QUANTUM GRAVITY AND BLACK HOLES

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ABSTRACT

At energies beyond the Planck mass gravitational interactions become fundamentally non-perturbative. The fact that the gravitational field of an ultra-energetic massless particle can be given in a simple closed form and vanishes nearly everywhere can be used to shed some light on the problem of "quantum gravity". It is argued that black holes must play an essential role in any successful theory for quantum gravity. A detailed introduction is given to the phenomenon of Hawking radiation but its derivation has shortcomings. We suggest a theory in which the set of states that build up the Hilbert space of black holes is entirely generated by the geometry of its horizon.

1. INTRODUCTION

The question how to reconcile the notion of General Relativity with that of Quantum Mechanics has bewildered physicists for more than half a century. And in spite of the recent claim that "Superstring theories" bear the promise of a complete unification of all interactions, including gravity, much of this problem is still shrouded in mystery. In these lectures we intend to show that there are not only mysteries but also *paradoxes*: questions to which known laws of physics seem to give conflicting answers. The apparent phenomenon of *Hawking radiation* [1] by a black hole provides us with such a paradox: is there a Schrodinger equation for phenomena at the Planck length scale or not? How should one catagorize states in Hilbert space?

In our first lecture we show that some questions *can* be answered by application of known laws of physics. We find that the scattering amplitude for particles can be computed exactly in the limit where the Mandelstam variable t stays much smaller than the Planck mass squared but s is in the order of the Planck mass squared or larger. The outcome shows some resemblance to the well-known Veneziano amplitude, but is in many important ways different. Indeed, at large s and small t, string theories cannot even be approximately right.

When s becomes very much larger than the Planck mass squared we expect black hole formation. It is here that the real difficulties arise. We quickly review the black hole solution to Einstein's equations in the next lecture, after which we show a detailed derivation of the phenomenon of Hawking radiation. As stated earlier, the result seems to defy common sense, because it suggests that purely quantum mechanical wave functions spontaneously turn into probabilistic mixtures of different states. Differently from other researchers in this field [1,2] we observe that

(i) such an outcome would suggest a probabilistic mixture of different Hamiltonians, whereas only one Hamiltonian can be acceptable if we want to maintain energy- and momentum conservation,

and (ii) the calculation ignored one extremely important effect: the gravitational interactions between in- and outgoing matter at the horizon of the black hole. This is a large s , small t interaction, which becomes dominant when in- and outgoing particles are separated by time scales larger than $\mathcal{O}(M \log M)$ in Planck units. This is to be compared to the life time of a black hole, which is of order M^3 .

Some speculative ideas on how to construct a black hole Hilbert space are given in the end.

2. THE GRAVITATIONAL FIELD OF A FAST, LIGHT PARTICLE

When a particle at rest is much lighter than the Planck mass, then at distance scales larger than the Planck length it generates a space-time metric very well approximated (in this chapter we take units such that G = 1), by

$$g_{\mu\nu} = \begin{bmatrix} -1+\frac{2m}{r} & & & 0 \\ & 1+\frac{2m}{r} & & \\ & & 1+\frac{2m}{r} & \\ 0 & & & 1+\frac{2m}{r} \end{bmatrix} , \tag{2.1}$$

(compared to the usual Schwarzschild metric, eqs (4.1), (4.5), the r coordinate here is shifted by an amount $2m$). In order to find the field of a fast moving particle we rewrite this in a covariant way introducing the covariant velocity u_μ :

$$u_\mu = \frac{1}{\sqrt{1-\vec{v}^2}} \begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} , \tag{2.2}$$

such that $u^2 = -1$. We can write (2.1), which holds for $\vec{v} = 0$, as

$$g_{\mu\nu} = \eta_{\mu\nu} \left[1+ \frac{2m}{r}\right] + \frac{4m}{r} u_\mu u_\nu ; \tag{2.3}$$

here r is now defined by

$$r = \sqrt{x^2 + (x.u)^2} . \tag{2.4}$$

Next, let us take the limit

$$u_\mu \to \infty \quad , \quad m \to 0 \quad , \quad mu_\mu \to p_\mu \quad \text{(fixed)} \quad , \quad p^2 \to 0 . \tag{2.5}$$

In this limit,

$$r \to |x.u| . \tag{2.6}$$

The infinitesimal line element ds becomes

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \to ds_0^2 + \frac{4m}{r} (u_\mu dx^\mu)^2 \to ds_0^2 + \frac{4(p_\mu dx)^2}{|p.x|} , \tag{2.7}$$

where ds is the flat metric. Now, using the definition (2.4) for r, we introduce two new sets of coordinates, $z_\mu^\pm$:

$$z_\mu^\pm = x_\mu \pm 2p_\mu \log r , \tag{2.8}$$

so that

$$dz_\pm^2 = dx^2 \pm 4(p_\mu dx^\mu)^2/(p.x) . \tag{2.9}$$

Notice that we are reproducing (2.7) apart from the absolute value signs. Apparently we have

$$\begin{aligned} &\text{at } (x.p) > 0 : \quad ds^2 \to dz_+^{\,2} , \\ &\text{at } (x.p) < 0 : \quad ds^2 \to dz_-^{\,2} . \end{aligned} \tag{2.10}$$

thus, both before and behind the plane $(x.p) = 0$ we have flat space, if the coordinates z_μ are used there. However, at the plane $(x.p) = (z_+.p) = (z_-.p) = 0$ itself these two flat spaces are glued together in such a way that

$$z^+_\mu = z^-_\mu + 2p_\mu \log \tilde{z}^2 , \qquad (2.11)$$

where $\tilde{z}_\mu$ is the transverse part of the coordinate z_μ (which is the same for z^+_μ as for z^-_μ). Because of the non-trivial $\tilde{z}$-dependence we have a δ-distributed Riemann curvature there.

As far as we are aware, this space-time was first described by Aichelburg and Sexl [3]. Our physical interpretation is that the plane $(z.p) = 0$ is a *shock wave*, carried along by the particle. Notice that multiplying the argument of the logarithm in (2.11) by a constant just corresponds to a relative shift between the coordinates z^+_μ and z^-_μ and hence does not affect the physical features of this space-time. Furthermore one can show that the effects of the first derivatives of the shift with respect to $\tilde{z}$ are also locally unobservable (they can be removed by a relative Lorentz transformation of the coordinates $z^\pm$). The locally observable amplitude of the shock wave therefore decreases as $1/\tilde{z}^2$.

The fact that energetic massless particles are surrounded by a space-time that is flat nearly but not quite everywhere is extremely important for understanding the problems in quantum gravity and some possible alleys towards their resolution.

3. GRAVITON DOMINANCE IN ULTRA-HIGH ENERGY SCATTERING [4]

Consider now two (electrically neutral) particles with rest masses $m^{(1)}$, $m^{(2)} \ll M_{\rm Planck}$. Let us first use a coordinate system in which the ingoing particle (1) is at rest or moves slowly. Let the second particle arrive from the right along the trajectory

$$\begin{bmatrix} x^{(2)} \\ y^{(2)} \end{bmatrix} \underset{\rm def}{=} \tilde{x}^{(2)} = 0 \quad ; \quad z^{(2)} = -t^{(2)} , \qquad (3.1)$$

with energy

$$\tfrac{1}{2}p_-^{(2)} = p_0^{(2)} = -p_3^{(2)} = \mathcal{O}(1/Gm^{(1)}) , \qquad (3.2)$$

where G is Newton's constant in units where $\hbar = c = 1$. Since we take $m^{(i)} \ll M_{\rm Planck}$ the velocity of particle (2) can be regarded to be that of light. The energy (3.2) is so tremendous that we can no longer ignore the gravitational field of particle (2). This field is due to the curvature found in the previous section. Two flat regions of space-time, $R^{(4)}_{(+)}$ ($t>-z$) and $R^4_{(-)}$ ($t<-z$), are glued together at the null plane $z=-t$, such that on this plane (see eq. 2.11),

$$\tilde{x}_{(+)} = \tilde{x}_{(-)} ,$$

$$z_{(+)} = z_{(-)} + 2Gp_o^{(2)}\log(\tilde{x}^2/C) , \qquad (3.3)$$

$$t_{(+)} = t_{(-)} - 2Gp_o^{(2)}\log(\tilde{x}^2/C) ,$$

where C is an irrelevant constant.

For simplicity we now take particle (1) to be spinless. In $R^4_{(-)}$ its wave function is

$$\psi^{(1)}_{(-)} = e^{i\tilde{p}^{(1)}\tilde{x} + ip_3^{(1)}z - ip^{(1)}t} =$$
$$= e^{i\tilde{p}^{(1)}\tilde{x} - ip_+^{(1)}u - ip_-^{(1)}v} , \qquad (3.4)$$

where $u = (t-z)/2$ and $v = (t+z)/2$ are lightcone coordinates.

Immediately after the shock wave went by we have the shifted wave function in $R^4_{(+)}$:

$$\psi^{(1)}_{(+)} = e^{i\tilde{p}^{(1)}\tilde{x}-ip_+^{(1)}(u+2Gp_o^{(2)}\log(\tilde{x}^2/C))} , \quad \text{at} \quad v = 0 . \qquad (3.5)$$

This we can expand in plane waves,

$$\psi^{(1)}_{(+)} = \int A(k_+,\tilde{k})\, dk_+ d^2\tilde{k}\, e^{i\tilde{k}\tilde{x} - ik_+u - ik_-v} , \qquad (3.6)$$

with

$$k_- = (\tilde{k}^2 + m^{(1)2})/k_+ . \qquad (3.7)$$

Clearly,

$$A(k_+,\tilde{k}) =$$
$$\delta(k_+-p_+^{(1)}) \frac{1}{(2\pi)^2} \int d^2\tilde{x}\, e^{i(\tilde{p}^{(1)}-\tilde{k}^{(1)})\tilde{x} - 2iGp_+^{(1)}p_o^{(2)} \log(\tilde{x}^2/C)} . \qquad (3.8)$$

The integral here is elementary:

$$\int d^2\tilde{x}\; e^{i\tilde{k}\tilde{x} - iB\log x} = \frac{\pi\Gamma(1-iB)}{\Gamma(iB)} \left[\frac{4}{\tilde{k}^2} \right]^{1-iB} . \tag{3.9}$$

In our case (see eq. (3.2)), $B = 2Gp_+^{(1)}p_o^{(2)} = -2G(p^{(1)}.p^{(2)}) = Gs$, where s is the usual Mandelstam variable. Notice furthermore that

$$dk_+ d\tilde{k} = \frac{k_+}{k_o}\, dk_3 d\tilde{k} . \tag{3.10}$$

Thus, concentrating only on particle (1,) we get

$$_{\rm out}\langle \vec{k}^{(1)} | \vec{p}^{(1)} \rangle_{\rm in} =$$

$$\frac{k_+}{4\pi k_o}\, \delta(k_+ - p_+)\, \frac{\Gamma(1-iGs)}{\Gamma(iGs)}\, C^{iGs} \left[\frac{4}{(\tilde{p}-\tilde{k}^2)} \right]^{1-iGs} . \tag{3.11}$$

No particle production or Bremsstrahlung is seen in any coordinate frame (as long as particle (1) is electrically neutral and $m^{(1)} \ll M_{\rm Planck}$), so the scattering is elastic. There is an exchange of momentum

$$q = k^{(1)} - p^{(1)} . \tag{3.12}$$

The Dirac delta in (3.11) is just energy conservation. Defining the Mandelstam variable $t = -q^2$, we find the elastic scattering amplitude to be (apart from the canonical factor $(k_+/k_o)\delta(\Sigma\, k - \Sigma\, p)$,

$$U(s,t) = \frac{\Gamma(1-iGs)}{4\pi\Gamma(iGs)}\, C^{iGs} \left[\frac{4}{-t} \right]^{1-iGs} , \tag{3.13}$$

from which the cross section follows,

$$\sigma(\tilde{p}^{(1)} \to \tilde{k}^{(1)})\, d^2\tilde{k} = \frac{4}{t^2} \left| \frac{\Gamma(1-iGs)}{\Gamma(iGs)} \right|^2 d^2k = 4G^2 \frac{s^2}{t^2}\, d^2\tilde{k} . \tag{3.14}$$

This resembles Rutherford scattering, except for the factor $s^2 \approx p_+^{(1)2}\, p_-^{(2)2}$. Such an extra energy dependence is of course to be expected from a theory of gravity. Apparently the cross section is just as if a single graviton were exchanged, but the amplitude, eq. (3.13), is more complicated. From the derivation it must be clear that the phase factor C^{iGs} is meaningless.

How exact is the amplitude (3.13)? Exchange of other massless particles would alter it. For instance, it is easily seen that electric charges $e^{(1)}$, $e^{(2)}$, just cause a shift:

$$Gs \rightarrow Gs - e^{(1)}e^{(2)}/4\pi \ , \tag{3.15}$$

in eqs (3.13) and (3.14) (this is found by considering the electric shock wave from a charged particle which is quite analogous to the gravitational shock wave). In many respects, eq. (3.13) can be seen to be an "eikonal approximation" [5]. However, we claim that other quantum field theoretic effects, for instance those due to exchange of scalar or massive particles, will be swamped by eq. (3.13) at sufficiently large Gs or small Gt. This is not only because of the obvious divergence at $t \rightarrow 0$. Take a particle (1) with a definite impact parameter b with respect to particle (2). If b is large we only have effects from the graviton (and other massless particles). But if b is small, we have the divergence of the logarithm in eqs. (2.11) and (3.3): particle (1) is being shifted along in the direction of $p_\mu^{(2)}$. Whatever it does, these effects will only be seen at much later times by any observeres in $R^4{}_{(+)}$.

Because of the above we suspect that the poles in our amplitude, which occur at

$$s = M^2_{\text{Planck}} \ (-Ni + k\alpha) \ , \tag{3.16}$$

where N is a positive integer, k is any integer, and α is the finestructure constant, may indicate the presence of new physical states. Their properties are subjects of further investigation.

Finally one may ask what happens if Gs becomes much larger than one. In that case it seems more appropriate to consider both particles (1) and (2) and their gravitational fields in the c.m. frame. The computation of the general relativistic effects when both shock waves collide is complicated however [6]. In general a spacelike singularity in space-time is expected. If the impact parameter is less than the Schwarzschild radius corresponding to the c.m. energy then one obviously expects a black hole to form, and classical gravitational waves will be emitted (in particle terms these are just coherent many-graviton modes). The corresponding computations are technically difficult, but should follow completely from the well known laws of general relativity, although we hasten to add to this that questions of a proper formulation of the initial conditions and the actual existence of solutions are far from settled mathematically [7]. In any case, non-trivial quantum field theoretical phenomena are well hidden behind the horizon.

4. THE BLACK HOLE

So at $s >> M^2_{\text{Planck}}$ we expect black hole formation to affect our amplitude (3.13), perhaps creating new poles (which, by estimates of the black hole life time one might expect to occur at $s/M^2_{\text{Planck}} \to A - iB$, with A large and B of order one.) At first sight this may also seem to be a doable calculation, for which only classical physics is needed. Black holes just look like solitons, and all we have to do is compute their first non-trivial quantum corrections. Unfortunately, this turns out to be impossible with our present knowledge. We would expect black holes to form a spectrum of states, extending the spectrum of all known, much lighter particles into the regime of ultra-high energies. After all, there should be no fundamental difference whatsoever between the "ordinary" particles and black holes, both carrying a gravitational field described by Einstein's equations, and both associated with De Broglie waves [8].

To see what is going on we first briefly resume the elementary mathematical aspects of black holes [10].

A spherically symmetric solution of Einstein's equation after matter has moved to the center ($T_{\mu\nu} = 0$) can be written as

$$ds^2 = - F(r)\ dt^2 + G(r)\ dr^2 + H(r)\ (d\theta^2 + \sin^2\theta\ d\varphi^2)\ , \tag{4.1}$$

but we still have the freedom to redefine r: $r \to r'$, such that after the redefinition,

$$H(r) = r^2\ . \tag{4.2}$$

The equations $R_{\mu\nu} = 0$ give three equations for F and G, but of these one is redundant because of the automatic Bianchi identity

$$2\partial_\mu R_{\mu\nu} = \partial_\nu R\ . \tag{4.3}$$

One finds successively

$$\partial_r(FG) = 0 \quad \to \quad F(r).G(r) = \text{Const.}\ , \tag{4.4}$$

which constant can be put equal to one by rescaling t, and

$$\partial_r(rF) = 1 \quad \to \quad F(r) = 1/G(r) = 1 - 2M/r\ . \tag{4.5}$$

Here, $2M$ is an arbitrary integration constantant. But one may observe that the function $F(r)$ corresponds directly to the gravitational red-shift, so that it can easily be identified as the gravitational potential, which is asymptotically

$$\sqrt{F(r)} \quad \rightarrow \quad 1 - M/r \ , \tag{4.6}$$

and one may conclude that

$$M = Gm \ , \tag{4.7}$$

where m is the black hole mass.

At the points

$$r = 2M \ , \tag{4.8}$$

this metric is singular, but this singularity is an artifact of the coordinates chosen. Consider the new time coordinate

$$\tilde{t}_+ = t + 2M \log(r-2M) \ , \tag{4.9}$$

then in the coordinates $(\tilde{t}_+, r, \theta, \varphi)$ the singularity disappears. These are the so-called "ingoing" Eddington-Finkelstein coordinates. The lines $\tilde{t}_+ + r =$ constant (at constant angles θ and φ) are the geodesics of infalling light rays. The region $0 < r < 2M$ can be reached from the outside. $\tilde{t}_+$ is real there, but t is complex. In that region (which will be called region III later), the local future light cone points entirely towards the singularity at $r = 0$.

One may also consider the "outgoing" Eddington-Finkelstein coordinates, replacing t by

$$\tilde{t}_- = t - 2M \log(r-2M) \ ; \tag{4.10}$$

in these coordinates the region $0 < r < 2M$ can be reached going backwards in time. The local future lightcone points outwards. We will call this region IV. Here also t is complex. However, if this metric is regarded as a solution of Einstein's equations of a black hole formed by collapse of matter at $t = t_1$, then region IV is unphysical: to reach it one would have to cross the region $-\infty < t < t_1$, where the black hole was not yet formed, and matter was present so that the vacuum Einstein equations were not valid there.

In spite of the fact that region IV is not present in such a "physical" black hole, it is still worthwhile to consider coordinates that show both regions III and IV. These are the so-called Kruskal coordinates (x,y,θ,φ), where x and y are defined by

$$\left[\frac{r}{2M} - 1 \right] e^{r/2M} = -xy \ , \tag{4.11}$$

$$e^{t/2M} = -x/y \ . \tag{4.12}$$

By differentiating one finds

$$4 \frac{dx\ dy}{xy} = \frac{1}{4M^2} \left[\frac{dr^2}{(1-2M/r)^2} - dt^2 \right] , \tag{4.13}$$

$$ds^2 = -2A(r)\ dx\ dy + r^2\ d\Omega^2 \ , \tag{4.14}$$

where

$$A(r) = \frac{16M^3}{r}\ e^{-r/2M} \ , \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\ d\varphi^2 \ . \tag{4.15}$$

Notice that in these coordinates the singularity at $r \to 2M$ totally disappears. The lines x = const., Ω = const., and the lines y = const., Ω = const., are light rays.

At every r,t we have two solutions for x and y (differing by a sign), so the regular region $r > 2M$ occurs twice in (x,y) space (to be called regions I and II). We indicate the regions I to IV in fig. 1.

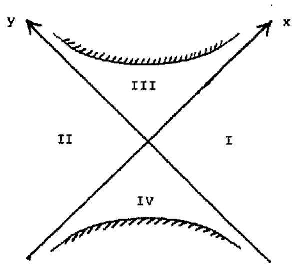

Fig. 1. The Kruskal coordinates.

The central region, $|x| \ll 1$, $|y| \ll 1$, is very important for understanding the quantum mechanics of the black hole. It is there where objects sent in in the far past, and particles that will emerge in the distant future meet each other. To describe that region, the *curvature* of space-time seems to be only of secondary importance. If we replace it by flat space we have the so-called *Rindler space*.

Consider a flat Minkowski space-time described in coordinates t, z and $\tilde{x} = (x,y)$. Let us then consider the new coordinates τ, ζ, and $\tilde{x}$, given by

$$z = \zeta \cosh \tau \ ,$$
$$t = \zeta \sinh \tau \ , \qquad (4.16)$$
$$\tilde{x} = \tilde{x} \ .$$

Of these coordinates, τ can be considered to be a *time* coordinate, because any shift of the form $\tau \to \tau + \tau_1$ is nothing but a Lorentz transformation, and hence leaves the laws of physics, as phrased in these "Rindler coordinates", invariant: the laws of physics do not change with time. A stationary observer in these coordinates has ζ = const., which is a curved trajectory in Minkowski space; hence such an observer feels a gravitational field which is constant in time. This field becomes infinitely strong at $\zeta = 0$. Clearly,. Rindler space is a model for a gravitational field. ζ and τ play the role of the Schwarzschild coordinates r and t. The Kruskal coordinates x and y correspond to the Minkowski lightcone coordinates $t \pm z$. By comparing (4.12) and (4.16) at the origin of x-y space we see that τ corresponds to $t/4M$.

5. FIELD THEORY IN RINDLER SPACE

Consider now a quantum field theory, possibly with interacting particles, in a flat Minkowski background metric. Usually its Hamiltonian, H_M, can be written as an integral over a Hamiltonian density:

$$H_M = \int \mathcal{H}(\vec{x})\ d^3\vec{x} \ . \qquad (5.1)$$

Now a time boost in Rindler space is generated by the operator

$$H_R = \int (\mathcal{H}(\vec{x})\ z - \mathcal{P}_z(\vec{x})\ t)\ d^3\vec{x} \quad , \qquad (5.2)$$

to be recognized as the generator of Lorentz transformations in Minkowski space. usually we consider it at $t = 0$,

$$H_R = \int \mathcal{H}(\vec{x}) z\ d^3\vec{x} \ . \qquad (5.3)$$

In contrast with the usual Hamiltonian (5.1) this is obviously not bounded from below. Let us write

$$H_R = H_1 - H_2 \ , \qquad (5.4)$$

with

$$H_1 = \int_{z>0} \mathcal{H}(\vec{x}) z \, d^3\vec{x} \,,$$

$$H_2 = \int_{z<0} \mathcal{H}(\vec{x}) \, |z| \, d^3\vec{x} \,. \tag{5.5}$$

We quickly find that quite generally

$$[\, H_1, H_2 \,] = 0 \,. \tag{5.6}$$

Consider namely two operators F_1 and F_2, defined by

$$F_i = \int f_i(\vec{x}) \, \mathcal{H}(x) \, d^3\vec{x} \,, \tag{5.7}$$

then

$$[F_1, F_2] \propto \int f_1(\vec{x}) \, \vec{\partial} f_2(\vec{x}) \ \ldots + \int f_2(\vec{x}) \, \vec{\partial} f_1(\vec{x}) \ \ldots \tag{5.8}$$

With $f_1 = z\theta(z)$ and $f_2 = -z\theta(-z)$ this vanishes.
Physically this result is understandable: H_1 governs the evolution within region I and H_2 the evolution in region II. No information can be transmitted between the two regions.

At first sight this situation is quite pleasing: all physics in region I is described by H_1 and H_1 alone. Unfortunately, there is a divergence. This divergence is seen more clearly if we substitute in (4.16),

$$\zeta = e^{\sigma} \,. \tag{5.9}$$

A Lagrangian of the form

$$\mathcal{L} \, d^3\vec{x} \, dt = (-1/2(\vec{\partial}\varphi)^2 + 1/2(\partial_t\varphi)^2 - \tfrac{1}{2}m^2\varphi^2) \, d^3\vec{x} \, dt \,, \tag{5.10}$$

then takes the form

$$[-1/2(\partial_\sigma\varphi)^2 + 1/2(\partial_\tau\varphi)^2 + e^{2\sigma}(-1/2(\tilde{\partial}\varphi)^2 - \tfrac{1}{2}m^2\varphi^2)] \, d^2\tilde{x} \, d\sigma \, d\tau \,. \tag{5.11}$$

At $\sigma \to -\infty$ the first two terms survive and describe plane waves. The problem is that there is an asymptotic region at $\sigma \to -\infty$ into which wave packets may disappear at $\tau \to +\infty$ or from which wave packets may come at $\tau \to -\infty$. If we would try to formulate scattering against the Rindler horizon we would need a boundary condition at $\sigma = -\infty$. We do not know what it is. *Hawking radiation* seems to be the "answer" at present: there is thermal radiation emerging from the region $\sigma = -\infty$. Let us now consider the derivation of this phenomenon.

Everything can be understood as a feature of the central region in Kruskal space, and indeed all we need is the Rindler coordinate transformation (4.16). We will only consider non-interacting scalar particles; other cases are not really different.

A scalar field Φ in Minkowski space $(\vec{r},t)$ can be written as

$$\Phi(\vec{r},t) = \int \frac{d^3\vec{k}}{\sqrt{2k_0(\vec{k})V}} \left[a(\vec{k})\, e^{i\vec{k}\vec{r} - ik_0 t} + a^\dagger(\vec{k})\, e^{-i\vec{k}\vec{r} + ik_0 t}\right] , \tag{5.12}$$

$$\dot{\Phi}(\vec{r},t) = \int \frac{d^3\vec{k}}{\sqrt{2k_0(\vec{k})v}} \left[-ik_0\, a(\vec{k})\, e^{ikx} + ik_0\, a^\dagger(\vec{k})\, e^{-ikx}\right] . \tag{5.13}$$

Here, $V = (2\pi)^3$, and we have

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3(\vec{k}-\vec{k}') , \tag{5.14}$$

and

$$[\dot{\Phi}(\vec{r}), \Phi(\vec{r}')] = -i\, \delta^3(\vec{r}-\vec{r}') , \tag{5.15}$$

etc.

First we make the transition to lightcone coordinates,

$$u = (t-z)/2 , \quad v = (t+z)/2 ,$$
$$k_+ = k_0 + k_3 , \quad k_- = k_0 - k_3 . \tag{5.16}$$

Then in Rindler time these evolve as

$$v \to v\, e^{\tau} ,$$
$$u \to u\, e^{-\tau} . \tag{5.17}$$

And we define new annihilation operators a_1:

$$a(\vec{k})\ \sqrt{k_o} = a_1(\tilde{k},k_+)\ \sqrt{k_+}\ , \tag{5.18}$$

which, because

$$\frac{\partial k_+}{\partial k_3}\ \Big|_{\tilde{k}} = \frac{k_+}{k_o}\ , \tag{5.19}$$

are now normalized by

$$\left[a_1(\tilde{k},k_+),a_1^\dagger(\tilde{k}'.k')\right] = \delta^2(\tilde{k}-\tilde{k}')\ \delta(k_+-k')\ . \tag{5.20}$$

So we can write

$$\Phi(\vec{r},t) = A(\vec{r},t) + A^\dagger(\vec{r},t)\ ; \tag{5.21}$$

$$A(\tilde{r},u,v) = \int_{k_+>0} \frac{d\tilde{k}\ dk_+}{\sqrt{2Vk_+}}\ a_1(\tilde{k},k_+)\ e^{i\tilde{k}\tilde{r}\ -\ ik_+u\ -\ ik_-v}\ . \tag{5.22}$$

Now in Rindler time, $\tilde{k}$ is constant and

$$\begin{aligned} k_+ &\to k_+\ e^{\tau}\ , \\ k_- &\to k_-\ e^{-\tau} \end{aligned} \tag{5.23}$$

If we Fourier transform the field Φ of eq. (5.21) with respect to τ then we may expect to get annihilation and creation operators corresponding to definite amounts of energy for the Rindler observer. Therefore we now choose to Fourier transform a_1 with respect to $\log k_+$:

$$a_1(\tilde{k},k_+)\sqrt{k_+} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega\ a_2(\tilde{k}.\omega)\ e^{-i\omega\ \ln(k_+/\mu)}\ , \tag{5.24}$$

where

$$\mu^2 = \tilde{k}^2 + m^2 = k_+k_-\ , \tag{5.25}$$

and the new annihilation operators a_2 are normalized as

$$[a_2(\tilde{k},\omega), a_2^\dagger(\tilde{k}',\omega')] = \delta^2(\tilde{k}-\tilde{k}')\ \delta(\omega-\omega') \ . \qquad (5.26)$$

The inverse of eq. (5.24) is

$$a_2(\tilde{k},\omega) = \int_0^\infty dk_+\ (2\pi k_+)^{-1/2}\ a_1(\tilde{k},k_+)\ e^{i\omega\ \ln(k_+/\mu)} \ . \qquad (5.27)$$

The Rindler Hamiltonian is

$$H_R = \int z\ \mathcal{H}_M(\vec{r},0)\ d^3\vec{r} \ , \qquad (5.28)$$

with

$$\mathcal{H}_M(\vec{r},t) = \tfrac{1}{2}\dot{\Phi}^2 + 1/2(\vec{\partial}\Phi)^2 + \tfrac{1}{2}m^2\Phi^2 \ . \qquad (5.29)$$

A straightforward calculation now yields:

$$H_R = \int d^2\tilde{k} \int_{-\infty}^{\infty} d\omega\ \omega\ a_2^\dagger(\tilde{k},\omega)\ a_2(\tilde{k},\omega) \ , \qquad (5.30)$$

so indeed in all respects, a_2 behaves as an annihilation operator corresponding to a Rindler energy ω.

Nevertheless, a_2 is *not* the annihilation operator we want to work with. We would like to split the operator H_R into two parts:

$$H_R = H_1 - H_2 \ , \qquad (5.31)$$

with

$$H_1 = \int z\theta(z)\ \mathcal{H}_M(\vec{r},0)\ d^3\vec{r} \ ; \quad H_2 = -\int z\theta(-z)\ \mathcal{H}_M(\vec{r},0)\ d^3\vec{r} \ . \qquad (5.32)$$

And now it is important to note that H_1 and H_2 do *not* split the integral (5.30) in the same manner.

To see what happens, let us write the field A in eq. (5.21) in terms of a_2:

$$A(\vec{r},t) = \int_{-\infty}^{\infty} d\omega \int \frac{d^2\tilde{k}}{\sqrt{4\pi V}}\ K(-\omega,\mu u,\mu v)\ e^{i\tilde{k}\tilde{r}}\ a_2(\tilde{k},\omega) \ . \qquad (5.33)$$

Here, u and v are the coordinates (5.16) and K is an integration kernel, which turns out to be

$$K(\omega,\alpha,\beta) = \int_0^\infty \frac{dx}{x}\, x^{i\omega}\, e^{-ix\alpha\ -i\beta/x} \tag{5.34}$$

We need some properties of K. For $\alpha < 0$ and $\beta > 0$ the integrand in (5.34) converges rapidly if $\mathrm{Im}(x) \geq 0$. Therefore we may rotate the integration contour by

$$x \to x\, e^{i\varphi}\ , \qquad 0 \leq \varphi \leq \pi\ . \tag{5.35}$$

Taking $\varphi = \pi$ gives us the identity

$$K(\omega,\alpha,\beta) = \int_0^\infty \frac{dx}{x}\, x^{i\omega}\, e^{-\pi\omega}\, e^{ix\alpha\ +\ i\beta/x} = e^{-\pi\omega}\, K^*(-\omega,\alpha,\beta)\ ,\ \alpha < 0\ ,\ \beta > 0\ . \tag{5.36}$$

When $\alpha > 0$ and $\beta < 0$ we have, using a similar contour shift,

$$K(\omega,\alpha,\beta) = e^{\pi\omega}\, K^*(-\omega,\alpha,\beta)\ , \qquad \alpha > 0\ ,\ \beta < 0\ . \tag{5.37}$$

We now split the integral (5.33) into two integrals for positive ω. If $\vec{r}$ is in region I we have $u < 0$ and $v > 0$. Therefore,

$$A(\vec{r},t) = \int_0^\infty d\omega \int \frac{d^2\tilde{k}}{\sqrt{4\pi V}}\, e^{i\tilde{k}\tilde{r}}\, \Big(K(-\omega,\mu u,\mu v)\ a_2(\tilde{k},\omega)\ + \\ +\ e^{-\pi\omega}\, K^*(-\omega,\mu u,\mu v)\ a_2(\tilde{k},-\omega)\Big)\ , \tag{5.38}$$

and

$$A^\dagger(\vec{r},t) = \int_0^\infty d\omega \int \frac{d^2\tilde{k}}{\sqrt{4\pi V}}\, e^{i\tilde{k}\tilde{r}}\, \Big(K^*(-\omega,\mu u,\mu v)\ a_2{}^\dagger(-\tilde{k},\omega)\ + \\ +\ e^{-\pi\omega}\, K(-\omega,\mu u,\mu v)\ a_2{}^\dagger(-\tilde{k},-\omega)\Big)\ . \tag{5.39}$$

Combining these now into the field Φ (see 5.21)), we see

$$\Phi(\vec{r},t) = \int_0^\infty d\omega\, \frac{d^2\tilde{k}}{\sqrt{4\pi V}}\, e^{i\tilde{k}\tilde{r}}\, \Big[K(-\omega,\mu u,\mu v)\big(a_2(\tilde{k},\omega) + e^{-\pi\omega}\, a_2{}^\dagger(-\tilde{k},-\omega)\big)\ + \\ K^*(-\omega,\mu u,\mu v)\big(a_2{}^\dagger(-\tilde{k},\omega) + e^{-\pi\omega}\, a_2(\tilde{k},-\omega)\big)\Big] \tag{5.40}$$

This prompts us to define an operator $a_{\mathrm{I}}(\tilde{k},\omega)$ as follows,

$$a_{I}(\tilde{k},\omega)\sqrt{1-e^{-2\pi\omega}} = a_2(\tilde{k},\omega) + e^{-\pi\omega}\, a_2{}^{\dagger}(-\tilde{k},-\omega) \ ; \tag{5.41}$$

Clearly, if $\tilde{r},t$ are in region I then $\Phi(\tilde{r},t)$ only depends on a_I and its Hermitean conjugate. Similarly, in region II we have a_{II}

$$a_{II}(\tilde{k},\omega)\sqrt{1-e^{-2\pi\omega}} = a_2(\tilde{k},-\omega) + e^{-\pi\omega}\, a_2{}^{\dagger}(-\tilde{k},\omega) \ . \tag{5.42}$$

The normalization factors are needed to get the commutation rules

$$\left[a_I(\tilde{k},\omega)\ ,\ a_I{}^{\dagger}(\tilde{k}',\omega')\right] = \delta^2(\tilde{k}-\tilde{k}')\ \delta(\omega-\omega') \ ; \tag{5.43}$$

similarly for $[a_{II}\ ,\ a_{II}{}^{\dagger}]$, and furthermore we have:

$$[a_I\ ,\ a_I] = [a_{II}\ ,\ a_{II}] = [a_I\ ,\ a_{II}] = [a_I\ ,\ a_{II}{}^{\dagger}] = 0 \ . \tag{5.44}$$

And now indeed,

$$\begin{aligned} H_1 &= \int_0^\infty d\omega \int d^2\tilde{k}\ \omega\ a_I{}^{\dagger} a_I + C \quad ; \\ H_2 &= \int_0^\infty d\omega \int d^2\tilde{k}\ \omega\ a_{II}{}^{\dagger} a_{II} + C \ , \end{aligned} \tag{5.45}$$

where C is common, irrelevant constant coming from the re-ordering process. It cancels in H_R, eq. (5.31).

From the commutation rules (5.43)-(5.44) we see that all observables in region II commute with all a_I, $a_I{}^{\dagger}$, and *vice versa*. Therefore, not the operators a_2, $a_2{}^{\dagger}$, but a_I and $a^{\dagger}{}_I$ are the proper annihilation and creation operators for Rindler observers in region I, and a_{II}, $a^{\dagger}{}_{II}$ in region II. Transformations such as (5.41) and (5.42) involving a and $a^{\dagger}$ are called "Bogolyubov transformations".

Let now $|\Omega\rangle$ be the vacuum state as defined by an observer in Minkowski space, i.e.

$$a|\Omega\rangle = a_1|\Omega\rangle = a_2|\Omega\rangle = 0 \ , \ \text{for all } \tilde{k},\omega \ . \tag{5.46}$$

It is now opportune to introduce as a basis for Hilbert space those states which at each set of values of $\pm\tilde{k}$ and ω have definite values for $n_I \underset{\text{def}}{=} a^{\dagger}{}_I(\tilde{k},\omega)\ a_I(\tilde{k},\omega)$ and for $n_{II} \underset{\text{def}}{=} a^{\dagger}{}_{II}\ a_{II}$. At each $(\pm\tilde{k},\omega)$ we label these states as $|n_I, n_{II}\rangle$. Clearly,

$$\prod_{\tilde{k},\omega} |0,0\rangle \neq |\Omega\rangle \ . \tag{5.47}$$

To express $|\Omega\rangle$ in our Rindler basis we use, from (5.41) and (5.42),

$$a_I(\tilde{k},\omega)\ |\Omega\rangle - e^{-\pi\omega}\ a_{II}{}^{\dagger}(-\tilde{k},\omega)\ |\Omega\rangle = 0 \ ; \tag{5.48}$$

$$a_{II}(\tilde{k},\omega)\ |\Omega\rangle - e^{-\pi\omega}\ a_I{}^{\dagger}(-\tilde{k},\omega)\ |\Omega. = 0 \ , \tag{5.49}$$

so that, when acting on $|\Omega\rangle$, we have

$$a^{\dagger}{}_I a_I = e^{-\pi\omega}\ a^{\dagger}{}_I a^{\dagger}{}_{II} = e^{-\pi\omega}\ a^{\dagger}{}_{II} a^{\dagger}{}_I = a^{\dagger}{}_{II} a_{II} \ . \tag{5.50}$$

Consequently, $|\Omega\rangle$ only consists of states with

$$n_I = n_{II} \ ; \tag{5.51}$$

$$|\Omega\rangle = \sum_n f_n\ |n,n\rangle \ . \tag{5.52}$$

We find f_n from (5.48):

$$\sum_n f_n \sqrt{n}\ |n-1,n\rangle = e^{-\pi\omega} \sum_n f_n \sqrt{n+1}\ |n,n+1\rangle \ ; \tag{5.53}$$

$$f_{n+1} = e^{-\pi\omega}\ f_n \ . \tag{5.54}$$

Conclusion:

$$|\Omega\rangle = \prod_{\tilde{k},\omega} \sqrt{1-e^{-\pi\omega}} \sum_{n=0}^{\infty} e^{-\pi n\omega}\ |n,n\rangle_{\pm\tilde{k},\omega} \ , \tag{5.55}$$

where the square root is a normalization factor. Notice that (5.51) implies

$$H_R|\Omega = 0 \ , \tag{5.56}$$

or: $|\Omega\rangle$ is Lorentz invariant. More surprising perhaps is that there are very many other Lorentz invariant states. These, as all elements $|n,m\rangle$ of our basis, must have divergent expectation values for their energy and momentum.

The probability that a Rindler observer in region I, while looking at the states $|\Omega\rangle$, observes n_{I} particles with energy ω and transverse momentum $\tilde{k}$ in region I is

$$P_{n_{\mathrm{I}}} = \sum_{n_{\mathrm{II}}} \langle\Omega|n_{\mathrm{I}},n_{\mathrm{II}}\rangle\langle n_{\mathrm{I}},n_{\mathrm{II}}|\Omega\rangle = |f_{n_{\mathrm{I}}}|^2 = (1-e^{-2\pi\omega})^{-1}\ e^{-2\pi n_{\mathrm{I}}\omega} \tag{5.57}$$

This we can write as

$$P_n = e^{-\beta(E-F)} \ , \tag{5.58}$$

where $E = n\omega$ is the energy, and $\beta = 1/T$ can be interpreted as a *temperature*. F is then the free energy. One concludes that the Rindler observer detects particles radiating in all directions at a temperature which is in his units of energy

$$T_{\mathrm{R}} = 1/2\pi \ . \tag{5.59}$$

Now consider the central region in the Kruskal space of a black hole. For the distant observer this is just Rindler space. Now as a matter of fact all observers must see matter here, namely the ingoing objects that produced the black hole, at a very early time. If at later times nothing more is thrown into the hole then the Rindler observer will see the *incoming* particles approach the state $|0,0\rangle$, which, as we remember, also corresponds to a highly energetic state in the local Minkowski space. But all these particles are *ingoing* particles. In the local Minkowski frame no particles are seen coming out. To describe those we need the state $|\Omega\rangle$. An observer at late times in the Kruskal region I will think he sees this state $|\Omega\rangle$. This is why we expect him to see radiation corresponding with a temperature T. Notice that when Rindler space is replaced by Kruskal space we must insert a factor $4M$ in the unit of time (see the end of sect. 4). Therefore, in natural units, the temperature of the expected radiation from a black hole is

$$T_{\mathrm{H}} = 1/8\pi M \ . \tag{5.60}$$

This is the Hawking temperature [1].

6. THE GRAVITATIONAL BACK REACTION

There is something very peculiar about this result of the previous section. It namely suggests that any observer who looks at a black hole long after the last objects have been thrown in, will see a thermodynamic mixture of quantum mechanical states, to be described by a density matrix,

$$\rho = N \prod_{\tilde{k},\omega} \sum |n_{\mathrm{I}}> e^{-\beta_{\mathrm{I}}\omega} <n_{\mathrm{I}}| \ , \tag{6.1}$$

where N is a normalization factor. After all, we must assume that the labels n_{II} corresponding to particles in region II are irrelevant to him. Only with this matrix ρ we can reproduce the probabilities (5.57). But what if we started with one pure quantum mechanical state describing imploding matter? Does then also the density matrix (6.1) evolve?

Suppose now that a black hole is simply the most compact, and in some sense the most general, object with a given total energy. Suppose that in all other respects black holes may be assumed to obey the ordinary rules of quantum mechanics. This then would imply that when the state of all particles that made up the black hole in some implosion process were completely specified, then also the state of all outgoing particles should be well determined, probably as a complicated linear superposition of many different "decay modes". In particular, it should not be a mixture of different states in a density matrix ρ unless

$$\mathrm{Tr}\ \rho^2 = \mathrm{Tr}\ \rho = 1 \ , \tag{6.2}$$

which means that it can be seen as a single pure state, if the initial state was pure as well.

Apparently, this is not what one finds when applying standard quantum field theory in the vicinity of the black hole horizon. One finds a thermal spectrum (6.1), to be normalized by

$$\mathrm{Tr}\ \rho = 1 \ , \tag{6.3}$$

so that eq. (6.2) cannot be obeyed. What is wrong in standard quantum field theory near a horizon?

First of all we note that the difference between pure states and mixed states will be more and more difficult to detect as the black hole becomes larger. The final state could be pure but so complicated that for all practical purposes it can be handled as a thermodynamical mixture. By the time a black hole is so large that observers can be sent in nobody will ever detect the difference.

For small black holes however the question of quantum mechanical purity is extremely important. It seems then that (6.1) can only be an "approximation". Is there a way to replace it by a single realtistic wave function?

What was ignored in the standard derivation was the gravitational interaction between ingoing and outgoing matter. But this interaction is crucial. Suppose a particle goes in with momentum p_1 at time $t = t_1$. After a long time, $t = t_2 >> t_1$, we look again at the black hole and observe a Hawking particle with momentum p_2 coming out. At some time t_0, roughly halfway between t_1 and t_2 the two particles must have met, that is, they were at the same distance from the horizon, one entering, the other leaving. Particles and shock waves collide at $t = t_0$. The center-of-mass energy with which this collision takes places can be easily estimated:

$$E^2_{\mathrm{c.m.}} = -s = \mathcal{O}\left[p_1 p_2 \, e^{(t_2-t_1)/4M}\right] , \tag{6.4}$$

increasing far beyond control when

$$t_2 - t_1 >> 4M . \tag{6.5}$$

Now if in (6.4) both p_1 and p_2 are large then the effect this has on the metric is so complicated that exact solutions probably do not exist. If only p_1 (or only p_2) is large then we have a δ-distributed Riemann curvature at the past horizon and, remarkably, the metric can be found analytically [10]. It corresponds to two Schwarzschild solutions, shifted with respect to each other at the past horizon by a shift δy in the Kruskal coordinate y. By imposing Einstein's equations

$$R_{\mu\nu} = 0 , \tag{6.6}$$

one finds for δy the equations

$$\frac{\partial}{\partial y}\,\delta y = 0 , \quad (\Delta_\Omega - 1)\;\delta y(\theta,\varphi) = C\,p_1\,\delta^2(\Omega-\Omega_{\mathrm{in}}) , \tag{6.7}$$

where Ω_{in} is the set of angles θ,φ at which the incoming particle entered, and C is a numerical constant.

$$\Delta_\Omega = \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2} \tag{6.8}$$

is the angular Laplacian. The Green function corresponding to eq. (6.7) can be found:

$$\delta y(\Omega_1) = C' \int_0^{\infty} \frac{\cos \frac{1}{2}\sqrt{3}\ s\ ds}{\sqrt{\cosh s - \cos(\Omega_1 - \Omega_2)}}\, p_1(\Omega_2) \ . \qquad (6.9)$$

where C' is another numerical constant. For Ω_1 close to Ω_2 this diverges as $\log(\Omega_1 - \Omega_2)$, just as in the flat space case.
Indeed, in reality collisions between the particles p_1 and p_2 with these energies do not take place, or rather, they are not seen. The ingoing particle does not see the outgoing particle but experiences a surrounding vacuum. However, as soon as we introduce a detector at $t = t_2$ that can distinguish different modes of outgoing particles, we have trouble at $t = t_o$, because the detector has split up the wave function in pieces that may contain these objects, hitting the ingoing particles with tremendous center-of-mass energies.

A black hole should be in a pure state if all particles that produced the hole in some distant past were in a pure state. We now notice that when observations are made at much later times, we are tempted to use elements of Hilbert space that are extremely singular in the past, and decomposition of the ingoing states into outgoing modes will be made extremely difficult because of this. Since the standard derivation of the Hawking effect ignores these gravitational self-interactions, we believe that the resulting density matrix ρ of eq. (6.1) cannot be trusted completely.

Consider a black hole that was formed by a collapse at $t = t_1$, and an observer at $t = t_2$ has decomposed the wave function as just described, by looking at particles emerging from the hole at various angles. If we follow these particles back in the past, we see that for a while they stick to the "past horizon", but then at $t \approx t_1$, they are released. By that time they have energies roughly described by eq. (6.4), and, coming from different directions they collide. The Schwarzschild radius corresponding to (6.4) is large, and hence a space-time singularity at $t = t_1$ is now unavoidable [11]. Thus there will be a time-reflected black hole in the past. This is sometimes called a "white hole".

Considering some Cauchy surface at $t = t_o$, we see that quantum mechanical superpositions must be allowed between states that contain different kinds of black hole singularities both in the future and the past. We also see that the distinction between "primordial" black holes and black holes that have been formed a relatively short time ago disappears, and our view upon black holes is entirely symmetric under time-reversal.

Since the arguments presented in this section are essentially independent of the assumptions mentioned earlier we believe that they provide further support to the idea that a black hole can decay entirely into "ordinary" particles. The concept of a naked remnant singularity [12] does not fit very well in this picture.

But the most difficult problem that the present discussion confronts us with is that not all configurations of particles surrounding a black hole late in its evolution should be accepted as independent, mutually orthogonal elements of Hilbert space. Most of these, in certain linear combinations, would produce the space-time singularity of a white hole in the (not very distant) past, whereas the real black hole should not have such a singularity.

How then should Hilbert space elements be labeled? The remainder of this section is an attempt.

Let us compare two states in Hilbert space. The second is the same as the first, except for one extra particle going in at $t = t_1$. That portion of the horizon $\mathcal{J}$ that corresponds to times $t < t_1$ in these two states is not in exactly the same position. The displacement can be accurately calculated. Let us write eq. (6.9) as

$$\delta u(\Omega) = f(\Omega,\Omega')\ p_{in}(\Omega')\ , \tag{6.10}$$

where p_{in} is the momentum of a particle coming in at solid angles Ω', in some suitable unitys; u is a Kruskal coordinate (the other Kruskal coordinate will be called v), but for simplicity we will soon replace u and v by Rindler coordinates.

Let now p_{out} be the momentum of an observed outgoing particle. Then naturally the displacement (6.10) will give its wave function an extra factor

$$e^{-ip_{out}(\Omega)\ \delta u(\Omega)} = e^{-ip_{out}(\Omega)\ f(\Omega,\Omega')\ p_{in}(\Omega')}\ . \tag{6.11}$$

If we suppose that the shift δu is all information we'll ever get back from the ingoing particle, then (6.11) must give the amplitude for the process. What have we found out? The functions $p_{in}(\theta,\varphi)$ describe the ingoing momenta, and $p_{out}(\theta,\varphi)$ the outgoing momenta. Suppose we can treat these as operators depending on θ and φ. The angles θ and φ are continuous; we'll quickly replace them by a dense but discrete lattice in Ω space. This will be necessary in order to make our expressions well-defined. Thus, any Dirac delta of the form $\delta(\Omega-\Omega')$ will have to be thought of as a Kronecker delta on some dense lattice.

In the usual Hilbert space of particles we expect

$$[p_{in}(\Omega)\ ,\ p_{in}(\Omega')] = 0\ , \tag{6.12}$$

and the same for the outgoing particles. The conjugated operators are $v_{in}(\Omega)$, $u_{out}(\Omega)$, with

$$[p_{in}(\Omega),\ v_{in}(\Omega')] = -i\delta(\Omega,\Omega')\ , \tag{6.13}$$

Now we see that eq. (6.11) suggests

$$u_{out}(\Omega) = -\int f(\Omega,\Omega')\ p_{in}(\Omega')\ d^2\Omega'\ , \tag{6.14}$$

$$v_{in}(\Omega) = \int f(\Omega,\Omega')\ p_{out}(\Omega')\ d^2\Omega'\ , \tag{6.15}$$

obtaining

$$\langle p_{out}(\Omega')|p_{in}(\Omega)\rangle = N\ e^{-i\int p_{out}(\Omega)\ f(\Omega,\Omega')\ p_{in}(\Omega')\ d\Omega\ d\Omega'}\ , \tag{6.16}$$

where N is a normalization factor.

Notice that (6.16) is an entirely acceptable unitary "scattering matrix". Unfortunately, the Hilbert space generated by (6.12) and (6.13), in which this matrix acts, is quite unnatural. It resembles a bit the Fock space of in- and out-particles in a mixed coordinate-momentum representation where the transverse coordinates Ω and the longitudinal momenta p_r are specified for each poarticle. What is unusual about it is that at *every* pair of values for the angles θ and φ we must have *exactly one* particle!

As physicists we might not be too much worried about this situation. After all, we already suspected that the black hole will be surrounded by particles, possibly swimming in a Dirac sea.
Suppose we had a dense lattice in Ω-space. Why not reshuffle those particles a little bit so that there is exactly one for each point in this Ω lattice? The answer is that it is not so easy to link this special Hilbert space to the space of real particles in the real world. In particular the situation far away from the black hole will be difficult to handle. Also, any such procedure will depend delicately upon the Ω cut-off procedure used.

There is another reason why the need for an Ω cut-off should not surprise us. Tiny values $\Delta\Omega$ can only be detected by particles with large transverse momentum. But these particles will not only shift the horizon as given by (6.10) but also cause shifts in the transverse direction. A difficulty here is that these shifts will produce more complicated forms of curvature in space-time. As long as we cannot handle this situation precisely we will stick to more crude *Ansätze* for a lattice cut-off.

A simple one-dimensional model for a quantum mechanically "coherent" black hole is constructed in ref. [13]. A single Dirac particle bounces back and forth against the horizon. It is found to display a discrete set of energy levels of the form

$$\omega_N \approx 2\pi N \, / \, \ln(Ng/m^2) \ , \qquad (6.17)$$

for large N, where g is a "gravitational" coupling constant.

7. CONCLUSION

Our present picture of the combined laws of quantum mechanics and general relativity is still beset by inconsistencies. In all other areas of physics we had frequently the good fortune that experimental observations eventually gave us the crucial pieces of information to sort out such problems, and invariably it would turn out that the correct answers given by nature are completely rational and logical. There is little doubt that a completely satisfactory description of quantum gravity including black holes should be possible. The question now is how to get there with at best only extremely indirect experimental evidence (such as the set of gauge groups, fermionic representations of these, and coupling constants). In this lecturer's opinion more *Gedanken*-experiments should be done, after which simply all conceivable answers should be compared. Our "scattering experiment" in section 3 is an example. We found that the result can be computed unambiguously. As for black holes, we think that some spectrum of states should exist. Perhaps it begins with the poles (3.16). Unfortunately these are not the poles given by string theory as was once hoped [11]. We remarked that the geometry of the horizon of a black hole could in principle generate its Hilbert space. This is the geometry of two-dimensional Riemann spaces, so there *could* be an important role for two-dimensional worlds in a future theory of quantum gravity.

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