, most often just a finite set, so that, in the Heisenberg picture, our ‘universe’ will be just in a finite number of distinct quantum states.

Generalizing this model to the case of more than two periodic degrees of freedom is straightforward. We see that, if the flow equations are allowed to be sufficiently generic

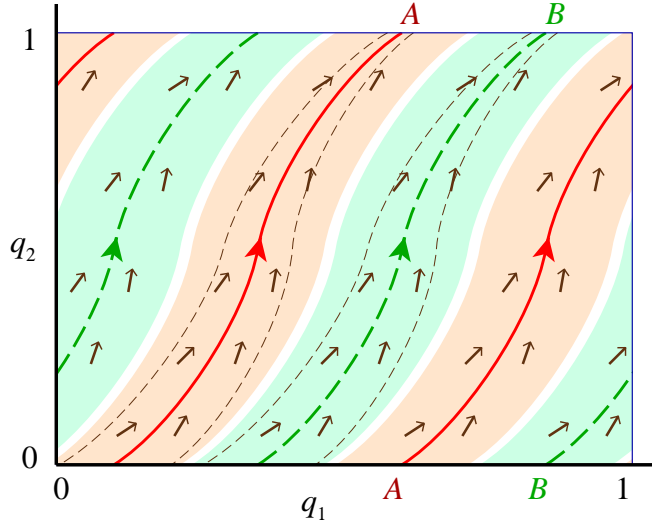


Fig. 2. Flow chart of a continuum model with two periodic variables, q_1 and q_2 . In this example, there are two stable limit cycles, A and B , representing the two ‘quantum states’ of this ‘universe’. In between, there are two orbits that would be stable in the time-reversed model.

(no constraints anywhere on the values of the Jacobians), then distinct stable limit orbits will arise. There is only one parameter that remains continuous, which is the global time coordinate. If we insert $H|\psi\rangle = 0$ for the entire universe, then the global time coordinate is no longer physically meaningful, as it obtains the status of an unobservable gauge degree of freedom.

Observe that, in the above models, what we call ‘quantum states’, coincides with Poincaré limit cycles of the universe. Just because our model universes are so small, we were able to identify these. When we glue tiny universes together to obtain larger and hence more interesting models, we get much longer Poincaré cycles, but also much more of them. Eventually, in practice, sooner or later, one has to abandon the hope of describing complete Poincaré cycles, and replace them by the more practical definitions of equivalence classes. At that point, when one combines mutually weakly interacting universes, the effective quantum states are just multiplied into product Hilbert spaces.

4. Gravity.

Models describing only a small number of distinct quantum states, such as all of the above, do not very clearly show the most salient difficulty encountered when one attempts to construct realistic models. This is the fact that our universe is known to be *thermodynamic stable*. A system in thermodynamic equilibrium is governed by Boltzmann factors $e^{-\beta E_i}$, where β is the inverse temperature. Stability is guaranteed only if the hamiltonian has a ground state. In the models above, only the zero eigenvalue of the hamiltonian plays a role, so we have to be more careful in our use of the notion of a hamiltonian. A thermodynamic treatment applies only to a hamiltonian describing some small subsystem of the universe. Apparently, one first must address the notion of *locality* before being able

to formulate the exact meaning of thermodynamic stability. Our definition of locality will be that the hamiltonian of the universe can be written as

$$H = \int d^3\mathbf{x} \mathcal{H}(\mathbf{x}), \quad (4.1)$$

where $\mathcal{H}(\mathbf{x})$ is a hamiltonian density, obeying

$$[\mathcal{H}(\mathbf{x}), \mathcal{H}(\mathbf{x}')] = 0 \quad \text{if} \quad |\mathbf{x} - \mathbf{x}'| > \varepsilon, \quad (4.2)$$

for some $\varepsilon > 0$. *Positivity* then means that $\mathcal{H}(\mathbf{x})$ is bounded from below:

$$\mathcal{H}(\mathbf{x}) > -\delta, \quad (4.3)$$

for some number δ . The value of δ may diverge as ε is sent to zero, but we do not really need to send ε all the way to zero; probably a small finite value will be good enough for us.

At first sight, it seems to be easy to realise (4.1)–(4.3) in a deterministic cellular automaton model⁷. In such a model, $\mathcal{H}(\mathbf{x})$ depends on only a finite number of states, and there is some freedom in defining \mathcal{H} , since the time variable is discrete. If the evolution law of the automaton is local, one naturally expects that the hamiltonian will be local as well, but unfortunately, the hamiltonian generated by a cellular automaton is not so simple. The point is that the hamiltonian is the logarithm of the evolution operator, and it can only be written as an infinite perturbation series in terms of the interactions. In calculating the outcome, one discovers divergences that contradict (4.1)–(4.3), and this is the only reason why our automaton models failed to serve as realistic models for a quantum field theory.⁷

We propose to circumvent these problems, by first returning to continuous time variables. As we see from the model of the previous section, it, in principle, relies on strictly continuous time, so there is a unique hamiltonian. Finiteness of the number of quantum states is then guaranteed by the mechanism explained, which is that information loss reduces the dimensionality of Hilbert space to be finite. The model one starts off with, may have continuous degrees of freedom, such as a classical field theory, but it must have information loss. We now propose that one takes a continuous, classical field theory with general coordinate invariance. Such models differ in various essential ways from the more naive cellular automaton models studied previously.

First, one observes that the time variable has now become a *local* gauge degree of freedom. The velocity of time evolution in various regions of the universe may differ, just as in the model of Sect. 3, and this difference is controlled by the gravitational field. The ratio of the speed of evolution at \mathbf{x} and \mathbf{x}' is $\sqrt{g_{00}(\mathbf{x})/g_{00}(\mathbf{x}')}$, and since $\sqrt{g_{00}}$ plays the role of a gravitational potential, this relative speed depends on the gravitational flux from \mathbf{x} to \mathbf{x}' . Indeed, the coupling introduced in Sect. 3 may be regarded as a ‘gravitational’ coupling.

Secondly, also the notion of locality is made more complicated as also the coordinates \mathbf{x} have become gauge degrees of freedom. This makes our study of the positivity constraint of the hamiltonian density much more difficult than before.

A third complication is the exact definition of what an hamiltonian actually is. We should distinguish the matter part of the hamiltonian from the gravitational part. It is the matter part which we want to be positive. The gravitational part, controlling the value of the gravitational fields, must be regarded separately, since in total they add up to zero: the hamiltonian density generates local time translations, which however are pure gauge transformations, under which the wave function does not change, hence

$$\mathcal{H}_{\text{matter}}(\mathbf{x}) + \mathcal{H}_{\text{grav}}(\mathbf{x}) = 0. \quad (4.6)$$

But “matter” should also include gravitons. The correct way to introduce the hamiltonian here is first to define Cauchy surfaces of equal time, and then to define the operator H that generates time evolution, that is, a mapping from one Cauchy surface to the next. This requires an external clock to be defined; we take as our clock the measurements made by observers far from the region studied. It is important, however, that this clock is part of the universe studied, and not external to it. It is the hamiltonian with this more subtle definition that has to be split into hamilton density functions $\mathcal{H}(\mathbf{x})$.

All of these aspects make gravity so much different from ordinary cellular automaton models that we have good hopes that the naive difficulties encountered with the cellular automata can now be resolved.

The most important distinction between gravitational and non-gravitational models is that, in gravitational models, information loss naturally occurs, since black holes may be formed. Indeed, it will be hard to avoid the development of coordinate singularities, but quite generally, one expects such singularities to be hidden behind horizons. So we have black holes. One may wonder whether black holes in our deterministic gravity models can emit any Hawking radiation¹⁰, since the latter is considered to be a typical quantum effect. The answer is that we do expect Hawking radiation, and the argument for this is that the usual derivation of this effect is still valid. One then may ask how it can be that a black hole can loose weight, since in classical theories black holes can only grow. The answer here is, presumably, as follows.

In gravity, the hamiltonian not only generates the evolution through the hamilton equations, but it is also the source of the gravitational field. In our deterministic model, the gravitational field already exists, whereas the logarithm of the evolution operator, at first sight, may have little to do with curvature. In writing the hamiltonian as an integral over hamiltonian densities, however, these two notions of energy get intertwined, and we end up with only one notion of energy. When a black hole looses energy, it is primarily because it absorbs negative amounts of “curvature energy”. Clearly then, our primordial model must allow for the presence of negative amounts of energy. Actually, this is obviously true for the quantum mechanical energy, because, after diagonalization, the total Hamiltonian has a zero eigenvalue. Prior to diagonalization of the total H , the hamilton density $\mathcal{H}(\mathbf{x})$ must have negative eigenstates. We now see that, since the black hole must loose weight, the primordial model must also have local fluctuations with negative “curvature energy”. Black holes absorb negative amounts of energy, allowing positive energy to escape to infinity.

It is due to the postulated thermodynamics stability that the fluctuations surviving

at spatial infinity may only have positive energy. Since the total energy balances out, the black hole will therefore receive only net amounts of negative energy falling in. Hence it loses weight and decays.

5. Viscosity.

To obtain some insight in continuum models with information loss, it is tempting to consider an example from macroscopic physics. Consider the Navier-Stokes equations for a fluid with viscosity¹¹. For simplicity, we take a pure, incompressible fluid with density ϱ equal to one, and viscosity η . As is well known, such fluids may develop turbulence, and turbulence occurs when the Reynolds number,

$$R = \varrho u \ell / \eta, \tag{5.1}$$

where u represents the velocities involved and ℓ the typical sizes, becomes larger than a certain critical value, R_{cr} . This is a dimensionless number, ranging between a few factors of 10 to something of the order of 10^3 .

Turbulence could be a nice example of the kind of chaotic behaviour to which one could apply our quantum mechanical philosophy. We see from the expression (5.1) for Reynold's number, that only if the viscosity η is sufficiently small compared to the dimensions of the system, instabilities arise that cause turbulence. Viscosity, for incompressible fluids, can be expressed in terms of cm^2/sec , so the distance scale at which turbulence can take place can be arbitrarily small, provided that the time scale decreases accordingly. This is why turbulence can cascade down to very tiny dimensions, until finally the molecular scale is reached, at which point the fluid equations no longer apply. Because of this divergence into the infinitesimally small scales, a viscous fluid cannot be treated with our Hilbert space methods.

In a relativistic classical field theory, the situation is likely to be very different. First of all, it is very difficult to introduce viscosity in a relativistically invariant way, since first derivatives in time must be linked to second derivatives in space. But, assuming that in sufficiently complicated systems, viscous yet Lorentz invariant terms can be introduced, one notices that there must be another distinction as well: if turbulence cascades down to smaller dimensions, it cannot be that the square of the distance scale divided by the time scale stays constant, because the limit of the ratio of the distance scale itself and the time scale is limited by the speed of light. Therefore, one may imagine that there is a lower limit to vortex size, and hence a natural smallest distance limit. It is necessary to have a smallest scale limit so as to have a workable cut-off leading to an effective quantization. Unfortunately, realistic relativistic classical field theories with viscosity were not (yet?) found, which is why perhaps information loss *via* black holes must be called upon.

6. The EPR paradox. A falsifiable prediction

The most serious objection usually raised against ideas of the kind discussed in this paper, is that deterministic theories underlying quantum mechanics appear to imply Bell's inequalities for stochastic phenomena¹², whereas it is well-known that many of these inequalities are violated in quantum mechanics. Clearly, we have to address these objections.

Bell's inequalities follow if one assumes deterministic equations of motion to be responsible for the behaviour of quantum mechanical particles at large scales. If one assumes that the x -component of an electron's spin exists, having some (unknown) value even while the z -component is measured, then the usual clashes our found. In our theory, however, the wave function has exactly the meaning and interpretation as in usual quantum mechanics; it describes the probability that something will or will not happen, given all other information of the system available to us. "Reality", as we perceive it, does not refer to the question whether an electron went through one slit or another. It is our belief that the true degrees of freedom are not describing electrons or any other particles at all, but microscopic variables at scales comparable to the Planck scale. Their fluctuations are chaotic, and no deterministic equation exists at all that describes the effects of these fluctuations at large scales. Thus, the behaviour of the things we call electrons and photons is essentially entirely unpredictable. It so happens, however, that some regularities occur within all these stochastic oscillations, and the *only* way to describe these regularities is by making use of Hilbert space techniques.

When we measure the spin of a photon, or the detection rate of particles by a counter, our measuring device is as much a chaotic object as the phenomena measured, and only at macroscopic scales can we detect statistical regularities that can in no other way be linked to microscopic behaviour than by assuming Schrödinger's equation. The idea that there might exist a deterministic law of physics underlying all of this essentially amounts to nothing more than the suggestion that there exists a 'primordial basis', a preferred basis of states in Hilbert space with the property that any operator that happens to be diagonal in this basis, will continue to be diagonal during the evolution of the system. *None* of the operators describing present-day atomic and subatomic physics will be completely diagonal in this basis. This enables us to accept *both* quantum mechanics with its usual interpretation *and* to assume that there is a deterministic physical theory lying underneath it.

Apparently, we are forced to deny the existence of electrons, and other microscopic objects, even if they appear to be obvious explanations of observed phenomena. Only macroscopic oscillations, such as the movements of planets and people, are undeniable realities (that is, approximately diagonal in the primordial basis), and it must be possible to recognise these 'realities' in terms of the microscopic, deterministic variables. This leads once again to a very serious objection, which is the following.

Quantum mechanics, as we know it, leads to many more phenomena that are at odds with classical deterministic descriptions. An example of this is the so-called quantum computer¹³. Using quantum mechanics, a device can be built that can handle information in a way no classical machine will ever be able to reproduce, such as the determination of

the prime factors of very large numbers in an amount of time not much more than what is needed to do multiplications and other basic arithmetic with these large numbers. If our theory is right, it should be possible to mimic such a device using a classical theory. This gives us a falsifiable prediction:

It will never be possible to construct a ‘quantum computer’ that can factor a large number faster, and within a smaller region of space, than a classical machine would do, if the latter could be built out of parts at least as large and as slow as the Planckian dimensions.

A somewhat stronger version of this prediction, based on the entropy formula for a black hole, would be:

“The classical machine may be thought of as being built of parts each of which occupy an area of at least one Planck length squared.”

If this would be true, it would not be the total volume but the total area that needs to be compared. We are less confident, however, of this latter version of our prediction, which is based on the holographic principle. The reason to doubt it is that the holographic principle follows from quantum mechanical arguments, hence refers to the number of equivalence classes, not the number of actual possible states, see Sect. 8. Therefore, a classical computer that is able to erase information, may have to use sites of Planckian dimension in a volume, not just on an area.

Quantum computers are known to suffer from problems such as ‘decoherence’. Often, it is claimed that decoherence is nothing but an annoying technical problem. In our view, it will be one of the essential obstacles that will forever stand in the way of constructing super powerful quantum computers, and unless mathematicians find deterministic algorithms that are much faster than the existing ones, factorization of numbers with millions of digits will not be possible ever.

7. Beables and changeables. Will the Copenhagen interpretation survive the 21st century?

In a way, our present approach does not really attack the Copenhagen interpretation. We attach to the wave function $|\psi\rangle$ exactly the same interpretation as the one taught at our universities. However, the Copenhagen interpretation also carries a certain amount of agnosticism: *We will never be able to determine what actually happened* during a physical experiment, and it is asserted that a deterministic theory is impossible. It is this agnosticism that we disagree with. There is a single ‘reality’, and physicists may be able to identify some of it. Of course, our physical universe is far too complex ever to be able to pinpoint in detail the actual sequence of events at tiny distance scales, but this situation is in no way different from our inability to follow individual atoms and molecules in a classical theory for gases and liquids. In a classical theory, we know that the atoms and molecules are there, we know their dynamics, but we are unable to trace individual entities, nor are we even interested in doing so; what we do want is to unravel the laws.

Thus, we add the following to the Copenhagen interpretation. In our theory, the

operators used for describing a system, will be divided in two types. If the representation of our Hilbert space is chosen to be such that the equivalence classes of the primordial states are chosen to form its basis elements, then we have *beables*, which, if expressed in this basis, multiply a state by a real or complex number referring to properties of the equivalence class our state is in, and *changeables*, which may replace a state by a different state, in a different equivalence class, possibly multiplying it also by a complex, state dependent, amplitude. Beables are operators which, in the Heisenberg representation, all commute with one another at all times. Changeables of course do not commute, in general. Operators that act non trivially on the different states within one equivalence class, are physically not meaningful, but could be used for mathematical purposes. We propose to refer to these as *ghost operators*. Conventional quantum mechanics results from the remarkable feature that, in describing systems of atomic sizes, we have become unable to distinguish the beables from the changeables. All operators known in the Standard Model of elementary particles are changeables. Beables may refer to features at the Planck scale, or to features at macroscopic scales, but in general they are not suitable for describing single particles at the atomic scale. Only if we manage to demonstrate that, under several restrictive conditions, diagonalizing a beable at the macroscopic scale corresponds to diagonalizing a changeable at the atomic scale, can we do a quantum experiment.

This, the author believes, does not contradict any of the usual findings concerning hidden variable theories and Bell's inequalities. These findings were based on the assertion that a theory describing the fluctuations at atomic scales should 'explain' these fluctuations in terms of laws *at the atomic scale* that go beyond ordinary quantum mechanics. In contrast, we now require such laws to exist *only* at the Planck scale. It will be the physicists' task in the next century to identify the beables that can be used at the Planck scale. They can clearly not include operators resmbing the ones we are used to at present, such as spin, positions or momenta. At the Planck scale, the introduction of the wave function will be nothing other than a mathematical trick enabling us to handle the equations statistically. Due to the powerful mathematics of linear algebras, this trick will allow us to perform renormalization group transformations towards the much lower energy scales and much larger distance scales of atomic physics. As a result, conventional quantum mechanics is the *only* way to describe the correlations at atomic scales.

Our theory does profoundly disagree with the so-called 'many world interpretation'. The unobserved outcomes of experiments are not realized in 'parallel universes' or anything of the sort. Every experiment has a single outcome that is true, and all other outcomes are not realized anywhere. The wave function only means something when it is used as a tool helping us to make statistical predictions. At atomic and molecular scales, it is the only tool we have; there will never be a better way to make predictions, but this does not mean that there will not be a better underlying theory.

We have no idea whether the Copenhagen interpretation, and in particular its agnostic elements, will survive the new century or not. This depends on human ingenuity which is impossible to predict. String theories and related approaches at present do not address at all the possibility of deterministic underlying equations. This does not mean that they would be wrong. It is quite conceivable that string theory is the only way to analyse our world to such detail that the underlying dynamical equations can be identified. Our paper

is a plea not to give up common sense while doing so.

8. Black holes and holography.

Dropping the requirement that information is preserved at the deterministic level, also settles an other vexing problem: the treatment of quantum mechanical black holes. The problem encountered in studying the theory for black holes is as follows.

Any sensible theory of matter and gravitation inevitably predicts that, given a sufficiently large amount of matter, gravitational collapse may occur and a black hole may form[†]. Consider now a large black hole. Its properties at moderately small scales, can be deduced unambiguously from invariance under general coordinate transformations. An elementary outcome of these considerations is that black holes emit particles of all kinds, in the form of thermal radiation. This result allows us to estimate the total number of possible quantum states of a black hole, and one finds that this number is essentially governed by the total area A of the black hole horizon¹⁰.

On the other hand, one can try to make a model of the black hole horizon, in order to attempt to describe these quantum states, in a statistical manner, in terms of local degrees of freedom residing at this horizon¹⁴. If quantum field theory is applied – *any* quantum field theory for particles in the background metric defined by the black hole – one finds the number of quantum states near the horizon to be strictly infinite. The difficulty is, that the quantum states of a field theory reside in a volume, not on a surface, and furthermore, the number of quantum states in a field theory is unlimited because of the freedom to perform unlimited Lorentz transformations at the extreme vicinity of a horizon.

It could be observed that what was needed is a ‘holographic principle’⁴. This principle states that the number of quantum states of the quantum field theory describing our world should not at all be as large as in conventional, non-gravitational systems; this number should, in fact, be bounded by an expression involving the total area of the boundary. This situation resembles what one gets if a holographic picture is taken of a scene in three spacelike dimensions, using a two-dimensional photographic plate. We give the photographic plate a resolution limited by one pixel per Planck length squared, approximately. This causes a slight blurring of our three-dimensional view, but, since it is Planckian dimensions that are involved, such a blurring is unobservable in ordinary physics. However, it appeared to be extremely difficult to construct a theory with ‘holography’ from first principles.

At this point, string theory and D -brane theory appear to come to the rescue¹⁵: beautiful studies provide for descriptions of black holes where, indeed, the quantum states are identified, counted, and their number is found to depend on the horizon area in a way that was expected from Hawking radiation. There appears to be just a small price to be paid. These theories do not tell us exactly how to handle the space-time transformations that relate the behaviour at a horizon to theories in the nearby volume. There are con-

[†] To see this, it is sufficient to study the Chandrasekhar limit.

jectures that describe the nature of these relations, but the physical implications of these conjectures are difficult to grasp.

String theory now asserts that a theory in $3 + 1$ dimensions must be equivalent to a conformal theory in lower dimensions³; this has to be the case if black holes are to be adequately described by these theories. However, it does raise all sorts of questions. In the real physical world, the number of space-time dimensions can be determined ‘experimentally’, and the outcome of such experiments should be either $3 + 1$ or $2 + 1$ or something else, but not two or more conflicting answers, except in the uninteresting case where inhabitants of this world cannot do their experiments because there are no usable inter particle interactions, or because the interactions in their world are severely non-local (Note that we are not referring to Kaluza-Klein compactification at this point, which of course would be an acceptable way to link theories with different dimensionalities). How can we get around these problems?

The theory in this paper gives a way out that is quite acceptable from a physical point of view. In our theory, quantum states are not the primary degrees of freedom. The primary degrees of freedom are deterministic states. Since, at a local level, information in these states is not preserved, the states combine into equivalence classes. By construction then, the information that distinguishes the different equivalence classes is absolutely preserved. Quantum states are equivalence classes, but in order to identify equivalence classes, the evolution of a system must be followed for a certain length of time, and this turns the definition of an equivalence class into a non-local one.

Black holes are nothing but extreme situations where information gets lost. Their equivalence classes comprise large sets of states that do look quite different for a local, ‘infalling’ observer, and this is why a black hole contains much fewer quantum states than the world seen by someone going in. But, as black holes are now truly large scale, composite objects, they cease to present elementary problems; they will take care of themselves in a natural manner; what remains to be done is the determination of the microscopic laws.

Just as all other structures in our theory, black holes will have to be described in terms of equivalence classes of states. States that have a different past, but identical future, will be joined in a single equivalence class. By construction, the evolution of these equivalence classes will be unitary, so the emerging description of black hole evolution will be as in standard quantum mechanics, but the exact formulation of the Rindler space transformation can only be given after the set of fundamental, primordial states for the vacuum fluctuations have been identified. The so-called ‘holographic principle’ will then turn out to be a feature of the effective quantum mechanical description of black holes, but is no longer needed for the description of the fundamental (deterministic) degrees of freedom of the world. What the holographic principle tells us is, that the number of equivalence classes of the deterministic theory will grow proportionally to the area of a black hole.

The fact that the number of equivalence classes depends only on the surface of the boundary may seem to be something quite natural, At the boundary, information can pour in and out. If we would keep the boundary fixed (including the vacuum fluctuations there), the finite system at the inside may eventually lose *all* of its information and turn into a

single Poincaré cycle (or into one of a small set of options). At closer inspection, however, this argument turns out to be insufficient. More investigation is needed for the mechanism that reduces the number of classes to an expression depending only on the area.

9. Conclusions and remarks.

In spite of the failure of macroscopic hidden variable theories, it may still be possible that the quantum mechanical nature of the phenomenological laws of nature at the atomic scale can be attributed to an underlying law that is deterministic at the Planck scale but with chaotic effects at all larger scales. In this picture, what is presently perceived as a wave function must be regarded as a mathematical device for computing probabilities for correlations in the chaos. This wave function does retain its usual Copenhagen interpretation, but identifying quantum states at the Planck scale will be impeded by the phenomenon of information loss at that scale. Due to information loss, Planck scale degrees of freedom must be combined into equivalence classes, and it is these classes that will form a special basis for Hilbert space, which we refer to as the ‘primordial basis’.

These considerations are of special importance for the description of black holes. The general coordinate transformation that underlies the definition of Rindler space, maps local degrees of freedom into local degrees of freedom. However, the fact that all information that disappeared into black holes must be considered as being lost, implies that the Rindler space transformation does not transform equivalence classes into equivalence classes, and therefore, this transformation is not a transformation of quantum states into quantum states.

Let us stress again that information loss in black holes only occurs at the classical level. Since, according to our philosophy, quantum states are identified with equivalence classes, quantum information is preserved, by construction. In our theory, however, we reestablish the fundamental nature of the *classical* states, and deprive the quantum states of their fundamental status of primary degrees of freedom. This way, the black hole information paradox may be resolved. The well-known black hole entropy formula tell us that the number of equivalence states for a black hole will grow as the exponent of the area in Planck units.

It is of interest to observe that, in constructing models with a deterministic interpretation for quantum states, the restriction to $1 + 1$ dimensions is usually quite helpful. This is a reason to suspect that a deterministic interpretation of string theory is possible. In Appendix A, a construction is shown. Here, we succeeded in producing a model in $3 + 1$ dimensions, but its ultraviolet cut-off is fairly artificial. In $1 + 1$ dimensions, the cut-off is straightforward.

Appendix A. Massless neutrinos are deterministic.

There is one system, actually realised to some extent in the real world, for which a *primordial basis* can be constructed. A primordial basis is a complete set of basis elements of Hilbert space that is such that any operator that happens to be diagonal now, will

continue to be diagonal in the future. Only if there is no information loss, the evolution of these elements is determined by local equations. The model constructed in this Appendix is first constructed in such a way that no information loss appears to occur, but it also appears to be not quite local. Then we restore locality (at the cost of a violation of Lorentz invariance) by introducing information loss (whether Lorentz invariance has to be broken in the real world, remains to be seen).

Consider massless, non-interacting chiral fermions in four space-time dimensions. We can think of neutrinos, although of course real neutrinos deviate slightly from the ideal model described here.

First, take the first-quantized theory. The hamiltonian for a Dirac particle is

$$H = \vec{\alpha} \cdot \vec{p} + \beta m, \quad \{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1. \quad (A.1)$$

Taking $m = 0$, we can limit ourselves to the subspace projected out by the operator $\frac{1}{2}(1 + \gamma_5)$, at which point the Dirac matrices become two-dimensional. The Dirac equation then reads

$$H = \vec{\sigma} \cdot \vec{p}, \quad (A.2)$$

where $\sigma_{1,2,3}$ are the Pauli matrices. We now consider the basis in which the following ‘primordial observables’ are diagonal:

$$\{\hat{p}, \quad \hat{p} \cdot \vec{\sigma}, \quad \hat{p} \cdot \vec{x}\}, \quad (A.3)$$

where \hat{p} stands for $\pm \vec{p}/|p|$, with the sign such that $\hat{p}_x > 0$. We do *not* directly specify the sign of \vec{p} .

Writing $p_j = -i \frac{\partial}{\partial x_j}$, one readily checks that these three operators commute, and that they continue to do so at all times. Indeed, the first two are constants of the motion, whereas the last one evolves into

$$\hat{p} \cdot \vec{x}(t) = \hat{p} \cdot \vec{x}(0) + \hat{p} \cdot \vec{\sigma} t. \quad (A.4)$$

The fact that these operators are complete is also easy to verify: in momentum space, \hat{p} determines the orientation; let us take this to be the z direction. Then, in momentum space, the absolute value of p , as well as its sign, are identified with its z -component, and it is governed by the operator $i\partial/\partial p_z = x_z = \hat{p} \cdot \vec{x}$. The spin is defined in the z -direction by $\hat{p} \cdot \vec{\sigma}$.

Mathematically, these equations appear to describe a *plane*, or a flat membrane, moving in orthogonal direction with the speed of light. Given the orientation (without its sign) \hat{p} , the coordinate $\hat{p} \cdot \vec{x}$ describes its distance from the origin, and the variable $\hat{p} \cdot \vec{\sigma}$ specifies in which of the two possible orthogonal directions the membrane is moving. Note that, indeed, this operator flips sign under 180° rotations, as it is required for a spin $\frac{1}{2}$ representation. This, one could argue, is what a neutrino really is: a flat membrane moving in the orthogonal direction with the speed of light. But we’ll return to that later: the theory can be further improved.

We do note, of course, that in the description of a single neutrino, the Hamiltonian is not bounded from below, as one would require. In this very special model, there is a remedy to this, and it is precisely Dirac's second quantization procedure. We consider a space with an infinite number of these membranes, running in all of the infinitely many possible directions $\hat{p} \cdot \vec{\sigma}$. In order to get the situation under control, we introduce a temporary cut-off: in each of the infinitely many possible directions \hat{p} , we assume that the membranes sit in a discrete lattice of possible positions. The lattice length a may be as small as we please. Furthermore, consider a box with length L , being as large as we please. The first-quantized neutrino then has a finite number of energy levels, between $-\pi/a$ and $+\pi/a$. The state we call 'vacuum state', has all negative energy levels filled and all positive energy levels empty. All excited states now have positive energy. Since the Dirac particles do not interact, their numbers are exactly conserved, and the collection of all observables (A.3) for all Dirac particles still correspond to mutually commuting operators.

In this very special model we thus succeed in producing a complete set of primordial observables, *i.e.*, operators that commute with one another at all times, whereas the hamiltonian is bounded from below. We consider this to be an existence proof, but it would be more satisfying if we could have produced a less trivial model. Unfortunately, our representation of neutrinos as infinite, strictly flat membranes, appears to be impossible to generalise so as to introduce mass terms and/or interactions. Also, the flat membranes appear to be irreconcilable with space-time curvature in a gravity theory. Quite likely, one has to introduce information loss. Suppose we may drop Lorentz invariance for the deterministic underlying theory[‡]. We then may add as physical variables also transverse coordinates \tilde{x} , orthogonal to \hat{p} . Particles are now described in terms of all three space coordinates \vec{x} , and a direction operator \hat{p} (reabsorbing $\hat{p} \cdot \vec{\sigma}$ to indicate its sign). In the direction of \hat{p} , the propagation is rigid. But in the orthogonal direction, the propagation is haphazard, such that information concerning the initial value of \tilde{x} is lost. All states with the same \hat{p} and $\hat{p} \cdot \vec{x}$, but with different \tilde{x} , will have to be put in the same equivalence class. Thus, it is the equivalence classes that form flat membranes, while the deterministic theory may be strictly local.

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[‡] At a later stage, this could lead to a tiny, in principle detectable, violation of Lorentz invariance for the quantum system.

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