QUANTUM GRAVITY

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1. Introduction

The gravitational force is by far the weakest elementary interaction between particles. It is so weak that only collective forces between large quantities of matter are observable at present, and it is elementary because it appears to obey a new symmetry principle in nature: the invariance under general coordinate transformations.

Ever since the invention of quantum mechanics and general relativity, physicists have tried to "quantize gravity"[1], and the first thing they realized is that the theory contains natural units of length (L), time (T) and mass (M). If

$$\kappa = 6.67 \cdot 10^{-11} \ m^3/kg\ sec^2$$

is the gravitational constant, then

$$L = \sqrt{\kappa\hbar/c^3} = 1.616 \cdot 10^{-35} \ m$$

$$T = \sqrt{\kappa\hbar/c^5} = 5.39 \cdot 10^{-44} \ sec$$

and $$M = \sqrt{\hbar c/\kappa} = 1.221 \cdot 10^{28} \ eV/c^2 = 2.177 \cdot 10^{-5} \ g$$

But then the theory contains a number of obstacles. First there are the conceptual difficulties: the meaning of space and time in Einstein's general relativity as arbitrary coordinates, is very different from that of space and time in quantum mechanics. The metric tensor $g_{\mu\nu}$, which used to be always fixed and flat in quantum field theory, now becomes a local dynamical variable.

Advances have been made, from different directions[2,3,4], to devise a language to formulate quantum gravity, but then the next problem arises: the theory contains essential infinities such that a field theorist would say: it is not renormalizable. This problem, discussed in detail in section 12, may be very serious. It may very well imply that there exists no well determined, logical, way to combine gravity with quantum mechanics from first principles. And then one is led to the question: should gravity be quantized at all? After all, such quantum effects would be small, too small perhaps to be ever measurable. Perhaps the truth is very different, both from quantum theory and from general relativity.

Whatever one should, or should not do, our present picture of what happens at a length scale L and a time scale T is incomplete, and we would like to improve it. We claim that it is very worthwhile to try and improve our picture step by step, as a perturbation expansion in κ. In the following it is shown how to apply the techniques of gauge field theory and gain some remarkable results.

The sections 2-4 deal with the conventional theory of general relativity, seen from the viewpoint of a gauge field theorist. In section 5 it is indicated how quantization could be carried out in principle, but in practice we need a more sophisticated formalism to ease calculations.

This formalism, the background field method[2,5], is explained in section 6-11 . In these sections we mainly discuss gauge theories, and gravity is hardly mentioned; gravity is just a special case here.

Back to gravity in section 12, where we discuss numerical results. It is shown there why only pure gravity is finite up to the one-loop corrections.

2. Gauge Transformations

The underlying principle of the theory of general relativity is invariance under general coordinate transformations,

$$x'^{\mu} = f^{\mu}(x) \quad . \tag{2.1}$$

It is sufficient to consider infinitesimal transformations,

$$x'^{\mu} = x^{\mu} + \eta^{\mu}(x) \text{ , } \eta \text{ infinitesimal.} \tag{2.2}$$

Or, in other words, a function A(x) is transformed into

$$A'(x) = A(x + \eta(x)) = A(x) + \eta^{\lambda}(x)\partial_{\lambda}A(x) \ . \tag{2.3}$$

If A does not undergo any other change, then it is called a scalar. We call the transformation (2.3) simply a gauge transformation, generated by the (infinitesimal) gauge function $\eta^\lambda(x)$, to be compared with Yang-Mills isospin transformations,generated by gauge functions $\Lambda^a(x)$.

For the derivative of A(x) we have

$$\partial_\mu A'(x) = \partial_\mu A(x) + \eta^\lambda{}_{,\mu}\partial_\lambda A(x) + \eta^\lambda \partial_\lambda \partial_\mu A(x) \ , \qquad (2.4)$$

where $\eta^\lambda{}_{,\mu}$ stands for $\partial_\mu \eta^\lambda$, the usual convention. Any object A_μ transforming the same way, i. e.

$$A'_\mu(x) = A_\mu(x) + \eta^\lambda{}_{,\mu} A_\lambda(x) + \eta^\lambda \partial_\lambda A_\mu(x) \ , \qquad (2.5)$$

will be called a covector. We shall also have contravectors $B^\mu(x)$ (note that the distinction is made by putting the index upstairs), which transform like

$$B^{\mu\prime}(x) = B^\mu(x) - \eta^\mu{}_{,\lambda} B^\lambda(x) + \eta^\lambda \partial_\lambda B^\mu(x) \ , \qquad (2.6)$$

by construction such that

$$A_\mu(x)\ B^\mu(x)$$

transforms as a scalar. Similarly, one may have tensors with an arbitrary number of upper and lower indices.

Finally, there will be density functions $\omega(x)$ that transform like

$$\omega'(x) = \omega(x) + \partial_\lambda\ [\eta^\lambda(x)\ \omega(x)] \ . \qquad (2.7)$$

They enable us to write integrals of scalars

$$\int \omega(x)\ A(x)\ d_4(x) \ ,$$

which are completely invariant under local gauge transformations (under certain boundary conditions).

For the construction of a complete gauge theory it is of importance that the gauge transformations form a group. Of course they do, and hence we have a Jacobi identity. Let u(i) be the gauge transformations generated by $\eta^\mu(i,x)$. Then if

$$[u(1),\ u(2)] = u(3) \ ,$$

then

$$\eta^\mu(3,x) = \eta^\lambda(2,x)\, \partial_\lambda \eta^\mu(1,x) - \eta^\lambda(1,x)\partial_\lambda \eta^\mu(2,x). \tag{2.8}$$

3. The Metric Tensor

In much the same way as in a gauge field theory[6)], we ask for a dynamical field that fixes the gauge of the vacuum by having a non-vanishing vacuum expectation value. (Contrary to the Yang-Mills case it seems to be impossible to construct a reasonable "symmetric" theory.) To this end we choose a two-index field, $g_{\mu\nu}(x)$, which is symmetric in its indices,

$$g_{\mu\nu} = g_{\nu\mu} \quad , \tag{3.1}$$

and its vacuum expectation value is

$$\langle g_{\mu\nu}(x) \rangle_o = \delta_{\mu\nu} \tag{3.2}$$

(our metric corresponds to a purely imaginary time coordinate).

With $g_{\mu\nu}$, or its inverse, $g^{\mu\nu}$, we can now define lengths and time-intervals at each point in space-time:

$$|\ell|^2 = g_{\mu\nu}\ell^\mu \ell^\nu \quad ,$$

and $g_{\mu\nu}$ can be used to raise or lower indices:

$$A^\mu = g^{\mu\nu}A_\nu \quad , \quad A_\nu = g_{\nu\mu}A^\mu \quad , \text{ etc.} \tag{3.3}$$

Just as in the Yang-Mills case, we can now define covariant derivatives:

$D_\mu A = \partial_\mu A$ (the derivative of a scalar transforms as a vector),

$$\begin{aligned} D_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha \, , \\ D_\mu B^\nu &= \partial_\mu B^\nu + \Gamma^\nu_{\mu\alpha} B^\alpha \, . \end{aligned} \tag{3.4}$$

The field $\Gamma^\alpha_{\mu\nu}$ is called the Christoffel symbol and is yet to be defined. First we write down how it should transform under a gauge transformation, such that the above-defined covariant derivatives be real tensors:

$$\Gamma^{\lambda\prime}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \text{(ordinary terms for 3-index tensor)} + \partial_\mu \partial_\nu \eta^\lambda . \tag{3.5}$$

We see that no harm is done by making the restriction that

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \quad , \tag{3.6}$$

because the symmetric part of Γ only is enough to make (3.4) covariant.

We now define the field Γ by requiring

$$D_\lambda g_{\mu\nu} = 0 \quad , \tag{3.7}$$

(from which follows: $D_\lambda g^{\mu\nu} = 0$). We see that we have exactly the right number of equations. By writing Eq. (3.7) in full we find that it is easy to solve

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \ . \tag{3.8}$$

Note that $\Gamma^\lambda_{\mu\nu}$ is not a covariant tensor.

The covariant derivative may be used just like ordinary derivatives when acting on a product:

$$D(XY) = (DX)Y + XDY \quad , \tag{3.9}$$

but two covariant differentiations do not necessarily commute:

$$D_\mu D_\nu A_\alpha \neq D_\nu D_\mu A_\alpha \quad . \tag{3.10}$$

Instead, we have:

$$D_\mu D_\nu A_\alpha - D_\nu D_\mu A_\alpha = R^\beta{}_{\alpha\nu\mu} A_\beta \tag{3.11}$$

with

$$R^\beta{}_{\alpha\nu\mu} = \Gamma^\beta{}_{\alpha\mu,\nu} - \Gamma^\beta{}_{\alpha\nu,\mu} + \Gamma^\beta{}_{\tau\nu}\Gamma^\tau{}_{\alpha\mu} - \Gamma^\beta{}_{\tau\mu}\Gamma^\tau{}_{\alpha\nu} \tag{3.12}$$

The comma denotes ordinary differentiation.
Since the l.h.s. of eq. (3.11) is clearly covariant and A_β is an arbitrary vector, $R^\beta{}_{\alpha\nu\mu}$ transforms as an ordinary tensor, in contrast with $\Gamma^\beta{}_{\alpha\mu}$. It is called the Riemann or curvature tensor (see the standard text books). Indices can be raised or lowered following (3.3). Without putting in any further dynamical equation, one finds the following identities,

$$\begin{aligned} &R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} = - R_{\alpha\beta\delta\gamma} \ , \\ &R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \ , \\ &R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} = 0 \ . \end{aligned} \tag{3.13}$$

The semicolon denotes covariant differentiation. Further, we define

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha} \quad , \quad R = R_{\mu\nu}\, g^{\mu\nu} \quad ,$$
$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} \quad , \tag{3.14}$$

which satisfy, according to (3.13):

$$R_{\mu\nu} = R_{\nu\mu} \quad , \qquad D_{\mu}G_{\mu\nu} = 0 \ . \tag{3.15}$$

The metric tensor also enables us to define a density function [see (2.7)],

$$\omega(x) = \sqrt{\det\,(g_{\mu\nu}(x))} \quad . \tag{3.16}$$

In the quantum theory we shall encounter a fundamental problem: instead of $g_{\mu\nu}$ we could go over to a new metric $g'_{\mu\nu}$ with, for instance,

$$g'_{\mu\nu} = f_1(R)g_{\mu\nu} + f_2(R)R_{\mu\nu} \ . \tag{3.17}$$

So in a curved space there is some arbitrariness in the choice of metric (Section 12).We bypass this problem here.

4. Dynamics

The question now is whether we can make the fields $g_{\mu\nu}$ propagate. Indeed we can, because we can construct a gauge invariant action integral

$$S = -\frac{c^2}{16\pi\kappa} \int \omega\, R\, d_4x \quad , \tag{4.1}$$

where κ is to be identified with the usual gravitational constant:

$$\kappa = 6{,}67 \cdot 10^{-11}\ m^3\ kg^{-1}\ sec^{-2} \ . \tag{4.2}$$

For simplicity we shall take the units in which

$$\frac{c^2}{16\pi\kappa} = 1 \tag{4.3}$$

At a later stage one could put κ back in the expressions to find that the expansion in numbers of closed loops will correspond to an expansion with respect to κ.

One can also add other fields in the Lagrangian, for instance

$$\mathcal{L} = \omega \{ - R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \} , \qquad (4.4)$$

where ϕ is a scalar field. We shall not repeat here the usual arguments to show that variation of the Lagrangian (4.4) really leads to the familiar gravitational interactions between masses, and to unfamiliar interactions between objects with a great velocity ("gravitational magnetism"). The equation for the gravitational field will be

$$G_{\mu\nu} = - \frac{1}{2} T_{\mu\nu} , \qquad (4.5)$$

where $T_{\mu\nu}$ is the usual energy-momentum tensor (Einstein's equation). The action (4.1) has much in common with the action

$$- \frac{1}{4} G^a_{\mu\nu} \quad G^a_{\mu\nu}$$

in Yang-Mills theories. As we shall indicate in the next section, a massless graviton with helicity $\pm$ 2 will propagate. Notice that we have been led to Einstein's theory of gravity almost automatically. It seems to be the simplest choice if we ask for a theory with invariance under general coordinate transformations.

5. Quantization

The first thing we must do is make a shift

$$g_{\mu\nu} = \delta_{\mu\nu} + A_{\mu\nu} , \qquad (5.1)$$

and consider $A_{\mu\nu}$ as the quantum fields. Here the problem mentioned in the introduction presents itself: what if we start with

$$g^{\mu\nu} = \delta^{\mu\nu} + B^{\mu\nu} \quad ? \qquad (5.2)$$

The answer is that as long as we take as our elementary field any local function of the $g_{\mu\nu}$, the obtained physical amplitudes will be the same. The transformation from one function (for example, $g_{\mu\nu}$) to the other (for example, $g^{\mu\nu}$) will be accompanied by a Jacobian, or closed loops of fictitious particles (see the Zinn-Justin lectures on gauge theories).

But the propagators of these particles are constants or pure polynomials in k, because the transformation is local. If we now turn on the dimensional regularization procedure, which has to be used in order to get gauge invariant results, then the integrals over polynomials,

$$\int d^n k \ \ \mathrm{Pol}\ (k)\ ,$$

vanish[7]. This is the reason why it makes no difference whether we start from Eq. (5.1) or Eq. (5.2). Non-local functions of $g_{\mu\nu}$ are not allowed. These non-local transformations would give rise to fictitious particles that do contribute in the cutting rules[8], and they are outlawed once unitarity has been established for the choice (5.1) or (5.2). By choosing a convenient gauge, comparable with the Coulomb gauge in QED, it is indeed not difficult to establish that the theory is unitary, in a Hilbert space with massless particles with helicity $\pm$ 2.

We can work out the bilinear part of the Lagrangian in Eq. (4.1) in terms of the fields $A_{\mu\nu}$:

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_{\alpha\beta})^2 + \tfrac{1}{8}(\partial_\mu A_{\alpha\alpha})^2 + \tfrac{1}{2}\, L_\mu^2 + \text{higher orders}, \qquad (5.3)$$

with

$$L_\mu \;=\; \tfrac{1}{2}\,\partial_\mu A_{\alpha\alpha} \;-\; \partial_\alpha A_{\mu\alpha}\ . \qquad (5.4)$$

For practical calculations it seems to be convenient to choose the gauge

$$\mathcal{L}^c \;=\; -\,\tfrac{1}{2}\,L_\mu^2 \qquad (5.5)$$

The Lagrangian in this gauge, $\mathcal{L} + \mathcal{L}^c$, has as a kinetic term

$$-\ \partial_\mu A_{\alpha\beta}\ W_{\alpha\beta|\gamma\delta}\ \ \partial_\mu A_{\gamma\delta} \qquad (5.6)$$

where W is a matrix built from δ-functions.
The propagator is then

$$\frac{1}{k^2 - i\varepsilon}\ (W^{-1})_{\alpha\beta|\gamma\delta}\ . \qquad (5.7)$$

Just in order to show the divergent character of the complications involved, we show here the Faddeev-Popov ghost[8] for this gauge, obtained by subjecting L_μ to an infinitesimal gauge transformation:

$$\mathcal{L}^{F.-P} = - \partial_\alpha \phi^*_\mu \, \partial_\alpha \phi^\mu + \phi^*_\mu \, [A_{\lambda\alpha,\alpha} \, \phi^\lambda_{\ ,\mu} +$$

$$A_{\mu\lambda} \phi^\lambda_{\ ,\alpha\alpha} + A_{\mu\lambda,\alpha} \phi^\lambda_{\ ,\alpha} + A_{\mu\alpha,\alpha\lambda} \phi^\lambda + A_{\mu\alpha,\lambda} \phi^\lambda_{\ ,\alpha}$$

$$- \tfrac{1}{2} A_{\alpha\alpha,\lambda} \phi^\lambda_{\ ,\mu} - \tfrac{1}{2} A_{\alpha\alpha,\mu\lambda} \phi^\lambda - A_{\lambda\alpha,\mu} \phi^\lambda_{\ ,\alpha}] \, . \qquad (5.8)$$

The Lagrangian (4.1), expanded in powers of $A_{\mu\nu}$, with the gauge-fixing term (5.5) and the ghost term (5.8), form a perfect quantum theory. It is, however, more complicated than necessary, because gauge invariance is given up right in the beginning by adding the bad terms (5.5) and (5.8). The background field method, discussed in the next sections, is much more elegant because gauge invariance is exploited in all stages of the calculations, thus simplifying things a lot.

6. A Prelude for the Background Field Technique: Gauge Invariant Source Insertions

The methods described in this and the following sections are not only suitable for quantum gravity, but have a very wide applicability, in particular in gauge theories, for instance for calculations of renormalization group coefficients. First, it is convenient to introduce the concept of a gauge invariant source insertion. This is an artificial term in the Lagrangian of the form

$$J(x) \, R(x) \; ,$$

where $J(x)$ is a c-number source function and $R(x)$ is some gauge invariant combination of fields, containing a linear part and quadratic or higher order corrections. Let us give some examples:

i) In a gauge theory with Higgs mechanism:

$$J_\mu \, \phi^* \cdot D_\mu \phi \quad , \qquad (6.1)$$

where ϕ is the Higgs field and $D_\mu \phi$ is its covariant derivative. We take $\partial_\mu J_\mu = 0$.

If

$$\langle \phi \rangle = F \quad ;$$

$$\phi = F + \psi \; ,$$

then (6.1) becomes

$$J_\mu \left[gF^* A_\mu F \quad + \quad g\,\psi^* A_\mu F \quad + \quad gF^* A_\mu \psi \quad + \quad \psi^* D_\mu \psi\right] \qquad (6.2)$$

The first term emits or absorbs single neutral vector particles. The other terms are higher order corrections, emitting two or three particles at once. In general one can find for all physical particles a similar gauge-invariant source that produces them predominantly and one by one.

ii) In a pure gauge theory (in momentum representation):

$$J^a_\mu(k)\; A^a_\mu(k) \;+\; g\, J^{ab}_{\mu\nu}\,(p,q)\; A^a_\mu(p)\; A^b_\nu(q) \;+\; \dots \qquad (6.3)$$

where $k_\mu J^a_\mu(k) = 0$. Integration over k, p and/or q is understood. The higher terms are determined by requiring gauge invariance under

$$A^{a'}_\mu(x) = A^a_\mu(x) + g\, f_{abc}\, \Lambda^b(x)\, A^c_\mu(x) - \partial_\mu\, \Lambda^a(x) .$$

This implies

$$ip_\mu(J^{bc}_{\mu\nu}(p,q) + J^{cb}_{\nu\mu}\,(q,p)) = J^a_\nu\,(p+q)\, f_{abc} \;, \text{ etc.} \qquad (6.4)$$

A source insertion that satisfies these conditions can be written in a closed form, in terms of an antisymmetric tensor source $J^a_{\mu\nu}(x)$:

$$J^a_{\mu\nu}(x)\quad T\left[\exp \int_{\text{path}}^{x} g\, A_\lambda(x')\, dx'^\lambda\right]_{ab}\; G^b_{\mu\nu}(x) \quad , \qquad (6.5)$$

where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f_{abc}\, A^b_\mu\, A^c_\nu$, and A_λ stands for the matrix

$$A^{ac}_\lambda = f_{abc} A^b_\lambda \quad .$$

The integral is along a path from infinity to x. The symbol T stands for time ordering along the path. Expanding (6.5) gives

$$J^a{}_{\mu\nu}(x)\; \left[G^a_{\mu\nu}(x) + g\, f_{abc} \int_{\text{path}}^{x} A^b_\lambda(x')\, dx'\; G^c_{\mu\nu}(x) + \dots\right] \qquad (6.6)$$

iii) In gravity one can do a similar thing:
a source $J^{\mu\nu}(x)$ satisfying

$$J^{\mu\nu} = J^{\nu\mu} \;; \qquad \partial_\mu\, J^{\mu\nu} = 0 \qquad (6.7)$$

can be coupled to $A_{\mu\nu}$ in eq. (5.1), and one can add higher order corrections to restore gauge invariance. The details are not very relevant for what follows.

The amplitude with which a gauge invariant source emits a single particle, obtains higher order corrections, see Fig. 1.

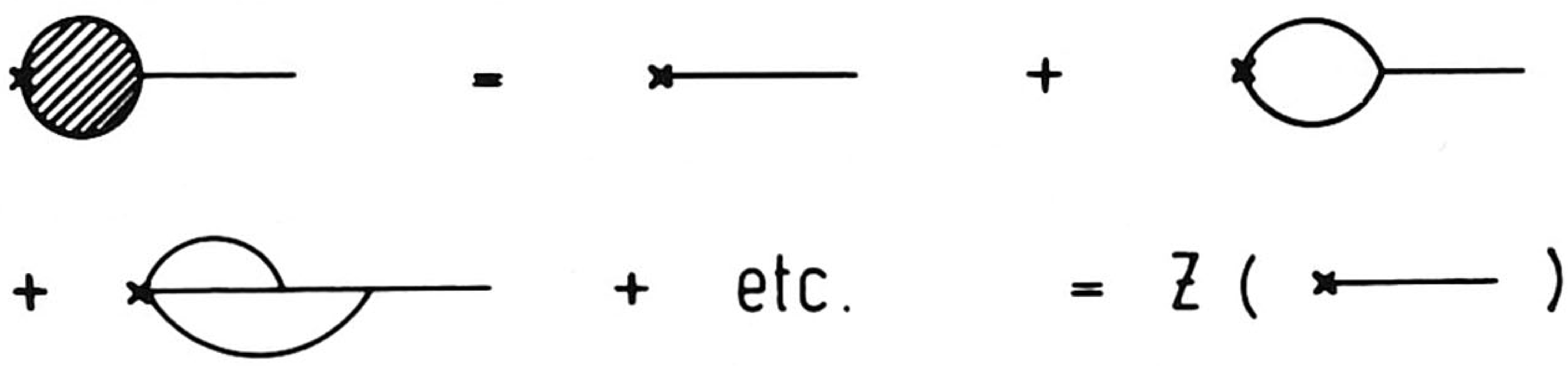

Fig. 1

A source can emit a single particle in different ways

The S-matrix can now readily be obtained in a gauge-independent way by considering vacuum-vacuum transitions in the presence of a gauge-invariant source. Then the external legs are amputated and put on mass-shell. From Fig. 1 it will be clear that in practice one can just as well calculate the amputated Green's functions directly and multiply them with some renormalization factor Z. This factor Z however may be gauge-dependent. The correct factor can be obtained by normalizing the imaginary part of the two-point function[8].

7. The Background Field Method

The background field method, useful for gauge theories, is practically indispensible for quantum gravity. Let the full Lagrangian be

$$\mathcal{L}(A) = \mathcal{L}^{inv}(A) + J\,R(A) - \tfrac{1}{2}\,C^2(A) + \mathcal{L}^{F.-P} \quad , \qquad (7.1)$$

where $\mathcal{L}^{inv}(A)$ is the complete gauge invariant Lagrangian for all fields A_i. As described in the previous section, R(A) is a gauge invariant field combination, and J is a c-number source function. C(A) is the gauge-fixing function and $\mathcal{L}^{F.-P}$ describes the associated Feynman-deWitt-Faddeev-Popov ghost[1,2,8]. All irrelevant indices have been suppressed. The gauge transformation law will be written as

$$A'_i = A_i + s^a_{ij}\,\Lambda_a\,A_j + t^a_i\,\Lambda^a \tag{7.2}$$

where $\Lambda^a(x)$ is the infinitesimal generator. t and s are coefficients built from numbers and the space-time derivative ∂_μ.

We now introduce the notion of a classical field A^{cl}, which is a function of the c-number sources J. Usually[2,5] one defines its J dependence by requiring that A^{cl} satisfies the classical equations of motion. This is sufficient as long as we are interested in diagrams with at most one closed loop. In these notes we shall make that restriction also, but it must be kept in mind that for applications of these techniques at still higher orders it will be more convenient to add quantum corrections to the equation of motion*. At this stage we require A^{cl} also to be in the gauge

$$C(A^{cl}) = 0 \ . \tag{7.3}$$

Next, we perform a shift:

$$A = A^{cl} + \phi \ , \tag{7.4}$$

where now ϕ is the new quantum field, and we rewrite the Lagrangian in terms of ϕ :

$$\mathcal{L}(A^{cl}+\phi) = \mathcal{L}_0(A^{cl},J) + \mathcal{L}_1(\phi,J,A^{cl}) + \mathcal{L}_2(\phi,J,A^{cl}) - \tfrac{1}{2}C^2(A^{cl}+\phi)$$
$$+ \mathcal{L}^{F.-P.} + \mathcal{O}(\phi^3). \tag{7.5}$$

Here $\mathcal{L}_1$ is linear in ϕ; $\mathcal{L}_2$ is quadratic in ϕ and $\mathcal{O}(\phi^3)$ is of higher order in ϕ.

Now, since A^{cl} satisfies the equation of motion

$$\frac{\delta\mathcal{L}}{\delta A^{cl}} = 0 \tag{7.6}$$

and $C(A^{cl}) = 0$, all terms linear in ϕ cancel:

$$\mathcal{L}_1(\phi) = 0 \ . \tag{7.7}$$

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* This way one can avoid the so-called Feynman baskets[1,2]. We leave the details to a possible future publication.

So the source J is not coupled to terms linear in ϕ, and there are no vertices with only one ϕ-line. Therefore, the ϕ-lines can only go around in loops, and if we confine ourselves to one-loop diagrams then we can neglect the terms $\mathcal{O}(\phi^3)$. One loop diagrams now only consist of a ϕ-loop, with bilinear insertions of classical sources depending on A^{cl} and J. But remember that A^{cl} is a function of J, which can be obtained by solving the classical equations of motion by iteration. This iteration process corresponds to adding all possible trees to the single loop. Thus we reproduce the original Feynman rules, with the only change that loop lines are called ϕ-lines, and tree lines are called A^{cl}-lines.

8. <u>The Background Field Gauge</u>

The relevant part of the Lagrangian (7.5) is

$$\mathcal{L}(\phi) = \mathcal{L}^{!}_{inv}(\phi,J,A^{cl}) - \tfrac{1}{2}C^2(A^{cl}+\phi) + \mathcal{L}^{F.-P.} \quad . \tag{8.1}$$

This is just an ordinary gauge field Lagrangian where $\mathcal{L}^{!}_{inv} = \mathcal{L}_2 + \mathcal{O}(\phi^3)$ is invariant under what we shall call gauge transformations of type Q:

$$\phi^{!}_i = \phi_i + s^a_{ij}\,\Lambda^a(A^{cl}_j + \phi_j) + t^a_i\,\Lambda^a \quad ;$$
$$A^{cl'} = A^{cl} \tag{8.2}$$

The gauge is fixed by the function $C(A^{cl}+\phi)$.

Now there is another invariance of $\mathcal{L}^{!}_{inv}$, also broken by this C term. We call this a gauge transformation of type C:

$$\phi^{!}_i = \phi_i + s^a_{ij}\,\Lambda^a\,\phi_j \quad ;$$
$$A^{cl'}_i = A^{cl}_i + s^a_{ij}\Lambda^a\,A^{cl}_j + t^a_i\,\Lambda^a \quad . \tag{8.3}$$

The power of the present formulation is that we can go over to a different gauge function $C(A^{cl},\phi)$ which breaks the Q-gauge invariance (as is necessary) but preserves gauge invariance of type C. For example, if ϕ^a_μ is the quantum part of the vector field A^a_μ, then the choice

$$C_a = D_\mu \phi^a_\mu \quad , \tag{8.4}$$

where D_μ is the covariant derivative in the classical (C) sence:

$$D_\mu \phi^a_\mu = \partial_\mu \phi^a_\mu + g\, f_{abc}\, A^{cl}_{\mu b}\, \phi^c_\mu \quad ,$$

clearly preserves C-invariance. Such a gauge is also possible in the case of gravity. If

$$g_{\mu\nu} = g^{cl}_{\mu\nu} + A_{\mu\nu} \quad ,$$

we can take

$$C_\alpha = \sqrt[4]{g^{cl}} \cdot t^{\alpha\mu} \left(\tfrac{1}{2} g^{\kappa\lambda}_{cl} D_\mu A_{\kappa\lambda} - g^{\kappa\lambda}_{cl} D_\kappa A_{\mu\lambda}\right) , \tag{8.5}$$

where

$$t^{\alpha\mu} t^{\alpha\nu} = g^{\mu\nu}_{cl} \quad .$$

Again, D_μ is the covariant derivative in the classical sence.

The one-loop (irreducible) vertex functions obtained in this gauge will be C-invariant also. It follows that they satisfy not only the Slavnov identities that describe the Q gauge symmetry, but in addition the much simpler Ward identities which are the direct generalizations of those in quantum electrodynamics.

Of course, the new Feynman rules in this background field gauge are independent of our original choice of the gauge C(A) in eq. (7.1). This implies that we can now also drop the gauge condition (7.3) for the classical fields since it can be replaced by another, arbitrary, gauge condition.

9. A Simple Example of the Background Field Gauge: Pure Yang-Mills Fields

Although we are mainly interested in quantum gravity, it is much more instructive to illustrate our methods in simpler field theories. Let us consider pure Yang-Mills fields. The invariant Lagrangian is

$$\mathcal{L}^{inv} = -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} , \tag{9.1}$$

with $\quad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f_{abc} A^b_\mu A^c_\nu \quad .$

The gauge invariance is

$$A_\mu^{a'} = A_\mu^a + g f_{abc} \Lambda^b A_\mu^c - \partial_\mu \Lambda^a . \qquad (9.2)$$

We leave aside the gauge invariance sources (sect. 6).

We shift

$$A_\mu^a = A_{\mu a}^{cl} + \phi_\mu^a ,$$

$$G_{\mu\nu}^a = G_{\mu\nu}^{a\,cl} + D_\mu \phi_\nu^a - D_\nu \phi_\mu^a + g f_{abc} \phi_\mu^b \phi_\nu^c ,$$

$$(\text{where } D = \partial + g f A^{cl}) \qquad (9.3)$$

Using

$$(D_\mu D_\nu)^{ac} - (D_\nu D_\mu)^{ac} = g f_{abc} G_{\mu\nu}^{b\,cl} , \qquad (9.4)$$

we get

$$\mathcal{L}^{inv} = \mathcal{L}^{inv}(A^{cl}) - \tfrac{1}{2} (D_\mu \phi_\nu)^2 + \tfrac{1}{2} (D_\mu \phi_\mu)^2$$

$$- g G_{\mu\nu}^{c\,cl} f_{abc} \phi_\mu^a \phi_\nu^b + \mathcal{O}(\phi^3) \qquad (9.5)$$

$$+ \text{ total derivative.}$$

A convenient background field gauge is

$$\mathcal{L}^c = -\tfrac{1}{2} C^2 = -\tfrac{1}{2} (D_\mu \phi_\mu^a)^2 . \qquad (9.6)$$

The ghost Lagrangian is then

$$\mathcal{L}^{F.-P.} = - D_\mu \psi_a^* D_\mu \psi_a , \qquad (9.7)$$

up to irrelevant interactions with ϕ_μ^a.
Of course the ghost is also C-invariant. Note that C-invariance permits us to write the interactions of ϕ_μ^a and ψ_a in a very condensed way. It is this feature that prevents overpopulation of indices in the case of gravity.

The one-loop infinities can be subtracted by a C-invariant counter term in the Lagrangian. The only candidate is

$$\alpha \cdot G^{a\,cl}_{\mu\nu} \; G^{a\,cl}_{\mu\nu} \quad .$$

The index α simultaneously governs the infinities of the two, three and four point vertices, and is therefore directly proportional to the Callan - Symanzik β-function[9]. To find this β-function one therefore only needs to investigate the two point function, contrary to the conventional formulation where also three point functions had to be calculated[10] in order to eliminate the Callan-Symanzik γ-function.

10. A Master Formula for all One-Loop Infinities

From the preceeding it will be clear that any one-loop amplitude* can be obtained from a Lagrangian bilinear in a set of fields ϕ_i. One can then rearrange the coefficients in such a way that

$$\mathcal{L} = \sqrt{g}\, \{ -\tfrac{1}{2}(\partial_\mu \phi_i + N^{ij}_\mu \phi_j) g^{\mu\nu} (\partial_\nu \phi_i + N^{ik}_\nu \phi_k) + \tfrac{1}{2}\phi_i \, X_{ij} \phi_j \} \; , \qquad (10.1)$$

where N, g and X are arbitrary functions of space time x. Further,

$$N^{ij}_\mu = - N^{ji}_\mu \; ; \qquad X_{ij} = X_{ji} \quad .$$

In dealing with gravity[3] we found it very convenient first to calculate the infinities of this general Lagrangian, in terms of N, g and X . Of course, the background metric is allowed to be curved. Afterwards one may substitute the details such as: the way N and X depend on the background fields; and the fact that the indices i, j actually stand for pairs of Lorentz-indices. In gauge theories the objects N are mostly background gauge fields and in gravity the N contain the Christoffel symbols Γ.

In ref. 3), we used dimensional regularization and considered the poles at $n \to 4$ (n is the number of space-time dimensions). From power counting arguments one easily deduces that the residues of these poles can consist only of a limited number of terms. This number is even more restricted if we use the observation that, whatever N, g or X are, there is a C-gauge symmetry:

$\mathcal{L}$ is invariant under

---

* provided a Feynman-like background field gauge is chosen.

$$\begin{aligned} \phi'(x) &= \phi(x) + \Lambda(x)\phi(x) \\ N'_\mu(x) &= N_\mu - \partial_\mu \Lambda + \Lambda N_\mu - N_\mu \Lambda \\ X'(x) &= X + \Lambda X - X\Lambda \end{aligned} \tag{10.2}$$

where Λ is an infinitesimal, antisymmetric matrix.

A straightforward calculation yields, that all one-loop infinities as $n \to 4$ are absorbed by the counter-Lagrangian[3]

$$\Delta\mathcal{L} = \frac{1}{8\pi^2(n-4)} \sqrt{g}\, \mathrm{Tr}\, \{ \frac{1}{24} Y^{\mu\nu} Y_{\mu\nu} + \frac{1}{4} X^2 - \frac{1}{12} RX + \frac{1}{120} R_{\mu\nu} R^{\mu\nu} I + \frac{1}{240} R^2 I \} \tag{10.3}$$

where

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + N_\mu N_\nu - N_\nu N_\mu \tag{10.4}$$

and

$\mathrm{Tr}\, I$ = number of fields.

$R_{\mu\nu}$ and R are the Riemann curvature tensors for the background metric.

Raising and lowering indices by use of the background $g^{\mu\nu}$ is understood.

The formula (10.3) can also be used to calculate a similar master formula in the case of Fermions[9].

11. Substituting Equations of Motion.

In some cases the resulting formula (10.3) is not the end of the story. One may still make use of the information that the background fields are not arbitrary, but satisfy equations of motion. If, for instance, one finds a contribution in $\Delta\mathcal{L}$ proportional to

$$(D_\mu A^{cl})^2 \, , \tag{11.1}$$

and the equation of motion is

$$D^2 A^{cl} = V(A^{cl}) \, , \tag{11.2}$$

then one may replace (10.1) by

$$- A^{cl} \, V(A^{cl}) \; . \tag{11.3}$$

Addition of infinitesimal terms in the Lagrangian that vanish if the equation of motion is fulfilled, corresponds exactly to making an infinitesimal field renormalization: if the equation of motion is

$$\frac{\delta \mathcal{L}}{\delta A} = 0 \quad , \tag{11.4}$$

then the terms in question must be

$$\varepsilon \cdot B \cdot \frac{\delta \mathcal{L}}{\delta A} \quad , \tag{11.5}$$

and we have

$$\mathcal{L}(A + \varepsilon B) = \mathcal{L}(A) + \varepsilon \, B \, \frac{\delta \mathcal{L}}{\delta A} \; . \tag{11.6}$$

$A' = A + \varepsilon B$ is a field renormalization.
Note that all terms in $\Delta\mathcal{L}$ are always infinitesimal, because we neglect everything that comes from two-loop diagrams.

12. Some Numerical Results. Conclusions.

The master formula (10.3) has been applied to calculate the infinity structure in different cases. For pure gravitation we used[3] the gauge (8.5) and found from the gravitons

$$\Delta \mathcal{L} = \frac{1}{\varepsilon} \sqrt{g} \, \left(\frac{7}{24} R^2 + \frac{7}{12} R_{\mu\nu} R^{\mu\nu}\right) \quad , \tag{12.1}$$

where

$$\frac{1}{\varepsilon} = 1/8\pi^2 (n-4) \; ,$$

and from the ghosts

$$\Delta \mathcal{L} = \frac{1}{\varepsilon} \sqrt{g} \, \left(- \frac{17}{60} R^2 - \frac{7}{30} R_{\mu\nu} R^{\mu\nu}\right) \tag{12.2}$$

Here $g_{\mu\nu}$ and $R_{\mu\nu}$ are the classical metric and curvature .(At this level it is never necessary to consider quantum fields in the counter terms. We are also not interested in the renormalization of the source terms). From power counting one would expect a third possible term of the form

$$\sqrt{g}\ R_{\alpha\beta\gamma\delta}\ R^{\alpha\beta\gamma\delta} \quad . \tag{12.3}$$

It indeed occurs, but we made use of the identity[2,3,11]

$$\sqrt{g}\ (R_{\alpha\beta\gamma\delta}\ R^{\alpha\beta\gamma\delta} - 4\ R_{\mu\nu}R^{\mu\nu} + R^2) = \text{total derivative} \quad , \tag{12.4}$$

which implies that terms of the form (12.3) can be eliminated.

So all together we have

$$\Delta\mathcal{L} = \frac{1}{\varepsilon}\sqrt{g}\ (\frac{1}{120}\ R^2 + \frac{7}{20}\ R_{\mu\nu}R^{\mu\nu}) \ . \tag{12.5}$$

However, we still have the equations of motion of the background field (see previous section), which is

$$R_{\mu\nu} = 0 \quad ; \qquad R = 0 \tag{12.6}$$

This means that the infinity vanishes in pure gravity. It is interesting to observe that the field renormalization that corresponds to the elimination of this infinity (see previous section) is of an unusual type:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \alpha\ R_{\mu\nu} + \beta\ R\ g_{\mu\nu} \quad . \tag{12.7}$$

This was the reason for our remark in the end of sect. 3. Note that eq. (12.4) was essential for this result.

Next we studied pure gravity with a (massless) Klein-Gordon field ϕ. The classical equations of motion are

$$\begin{aligned} D_\mu D^\mu \phi &= 0 \ ; \\ R_{\mu\nu} &= -\frac{1}{2}\ D_\mu\phi D_\nu\phi \\ R &= -\frac{1}{2}\ (D\phi)^2 \end{aligned} \tag{12.8}$$

There is one type of counterterm which is allowed by power counting and cannot be eliminated by substitution of the equations of motion:

$$\Delta\mathcal{L} = \frac{\sqrt{g}}{\varepsilon} \cdot \frac{203}{80} \cdot R^2 \tag{12.9}$$

The next thing that has been tried is pure gravity with in addition

Maxwell fields[12].
The Lagrangian is

$$\mathcal{L} = \sqrt{g}\ (-R - \frac{1}{4} F_{\mu\nu}F_{\alpha\beta}\ g^{\mu\alpha}g^{\nu\beta}).$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \qquad (12.10)$$

The equations of motion are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} T_{\mu\nu} \ ;$$

$$T_{\mu\nu} \equiv F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta}F_{\alpha\beta} \ ,$$

$$D_\alpha F^{\alpha\beta} = 0 \ ,$$

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0 \ . \qquad (12.11)$$

From which

$$R = \frac{1}{2} T^{\alpha}{}_{\alpha} = 0 \ .$$

In principle one may expect

$$\Delta\mathcal{L} = \frac{1}{\varepsilon} \sqrt{g} \left[\alpha_1 R_{\mu\nu}R^{\mu\nu} + \alpha_2 (F_{\alpha\beta}F^{\alpha\beta})^2 + \alpha_3 R_{\mu\nu\alpha\beta} F^{\mu\nu}F^{\alpha\beta} \right] \ . \qquad (12.12)$$

Explicit calculation shows however:

$$\Delta\mathcal{L} = \frac{1}{\varepsilon} \sqrt{g} \cdot \frac{137}{60} R_{\mu\nu}R^{\mu\nu}. \qquad (12.13)$$

So we see that there are some cancellations:

$$\alpha_2 = \alpha_3 = 0 \ . \qquad (12.14)$$

Indeed, they occur in a miraculous way during the calculation and have not yet been explained.
In the coupled Einstein-Yang-Mills system [13] these cancellations persist and many new cancellations occur. Starting from the obvious generalization of the Abelian Lagrangian (12.10), one finds

$$\Delta\mathcal{L} = \frac{1}{\varepsilon} \sqrt{g} \left\{ \left[\frac{137}{60} + \frac{r-1}{10} \right] R_{\mu\nu}R^{\mu\nu} + f^2 C_2 \frac{11}{12} F_{\alpha\beta}F^{\alpha\beta} \right\} \qquad (12.15)$$

where f is the gauge coupling constant,

$$C_2 \delta_{ab} = f_{apq} f_{bpq} ,$$

$$C_2 r = f_{apq} f_{apq} \qquad (12.16)$$

Five other coefficients each happen to vanish. The second term in (12.15) is of the renormalizable type. It renormalizes the gauge coupling constant and fixes the Callan-Symanzik β-function.

Finally, also the combined Einstein-Dirac system has been investigated and nonrenormalizable infinities have been found also[14].

The fact that in all these systems where matter in some form is added to pure gravity infinities of the nonrenormalizable dimension survive really means that these theories cannot be renormalized in the perturbation expansion. In the case of pure gravity the infinities have been shown to be non-physical up to the one-loop level. No calculations have been performed to investigate renormalizability in the order of two loops. The calculations of Nieuwenhuizen and Deser show that "miraculous" cancellations often occur . Perhaps this is an indication of a new sort of symmetry that we are not aware of. Investigation of this symmetry could reveal new renormalizable models with gravity.

Even so, a renormalized perturbation expansion would only be a small step forward. At very small distances the gravitational effects must be large, because of the dimension of the gravitational constant, so the expansion would break down at small distances anyhow. We have the impression that not only a better mathematical analysis is needed, but also new physics. What we learned (see eq. (12.7) and the remarks in the end of sect.3) is that in such a theory the metric tensor might not at all be such a fundamental concept. In any case, its definition is not unambiguous.

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